

On the Strong Coupling Case for Spin-Dependent Interactions in Scalar- and Vector-Pair Theories

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1. INTRODUCTION

AS a competitor to the meson theory, in which the interaction energy with the heavy particles (nucleons) is in analogy to electrodynamics assumed to be linear in the meson field, the so-called pair theory has been developed¹ in which the interaction energy of the nucleons is bilinear in the field variables and describes processes in which a pair of particles with opposite electric charges is emitted or absorbed. The field was originally assumed to be that of electrons and positrons, while later on heavier rest mass of the particles described by the field was also considered. Finally the pair theory was even generalized for particles with spin 0 (scalar field) and spin 1 (vector field). While a spin-independent interaction could be treated rigorously for arbitrary values of the coupling constants, provided that a finite shape (radius a) of the sources is introduced, the spin-dependent interactions have been treated until now only with perturbation methods (weak coupling). In the spin-dependent case, which alone is of immediate physical interest, it turns out, both for spin $\frac{1}{2}$ and for spin 0 or 1 of the field particles (mesons), that if the perturbation method (development in powers of the coupling constant) is valid and if the radius a of the nucleon is supposed to be smaller than the range of the resulting nuclear forces, the coupling constant must be so small that the nuclear interaction becomes much smaller than the empirical one.²

It seems, therefore, interesting to investigate for pair theories the strong coupling case of spin-dependent interactions, which cannot be treated rigorously. This is done in this paper for the scalar theory and the vector theory. In the first case the interaction of one nucleon at rest in the origin of the coordinate system with the meson field was assumed to be

$$H_{\text{int}} = 4\pi f \sigma i \left[\int U(x) \nabla \varphi^*(x) d\mathbf{x} \times \int U(x) \nabla \varphi(x) d\mathbf{x} \right] + 4\pi g \int U(x) \nabla \varphi^*(x) d\mathbf{x} \cdot \int U(x) \nabla \varphi(x) d\mathbf{x},$$

where $U(x)$ is the source function, $\varphi(x)$ a complex scalar field describing charged particles with spin 0, f and g coupling constants with the dimension of a volume (if units corresponding to $\hbar=c=1$ are used), and σ the spin of the nucleon. The second term was added because for strong coupling it turns out that only for $g > f$ reasonable results can be expected because, otherwise, the Hamiltonian is not positive definite, and the eigenvalues of the energy are not all positive and do not any longer have a lower bound. For $g=0$ the weak coupling case was treated by Jauch and Lopes. Their result for the interaction between two nucleons A, B at a distance $r = |\mathbf{x}_A - \mathbf{x}_B|$ can be generalized for $g \neq 0$ and can for distances r of the nucleons large in comparison with their radius a , be written in

¹ Electron-pair theory: G. Gamow and E. Teller, *Phys. Rev.* **51**, 289 (1937); G. Wentzel, *Helv. Phys. Acta* **10**, 107 (1937). Meson-pair theory (spin $\frac{1}{2}$, perturbation method): R. E. Marshak, *Phys. Rev.* **57**, 1101 (1940); R. E. Marshak and V. Weisskopf, *Phys. Rev.* **59**, 130 (1941). Exact treatment of spin-independent interaction, one-body problem: E. Wigner, C. L. Critchfield, and E. Teller, *Phys. Rev.* **56**, 530 (1939). Interaction between nucleons: C. L. Critchfield and W. E. Lamb, Jr., *Phys. Rev.* **58**, 46 (1940); C. L. Critchfield, *Phys. Rev.* **59**, 48 (1941). Exact treatment of spin-independent coupling for spin $\frac{1}{2}$ particles: J. M. Jauch, *Helv. Phys. Acta* **15**, 175 (1942); A. Houriet, *Helv. Phys. Acta* **16**, 529 (1943). Exact treatment of spin-independent interaction, scalar theory: G. Wentzel, *Zeits. f. Physik* **118**, 277 (1941); *Helv. Phys. Acta* **15**, 111 (1942) (quoted as "I"). Spin-dependent interaction, perturbation theory: (a) Spin 0 mesons: J. M. Jauch and J. Leite Lopes, *Anais da Academia Brasileira de Ciencias* **16**, 281 (1944) (quoted as "II"); (b) Spin 1 mesons: O. Klein, *Arkiv för Matematik, Astronomi och Fysik* **30A**, No. 3 (1944) (quoted as "III").

² In the case of spin-dependent interactions, the criterion for weak coupling was assumed in analogy to cases which can be treated rigorously (compare II).

the form

$$V_{AB}(r) = -\frac{\mu}{\pi r^6} \left\{ g^2 g(2\mu r) + \frac{1}{3} f^2 F(2\mu r) (\boldsymbol{\sigma}_A \cdot \boldsymbol{\sigma}_B) + \frac{1}{3} f^2 f(2\mu r) S_{AB} \right\},$$

with μ the rest mass of the mesons in units $\hbar = c = 1$ and

$$S_{AB} = (\boldsymbol{\sigma}_A \cdot \boldsymbol{\sigma}_B) - 3(\boldsymbol{\sigma}_A \cdot \mathbf{n})(\boldsymbol{\sigma}_B \cdot \mathbf{n}); \quad \mathbf{n} = (\mathbf{x}_A - \mathbf{x}_B)/r.$$

The functions $g(x)$, $F(x)$, $f(x)$ can be expressed by Henkel functions and behave as x^{-1} for small x and as $x^{7/2}e^{-x}$, $x^{5/2}e^{-x}$, $x^{5/2}e^{-x}$, respectively, for large x .³ The function $g(x)$ is given explicitly below in Section 4, Eqs. (84a) and (84b). Terms proportional to the product of the coupling constants do not occur. The case $f=0$ (spin-independent coupling) can be treated rigorously (Section 5) and gives for $r \gg a$

$$V_{AB} = -\frac{1}{(g^{-1} + A)^2} \frac{\mu}{\pi r^6} g(2\mu r),$$

where A is of the order a^{-3} and depends only on the source function of one nucleon.

The main result of this paper, derived in Section 4, is that *the resulting interaction energy for $r \gg a$ is spin independent and given by*

$$V_{AB} = -\frac{\mu}{\pi A^2 r^6} g(2\mu r),$$

if the criterion for strong coupling

$$(g+f)a^3 \gg 1 \quad \text{and} \quad (g-f)a^3 \gg 1$$

is fulfilled, the spin-dependent part of V being at most of the relative order $[(g+f)A]^{-1}$ or $[(g-f)A]^{-1}$ in comparison with the spin-independent part.

This result is not satisfactory in view of our empirical knowledge of the interaction of proton and neutron in the deuteron. The high negative power r^{-7} for small distances ($a \ll r \ll \mu^{-1}$) which makes the range of the forces not much larger than a is another argument against this theory.

The discussed result is derived with a method to split the Hamiltonian in three parts H_0 , H' , and Ω (Section 2) from which H_0 gives no interaction between nucleons for $r > a$ (Section 3),⁴ H' gives the result mentioned, and Ω can be treated as perturbation energy in the strong coupling case and gives there a contribution to the interaction energy of smaller order of magnitude.

Very analogous to this scalar-pair theory is the vector theory for which the interaction-energy with one nucleon at rest is given by

$$H_{\text{int}} = 4\pi \bar{f} \left[\int U(x) \phi^*(x) d\mathbf{x} \times \int U(x) \phi(x) d\mathbf{x} \right] + 4\pi \bar{g} \left(\int U(x) \phi^*(x) d\mathbf{x} \cdot \int U(x) \phi(x) d\mathbf{x} \right),$$

where \bar{f} and \bar{g} have the dimensions of a length, $\bar{f}\mu^{-2}$, $\bar{g}\mu^{-2}$ playing now a role similar to earlier f and g . The interaction energy between two nucleons in the weak coupling case was calculated by Klein⁵

³ In the quoted paper "II" (Jauch and Lopes) the factor $(2\pi)^{-3}$ is missing in the final result; moreover, some errors are contained in the numerical coefficients of their Eqs. (11) and (13). In our notations (the coupling constant λ_2 in II is equal to our $4\pi f$) the functions $F(x)$, $f(x)$ introduced in the expression for $V_{AB}(r)$ are given by

$$F(x) = -(20 + (7/2)x^2)K(x) - (10x + \frac{1}{2}x^3)K_0(x) \\ f(x) = -\frac{1}{2}(35 + 5x^2)K(x) - \frac{1}{4}(35x + x^3)K_0(x).$$

$K(x)$, $K_0(x)$ are Watson's functions connected with Hankel's cylinder functions (compare their definition given below in Section 4).

⁴ For $r < a$ the contribution of H_0 to the interaction energy is not zero and proportional to the square root of the coupling constants. The analogy to this " H_0 -problem" for mesons with spin $\frac{1}{2}$ was discussed by J. R. Oppenheimer and E. Nelson, Phys. Rev. A61, 202 (1942). (For spin $\frac{1}{2}$ mesons the corresponding result for $r < a$ is linear in the coupling constant.) For $r \gg a$ where the contribution to the interaction energy of H_0 vanishes, these authors, however, do not consider the interaction-energy which springs from the part of the Hamiltonian analogous to our H' , which is independent of the coupling constant.

⁵ Compare note (1), paper "III." In Klein's notation $\gamma e^2/\mu c^2$ is identical with our $4\pi \bar{f}$.

for $\bar{g}=0$ and can, analogous to the scalar theory, be written in the form

$$V_{AB} = -\frac{1}{\pi\mu^3 r^6} \{ \bar{g}^2 \bar{g}(2\mu r) + \frac{1}{3} \bar{f}^2 \bar{F}(2\mu r) (\boldsymbol{\sigma}_A \cdot \boldsymbol{\sigma}_B) + \frac{1}{3} \bar{f}^2 \bar{f}(2\mu r) S_{AB} \},$$

where $\bar{g}(x)$, $\bar{F}(x)$, $\bar{f}(x)$ are again of the order x^{-1} for small x and of the order $x^{7/2}e^{-2}$ for large x . For the spin-independent interaction given by $\bar{f}=0$ the problem can also here be solved rigorously again leading to the result, valid for $r \gg a$

$$V_{AB} = -\frac{1}{(\mu^2/\bar{g} + \bar{A})^2} \frac{\mu}{\pi} \bar{g}(2\mu r),$$

where \bar{A} is a quantity similar to A and also of the order of magnitude and dimension of a^{-3} .

