REVIEWS OF MODERN PHYSICS VOLUME 17, NUMBERS 2 AND 3 APRIL–JULY, 1945

Theory of Cosmic-Ray Mesons

W. HEITLER AND P. WALSH
Dublin Institute for Advanced Studies, Dublin, Ireland

1. INTRODUCTION AND GENERAL THEORY

IN a recent paper[1] Hamilton, Peng, and one of us (H.) gave a theory for the production of mesons by proton-nucleon collisions in the high atmosphere and various phenomena connected with this process. This theory is based on the divergence-free quantum theory of damping (references in HHP) which, for collision processes, leads to an integral equation. For the process in question this integral equation is extremely difficult to solve and it was therefore necessary to use the simpler but cruder method of Weizsäcker-Williams. It has since been realized that the application of this method requires particular care in the case where the two colliding particles have equal masses and is, in this case far less accurate than if one of the particles is relatively very heavy. In fact the way in which the method was applied in HHP has turned out to be defective and the results require re-examination.

The point in question is this: Let a fast particle collide with a particle at rest and consider the emission of a quantum (light quantum or meson). If the moving particle is sufficiently fast its field is equivalent to a spectrum of quanta with various energies ϵ. One of these virtual quanta ϵ can then be scattered by the particle at rest (the scattered quantum having energy ϵ'). The process appears as emission of a quantum of energy ϵ', while the energy ϵ is lost by the fast particle and $\epsilon-\epsilon'$ transferred to the particle at rest. In order that this way of computing the cross section of emission of ϵ' be valid it is necessary that, throughout the collision, the fast particle (energy E) can be considered as practically undisturbed. The condition for this to be true is

$$\epsilon \ll E. \tag{1}$$

There are, of course, also collisions in which (1) is not satisfied. In HHP these were taken into account by extrapolating the results up to $\epsilon=E$. It is this point which is incorrect. We may consider in particular the "opposite case" to (1) where ϵ is comparable with E but the energy transfer to the particle at rest is small, $\epsilon-\epsilon'\ll M$ (M being the rest energy of the particle at rest). This case can also be treated by the present method, namely by considering the process from the opposite Lorentz system in which the fast particle (mass m) is brought to rest. The particle M remains then practically undisturbed and its field is therefore equivalent to a beam of virtual mesons ϵ^*. The condition $\epsilon-\epsilon'\ll M$ becomes $\epsilon^*\ll ME/m=$ energy of the particle M in the opposite Lorentz system. The contribution to the effect from these collisions can thus be obtained by working first in the "opposite Lorentz system" and transforming the *result* back to the original system. In the following we call the two cases $\epsilon\ll E$ and $\epsilon-\epsilon'\ll M$ contributions I and II, respectively. It is in fact well known that contribution II is the only one of importance if $M\gg m$ (cf. bremsstrahlung emitted by an electron passing through an atomic field), because the probability of scattering of a quantum by a very heavy particle is negligible.

If $m=M$ it might be thought that little is gained by such a transformation since contribution II appears in the opposite Lorentz system precisely as the same process as I in the original system. It is true, indeed, that the total number of quanta obtained from I and II are the same, but when transforming the results for contribution II back to the original system the energy distribution changes decisively. There are naturally also cases in which neither $\epsilon\ll E$ nor $\epsilon-\epsilon'\ll M$. These cannot be treated accurately by the Weizäcker-Williams method. It might be expected, however, that a fair if not very accurate picture of the process is obtained by adding up the two limiting cases I and II and interpreting the condition (1) in the sense, say, $\epsilon<E/2$. It must be pointed out, however, that the results are essentially less accurate than in the well-

[1] Hamilton, Heitler, and Peng, Phys. Rev. **64**, 78 (1943). This paper will be referred to as HHP. For the theoretical foundation of the present paper see the references in HHP.

known cases where $m \ll M$, and we may very well expect errors of the order of magnitude of a factor 2 for the energy spectrum or total number of mesons. In Section 4 we shall consider a phenomenon which depends explicitly on the *ratio* of the contributions I and II. For this we must expect even larger errors and can hope for no more than to obtain the correct order of magnitude.

The two contributions I and II for meson emission in nucleon-nucleon collisions have been worked out by one of us (H.),[2] and we quote the results. The form of the meson theory used is the same as in HHP, i.e., a charge-symmetrical mixture of pseudoscalar and vector fields. So far no conclusive evidence as to whether the masses of these two kinds of mesons are equal or not exists and we assume for simplicity that they are equal *viz.*, $\frac{1}{10}$ of the proton mass.[3] It is hardly to be expected that the results obtained in this paper will change much if the two masses are different. We use the same natural meson units as in HHP $c=\hbar=\mu=1$. Cross sections are therefore in units of $(\hbar/\mu c)^2=4.3\cdot 10^{-26}\,\text{cm}^2$ and energies in units of $\mu c^2=0.94\times 10^8$ ev. The cross section for production of a meson of energy ϵ in a collision where the fast nucleon has energy E is then, according to H:

$$\left.\begin{aligned}\overset{\mathrm{I}}{\Phi}_{pNR}d\epsilon&=\frac{4}{3}f^2(D_t+D_p)\frac{d\epsilon}{\epsilon^3}\left(1-\frac{\epsilon}{M}\right) &&\text{(2a)}\\ \overset{\mathrm{I}}{\Phi}_{tNR}d\epsilon&=2\overset{\mathrm{I}}{\Phi}_{pNR}d\epsilon &&\text{(2b)}\end{aligned}\right\}(1/f<\epsilon<M/2),$$