In the strong coupling case, the criteria of which are here the conditions

$$(\bar{g} + f)\mu^{-2}\bar{A} \gg 1 \quad \text{and} \quad (\bar{g} - \bar{f})\mu^{-2}\bar{A} \gg 1,$$

the interaction-energy is here, too, spin-independent as in the scalar theory and given by

$$V_{AB} = -\frac{\mu}{\pi\bar{A}^2} \bar{g}(2\mu r),$$

as is derived in Section 6. All objections against the scalar theory also hold against the vector theory. The condition $\bar{g} > \bar{f}$ is necessary to make the Hamiltonian positive definite and the eigenvalues of the energy discrete and positive.

The analogous treatment of the strong coupling case of a spin-dependent interaction in the pair theory for mesons with spin $\frac{1}{2}$ will be given in a thesis by J. M. Blatt. This case is simpler than the case of mesons with integer spin because, owing to the exclusion principle for the mesons with spin $\frac{1}{2}$, there is no necessity to make the Hamiltonian positive, and a single coupling constant is sufficient also for strong coupling. Moreover, the problem analogous to our H_0 can be treated rigorously, and the magnetic moment of a nucleon according to this theory can be discussed.

2. SCALAR-PAIR THEORY. SPLITTING OF THE FIELD

As the simplest spin-dependent interaction of pairs of scalar (or pseudoscalar) mesons with nucleons, we consider the Hamiltonian

$$H = \int [\pi\pi^* + \nabla\varphi^*\nabla\varphi + \mu^2\varphi^*\varphi] d\mathbf{x} + 4\pi f \sum_A \boldsymbol{\sigma}_A \cdot i \left[\int U(\mathbf{x} - \mathbf{z}_A) \nabla\varphi^*(\mathbf{x}) d\mathbf{x} \times \int U(\mathbf{x} - \mathbf{z}_A) \nabla\varphi(\mathbf{x}) d\mathbf{x} \right] + 4\pi g \sum_A \left(\int U(\mathbf{x} - \mathbf{z}_A) \nabla\varphi^*(\mathbf{x}) d\mathbf{x} \cdot \int U(\mathbf{x} - \mathbf{z}_A) \nabla\varphi(\mathbf{x}) d\mathbf{x} \right). \quad (1)$$

Here $\varphi(\mathbf{x})$ is the complex scalar field describing positively and negatively charged particles with restmass μ , its canonical conjugate satisfying the commutation rule,

$$i[\pi(\mathbf{x}), \psi(\mathbf{x}')] = i[\pi^*(\mathbf{x}), \psi^*(\mathbf{x}')] = \delta(\mathbf{x} - \mathbf{x}'). \quad (2)$$

Capital Roman indices denote the different nucleons, the motion of which is neglected, \mathbf{z}_A the coordinates of their centers, $\boldsymbol{\sigma}_A$ their spins, $U(\mathbf{x})$ the source function, supposed to be spherical symmetrical and normalized according to

$$\int U(\mathbf{x}) d\mathbf{x} = 1. \quad (3)$$

The reason for the assumption of the second spin-independent interaction term (proportional to g) will be explained later. The factors 4π are only conventional to facilitate the comparison of later

results with those of other theories. We are using the units $\hbar=c=1$ in which the dimension of H is cm^{-1} , hence:

$$[\pi] = \text{cm}^{-2}, \quad [\psi] = \text{cm}^{-1}, \quad [f] = [g] = \text{cm}^{-3}. \quad (4)$$

The electric charge of the nucleon never changes in the pair theories; therefore, the total charge of the meson field,

$$e = i \int (\varphi^* \pi^* - \varphi \pi) d\mathbf{x}, \quad (5)$$

is conserved. With two real fields φ_1, φ_2 instead of one complex field defined in the well-known way by

$$\pi = \frac{1}{\sqrt{2}}(\pi_1 + i\pi_2), \quad \varphi = \frac{1}{\sqrt{2}}(\varphi_1 - i\varphi_2), \quad (6)$$

$$\pi^* = \frac{1}{\sqrt{2}}(\pi_1 - i\pi_2), \quad \varphi^* = \frac{1}{\sqrt{2}}(\varphi_1 + i\varphi_2), \quad (7)$$

the Hamiltonian acquires the form

$$H = \frac{1}{2} \sum_{\alpha=1,2} \int [\pi_\alpha^2 + (\nabla \varphi_\alpha)^2 + \mu^2 \varphi_\alpha^2] d\mathbf{x} + 4\pi f \sum_A \sigma_A \left[\int U(x-z_A) \nabla \varphi_1 d\mathbf{x} \times \int U(x-z_A) \nabla \varphi_2 d\mathbf{x} \right] + 4\pi g \cdot \frac{1}{2} \sum_{A,\alpha} \left(\int U(x-z_A) \nabla \varphi_\alpha d\mathbf{x} \right)^2, \quad (8)$$

with the charge of the meson field

$$e = \int (\varphi_1 \pi_2 - \varphi_2 \pi_1) d\mathbf{x}. \quad (9)$$

It will be convenient to introduce the momentum space instead of the \mathbf{x} -space, which is done in the usual way by

$$\varphi_\alpha(x) = (2\pi)^{-\frac{3}{2}} \int q_\alpha(k) \exp(i\mathbf{k}\mathbf{x}) d\mathbf{k}, \quad \pi_\alpha(x) = (2\pi)^{-\frac{3}{2}} \int p_\alpha(k) \exp(-i\mathbf{k}\mathbf{x}) d\mathbf{k}, \quad (10a)$$

$$q_\alpha(k) = (2\pi)^{-\frac{3}{2}} \int \varphi_\alpha(x) \exp(-i\mathbf{k}\mathbf{x}) d\mathbf{x}, \quad p_\alpha(k) = (2\pi)^{-\frac{3}{2}} \int \pi_\alpha(x) \exp(i\mathbf{k}\mathbf{x}) d\mathbf{x}, \quad (10b)$$

$$U(x) = (2\pi)^{-3} \int v(k) \exp(i\mathbf{k}\mathbf{x}) d\mathbf{k}, \quad v(k) = \int U(x) \exp(-i\mathbf{k}\mathbf{x}) d\mathbf{x}. \quad (11)$$

$$v(0) = 1, \quad (12)$$

$$q_\alpha^*(k) = q_\alpha(-k), \quad p_\alpha^*(k) = p_\alpha(-k), \quad v^*(k) = v(-k); \quad (13)$$

$$i[p_\alpha(k), q_\beta(k')] = \delta_{\alpha\beta} \delta(\mathbf{k} - \mathbf{k}'), \quad (14)$$

and with the usual abbreviation $k_0^2 = k^2 + \mu^2$,

$$H = \frac{1}{2} \sum_\alpha \int [p_\alpha(k) p_\alpha(-k) + k_0^2 q_\alpha(k) q_\alpha(-k)] d\mathbf{k} + f \sum_A \sigma_A (2\pi^2)^{-1} \left[\int v(-k) i\mathbf{k} \exp(i\mathbf{k}\mathbf{z}_A) q_1(k) d\mathbf{k} \times \int v(k) i\mathbf{k} \exp(i\mathbf{k}\mathbf{z}_A) q_2(k) d\mathbf{k} \right] + \frac{1}{2} g \sum_A (2\pi^2)^{-1} \left(\int v(-k) i\mathbf{k} \exp(i\mathbf{k}\mathbf{z}_A) q_\alpha(k) d\mathbf{k} \right)^2, \quad (15)$$

$$e = \int [q_1(k) p_2(k) - q_2(k) p_1(k)] d\mathbf{k}. \quad (16)$$

For the following considerations we split the field into a "zero field" which alone appears in the interaction energy and a residual field $p_{\alpha}'(k)$, $q_{\alpha}'(k)$. The latter can be chosen in such a way that in the Hamiltonian, cross terms between the zero field and the residual field only appear for $q_{\alpha}(k)$ but not for $p_{\alpha}(k)$. This is fulfilled if we put

$$\int v(-k)\mathbf{k} \exp(i\mathbf{kz}_A)q_{\alpha}'(k)d\mathbf{k}=0, \quad (17)$$

$$\int v(k)\mathbf{k} \exp(-i\mathbf{kz}_A)p_{\alpha}'(k)d\mathbf{k}=0, \quad (17a)$$

$$p_{\alpha}(k)=\frac{1}{\pi\sqrt{2}}i\sum_A v(-k)\exp(i\mathbf{kz}_A)(\mathbf{k}\cdot\mathbf{P}_{\alpha A})+p_{\alpha}'(k), \quad (18)$$

$$\mathbf{Q}_{\alpha A}=\frac{1}{\pi\sqrt{2}}i\int v(-k)\mathbf{k} \exp(i\mathbf{kz}_A)q_{\alpha}(k)d\mathbf{k}. \quad (19)$$

From (17a) and (18) follows, with the abbreviation

$$G(k)=v(k)v(-k)=|v(k)|^2, \quad (20)$$

$$N_{Ai,Bj}=N_{Bj,Ai}=\frac{1}{2\pi^2}\int G(k)k_i k_j \exp(i\mathbf{k}(\mathbf{z}_A-\mathbf{z}_B))d\mathbf{k}=4\pi\int\frac{\partial U(x-z_A)}{\partial x^i}\frac{\partial U(x-z_B)}{\partial x^j}d\mathbf{x}, \quad (21)$$

$$\frac{-i}{\pi\sqrt{2}}\int p_{\alpha}(k)v(k)\exp(-i\mathbf{kz}_B)k_j d\mathbf{k}=\sum_{A,i}P_{\alpha A,i}N_{Ai,Bj}.$$

If M is the reciprocal matrix to N defined by

$$\sum_{C,l}N_{Ai,Cl}M_{Cl,Bj}=\delta_{AB}\delta_{ij}, \quad (22)$$

one has therefore

$$q_{\alpha}(k)=\frac{-i}{\pi\sqrt{2}}\sum_{A,B,i,j}v(k)k_i \exp(-i\mathbf{kz}_A)M_{Ai,Bj}Q_{Bj}+q_{\alpha}'(k), \quad (18a)$$

$$P_{\alpha A,i}=\frac{-i}{\pi\sqrt{2}}\left(\int p_{\alpha}(k)v(k)\exp(-i\mathbf{kz}_B)k_j d\mathbf{k}\right)M_{Bj,Ai}. \quad (19a)$$