$$\left.\begin{aligned}\overset{\mathrm{I}}{\Phi}_{pER}d\epsilon&=\frac{64}{5}f^2D'\frac{d\epsilon}{(E)^{\frac{1}{2}}}[(1/\epsilon)^{3/2}-\epsilon(2/E)^{5/2}] &&\text{(2c)}\\ \overset{\mathrm{I}}{\Phi}_{tER}d\epsilon&=\frac{16}{3}\frac{f^2}{M}D_td\epsilon[1/\epsilon^2-\epsilon(2/E)^3] &&\text{(2d)}\end{aligned}\right\}(M/2<\epsilon<E/2),$$

$$\overset{\mathrm{II}}{\Phi}_{pNR}d\epsilon=\frac{2}{9}Mf^2(D_t+D_p)\frac{E-\epsilon}{E^2}d\epsilon\cdot\begin{cases}\left(f^3-\dfrac{8}{M^3}\right) & \text{if } 1<\epsilon<2E/Mf\\ \dfrac{8}{M^3}((E/\epsilon)^3-1) & \text{if } 2E/Mf<\epsilon<E,\end{cases} \tag{2e}$$

$$\overset{\mathrm{II}}{\Phi}_{tNR}d\epsilon=2\overset{\mathrm{II}}{\Phi}_{pNR}d\epsilon, \tag{2f}$$

$$\left.\begin{aligned}\overset{\mathrm{II}}{\Phi}_{pER}d\epsilon&=\frac{64\sqrt{2}f^2}{M}D'\frac{d\epsilon}{E^3}(E-\epsilon)[((E-\epsilon)/M)^{\frac{1}{2}}-1] &&\text{(2g)}\\ \overset{\mathrm{II}}{\Phi}_{tER}d\epsilon&=\frac{32f^2}{M}D_t\frac{d\epsilon}{E^3}(E-\epsilon)((E-\epsilon)/M-1) &&\text{(2h)}\end{aligned}\right\}1<\epsilon<E-M.$$

Here Φ^{I} and Φ^{II} are the contributions I and II to the cross section. Each formula is valid for the region of ϵ indicated; these regions partly overlap and in this case the various contributions have to be added. Expression (2) is valid for $E\gg M$, but may still be used if E is not much larger than M; then E is the momentum or total energy of the nucleon but not the kinetic energy (total energy minus rest energy). The constants are the same as in HHP.

$$D_t=165,\quad D_t+D_p=200,\quad D_t+2D_p=230,\quad D'=50,\quad f^2=0.13. \tag{3}$$

The total cross section for the production of a

[2] W. Heitler, Proc. Roy. Ir. Ac. (1945), in the press. This paper will be referred to as H.

[3] We do this also because only in this case the numerical values of all the coupling constants are sufficiently well known. The case $\mu_v \neq \mu_p$ (masses of vector and pseudoscalar mesons) has recently been investigated by J. M. Jauch and N. Hu (Phys. Rev. **65**, 289 (1944)) with the view of giving an accurate account of the quadrupole moment of the deuteron. Since, however, the latter is a small effect and depends also on higher approximations of the proton-neutron potential (especially on the velocity dependent terms) it has not been possible to determine all the constants conclusively, i.e., the two coupling constants g, f and the mass ratio μ_p/μ_v.

meson is

$$\Phi=\int_1^E(\Phi_p+\Phi_t)d\epsilon=10=4.3\cdot 10^{-25}\ \mathrm{cm}^2. \quad (4)$$

The energy distribution of Φ^{II} is much more in favor of large ϵ than that of Φ^{I}. Consequently, the energy loss will be greater than according to HHP. We use the same unit for the thickness of matter traversed as in HHP, i.e., the thickness which a fast particle has to travel in order to lose the energy μc^2 (meson rest energy). In these units the height of the atmosphere is 22. We call these units x-units. One x-unit$=3.45$ cm Hg. According to H the distance travelled by a fast nucleon while losing the energy from E_0 to E is

$$x_{E_0,E}=\tfrac{1}{3}\log(E_0/E), \quad (5)$$

(according to HHP $x_{E_0,E}$ would increase like $E_0/\log E_0$). Equation (5) actually holds only for large E_0 and E, but it was shown in H to be still valid for $E_0=3M$ and $E=2M$.

Below we shall also require the total average number of mesons $\bar{n}$ produced by a nucleon travelling the distance $x_{E_0,E}$ and losing energy from E_0 to E. For this figure a table is given in H. It is seen that $\bar{n}$ increases about logarithmically with E_0 (the leading term of the formula is $\sim\log(E_0/E)$). Actually the data in question are well represented by

$$\bar{n}=a\log\frac{E_0}{E}, \quad a=2.5\text{—}3, \quad (6)$$

where $a=2.5$ holds for $E_0=2M$—$5M$ and $a=3$ for larger E_0. This formula, like (5), holds only if E as well as E_0 is larger than M. Meson production ceases when $E<M$ as was pointed out in HHP and H. For E in the neighborhood of M (6) in inaccurate. Equation (6) includes pseudoscalar and transverse charged mesons of all energies. We note that exactly half of the values (6) is due to each of the contributions I and II.