The commutation rules are

$$i[P_{\alpha A,i},Q_{\beta B,j}]=\delta_{\alpha\beta}\delta_{AB}\delta_{ij},$$

$p_{\alpha}'(k)$, $q_{\alpha}'(k)$ commute with $P_{\alpha A,i}$ and $Q_{\alpha A,i}$; and in accordance with (18), (18a),

$$i[p_{\alpha}'(k),q_{\beta}'(k')]=\delta_{\alpha\beta}\left[\delta(\mathbf{k}-\mathbf{k}')-\sum_{A,\beta,i,j}\frac{1}{2\pi^2}v(-k)\exp(i(\mathbf{kz}_B-\mathbf{k}'z_A))v(k')\cdot k_j k_i' M_{Ai,Bj}\right]. \quad (23)$$

Inserting (18), (18a) in the Hamiltonian one obtains by taking into account (17), (17a)

$$H=H_0+H'+\Omega, \quad (24)$$

with

$$H_0=\frac{1}{2}\sum_{\alpha,A,B,i,j}[N_{Ai,Bj}P_{\alpha A,i}P_{\alpha B,j}+d_{Ai,Bj}Q_{\alpha A,i}Q_{\alpha B,j}]+f\sum_A\sigma_A[\mathbf{Q}_{1A}\times\mathbf{Q}_{2A}]+\frac{1}{2}g\sum_{\alpha,A,i}Q_{\alpha A,i}^2, \quad (25)$$

where

$$d_{Ai,Bj}=d_{Bj,Ai}=\sum_{C,D,m,n}M_{Cm,Ai}M_{Dn,Bj}\frac{1}{2\pi^2}\int G(k)k_0^2 k_m k_n \exp(i\mathbf{k}(\mathbf{z}_C-\mathbf{z}_D))d\mathbf{k}, \quad (26)$$

$$H'=\frac{1}{2}\sum_{\alpha}\int[p_{\alpha}'(k)p_{\alpha}'(-k)+k_0^2 q_{\alpha}'(k)q_{\alpha}'(-k)]d\mathbf{k}, \quad (27)$$

$$\Omega=\sum_{\alpha,A,B,i,j}M_{Ai,Bj}Q_{\alpha B,j}\frac{1}{\pi\sqrt{2}}\int v(-k)ik_i \exp(i\mathbf{kz}_A)k_0^2 q_{\alpha}'(k)d\mathbf{k}. \quad (28)$$

For the electric charge one has without cross terms $e = e_0 + e'$

$$e_0 = \sum_A (\mathbf{Q}_{1A} \cdot \mathbf{P}_{2A} - \mathbf{Q}_{2A} \cdot \mathbf{P}_{1A}), \quad (29)$$

$$e' = \int [q_1'(k)p_2'(k) - q_2'(k)p_1'(k)] d\mathbf{k}. \quad (30)$$

These expressions simplify considerably if the different nucleons do not overlap; in other words, if the distance $r_{AB} = |z_A - z_B|$ between different nucleons is large in comparison with the dimension of one nucleon, which in the usual way can be characterized by

$$\frac{1}{a} = \frac{1}{2\pi^2} \int G(k) k^{-2} d\mathbf{k} = \frac{2}{\pi} \int_0^\infty G(k) dk.$$

In this case,

$$r_{AB} \gg a, \quad (31)$$

the coefficients N_{A_i, B_j} defined by (21) can be neglected for $A \neq B$; moreover one has for $A = B$, because of the spherical symmetry of $U(x)$,

$$N_{A_i, A_j} = N \delta_{ij}, \quad (32)$$

with

$$3N = \frac{2}{\pi} \int_0^\infty G(k) k^4 dk = 4\pi \int (\nabla U)^2 d\mathbf{x}. \quad (33)$$

In this approximation one has therefore

$$M_{A_i, B_j} = \delta_{ij} \delta_{AB} \frac{1}{N}, \quad (34)$$

and for similar reasons for the coefficient d_{A_i, B_j} defined in (25)

$$d_{A_i, B_j} = \delta_{AB} \delta_{ij} d, \quad (35)$$

$$3d = \frac{1}{N^2} \frac{2}{\pi} \int_0^\infty G(k) k_0^2 k^4 dk. \quad (36)$$

Instead of (25), (28) we obtain then

$$H_0 = \frac{1}{2} \sum_{\alpha A} \{ N P_{\alpha A}^2 + f \sigma_A [\mathbf{Q}_{1A} \times \mathbf{Q}_{2A}] + \frac{1}{2} (g + d) \mathbf{Q}_{\alpha A}^2 \}, \quad (37)$$

$$\Omega = \frac{1}{N} \sum_{\alpha, A, i} Q_{\alpha A, i} \frac{1}{\pi \sqrt{2}} \int v(-k) i k_i \exp(i \mathbf{k} z_A) k_0^2 q_\alpha'(k) d\mathbf{k}. \quad (38)$$

Obviously the Eqs. (32) to (38) are exactly true if there is only one nucleon present.

3. THE H_0 -PROBLEM

We consider now more in detail the problem defined by the Hamiltonian which is generally given by (25). We shall first investigate the particular case of non-overlapping sources [see condition (31)], in which, according to (37), H_0 decomposes into separate parts H_{0A} from which each contains only variables connected with *one* nucleon. The eigenvalues of H_0 for nucleons at distances larger than their sizes are therefore independent of these distances and of the orientations of their spins. Therefore the part H_0 of the Hamiltonian does not give rise to nuclear forces in distances where the forces do not overlap.

In this region it is sufficient to consider only the part of H_0 which is connected with a single nucleon and to omit the index A . Moreover we shall be interested in the strong coupling case in which d is small in comparison with g and can be neglected. Indeed, according to (33), (36) one finds that d is of the order a^3 , and the inequality,

$$g \gg a^3, \quad (39)$$

will turn out to be one of the strong coupling conditions. We have, therefore,

$$H_0 = \frac{1}{2}N(\mathbf{P}_1^2 + \mathbf{P}_2^2) + \frac{1}{2}g(\mathbf{Q}_1^2 + \mathbf{Q}_2^2) + f\sigma[\mathbf{Q}_1 \times \mathbf{Q}_2]. \quad (38')$$

Besides the charge

$$e_0 \equiv T = \mathbf{Q}_1 \cdot \mathbf{P}_2 - \mathbf{Q}_2 \cdot \mathbf{P}_1, \quad (29')$$

there exists the angular momentum integral

$$\mathbf{J} = \mathbf{L} + \frac{1}{2}\sigma, \quad (40)$$

with

$$\mathbf{L} = [\mathbf{Q}_1 \times \mathbf{P}_1] + [\mathbf{Q}_2 \times \mathbf{P}_2]. \quad (41)$$

We found it convenient to use coordinates introduced by Pauli and Dancoff⁶ which describe the two vectors \mathbf{Q}_1 , \mathbf{Q}_2 by two positive scalars and four angles, one of which, denoted by θ is canonically conjugate to the charge T . The three other angles define a system of three orthogonal unit vectors $\mathbf{n}^{(r)}$ ($r=1, 2, 3$) in such a way that

$$\mathbf{Q}_1 = Q_1 \cos \theta \mathbf{n}^{(1)} - Q_2 \sin \theta \mathbf{n}^{(2)}, \quad \mathbf{Q}_2 = Q_1 \sin \theta \mathbf{n}^{(1)} + Q_2 \cos \theta \mathbf{n}^{(2)}; \quad (42)$$

hence,

$$[\mathbf{Q}_1 \times \mathbf{Q}_2] = Q_1 Q_2 \mathbf{n}^{(3)} \quad \text{with} \quad \mathbf{n}^{(3)} = [\mathbf{n}^{(1)} \times \mathbf{n}^{(2)}]. \quad (43)$$

In other words $\mathbf{n}^{(1)}$ and $\mathbf{n}^{(2)}$ are in the plane spanned by \mathbf{Q}_1 , \mathbf{Q}_2 , their direction being fixed by the condition

$$Q_1^{(1)}Q_1^{(2)} + Q_2^{(1)}Q_2^{(2)} = 0, \quad (44)$$

where here and below the components of a vector \mathbf{X} parallel to $\mathbf{n}^{(r)}$ are denoted by $X^{(r)} = (\mathbf{X} \cdot \mathbf{n}^{(r)})$. In the paper referred to it was shown that

$$\mathbf{P}_1^2 + \mathbf{P}_2^2 = P_1^+ P_1 + P_2^+ P_2 + \frac{(L^{(3)} + T)^2}{2(Q_1 - Q_2)^2} + \frac{(L^{(3)} - T)^2}{2(Q_1 + Q_2)^2} + \frac{(L^{(2)})^2}{Q_1^2} + \frac{(L^{(1)})^2}{Q_2^2}. \quad (45)$$

The operator P_α^+ is hermitian conjugate to P_α with respect to the density $\rho = Q_1 Q_2 (Q_1 + Q_2)(Q_1 - Q_2)$; that means

$$P_\alpha \Psi = -i \frac{\partial \Psi}{\partial Q_\alpha}, \quad P_\alpha^+ \Psi = -i \frac{1}{\rho} \frac{\partial}{\partial Q_\alpha} (\rho \Psi).$$

Hence we obtain finally

$$H_0 = \frac{1}{2}N \left[P_1^+ P_1 + P_2^+ P_2 + \frac{(L^{(3)} + T)^2}{2(Q_1 - Q_2)^2} + \frac{(L^{(3)} - T)^2}{2(Q_1 + Q_2)^2} + \frac{(L^{(2)})^2}{Q_1^2} + \frac{(L^{(1)})^2}{Q_2^2} \right]. \quad (46)$$

As was shown in the paper referred to,⁷ it is also possible to make an S -transformation which brings $\sigma^{(r)}$ to its normal form σ_r and at the same time $L^{(r)}$ to $J^{(r)} - \sigma^{(r)}$, where $J^{(r)}$ as an operator applied to the new wave function has the same form as $L^{(r)}$ had before the transformation, so that (46)

⁶ W. Pauli and S. M. Dancoff, Phys. Rev. **62**, 85 (1942); compare especially, Section 8, Eqs. (115), (116), (50c). For the definition of three Euler angles ϑ , φ , ψ compare Appendix, Eq. (1), the quantities A_{rk} being identical with the components of $\mathbf{n}^{(r)}$. The expressions of $L^{(r)} = (\mathbf{L} \cdot \mathbf{n}^{(r)})$ are given by Appendix, Eq. (7).