A further remark is to be made in connection with (5). The term $x_{E_0,E}$ is the average distance traveled by the fast nucleon provided that all the nucleons of the matter traversed are distributed at random. It has been pointed out by Janossy[4] that the fact that the nucleons are concentrated in nuclei leads to an actual mean free path of the nucleon quite different from (5). The actual absorption in air or any other material except hydrogen is quite different from the range relation (5) as will be shown in Section 6. If we are, however, interested in the meson intensity below a thickness, large compared with the actual mean free path of the primary nucleons (in the atmosphere a few cm Hg), the actual law of absorption makes no difference and (4) can be used instead, because it does not matter then at which point precisely (within the distance of the mean free path) the mesons are produced.[5]

[4] L. Janossy, Phys. Rev. **64**, 345 (1943).

2. ENERGY SPECTRUM AND TOTAL NUMBER OF MESONS

We assume that a primary spectrum of protons with an energy distribution $F(E_0)dE_0$ is falling on the top of the atmosphere and consider the diffusion of the mesons produced through the atmosphere. The latter are absorbed through ionization and β-decay. With the same assumptions as made in HHP the number of mesons $f(\epsilon, x)d\epsilon$ of energy ϵ at a depth x below the top of the atmosphere (in x-units) is given by

$$\frac{\partial f}{\partial x}=\frac{\partial f}{\partial \epsilon}-\frac{b}{\epsilon x}f+S, \quad (7)$$

where

$$S=\int_\epsilon^\infty \Phi(\epsilon, E)F(E, x)dE, \quad (8)$$

Φ being the cross section for production of a meson ϵ by a nucleon with energy E and $F(E, x)$ the number of nucleons found with energy E at the depth x. b is connected with the proper life time of the meson at rest by

$b=x/c\tau$ (height of atmosphere in cm reduced to normal pressure). (9)

[5] In order that this be true the assumption has to be made that during the passage of a fast nucleon through a nucleus all the nuclear particles in the nucleus contribute independently to meson production in spite of the fact that the cross section for meson production is even larger than the average distance between two nucleons. It seems that this assumption is justified in view of the small binding energy of the nucleus, but may require further examination.

The solution of (7) is:

$$f(\eta, x)=\left(\frac{\eta-x}{x}\right)^{b/\eta}\int_0^x\left(\frac{\xi}{\eta-\xi}\right)^{b/\eta}$$

$$\int_\eta^\infty \Phi(\eta-\xi, E)F(E, \xi)dEd\xi, \quad (10)$$

$$\eta=\epsilon+x. \quad (11)$$

We assume that the primary spectrum (i.e., $F(E_0, 0)\equiv F(E_0)dE_0$) is of the form

$$F(E_0)=A/E_0^{\alpha+1}. \quad (12)$$

All the experimental evidence points to the fact that α is fairly constant over a wide range of E_0. It will be seen that (12) with a constant α represents the experimental facts fairly well, at any rate within the accuracy of the present theory. The number of nucleons at a depth ξ with energy E follows then from (5)

$$F(E, \xi)dE=Ae^{-3\alpha\xi}dE/E^{\alpha+1}. \quad (13)$$

It is seen from (13) that only very small values of ξ contribute to (11). For $\eta=\epsilon+x\geqslant 3$, say, we can neglect ξ against η and besides extend the integration over ξ in (10) to infinity. Equation (10) then becomes

$$f(\eta, x)=A\left(\frac{\epsilon}{\eta x}\right)^{b/\eta}\frac{\Gamma(1+b/\eta)}{(3\alpha)^{1+b/\eta}}$$

$$\times\int_\eta^\infty\Phi(\eta, E)E^{-\alpha-1}dE. \quad (14)$$

For Φ the cross sections (2) have to be inserted. The limits $\eta\cdots\infty$ have to be replaced, in some cases, by narrower limits, according to the regions of validity of the formulae (2). The integration is elementary. f consists of 4 contributions $f_{NR}{}^{\mathrm{I}}$, $f_{ER}{}^{\mathrm{I}}$, $f_{NR}{}^{\mathrm{II}}$, $f_{ER}{}^{\mathrm{II}}$, but for values $x\geqslant 5$ to which we shall confine ourselves ($\sim$17 cm Hg), $f_{NR}{}^{\mathrm{I}}=0$. We assume that the mesons observed at sea level are pseudoscalar mesons and that the transverse mesons have a very short lifetime.[6] Thus, for pseudoscalar mesons we find:

$$f_{ER}{}^{\mathrm{I}}=X(\eta, x)\cdot\frac{32D'}{M}\frac{1}{\eta}\frac{1}{2^{\alpha+\frac{1}{2}}(\alpha+\frac{1}{2})(\alpha+3)},$$

$$(\eta>M/2), \quad (15a)$$

$$f_{NR}{}^{\mathrm{II}}=X(\eta, x)\frac{2M(D_t+D_p)}{9}\left\{\left(f^3-\frac{8}{M^3}\right)\right.$$

$$\times\left(\frac{2}{Mf}\right)^{\alpha+1}\left(\frac{1}{\alpha+1}-\frac{1}{\alpha+2}\frac{2}{Mf}\right)$$

$$+\frac{8}{M^3}\left[\frac{1}{\alpha-2}(1-(2/Mf)^{\alpha-2})\right.$$

$$-\frac{1}{\alpha-1}(1-(2/Mf)^{\alpha-1})$$

$$-\frac{1}{\alpha+1}(1-(2/Mf)^{\alpha+1})$$

$$\left.\left.+\frac{1}{\alpha+2}(1-(2/Mf)^{\alpha+2})\right]\right\}, \quad (15b)$$

$$f_{ER}{}^{\mathrm{II}}=X(\eta, x)\frac{64\sqrt{2}D'}{M\eta^{\frac{1}{2}}}\left\{\frac{3}{4}\left(\frac{\pi}{M}\right)^{\frac{1}{2}}\frac{\Gamma(\alpha+\frac{3}{2})}{\Gamma(\alpha+4)}\right.$$

$$-\frac{1}{M^{\frac{1}{2}}}\int_0^{M/\eta}\frac{y^{\frac{1}{2}}dy}{(y+1)^{\alpha+4}}$$

$$\left.-\frac{\eta^{\alpha+\frac{3}{2}}}{(\eta+M)^{\alpha+2}}\left(\frac{1}{\alpha+2}-\frac{\eta}{\eta+M}\frac{1}{\alpha+3}\right)\right\}, \quad (15c)$$

where

$$X(\eta, x)=Af^2\left(\frac{\eta-x}{\eta x}\right)^{b/\eta}\frac{\Gamma(1+b/\eta)}{\eta^{\alpha+1}}(3\alpha)^{-1-b/\eta}. \quad (15d)$$

For large values of x or ϵ (large η) f decreases like $\eta^{-\alpha-1}$ but there are also contributions behaving like $\eta^{-\alpha-2}$ and $\eta^{-\alpha-\frac{3}{2}}$. The tail end of the energy spectrum decreases therefore like $\epsilon^{-\alpha-1}$. It is known from experiments that the decrease is about $\epsilon^{-2.5}$. It is also known that under large thicknesses the total intensity decreases like $x^{-1.8}$. The total intensity being the integral $\int^\infty f d\epsilon$ this suggests a value $\alpha=1.5$ or 1.8. For the following we have assumed $\alpha=1.5$ but have also worked out the spectrum at sea level for $\alpha=1.8$. We found that the difference is not great and is within the accuracy of the present theory.

[6] A controversy has recently arisen about this point and arguments have been put forward to show that the mesons at sea level may be vector mesons. It does not seem that these arguments are very conclusive yet, either way, but it seems to us that it is more probable that the mesons at sea level are pseudoscalar. No doubt the possibility has to be kept in mind that the roles of the two kinds of mesons have to be reversed. This will hardly cause any serious alterations in our theory since the formulae for transverse mesons are not very different from those for pseudoscalar mesons. Cf. S. Kusaka, Phys. Rev. **64**, 256 (1943); S. K. Chakrabarty and R. C. Majumdar, Phys. Rev. **65**, 206 (1944), and literature quoted there.

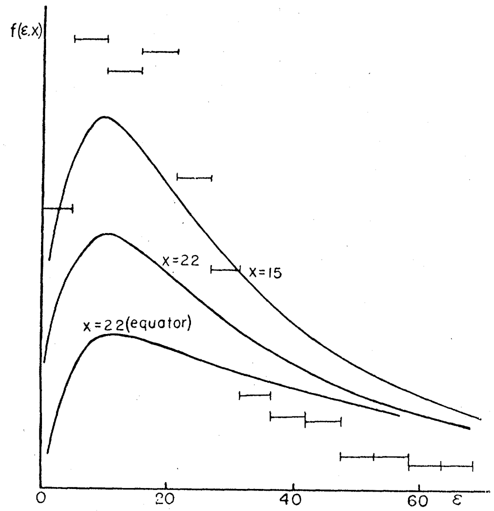

FIG. 1. Energy spectrum of mesons at sea level ($x=22$) and mountain heights ($x=15$) theoretical. The same for sea level experimental. Energy spectrum at sea level at the geomagnetic equator (theoretical).

The value of α presumably decreases with decreasing E_0 and it seems likely that for the sea level measurements $\alpha=1.5$ is a better value than 1.8.

To find b we have assumed $\tau=2.7\times10^{-6}$ sec. and a meson mass of 184 m, b then is[7]

$$b=9.5. \tag{16}$$

All the constants being fixed f can easily be evaluated. (The integral occurring in (15c) is an "incomplete beta-function" which is tabulated.[8]) The results for f in the region of medium energies $\epsilon=5$—70 depend rather sensitively on b. This constant is only known to within 10–20 percent (some measurements disagree with the value (16) still more).

In the present calculations we have neglected the contribution to the meson component arising from fast secondary nucleons. As was pointed out in HHP fast recoil nucleons are also produced together with the mesons and these nucleons produce further mesons, etc. It was estimated[9] that in this way the total number of mesons is increased by about a factor 2. In the present version of the theory this effect will probably be relatively smaller because fast recoil nucleons arise only from contribution I, i.e., in half the number of all collisions. The mesons produced by secondary, tertiary, etc., nucleons are, of course, mainly slow mesons. We may well expect an increase of the number of slow mesons by a factor 2 and of the total number by 50 percent or so. The effect will also become increasingly important with increasing height. (Compare also Section 6.)