⁷ W. Pauli and S. M. Dancoff, see reference 6, Section 8, Eqs. (65a), (69).

is equivalent to

$$H_0 = \frac{1}{2}N \left[P_1^2 + P_1 + P_2^2 + P_2 + \frac{(J^{(3)} - \frac{1}{2}\sigma_3 + T)^2}{2(Q_1 - Q_2)^2} + \frac{(J^{(3)} - \frac{1}{2}\sigma_3 - T)^2}{2(Q_1 + Q_2)^2} + \frac{(J^{(2)} - \frac{1}{2}\sigma_2)^2}{Q_1^2} + \frac{(J^{(1)} - \frac{1}{2}\sigma_1)^2}{Q_2^2} \right] + \frac{1}{2}g(Q_1^2 + Q_2^2) + f\sigma_3 Q_1 Q_2,$$

or with

$$\begin{aligned} R &= \frac{1}{\sqrt{2}}(Q_1 + Q_2), & S &= \frac{1}{\sqrt{2}}(Q_1 - Q_2), \\ P_R &= \frac{1}{\sqrt{2}}(P_1 + P_2), & P_S &= \frac{1}{\sqrt{2}}(P_1 - P_2), \end{aligned} \quad (47)$$

$$H_0 = \frac{1}{2}N \left[P_R^2 + P_R + P_S^2 + P_S + \frac{(J^{(3)} - \frac{1}{2}\sigma_3 + T)^2}{4S^2} + \frac{(J^{(3)} - \frac{1}{2}\sigma_3 - T)^2}{4R^2} + 2 \frac{(J^{(2)} - \frac{1}{2}\sigma_2)^2}{(R+S)^2} + 2 \frac{(J^{(1)} - \frac{1}{2}\sigma_1)^2}{(R-S)^2} \right] + \frac{1}{2}(g + f\sigma_3)R^2 + \frac{1}{2}(g - f\sigma_3)S^2. \quad (48)$$

If $R, S, R+S, R-S$ (in other words $Q_1, Q_2, Q_1+Q_2, Q_1-Q_2$) are large, the terms with the denominators can be neglected and the eigenvalues of σ_3 being ± 1 , we obtain a system of oscillators with the frequencies

$$\nu_1 = [(g+f)N]^{\frac{1}{2}}, \quad \nu_2 = [(g-f)N]^{\frac{1}{2}}, \quad (49)$$

each of them double-degenerated, provided that

$$g > f. \quad (50)$$

In the opposite case the Hamiltonian is not definitely positive, and we obtain a system which partly corresponds to repulsive forces and for which a continuous spectrum with any eigenvalues of H_0 between $-\infty$ and $+\infty$ has to be expected. This conclusion seems to be confirmed by the discussion of the behavior of the eigenfunctions for large values of $R, S, R+S$, and $R-S$. This result seems to be in such a disagreement with the empirical properties of particles that we postulate the inequality (50) as a necessary condition to be fulfilled by our Hamiltonian.

The actual discussion of the different eigenstates of H_0 can better be made with help of (46) without the S -transformation than with (48). The ground state $T=0, j=\frac{1}{2}$ [where j is defined by $J^2 = j(j+1)$] for instance leads to two simultaneous differential equations of the second order for two wave functions Ψ and ϕ which, however, do not have a simple analytic solution. We have not carried through a detailed numerical discussion of this eigenvalue problem because it was not necessary for our purpose.

Before concluding this section we discuss the case $f=0$, but for small distances where the sources overlap, in order to prove that in this region the H_0 -problem actually does give rise to interaction forces. Neglecting the coefficients d_{A_i, B_j} for the same reasons as before the d , we obtain for $f=0$ from (25),

$$H_0 = \frac{1}{2} \sum_{\alpha, A, B, i, j} N_{A_i, B_j} P_{\alpha A, i} P_{\alpha B, j} + \frac{1}{2}g \sum_{\alpha, A, i} Q_{\alpha A, i}^2.$$

As there is complete separation of the variables for $\alpha=1$ and $\alpha=2$ we can neglect in the following the index α and write:

$$H_0 = \frac{1}{2} \sum_{A, B, i, j} N_{A_i, B_j} P_{A, i} P_{B, j} + \frac{1}{2}g \sum_{A, i} Q_{A, i}^2.$$

The Hamiltonian being a quadratic form, this leads to a system of oscillators which can be obtained by searching the classical periodical solutions, with the frequencies ω to be determined. As we have

$$\dot{P}_{A, i} = -\frac{\partial H}{\partial Q_{A, i}} = -gQ_{A, i}, \quad \dot{Q}_{A, i} = \frac{\partial H}{\partial P_{A, i}} = \sum_{B, j} N_{A_i, B_j} P_{B, j};$$

hence,

$$-\ddot{P}_{A,i} = g \sum_{B,j} N_{A_i, B_j} P_{B_j}.$$

We obtain for the frequency the determinant condition

$$\| -\omega^2 \delta_{A_i, B_j} + N_{A_i, B_j} \| = 0. \quad (51)$$

This can be simplified for the case of two nucleons according to the definition (21) of N_{A_i, B_j} and the central symmetry of the source function. Using (32) for $A=B$ and choosing a coordinate system where the x_3 axis is parallel to the line joining the two nucleons, one easily finds for $A \neq B$, $N_{A_i, B_j} = 0$ for $i \neq j$, $N_{A_1, B_1} = N_{A_2, B_2}$; hence,

$$\| -\omega^2 \delta_{A_i, B_j} + N_{A_i, B_j} \| = \left\| \begin{array}{cc} -\omega^2 + N, & N_{A_3, B_3} \\ N_{A_3, B_3}, & -\omega^2 + N \end{array} \right\| \cdot \left\| \begin{array}{cc} -\omega^2 + N, & N_{A_1, B_1} \\ N_{A_1, B_1}, & -\omega^2 + N \end{array} \right\|^2.$$

The roots for ω^2 are, therefore,

$$\omega^2 = N \pm N_{A_3, B_3}, \quad N \pm N_{A_1, B_1}, \quad N \pm N_{A_1, B_1},$$

and the zero-point energy

$$E = \frac{1}{2} \sum \omega = \frac{1}{2} [(N + N_{A_3, B_3})^{\frac{1}{2}} + (N - N_{A_3, B_3})^{\frac{1}{2}}] + [N + N_{A_1, B_1}]^{\frac{1}{2}} + [N - N_{A_1, B_1}]^{\frac{1}{2}}. \quad (52)$$

For infinite distances one has $E = 3(N)^{\frac{1}{2}}$; hence,

$$V_{AB} = \frac{1}{2} [(N + N_{A_3, B_3})^{\frac{1}{2}} + (N - N_{A_3, B_3})^{\frac{1}{2}} - 2N^{\frac{1}{2}}] + (N + N_{A_1, B_1})^{\frac{1}{2}} + (N - N_{A_1, B_1})^{\frac{1}{2}} - 2N^{\frac{1}{2}}, \quad (53)$$

which actually depends on the distance r_{AB} as soon as the sources overlap. Because of the presence of two real fields ($\alpha = 1, 2$), one has still the result given by (53) to multiply with two.

4. THE H' -PROBLEM

We have now to investigate the problem defined by the Hamiltonian H' Eq. (27), and the extra conditions (17), (17a). As there is complete separation between the two fields with $\alpha = 1$ and $\alpha = 2$, we omit the index α and write

$$H' = \frac{1}{2} \int [p'(k)p'(-k) + k_0^2 q'(k)q'(-k)] d\mathbf{k}, \quad (54)$$

$$\int v(-k) \mathbf{k} \exp(i\mathbf{k} \cdot \mathbf{z}_A) q'(k) d\mathbf{k} = 0, \quad (55)$$

$$\int v(k) \mathbf{k} \exp(-i\mathbf{k} \cdot \mathbf{z}_A) p'(k) d\mathbf{k} = 0. \quad (56)$$

Assuming that the sources do not overlap [inequality (31)] we can write the commutation rule (23) according to (34) in the form

$$i[p'(k), q'(k')] = \delta(\mathbf{k} - \mathbf{k}') - \frac{1}{N} \frac{1}{2\pi^2} v(-k)v(k') (\mathbf{k} \cdot \mathbf{k}') \sum_A \exp(i(\mathbf{k} - \mathbf{k}') \cdot \mathbf{z}_A). \quad (57)$$

One can see immediately that neither the coupling constant nor the spin of the nucleons enter into the formulation of the problem.

Applying the rule $\dot{F} = i[H', F]$ for $F = q'(k)$ and $F = p'(-k)$ one obtains, with help of (56), (57), the equations of motion

$$\dot{q}'(k) = p'(-k), \quad -\dot{p}'(-k) = -\dot{q}'(k) = k_0^2 q'(k) - \frac{1}{\pi\sqrt{2}} v(k) \sum_A \exp(-i\mathbf{k} \cdot \mathbf{z}_A) \mathbf{k} \lambda_A, \quad (58)$$

with

$$\lambda_A = \frac{1}{N} \frac{1}{\pi\sqrt{2}} \int v(-k) k_0^2 \mathbf{k} \exp(i\mathbf{k} \cdot \mathbf{z}_A) q'(k) d\mathbf{k}, \quad (59)$$

or also according to (55)

$$\lambda_A = \frac{1}{N} \frac{1}{\pi\sqrt{2}} \int v(-k) k^2 \mathbf{k} \exp(i\mathbf{k} \cdot \mathbf{z}_A) q'(k) d\mathbf{k}. \quad (59a)$$

The same result follows, of course, with the method of Lagrangian multipliers.

As the equations of motions are linear, it is sufficient to treat these equations classically by searching its periodic solutions. It will turn out, however, that the problem is only uniquely defined if we first make the \mathbf{k} -space discrete, assuming for instance $k_i = \epsilon n_i$ (n_i integers) and replacing $\int d\mathbf{k}$ by $\epsilon^3 \sum_{n_i}$. Only at the end of the calculation shall we perform the limiting process $\epsilon \rightarrow 0$ again. For instance, we have then:

$$\lambda_A = \frac{1}{N} \frac{1}{\pi\sqrt{2}} \epsilon^3 \sum_n v_n k_n^2 \mathbf{k}_n \exp(i\mathbf{k}_n \cdot \mathbf{z}_A) q_n'. \quad (59')$$

For a periodic solution with a frequency ω which we may put equal

$$\omega = (l^2 + \mu^2)^{\frac{1}{2}}, \quad (60)$$

we obtain from (58)

$$(-l^2 + k_n^2) q_n' - \frac{1}{N} \frac{1}{\pi\sqrt{2}} \sum_A \exp(-i\mathbf{k}_n \cdot \mathbf{z}_A) \mathbf{k}_n \lambda_A = 0. \quad (61)$$

Hence, either

$$\lambda_A = 0, \quad l^2 = k_n^2, \quad (62a)$$

or

$$q_n' = \frac{1}{N} \frac{1}{\pi\sqrt{2}} \sum_A \exp(-i\mathbf{k}_n \cdot \mathbf{z}_A) \frac{v_n \mathbf{k}_n}{-l^2 + k_n^2} \lambda_A. \quad (62b)$$

From (55) we therefore obtain, using (20),

$$\epsilon^3 \frac{1}{2\pi^2} \sum_{n, B, j} G(k_n) \exp(i\mathbf{k}_n \cdot (\mathbf{z}_A - \mathbf{z}_B)) \frac{k_{in} k_{jn}}{-l^2 + k_n^2} \lambda_{Bj} = 0, \quad (63)$$

which leads to the determinant condition

$$\left\| \epsilon^3 \frac{1}{2\pi^2} \sum_n G(k_n) \exp(i\mathbf{k}_n \cdot (\mathbf{z}_A - \mathbf{z}_B)) \frac{k_{in} k_{jn}}{-l^2 + k_n^2} \right\| = 0. \quad (64)$$

If N denotes the number of nucleons present, the determinants (the elements of which are written with the double-indices Ai, Bj) have $3N$ rows and $3N$ columns.

While the values of the roots of this equation essentially depend on the exact discrete values of the k_{in} , symmetrical functions of its roots can be computed by an elegant method given by Wentzel,⁸ which makes it possible to perform again the limiting process to the continuous \mathbf{k} -space. We define a function $\varphi(t)$ in the complex t -plane by

$$\varphi(t) \equiv \left\| \epsilon^3 \frac{1}{2\pi^2} \sum_n G(k_n) \exp(i\mathbf{k}_n \cdot (\mathbf{z}_A - \mathbf{z}_B)) \frac{k_{in} k_{jn}}{-t + k_n^2} \right\|. \quad (65)$$

The zeros of $\varphi(t)$ are just the eigenvalues of $l_n^2 = \omega_n^2 - \mu^2$ we are searching for. As the Hamiltonian is positive definite, the eigenvalues of ω^2 are certainly positive, and the eigenvalues of $l^2 = \omega^2 - \mu^2$, therefore, certainly larger than $-\mu^2$. The poles of $\varphi(t)$ are obviously $t = k_n^2$. If now $f(t)$ is any function

⁸ I (see note 1); compare the figure on p. 115.

which is regular on the real axis and its environment for $t > -\mu^2$ one has

$$\sum_{l_n} f(l_n^2) - \sum_n f(k_n^2) = \frac{1}{2\pi i} \oint dt f(t) \frac{d \log \varphi(t)}{dt}. \quad (66)$$

On the left side the first sum has to be taken over all eigenvalues; on the right side the path of integration is a loop in the positive sense around the real axis from $+\infty + i\epsilon$ over $t_0 + i\epsilon$ and $t_0 - i\epsilon$ to $+\infty - i\epsilon$, where the choice of t_0 lets the point $t = -\mu^2$ to the left. By partial integration the expression (66) is transformed into

$$\sum_{l_n} f(l_n^2) - \sum_n f(k_n^2) = -\frac{1}{2\pi i} \oint dt f'(t) \log \varphi(t). \quad (67)$$

(It can be shown that the upper limits together do not give any contribution if $G(k)$ decreases sufficiently with increasing k .) For $f(t) = (t + \mu^2)^{\frac{1}{2}}$, we obtain in this way just the change E of the total zero-point energy of our oscillators owing to the nucleons and their coupling with the meson field (where the existence of two mesons corresponding to $\alpha = 1, 2$ has been taken into account). Hence we have

$$E = -\frac{1}{4\pi i} \oint \frac{dt}{(t + \mu^2)^{\frac{1}{2}}} \log \varphi(t). \quad (68)$$

It may be noted that the solutions satisfying (62a) automatically make no contribution to this difference.

The remarkable fact is that it is possible to go to the limit of a continuous \mathbf{k} -space in (65), namely

$$\varphi(t) \equiv \left\| \frac{1}{2\pi^2} \int G(k) \exp(i\mathbf{k}(\mathbf{z}_A - \mathbf{z}_B)) \frac{k_i k_j}{-t + k^2} d\mathbf{k} \right\|. \quad (69)$$

The integrals are defined if t is not on the positive real axis. There, however, one can still define the two limiting values

$$\varphi_+(x) = \lim_{y \rightarrow 0} \varphi(x + iy), \quad \varphi_-(x) = \lim_{y \rightarrow 0} \varphi(x - iy),$$

which both exist for x, y real, y positive, but turn out to be different from each other.

It is convenient now to separate the contribution of possible existing roots of $\varphi(t)$ for real t in the interval $(-\mu^2, 0)$ from the contribution to (68) from t -values with a positive real part. The former, if present at all, remain discrete even if the \mathbf{k} -space becomes continuous, and may be denoted by

$$t_n = (\omega_n^2 - \mu^2) < 0, \quad \varphi(t_n) = 0. \quad (70)$$

Introducing $l = (t_n)^{\frac{1}{2}}$ as variable of integration for the other part with positive real part of t we obtain

$$E = \sum_n \omega_n - \frac{1}{2\pi i} \oint \frac{l dl}{(l^2 + \mu^2)^{\frac{1}{2}}} \log \varphi(l^2). \quad (71)$$

Going with the part of integration near to the real axis from above and below, one obtains from (71) immediately

$$E = \sum_n \omega_n + \frac{1}{2\pi i} \int_0^\infty \frac{l dl}{(l^2 + \mu^2)^{\frac{1}{2}}} \log \frac{\varphi_+(l^2)}{\varphi_-(l^2)}. \quad (72)$$

We shall now compute the determinant (69) defining $\varphi(t)$ for the case of two nucleons present, for which we are here mainly interested. There are first the elements corresponding to $A = B$ which we can write $I \delta_{ij}$ with

$$I = \frac{1}{3} \frac{2}{\pi} \int_0^\infty \frac{k^4}{-t + k^2} G(k) dk. \quad (73)$$

The elements for $A \neq B$ are symmetrical in A and B and can first be written, if we use the abbreviation

$$\mathbf{z} = \mathbf{z}_A - \mathbf{z}_B, \quad r = |\mathbf{z}|, \quad (74)$$

$$J_{ij} = -\frac{\partial^2 J}{\partial z_i \partial z_j}, \quad (75)$$

$$J = \frac{1}{2\pi^2} \int \frac{\exp(i\mathbf{kz})}{-t+k^2} G(k) d\mathbf{k} = \frac{1}{r} \frac{2}{\pi} \int_0^\infty \frac{k \sin kr}{-t+k^2} G(k) dk. \quad (76)$$

If we choose the 3-axis of our coordinate system as being parallel to the line joining the two nucleons, we easily obtain from these expressions

$$J_{ij} = 0 \text{ for } i \neq j, \quad J_{11} = J_{22} = -\frac{1}{r} \frac{dJ}{dr}, \quad J_{33} = -\frac{d^2 J}{dr^2}. \quad (77)$$

For the determinant in (69) one obtains now immediately

$$\varphi(t) = \left[I^2 - \left(\frac{1}{r} \frac{dJ}{dr} \right)^2 \right]^2 \left[I^2 - \left(\frac{d^2 J}{dr^2} \right)^2 \right],$$

or

$$\varphi(t) = [\varphi_1(t)]^2 [\varphi_2(t)]^2 \varphi_3(t), \quad (78)$$

with

$$\varphi_1(t) = I^3, \quad (78_1)$$

being the expression for $\varphi(t)$ if only one nucleon is present and

$$\varphi_2(t) = 1 - \left(\frac{1}{I} \frac{1}{r} \frac{dJ}{dr} \right)^2, \quad (78_2)$$

$$\varphi_3(t) = 1 - \left(\frac{1}{I} \frac{d^2 J}{dr^2} \right)^2. \quad (78_3)$$

We can now compute the limiting values $\varphi_+(x)$, $\varphi_-(x)$ by using the formula

$$\lim_{y \rightarrow 0} \int_0^\infty \frac{F(k)}{-(l^2 \pm iy) + k^2} dk = \int_P \frac{F(k)}{-l^2 + k^2} dk \pm i\pi \frac{F(l)}{2l}, \quad (79)$$

which holds for positive y and every $F(k)$ which is regular on the positive real axis and its environment. The symbol P under the integral sign denotes the principal value. Transforming the expression for I into

$$3I(l^2) = \frac{2}{\pi} \int_0^\infty k^2 G(k) dk + l^2 \cdot \frac{2}{\pi} \int_0^\infty G(k) dk + l^4 \cdot \frac{2}{\pi} \int_0^\infty \frac{G(k)}{-l^2 + k^2} dk,$$

we obtain with the abbreviation

$$3A = \frac{2}{\pi} \int_0^\infty G(k) k^2 dk, \quad \frac{1}{a} = \frac{2}{\pi} \int_0^\infty G(k) dk,$$

$$3I = 3A + l^2 \cdot \frac{1}{a} + l^4 \cdot \frac{2}{\pi} \int_0^\infty \frac{G(k)}{-l^2 + k^2} dk.$$

Obviously the order of magnitude of A is a^{-3} . For the following it will be sufficient to evaluate the integral for

$$l \ll a^{-1}. \quad (80)$$

In this case it is permitted to put $G(k) = 1$ in the last integral and we obtain⁹

$$3I_{\pm}(l^2) = 3A + l^2 \cdot \frac{1}{a} \pm il^3. \tag{81}$$

Under our assumption $r \gg a$ [see (31)], and taking (80) into account we can put $G(k) = 1$ in the definition (76) for J . In this way one easily gets¹⁰

$$J_{\pm}(l^2) = e^{\pm ilr}/r. \tag{82}$$

Inserting now (78) into (72) one sees that $\varphi_1(t)$ gives rise to the zero-point energy and that for the interaction energy we have

$$V_{AB} = \frac{1}{2\pi i} \int_0^{\infty} \frac{ldl}{(l^2 + \mu^2)^{\frac{1}{2}}} \cdot 2 \log \frac{\varphi_{2,+}(l^2)}{\varphi_{2,-}(l^2)} + \frac{1}{2\pi i} \int_0^{\infty} \frac{ldl}{(l^2 + \mu^2)^{\frac{1}{2}}} \log \frac{\varphi_{3,+}(l^2)}{\varphi_{3,-}(l^2)}.$$