The energy spectrum $f(\epsilon, x)$ is plotted in Fig. 1 for sea level $x=22$ and mountain heights $x=15$. The experimental points are those of Hughes,[10] who has measured $f(\epsilon, x)$ in energy intervals of 0.5×10^9 ev. The agreement in the general shape of the curve is as good as can be expected in the circumstances. The experimental curve falls down more steeply than the theoretical one. This is just what was to be expected and the relatively larger number of slow mesons found experimentally may largely be due to the secondary etc. nucleons. A value of b somewhat smaller (8 say, i.e., $\tau=3\times10^{-6}$ sec.) would also give a curve falling down more steeply between $\epsilon=15-70$.

We obtain the total number of mesons by integration (graphical) over ϵ. This number we can compare with the total number of mesons observed and normalized to, say, 100 incident protons. The number of incident protons can only be obtained by extrapolation of the Regener-Pfotzer curve or Schein's experiments in the high atmosphere. It is only known to within about 10–20 percent.[11] We assume that the number of mesons at sea level is 4 for 100 incident particles (at the latitude where these measurements are made). The normalization in our theory depends on the value of the geomagnetic cut-off energy. We assume an average value of $P=3\times10^9$ ev, P being the momentum of the incident particles. As was mentioned above we have to equate E in all our formulae to P or the energy+rest energy, when E approaches the

[7] The value $b=13$ given in HHP was erroneous.

[8] Pearson, *Tables of the Incomplete Beta-Function* (Cambridge University Press, 1934).

[9] Peng, Proc. Roy. Ir. Acad. **49A**, 245 (1944).

[10] D. J. Hughes, Phys. Rev. **57**, 594 (1940).

[11] We do not think that the calculations of Bowen, Millikan, and Neher (Phys. Rev. **53**, 855 (1938)) which are based on the total ionization produced in the atmosphere lead to a reliable estimate of the total number of incident protons, because a considerable fraction of the incident energy is transferred to neutrons, and possibly neutrettos.

value M. Thus we have for the cut-off energy $E=33.5$, in our energy units. (In HHP the value $E=22$ referred to $E-M$.) This gives $A=100\alpha(33.5)^{\alpha}$. In this way Table I is obtained. The increase with height is better compared with the experiments by normalizing the theoretical number at sea level to agree with the experimental figure. These are the figures in brackets. The total number at sea level 2.8 agrees well with the experimental 4 considering the inaccuracies of theory and experiment. The increase with height is somewhat less rapid than the experimental figures. But this is to be expected. The experimental figures refer to the number of penetrating particles which undoubtedly include protons. As was pointed out in HHP a proton no longer produces mesons when its energy is less than M. Such protons penetrate through a considerable depth, roughly half the atmosphere. In the high atmosphere we must therefore expect to find an increasing number of the primary protons whose energy has, by meson emission, degenerated below the value M. It is true that roughly half of them have become neutrons and are not detected by counters, but, on the other hand, recoil protons are also produced. Many of them have sufficient energy to be classed as "penetrating particles." There can therefore be little doubt that a large fraction of what is called penetrating component consists of protons in the high atmosphere and their relative importance increases with height. All experimental facts show, indeed, the rapid increase of the proton component with height. The mesons produced by secondary etc., nucleons also, of course, add to the penetrating component and their relative importance increases also with height.

3. LATITUDE EFFECT

In the preceding calculations no account was taken of the latitude cut-off for the evaluation of the meson component. The effect is entirely negligible for latitudes higher than 48° (cut-off momentum 3×10^9 ev) since meson production ceases when E is less than M and the primaries between $E=M$ and $3M$ contribute next to nothing to the mesons found at any depth $x\geqslant5$, say. No increase of the meson intensity with increasing latitude is therefore to be expected even at great heights. ("Knee" of the latitude effect.)

TABLE I. Total number of mesons (for 100 incident protons) for various heights. Experimental number of penetrating particles.

x	22	15	10	5
number of mesons (theor.)	2.8 (4)	3.8 (5.4)	4.8 (6.9)	7.5 (10.7)
number of penetrating particles (exp.)	4	5.7	9.8	20

We now evaluate the difference in meson intensity at the equator. If E_θ is the cut-off energy at the latitude θ the difference of the meson intensity at the depth x and energy ϵ is obtained from (10) and (13) by a change of the limits of integration:

$$\Delta_\theta f(\eta,x)=A\left(\frac{\eta-x}{x}\right)^{b/\eta}\int_\eta^{E_\theta}\frac{\Phi(\eta,E)dE}{E^{\alpha+1}}\times\int_0^{\frac{1}{3}\log E_\theta/E}\xi^{b/\eta}e^{-3\alpha\xi}d\xi. \quad (17)$$

Again the limits η, E_θ are in some cases superseded by stricter limits for the various contributions of f. The integral over ξ is an incomplete gamma-function which is tabulated.[12] The integration over E is then very easily performed graphically. In Fig. 1 the energy spectrum to be expected at the equator at sea level is also plotted. The spectrum is much flatter than at higher latitudes. To our knowledge no measurements of the spectrum exist.

The decrease in total intensity is obtained by integration over ϵ. The result is given in Table II, and there compared with the experiments. The experimental figure (12 percent at sea level) refers to the total intensity of cosmic radiation (hard+soft) but is probably not much different for mesons alone. The discrepancy of this figure with the theoretical (21 percent) is within the accuracy of the theory.

TABLE II. Latitude effect in percent.

x	22	15	10	5
$\Delta f/f$ (theor.)	21	31	39	51
$\Delta f/f$ (exp.)	12	—	—	—

[12] Pearson, *Tables of the Incomplete Gamma-Function* (Cambridge University Press, 1934).