From (81) and (82) it follows that for $r \gg a$, $\varphi_{2,\pm}$ and $\varphi_{3,\pm}$ are small in comparison with 1 (they are therefore never zero and the discrete term in (72) can be omitted) and we can put

$$\begin{aligned} \log \varphi_{2,\pm} &= \log \left[1 - \left(\frac{1}{I_{\pm}} \frac{1}{r} \frac{dJ_{\pm}}{dr} \right)^2 \right] = - \left(\frac{1}{I_{\pm}} \frac{1}{r} \frac{dJ_{\pm}}{dr} \right)^2, \\ \log \varphi_{3,\pm} &= \log \left[1 - \left(\frac{1}{I_{\pm}} \frac{d^2 J_{\pm}}{dr^2} \right)^2 \right] = - \left(\frac{1}{I_{\pm}} \frac{d^2 J_{\pm}}{dr^2} \right)^2. \end{aligned}$$

As in the analogous case of Wentzel only the region $l \ll a^{-1}$ contributes appreciably to the Fourier integral;¹¹ therefore we can simply substitute A for I_{\pm} and get

$$V_{AB} = - \frac{1}{2\pi i} \frac{1}{A^2} \int_0^{\infty} \frac{ldl}{(l^2 + \mu^2)^{\frac{1}{2}}} \left[\frac{2}{r^2} \left(\frac{dJ_+}{dr} \right)^2 - \frac{2}{r^2} \left(\frac{dJ_-}{dr} \right)^2 + \left(\frac{d^2 J_+}{dr^2} \right)^2 - \left(\frac{d^2 J_-}{dr^2} \right)^2 \right]. \tag{83}$$

Inserting (82) we obtain the final result

$$V_{AB} = - \frac{1}{2\pi i} \frac{1}{A^2 r^6} \int_0^{\infty} \frac{ldl}{(l^2 + \mu^2)^{\frac{1}{2}}} e^{2ilr} [2(-1 + ilr)^2 + (2 - 2ilr - l^2 r^2)^2] + \text{conj. compl.}$$

which can be written

$$V_{AB} = - \frac{1}{\pi A^2 r^6} \int_0^{\infty} \frac{ldl}{(l^2 + \mu^2)^{\frac{1}{2}}} [\sin 2lr \cdot (6 - 10l^2 r^2 + l^4 r^4) + 4 \cos 2lr \cdot (-3lr + l^3 r^3)]. \tag{84}$$

The result can be expressed by the function

$$-\frac{\pi}{2} H_1^{(1)}(ix) = K(x) = \int_0^{\infty} \frac{z \sin zx}{(1+z^2)^{\frac{1}{2}}} dz = \int_0^{\infty} \exp[-z(s^2+1)^{\frac{1}{2}}] ds$$

⁹ For $t < 0$ one obtains, putting $t = -l^2$,

$$3I(-l^2) = 3A - l^2 \cdot \frac{1}{a} + l^3.$$

¹⁰ For $t < 0$ one obtains, putting again $t = -l^2$,

$$J(-l^2) = e^{-lr}/r.$$

¹¹ In the usual way we use, in the following, for integrals with oscillating integrands $f(x)$ which are not convergent in the proper sense, the definition

$$\int_0^{\infty} f(x) dx = \lim_{\epsilon \rightarrow 0} \int_0^{\infty} f(x) e^{-\epsilon x} dx.$$

[Compare W. Pauli, Rev. Mod. Phys. 15, 175 (1943), Eq. (190)].

and its derivatives. We also notice the connection with

$$\frac{i\pi}{2}H_0'(ix) = K_0(x) = \int_0^\infty \frac{\cos zx}{(1+z^2)^{\frac{3}{2}}} dz = \int_0^\infty \frac{\exp[-z(s^2+1)^{\frac{1}{2}}]}{(s^2+1)^{\frac{3}{2}}} ds,$$

given by

$$K(x) = -K_0'(x), \quad K_0(x) = -K'(x) - \frac{1}{x}K(x).$$

From (84) one gets the expression

$$V_{AB} = -\frac{\mu}{\pi A^{2\gamma^6}} g(2\mu r) \quad (84a)$$

with

$$g(x) = 6K(x) + \frac{5}{2}x^2K''(x) + \frac{1}{16}x^4K^{iv}(x) - 6xK'(x) - \frac{1}{2}x^3K'''(x), \quad (84b)$$

which can be transformed into

$$g(x) = \left(\frac{43}{2} + \frac{63}{16}x^2 + \frac{1}{16}x^4\right)K(x) + \left(\frac{43}{4}x + \frac{5}{8}x^3\right)K_0(x). \quad (84c)$$

Using $K(x) = 1/x$ for $x \ll 1$ and $K(x) = (\pi/2x)^{\frac{1}{2}}e^{-x}$ for $x \gg 1$, we obtain

$$V_{AB} = -\frac{43}{4\pi} \frac{1}{A^{2\gamma^7}} \quad \text{for } \mu r \ll 1, \quad (85a)$$

$$V_{AB} = -\frac{1}{2\sqrt{\pi}} \frac{\mu^7}{A^2} \frac{1}{(\mu r)^{5/2}} e^{-2\mu r} \quad \text{for } \mu r \gg 1. \quad (85b)$$

5. RIGOROUS TREATMENT OF THE SPIN-INDEPENDENT CASE $f=0$. ESTIMATION OF THE CONTRIBUTION OF Ω IN THE SPIN-DEPENDENT CASE

To prepare the estimation of the part Ω given by (28), to the interaction energy in the general case, we first discuss briefly the rigorous treatment of the spin-independent Hamiltonian¹² obtained from (15) by putting $f=0$ and omitting the index α , namely

$$H = \frac{1}{2} \int [p(k)p(-k) + k_0^2 q(k)q(-k)] dk + \frac{1}{2} g \sum_A \mathbf{Q}_A^2, \quad (86)$$

where again, as in (19), \mathbf{Q}_A is given by

$$\mathbf{Q}_A = \frac{1}{\pi\sqrt{2}} i \int v(-k) \mathbf{k} \exp(i\mathbf{k} \cdot \mathbf{z}_A) q(k) dk. \quad (87)$$

From the canonical equations of motion one easily obtains

$$\ddot{q}(k) + k_0^2 q(k) + g \frac{-i}{\pi\sqrt{2}} v(k) \mathbf{k} \sum_A \exp(-i\mathbf{k} \cdot \mathbf{z}_A) \mathbf{Q}_A = 0.$$

As these equations are linear it is sufficient to search their periodic solutions and if we put, as in the last section, for the square of their frequency,

$$\omega^2 = l^2 + \mu^2,$$

¹² The analogous case, with the scalar

$$Q_A = \frac{1}{\pi\sqrt{2}} \int v(-k) \exp(i\mathbf{k} \cdot \mathbf{z}_A) q(k) dk$$

instead of the vector \mathbf{Q}_A in the Hamiltonian was treated by Wentzel, see reference 1 "I."

and introduce discrete k -values, we obtain the system of linear equations

$$(-l^2 + k_n^2)q_n + g \frac{1}{2\pi^2} \epsilon^3 \sum_{i,A} v(k_n) k_{ni} \exp(-i\mathbf{k}_n \mathbf{z}_A) \sum_{n'} v(-k_{n'}) \exp(i\mathbf{k}_{n'} \mathbf{z}_A) q_{n'} = 0. \quad (88)$$

Hence we have either $l^2 = k_n^2$ and $\mathbf{Q}_A = 0$ or

$$\left\| \delta_{nn'} + g \frac{1}{2\pi^2} \epsilon^3 \sum_{i,A} \frac{v(k_n)}{-l^2 + k_n^2} k_{ni} \exp(-i\mathbf{k}_n \mathbf{z}_A) v(-k_{n'}) \exp(i\mathbf{k}_{n'} \mathbf{z}_A) \right\| = 0.$$

This determinant can be transformed with the help of the following algebraic theorem

$$\left\| \delta_{nn'} + \sum_{i=1}^N f_i(n) g_i(n') \right\| = \left\| \delta_{ij} + \sum_n f_i(n) g_j(n) \right\|,$$

the latter determinant having N rows and N columns, whatever the corresponding number for the first determinant may be. The proof of the theorem is given in the appendix. In our case one has to replace i, j by the double indices Ai, Bj , and N by $3N$, if N is the number of nucleons present. The determinant condition then gets the form

$$\left\| \delta_{Ai, Bj} + g \frac{1}{2\pi^2} \epsilon^3 \sum_n k_{ni} k_{nj} \frac{G(k_n)}{-l^2 + k_n^2} \exp[i\mathbf{k}_n \cdot (\mathbf{z}_A - \mathbf{z}_B)] \right\| = 0. \quad (89)$$

The subsequent calculations can be made in exactly the same way as in the last section, by replacing in (68), (71), (72) the previously used function $\varphi(t)$ by the new one $\psi(t)$ defined by

$$\psi(t) = \left\| \delta_{Ai, Bj} + g \frac{1}{2\pi^2} \int G(k) \frac{k_i k_j}{-t + k^2} \exp[i\mathbf{k} \cdot (\mathbf{z}_A - \mathbf{z}_B)] d\mathbf{k} \right\|, \quad (90)$$

instead of (69). Again this definition fails on the positive real axis, where $\psi(t)$ has different values if the approach is made from the upper or the lower half of the complex plane. From (78₁) to (78₃) one gets the corresponding expressions for $\psi(t)$ by substituting $1 + gI$ and gJ for I and J , respectively. Hence

$$\psi_1(t) = 1 + g^3 I^3, \quad \psi_2(t) = 1 - \left(\frac{1}{g^{-1} + I} \frac{1}{r} \frac{dJ}{dr} \right)^2, \quad \psi_3(t) = 1 - \left(\frac{1}{g^{-1} + I} \frac{d^2 J}{dr^2} \right)^2. \quad (91)$$

For $r_{AB} \gg a$ the final result for the interaction energy V_{AB} is therefore exactly the same as that for the strong coupling case which was given by (84) if the factor $1/A^2$ is replaced by $g^2/(1+gA)^2$. While in the strong coupling case the result holds if only the coupling constants fulfill certain inequalities which will be derived, the generalized result for arbitrary coupling only holds for $f=0$. In this particular case the relative error of the result (84) is obviously of the order of magnitude $1/gA$.

We are now in a position to estimate the corresponding error in the general case, where both f and g are different from zero. For $r_{AB} \gg a$, we can use the expression (38) which we can write, using the notation λ_A introduced in (59)

$$\Omega = \sum_{\alpha, A} \mathbf{Q}_{\alpha A} \cdot \lambda_{\alpha A}. \quad (38')$$

In order to estimate the contribution of Ω to the interaction energy (or self-energy) we use the well-known formula of the perturbation theory

$$\Delta E_0 = - \sum_r \frac{|\Omega_{0r}|^2}{E_r - E_0}, \quad (92)$$

where r are the excited states, 0 the ground states, and E_r, E_0 the corresponding energies. In our cases the excited states are those where one of the oscillators of the H' -problem and one state of the H_0 -problem are excited, the former corresponding to the matrix element of $\lambda_{\alpha A, i}$ (which are

independent of the coupling constant), the latter to those of the $Q_{\alpha A, i}$. In order to estimate the order of magnitude of the matrix elements of the Q we can, according to (47), (48), (49) approximately replace H_0 by a system of oscillators with the frequencies (49), namely

$$\nu_1 = [(g+f)N]^{\frac{1}{2}}, \quad \nu_2 = [(g-f)N]^{\frac{1}{2}},$$

the order of magnitude of the corresponding matrix elements of $Q_{\alpha, i}$ is, according to (47), (48)

$$(0|Q_{\alpha A, i}|r) \sim (N/\nu_1)^{\frac{1}{2}} \quad \text{and} \quad (N/\nu_2)^{\frac{1}{2}}.$$

In the denominator of (92) the excitation energy ν_1 or ν_2 of H_0 will in the strong-coupling case always be large in comparison with the excitation energy of H' (the l 's being cut off by a^{-1}); hence

$$\Delta E_0 \sim \frac{N}{\nu^2} \langle \lambda^2 \rangle_{Av} \sim \frac{1}{g \pm f} \langle \lambda^2 \rangle_{Av},$$

where $\langle \lambda^2 \rangle_{Av}$ means some average, this second factor being certainly independent of the coupling constant.

The comparison with the above result for $f=0$ let us also expect in the general case errors of the relative order

$$\frac{\Delta E_0}{E_0} \sim \frac{1}{(g+f)A} \quad \text{and} \quad \frac{1}{(g-f)A}. \quad (93)$$

The strong coupling conditions are therefore

$$(g+f)A \gg 1 \quad \text{and} \quad (g-f)A \gg 1, \quad (94)$$

or, as $A \sim a^{-3}$

$$(g+f) \gg a^3, \quad (g-f) \gg a^3, \quad (94')$$

as was stated in the introduction. Moreover one sees again the importance of the inequality $g > f$ [stated in (50)] for the consistency of our approximation. Moreover, if (94) holds, the spin-dependent part of the interaction energy will be smaller than the spin-independent part given by (84), by a factor whose order of magnitude is at most that indicated in (93).

6. VECTOR-PAIR THEORY

In close analogy to the scalar-pair theory we discuss in the following the vector-pair theory, in which a pair of mesons with spin 1 and opposite charge interacts with the nucleons. The charged mesons with spin 1 are described in the usual way by a complex vector field $\phi(x)$ or two real vector fields $\phi_\alpha(x)$ with $\alpha=1, 2$ and the simplest spin-dependent interaction of pair type (that means quadratic in the meson field) with the nucleons is characterized by the following Hamiltonian¹³ which is analogous to (1) and (8):

$$H = \frac{1}{2} \sum_{\alpha} \int \left\{ \pi_{\alpha}^2 + \frac{1}{\mu^2} (\nabla \cdot \pi_{\alpha})^2 + [\nabla \times \phi_{\alpha}]^2 + \mu^2 \phi_{\alpha}^2 \right\} d\mathbf{x} + 4\pi f \sum_A \sigma_A \left[\int U(x-z_A) \phi_1(x) d\mathbf{x} \right. \\ \left. \times \int U(x-z_A) \phi_2(x) d\mathbf{x} \right] + \frac{1}{2} \cdot 4\pi g \sum_{\alpha, A} \left(\int U(x-z_A) \phi_{\alpha} d\mathbf{x} \right)^2, \quad (95)$$

with the canonical commutation relations

$$i[\pi_{\alpha i}(x), \varphi_{\beta j}(x')] = \delta_{\alpha\beta} \delta_{ij} \delta(\mathbf{x} - \mathbf{x}'). \quad (96)$$

Transforming into momentum space as in Eqs. (10) to (14), and again putting $k_0^2 = k^2 + \mu^2$ we obtain

¹³ For the force-free part in which the fourth component of the four vector and its time derivative is eliminated, compare N. Kemmer, Proc. Roy. Soc. A166, 127 (1938); H. J. Bhabha, Proc. Roy. Soc. A166, 501 (1938); and W. Pauli, Rev. Mod. Phys. 13, 203 (1941), Section 2a.

as in (15),

$$\begin{aligned}
 H = & \frac{1}{2} \sum_{\alpha, i, j} \int \left\{ \left(\delta_{ij} + k_i k_j \frac{1}{\mu^2} \right) p_{\alpha i}(k) p_{\alpha j}(-k) + (k_0^2 \delta_{ij} - k_i k_j) q_{\alpha i}(k) q_{\alpha j}(-k) \right\} d\mathbf{k} \\
 & + f \sum_A \sigma_A \frac{1}{2\pi^2} \left[\int v(-k) \exp(i\mathbf{k} \cdot \mathbf{z}_A) \mathbf{q}_1(k) d\mathbf{k} \times \int v(-k) \exp(i\mathbf{k} \cdot \mathbf{z}_A) \mathbf{q}_2(k) d\mathbf{k} \right] \\
 & + \frac{1}{2} g \sum_A \frac{1}{2\pi^2} \left(\int v(-k) \exp(i\mathbf{k} \cdot \mathbf{z}_A) \mathbf{q}_\alpha(k) d\mathbf{k} \right)^2. \quad (97)
 \end{aligned}$$

The coupling constants f and g have now the dimensions of a length. The split of the field in a "zero field" and a residual field analogous to (17) to (19), which avoids cross terms between the $\mathbf{P}_{\alpha A}$ and the $\mathbf{p}_{\alpha A}'$ in the Hamiltonian is here given by

$$\int v(-k) \exp(i\mathbf{k} \cdot \mathbf{z}_A) \mathbf{q}_{\alpha'}(k) d\mathbf{k} = 0, \quad (98)$$

$$\sum_j \int v(k) \exp(-i\mathbf{k} \cdot \mathbf{z}_A) \left[p_{\alpha i}'(k) + \frac{1}{\mu^2} k_i k_j p_{\alpha j}'(k) \right] d\mathbf{k} = 0, \quad (98a)$$

$$\mathbf{p}_\alpha(k) = \frac{1}{\pi\sqrt{2}} \sum_A v(-k) \exp(i\mathbf{k} \cdot \mathbf{z}_A) \mathbf{P}_{\alpha A} + \mathbf{p}_{\alpha'}(k), \quad (99)$$

$$\mathbf{Q}_{\alpha A} = \frac{1}{\pi\sqrt{2}} \int v(-k) \exp(i\mathbf{k} \cdot \mathbf{z}_A) \mathbf{q}_\alpha(k) d\mathbf{k}. \quad (100)$$

We restrict ourselves here to the case where the sources do not overlap; then it follows from it

$$q_{\alpha i}(k) = \frac{1}{\bar{N}} \frac{1}{\pi\sqrt{2}} v(k) \sum_{A, j} \exp(-i\mathbf{k} \cdot \mathbf{z}_A) \left(Q_{\alpha A, i} + \frac{1}{\mu^2} k_i k_j Q_{\alpha A, j} \right) + q_{\alpha i}'(k), \quad (101)$$

$$P_{\alpha A, i} = \frac{1}{\bar{N}} \frac{1}{\pi\sqrt{2}} \sum_j \int v(k) \exp(-i\mathbf{k} \cdot \mathbf{z}_A) \left\{ p_{\alpha i}(k) + \frac{1}{\mu^2} k_i k_j p_{\alpha j}(k) \right\} d\mathbf{k}, \quad (102)$$

where \bar{N} is in analogy to (32), (33) given by

$$\bar{N} \delta_{ij} = \frac{1}{2\pi^2} \int G(k) \left(\delta_{ij} + \frac{k_i k_j}{\mu^2} \right) d\mathbf{k}, \quad (103)$$

or

$$\bar{N} = \frac{2}{\pi} \int_0^\infty G(k) \left(1 + \frac{1}{3} \frac{k^2}{\mu^2} \right) k^2 dk. \quad (103')$$

We remark that for $\mu a \ll 1$ the difference of \bar{N} and N/μ^2 is small, of the relative order $(\mu a)^2$. The commutation relations are

$$i[P_{\alpha A, i}, Q_{\beta B, j}] = \delta_{\alpha\beta} \delta_{AB} \delta_{ij},$$

as before, the $p_{\alpha i}'(k)$, $q_{\alpha i}'(k)$ commute with $P_{\alpha A, i}$ and $Q_{\alpha A, i}$ and in accordance with (98), (98a) one has

$$\begin{aligned}
 i[p_{\alpha i}'(k), q_{\beta j}'(k')] = & \delta_{\alpha\beta} \left[\delta(\mathbf{k} - \mathbf{k}') \delta_{ij} - \frac{1}{\bar{N}} \frac{1}{2\pi^2} \sum_A \exp[i(\mathbf{k} - \mathbf{k}') \cdot \mathbf{z}_A] \right. \\
 & \left. \cdot v(-k)v(k') \cdot \left(\delta_{ij} + \frac{1}{\mu^2} k_i k'_j \right) \right]. \quad (104)
 \end{aligned}$$