On the whole it is seen that the present theory gives results not much different from those found according to the earlier version (HHP) of the theory. One might be surprised at this fact since the cross sections for meson production are very different in the two cases. The fact can, however, easily be understood: According to HHP a very fast primary could penetrate to a considerable depth of the atmosphere and although those primaries are very rare they produce a very large number of mesons. These mesons, although having comparatively small energies, can reach sea level easily as they have to travel only a short distance. According to the present theory *all* mesons are produced near the top of the atmosphere, but their energies are on the average much higher and their chance of reaching sea level turns out to be about the same as before.

4. POSITIVE EXCESS

All measurements of the energy spectrum at sea level agree in that the number of positive mesons is higher than the number of negative mesons. The difference $\Delta_+ f/f$, say, is found to be of the order of magnitude of 7–10 percent. In trying to understand the positive excess according to our theory we shall find that we are dealing here with an effect which depends on the separation of the meson emission into contributions I and II. In fact the positive excess will be seen to be proportional to $f^{\mathrm{I}}/(f^{\mathrm{I}}+f^{\mathrm{II}})$. We must realize that this separation is a rather artificial device, for the purpose of calculation, and that no sharp distinction between the two contributions can be made in reality. The Weizsäcker-Williams method must be considered too crude for this purpose and we cannot hope to obtain more than the correct order of magnitude of the effect.

We consider a fast nucleon passing through an oxygen or nitrogen nucleus. The contribution I is then to be understood as follows: The fast nucleon *loses* a meson which is scattered by the nucleus. The meson is therefore positive/negative if the fast nucleon is a proton/neutron. On the other hand, if the meson arises from contribution II it is a particle belonging to the nucleus which emits the meson, the latter being scattered by the fast nucleon. Since the nuclei in question consist of an equal number of protons and neutrons, the meson will be with probabilities $\frac{1}{2}$ positive/negative. A difference in the number of positive and negative mesons arises therefore only from contribution I.

Yet it would not be correct to say that $f^{\mathrm{I}}/(f^{\mathrm{I}}+f^{\mathrm{II}})$ gives the positive excess. This would be true only if the fast nucleon were always a proton. However, a primary proton emits not only one single meson but several mesons in succession. As soon as one meson is emitted by contribution I the proton changes into a neutron. The second meson emitted by contribution I will therefore be negative. The third meson (always considering only contribution I) will be positive again and so forth. A particular meson emitted by a primary with energy E at the depth ξ below the top of the atmosphere will therefore be positive/negative if (i) it is emitted by contribution I and (ii) if the primary has emitted *before* (i.e., in traveling the distance ξ) an even/odd number of mesons by contribution I (no matter whether any of these mesons reach sea level or not). The positive excess depends therefore on the probability that a primary at depth ξ has already emitted a certain number n of mesons by contribution I. We call this probability γ_n. The quantity required is obviously

$$\gamma_0+\gamma_2+\cdots-(\gamma_1+\gamma_3+\cdots)\equiv\gamma,$$

γ, of course depends on ξ (it will be seen that it is a function of ξ only).

The number of mesons found at sea level is given by (10). The integrand of the double integral has the following physical meaning: $\Phi\cdot F$ is the number of mesons produced at depth ξ by the primaries. $\{(\eta-x)\xi/x\eta\}^{b/\eta}$ is the probability that these mesons reach sea level without decaying. To obtain the difference between the numbers of positive and negative mesons we have therefore to:

(i) substitute Φ^{I} for Φ in (10),
(ii) insert the factor $\gamma(\xi)$ under the double integral (10).

The total average number of mesons emitted is given by (6). Comparing (6) with (5) we see that

$$\bar{n}=3a\xi=7.5\xi, \tag{18}$$

(writing the variable ξ for x). Exactly half of this number is due to contribution I, as was mentioned in Section 1. We wish to know what the probability γ_n is for n mesons having been emitted if the average is $\frac{1}{2}\bar{n}^1$. Since most of these mesons have comparatively little energy we may assume that the emission of individual mesons are independent events. No doubt this assumption is not strictly correct but it would be very hard to take the statistical inter-dependence of the various acts of meson emission into account. This point further adds to the crudeness of our estimate. If we accept this assumption, the Poisson law gives

$$\gamma_n = \frac{(\bar{n}/2)^n e^{-\bar{n}/2}}{n!},$$

and hence

$$\gamma(\xi) = \gamma_0 + \gamma_2 + \cdots - (\gamma_1 + \gamma_3 + \cdots) = e^{-\bar{n}} = e^{-7.5\xi}. \quad (19)$$

Inserting this factor into (10) the integration over ξ gives $\Gamma(1+b/\eta)/(3\alpha+7.5)^{1+b/\eta}$ which differs by the factor $[3\alpha/(3\alpha+7.5)]^{1+b/\eta}$ from the result obtained otherwise for f. Thus, the positive excess (difference of positive and negative mesons divided by the total) is

$$\Delta_+ f/f = \frac{f^{\mathrm{I}}}{f^{\mathrm{I}} + f^{\mathrm{II}}} \left(\frac{3\alpha}{3\alpha+7.5} \right)^{1+b/\eta}. \quad (20)$$

The calculations of Section 2 give f^{I} and f^{II} separately. It is found, for instance, that at sea level and at the maximum of the energy spectrum ($\epsilon = 13$) f^{I} is 1/15th of the total. The second factor of (20) is for $\alpha = 1.5$ and $\eta = 13+22 = 35$, 1/3.5. Hence

$$\Delta_+ f/f = 2 \text{ percent}. \quad (21)$$

For larger energies this figure decreases slightly, to 1 percent at $\epsilon = 50$. It increases somewhat with height.