The Hamiltonian splits again into

$$H = H_0 + H' + \Omega.$$

Here H_0 is again given by (37) with a slightly different meaning of d , which coefficient however can also here be neglected in the strong coupling case. H' has the same form as the force-free function

$$H' = \frac{1}{2} \sum_{\alpha, i, j} \int \left\{ \left(\delta_{ij} + \frac{1}{\mu^2} k_i k_j \right) p_{\alpha i}'(k) p_{\alpha j}'(-k) + (k_0^2 \delta_{ij} - k_i k_j) q_{\alpha i}'(k) q_{\alpha j}'(-k) \right\} d\mathbf{k}, \quad (105)$$

but with the subsidiary conditions (98), (98a). Using

$$\sum_l \left(\delta_{il} - \frac{k_i k_l}{k_0^2} \right) \left(\delta_{lj} + \frac{k_l k_j}{\mu^2} \right) = \delta_{ij},$$

one obtains finally for Ω the expression

$$\Omega = \frac{1}{N} \sum_{\alpha, A, i} \mathbf{Q}_{\alpha A} \cdot \frac{1}{\pi\sqrt{2}} \int v(-k) \exp(i\mathbf{k} \cdot \mathbf{z}_A) k_0^2 \mathbf{q}_{\alpha}'(k) d\mathbf{k}. \quad (106)$$

In view of (98) in the integral k_0^2 can be replaced by k^2 , hence putting in analogy to (59)

$$\lambda_{\alpha A} = \frac{1}{\pi\sqrt{2}} \frac{1}{N} \int v(-k) k^2 \exp(i\mathbf{k} \cdot \mathbf{z}_A) \mathbf{q}_{\alpha}'(k) d\mathbf{k}, \quad (107)$$

we have as in (38')

$$\Omega = \sum_{\alpha, A} \mathbf{Q}_{\alpha A} \cdot \lambda_{\alpha A}. \quad (38')$$

The H' -problem can be treated as in Section 4. Instead of (58) we obtain from (105), using (104), (98), (98a) and omitting the index α ,

$$\begin{aligned} \dot{q}_i'(k) &= \sum_j \left(\delta_{ij} + \frac{1}{\mu^2} k_i k_j \right) \dot{p}_j'(-k), \\ -\dot{p}_j(-k) &= \sum_l (k_0^2 \delta_{jl} - k_j k_l) \dot{q}_l(k) - \frac{1}{\pi\sqrt{2}} v(k) \sum_A \exp(-i\mathbf{k} \cdot \mathbf{z}_A) \lambda_{Aj}, \end{aligned}$$

where λ_A is defined by (107). For $\ddot{q}_i'(k)$ one obtains

$$-\ddot{q}_i'(k) = k_0^2 \dot{q}_i'(k) - \frac{1}{\pi\sqrt{2}} v(k) \sum_{A, j} \exp(-i\mathbf{k} \cdot \mathbf{z}_A) \left(\delta_{ij} + \frac{1}{\mu^2} k_i k_j \right) \lambda_{Aj}. \quad (108)$$

The only changes necessary in the computations of Section 4 for vector mesons are therefore the replacements of $k_i k_j$ by $\delta_{ij} + (1/\mu^2) k_i k_j$ from Eq. (63) on. Instead of I we get

$$\bar{I} = \frac{2}{\pi} \int_0^\infty \frac{k^2 (1 + \frac{1}{3} k^2 / \mu^2)}{-l^2 + k^2} G(k) dk = \frac{2}{\pi} \int_0^\infty \frac{k^2}{-l^2 + k^2} G(k) dk + \frac{1}{\mu^2} I, \quad (109)$$

and instead of J_{ij}

$$\bar{J}_{ij} = J \delta_{ij} + \frac{1}{\mu^2} J_{ij}, \quad (110)$$

where J_{ij} and J were defined by (75), (76). As

$$\frac{2}{\pi} \int_0^\infty \frac{k^2}{-l^2 + k^2} G(k) dk = \frac{1}{a} + l^2 \cdot \frac{2}{\pi} \int_0^\infty \frac{G(k)}{-l^2 + k^2} dk,$$

we have for $l \gg a^{-1}$

$$\bar{I}_\pm = \frac{1}{\mu^2} I_\pm + \frac{1}{a} \pm il, \quad (111)$$

or according to (81) with

$$\bar{A} = A + \frac{1}{3} \frac{\mu^2}{a}, \tag{112}$$

$$3\mu^2 I_{\pm} = 3\bar{A} + (l^2 + 3\mu^2) \frac{1}{a} \pm il(l^2 + 3\mu^2). \tag{111a}$$

We notice that for $\mu a \ll 1$ the difference between \bar{A} and A is negligible.

The computation of \bar{V}_{AB} for the vector theory follows the same line as for the pseudoscalar theory. According to (110) one obtains for $r \gg a$ instead of (83)

$$\bar{V}_{AB} = -\frac{1}{2\pi i} \frac{1}{\bar{A}^2} \int_0^\infty \frac{ldl}{(l^2 + \mu^2)^{\frac{3}{2}}} \left[2 \left(-\frac{1}{r} \frac{dJ_+}{dr} + \mu^2 J_+ \right)^2 - 2 \left(-\frac{1}{r} \frac{dJ_-}{dr} + \mu^2 J_- \right)^2 + \left(-\frac{d^2 J_+}{dr^2} + \mu^2 J_+ \right)^2 - \left(-\frac{d^2 J_-}{dr^2} + \mu^2 J_- \right)^2 \right], \tag{113}$$

or

$$\bar{V}_{AB} = -\frac{1}{2\pi i} \frac{1}{\bar{A}^2 r^6} \int_0^\infty \frac{ldl}{(l^2 + \mu^2)^{\frac{3}{2}}} e^{2ilr} [2(-1 + ilr - \mu^2 r^2)^2 + (2 - 2ilr - l^2 r^2 - \mu^2 r^2)^2] + \text{conj. compl.} \tag{114}$$

One finds by comparison with (84)

$$\bar{V}_{AB} = \frac{A^2}{\bar{A}^2} V_{AB} - \frac{1}{\pi \bar{A}^2 r^6} \int_0^\infty \frac{ldl}{(l^2 + \mu^2)^{\frac{3}{2}}} \sin 2lr \cdot (2\mu^2 l^2 + 3\mu^4) r^4. \tag{115}$$

With the help of the function $K(x)$ already used, we obtain

$$\bar{V}_{AB} = -\frac{\mu}{\pi \bar{A}^2 r^6} \bar{g}(2\mu r), \tag{115a}$$

with

$$\bar{g}(x) = g(x) + \left(\frac{x}{2}\right)^4 [-2K''(x) + 3K(x)], \tag{115b}$$

which can be transformed into

$$\bar{g}(x) = \left(\frac{43}{2} + \frac{59}{16}x^2 + \frac{1}{8}x^4\right)K(x) + \left(\frac{43}{4}x + \frac{1}{2}x^3\right)K_0(x). \tag{115c}$$

The evaluation for $\mu r \ll 1$ shows that there the second term in (115) is small compared to the first by the relative order $(\mu r)^2$; hence in view of (85a)

$$\bar{A}_{AB} = \frac{A^2}{\bar{A}^2} V_{AB} = -\frac{43}{4\pi} \frac{1}{\bar{A}^2 r^7} \quad \text{for } \mu r \ll 1. \tag{116a}$$

For $\mu r \gg 1$, the second term turning out to be equal to the first; hence,

$$\bar{V}_{AB} = 2 \frac{A^2}{\bar{A}^2} V_{AB} = -\frac{1}{\sqrt{\pi}} \frac{\mu^7}{\bar{A}^2} \frac{1}{(\mu r)^{5/2}} e^{-2\mu r} \quad \text{for } \mu r \gg 1. \tag{116b}$$

Finally, in analogy to the development of Section 5, the spin-independent case $\bar{f} = 0$ can be treated rigorously and leads to the result that for \bar{I} there has to be substituted $1/g + \bar{I}$. Thus for $r_{AB} \gg a$, V_{AB} is again given by (115) if the factor $1/\bar{A}^2$ is replaced by $[(\mu^2/\bar{g}) + \bar{A}]^{-2}$. The relative error of the result (115) for strong coupling is here therefore of the order $\mu^2/\bar{g}\bar{A}$. In the general case this relative error will therefore be

$$\mu^2 [(\bar{g} + \bar{f})\bar{A}]^{-1}, \quad \text{and} \quad \mu^2 [(\bar{g} - \bar{f})\bar{A}]^{-1}, \tag{117}$$

and the strong coupling condition

$$(\bar{g} + \bar{f})\bar{A} \gg \mu^2, \quad (\bar{g} - \bar{f})\bar{A} \gg \mu^2, \tag{118}$$

or for $\mu a \ll 1$,

$$(\bar{g} + \bar{f})a^3 \gg \mu^2, \quad (\bar{g} - \bar{f})a^3 \gg \mu^2. \tag{118'}$$

One has to remember that for vector mesons, \bar{g} and \bar{f} have the dimension of a length and not of a volume as in the scalar theory.

APPENDIX

Transformation of a Determinant

In Section 5 we had to evaluate a determinant of the form

$$\|\phi_{nn'}\| = \left\| \delta_{nn'} + \sum_{i=1}^N f_i(n)g_i(n') \right\|.$$

On the number of rows or columns we only suppose that it is larger or equal to N (it may even be infinite, if the functions $f_i(n)$, $g_i(n)$ are decreasing with increasing n with a sufficient degree). We now make an S -transformation

$$\phi' = S^{-1}\phi S,$$

which certainly does not change the determinant:

$$\|\phi'\| = \|\phi\|,$$

by choosing S in the following way:

$$(n|S|m) \begin{cases} = f_m(n) \text{ for } m=1, 2, \dots, N \\ \text{is undetermined for } m > N. \end{cases}$$

Then we obtain

$$\phi'_{mm'} = \delta_{mm'} + \sum_{n, n'} (m|S^{-1}|n) \left[\sum_{i=1}^N (n|S|i)g_i(n') \right] (n'|S|m')$$

and because of

$$\sum_n (m|S^{-1}|n)(n|S|i) = \delta_{mi},$$

$$\phi'_{mm'} = \delta_{mm'} + \sum_{n'} \sum_{i=1}^N \delta_{mi}g_i(n')(n'|S|m').$$

Hence,

$$\phi'_{ij} = \delta_{ij} + \sum_n g_i(n)f_j(n) \text{ for } i \leq N, j \leq N,$$

$$\phi'_{im'} \text{ is undetermined for } i \leq N, m' > N,$$

$$\phi'_{mm'} = \delta_{mm'} \text{ for } m > N \text{ and all } m'.$$

The determinant has therefore the form

	N	
N	$\delta_{ij} + \sum_n g_i(n)f_j(n)$	undetermined
	0	1

where 1 means the unit matrix. Hence it is equal to the smaller determinant with N rows and N columns:

$$\|\delta_{ij} + \sum_n g_i(n)f_j(n)\| = \|\delta_{ij} + \sum_n f_i(n)g_j(n)\|,$$

as was stated in the text.