The value (21) is too small by a factor 3–5, although of the correct order of magnitude. The cause of the discrepancy is probably the sharp distinction of contributions I and II. If only in a small number of cases the charge of the meson emitted by contribution II were determined by the fast primary instead of by the nucleus a larger figure for $\Delta_+ f/f$ would be obtained. The crudeness of the Weizsäcker-Williams method applied to collisions of two *equal* particles does not permit a more accurate determination of the effect.

5. MESONS PRODUCED IN MULTIPLES, MEAN FREE PATH

It has been pointed out by Janossy[4] that the total cross section (4) for production of a meson in a nucleon-nucleon collision is larger than the average area occupied by a single nucleon in a nucleus. This means that in the passage of a fast nucleon through, or near, a nucleus more than one meson is on the average produced. Consequently, the mean free path l traveled by a fast nucleon before emitting at least one meson is larger than the "average range" (5) calculated on the assumption that the nucleons in the matter traversed are distributed at random. If we are interested in the meson intensity at a depth $x \gg l$ the difference between mean free path and average range is of no importance (compare, however, footnote 5) but for the top layer of the atmosphere and in particular for the absorption of the primaries themselves it is, of course, important to examine more closely what happens in the passage of the fast nucleon through a nucleus.

We introduce a radius r_m such that $\pi r_m^2 = \Phi$ is the total cross section for production of a meson by one nuclear particle. According to (4) $r_m = 3.7 \times 10^{-13}$ cm. Let r_A be the radius of the nucleus with atomic weight A. We can represent the primary nucleon by a disk with area πr_m^2. A fast nucleon passing at a distance x from the center of the nucleus will then produce so many mesons (n, say) as is the number of nuclear particles through which the disk passes.

It is clear that the total cross section of the nucleus for production of at least one meson must be (according to Janossy) $\pi(r_A + r_m)^2$, in fact a little less because the nuclear particles are on the average a certain distance inside the surface of the nucleus. We are interested in the probabilities w_n for the number of mesons produced being 1, 2, 3, $\cdots$. These probabilities can be obtained as follows:

We assume that the nucleus is a sphere with radius r_A filled with nuclear matter of constant

TABLE III. Total cross section (in 10^{-25} cm^2) and relative probabilities (in percent) for the emission of n mesons. πr_A^2 is the geometrical cross section.

	Φ	πr_A^2	w_1	w_2	w_3	w_4	w_5
nitrogen	12	4.3	20	14	11	9	7.6
lead	58	38	11	7	5	4	3.7

density. We imagine the path of the nucleon to be the central axis of a cylinder with radius r_m. This cylinder will cut out a volume v say, of the sphere $4\pi r_A^3/3$, v being a function of x. Then the number of mesons produced is obviously

$$n(x)=3Av(x)/4\pi r_A^3. \qquad (22)$$

n is here, of course, not an integer.

The evaluation of v leads to an elliptic integral, but for $x \geqslant r_A$, which is the most interesting case, a good approximation, with a maximum error of 15 percent is:

$$v=\frac{2\pi}{3}\frac{r_A^2(x+r_m)(r_A+r_m-x)-\frac{3}{8}(r_A^2-(x-r_m)^2)-\frac{1}{2}(x^2+r_m^2)(r_A^2-(x-r_m)^2)}{[4xr_m-\frac{1}{2}r_A^2+\frac{1}{2}(x-r_m)^2]^{\frac{1}{2}}}. \qquad (23)$$

Having thus obtained n as a function of x which, of course, monotonically increases with decreasing x, one can mark off the points where $n(x)$ lies for instance between $n+\frac{1}{2}$ and $n-\frac{1}{2}$ and thus has the average value of the integer n. Let these values be $x_{n+\frac{1}{2}}$ and $x_{n-\frac{1}{2}}$. There will be then, of course, an outside fringe, where $n<\frac{1}{2}$, namely $x>x_{\frac{1}{2}}$. This limit is nearly the same for all nuclei. The fringe $x>x_{\frac{1}{2}}$ is practically ineffective for meson production and can be neglected. We therefore put the total cross section equal to $\pi x_{\frac{1}{2}}^2$ for which we found approximately:[13]

$$\Phi=\pi(r_A+r_m-\bar{r})^2, \quad \bar{r}=1.1\times 10^{-13}\text{ cm}, \qquad (24)$$

which, as was to be expected is slightly smaller than $\pi(r_A+r_m)^2$. The probabilities for emission of n mesons are then

$$w_n=\frac{x_{n-\frac{1}{2}}^2-x_{n+\frac{1}{2}}^2}{x_{\frac{1}{2}}^2}. \qquad (25)$$

In this way Table III was obtained. We have used the radii for nitrogen $r_N=3.7$ and for lead $r_{Pb}=11$ (in 10^{-13} cm). r_N happens to be about equal to r_m.

A remarkable feature is the comparatively large probability for small n's. For N, in about 45 percent of all cases where mesons are at all emitted only 1, 2 or 3 mesons are produced.

When x is equal to r_A, n is about 6 and increases up to a maximum value of 10–15 for $x=0$. This is so for all nuclei heavier than nitrogen (i.e. for which $r_m \leqslant r_A$) since, clearly, the maximum value of $v(x)$ is of the order of magnitude πr_m^3. For lighter nuclei the maximum is A. We can therefore divide the total cross section into an outside fringe with only a small number of mesons emitted and an inner core, which is a little larger than the nucleus itself, where n is large, i.e., 5–15.

The results for the total cross section for production of at least one meson can be compared directly with the experiments. Schein, Iona, and Tabin,[14] for instance found, that in the high atmosphere, at 6 cm Hg (or 1.7 x-units), 25 percent or 50 percent of all penetrating particles produce secondaries in a layer of 5-cm or 10-cm paraffin respectively. In paraffin no other process (knock-on showers or cascades) can be appreciable in such a thin layer and the only process in question is meson production by protons. According to these data the mean free path of the proton must be about 10 cm, because, on the one hand, the number of secondaries still increases between 5 and 10 cm, and on the other hand at this height the remaining 50 percent of ineffective particles are almost certainly mesons and protons with $E<M$.

The theoretical cross section of a CH_2 group is

$$\Phi_{CH_2}=\pi(r_C+r_m-\bar{r})^2+2\pi r_m^2 = 2\times 10^{-24}\text{ cm}^2, \qquad (26)$$

if we assume $r_C=3.5\times 10^{-13}$ cm. Nearly half of this cross section is due to the hydrogen. In hydrogen-containing substances the mean free

[13] Janossy puts $\bar{r}$ equal to the average distance of two nuclear particles, *viz.*, $\bar{r}=1.5\times 10^{-13}$ cm which comes much to the same.

[14] Schein, Iona, and Tabin, Phys. Rev. **64**, 253 (1943).

path of a fast nucleon is thus comparatively much shorter than in substances containing only higher nuclei. The mean free path is, if we assume 3.9×10^{22} CH_2-groups per cm^3, $l=13$ cm, in excellent agreement with the experiments.

Although this agreement may, to a certain extent, be accidental, considering the crude character of our theory, we may safely conclude that the present theory of meson production is substantially correct, within its accuracy.

6. ABSORPTION OF THE PRIMARIES

We consider now the way in which the primary fast nucleons are absorbed in passing through matter. Since it is difficult to say at what rate a nucleon loses energy if $E<M$ we consider only nucleons with energy $>M$. We wish to know the number of nucleons (with $E>M$) found at a depth x for a given primary spectrum, for instance the spectrum (12).

The mean free path, in normal air (or nitrogen) and Pb, follows from (24)

$$l_{\text{air}}=1.4 \text{ cm Hg}=0.4\ x\text{-units}. \tag{27}$$

At a depth x, the probability for a primary to have altogether suffered ν collisions is

$$W_\nu=e^{-x/l}\left(\frac{x}{l}\right)^\nu\frac{1}{\nu!}. \tag{28}$$

Of course, not all the primaries which have suffered a collision (or several collisions) are "lost." The chance is quite big that only a small number of mesons is emitted in each collision and that therefore the primary has only lost a small amount of energy. The probabilities w_n for the number of mesons emitted being n in *one* collision are given in Table III. Let $\omega_{\bar{n}}$ be the probability that a number $\bar{n}$ of mesons is emitted up to the depth x in *all* collisions. Then obviously

$$\begin{aligned}
\omega_0&=W_0,\quad \omega_1=W_1w_1,\\
\omega_2&=W_1w_2+W_2w_1^2,\\
\omega_3&=W_1w_3+2W_2w_1w_2+W_3w_1^3,\\
\omega_4&=W_1w_4+W_2(w_2^2+2w_1w_3)\\
&\qquad+3W_3w_1^2w_2+W_4w_1^4,\\
\omega_5&=W_1w_5+2W_2(w_1w_4+w_2w_3)\\
&\qquad+3W_3(w_1^2w_3+w_2^2w_1),\\
&\qquad+4W_4w_1^3w_2+W_5w_1^5.
\end{aligned} \tag{29}$$

TABLE IV. Number of primaries penetrating to a depth x of the atmosphere.

x	0	0.4	0.8	1.2	1.6
cm Hg	0	1.4	2.8	4.2	5.6
$F(x)$	100	64	37	20	11

The $\omega_{\bar{n}}$ can easily be worked out from Table III and (28) for any given x/l.

If $\bar{n}$ mesons are emitted altogether, the primary will have lost a certain amount of energy. The average energy lost is given by (6) namely (we choose the value of a for the larger $\bar{n}$'s)

$$E=E_0\exp(-\bar{n}/3). \tag{30}$$

We disregard the fluctuations in the loss of energy (in most cases the mesons have small energy) and assume that (30) represents the exact amount of the energy change. Then a primary is found to have energy $E>M$ after having emitted $\bar{n}$ mesons if it has had originally an energy $E_0\geqslant Me^{\bar{n}/3}$. For the primary spectrum (12) the number of protons capable of emitting $\bar{n}$ mesons and still retaining an energy $E>M$ is then:

$$\gamma_{\bar{n}}=A\int_{Me^{\bar{n}/3}}^\infty dE_0/E_0^{\alpha+1}\ \text{or}\ A\int_{33.5}^\infty dE_0/E_0^{\alpha+1} \tag{31}$$

according to whether $Me^{\bar{n}/3}$ is larger than the critical latitude energy or not. Normalizing the total number of primaries to 100 the $\gamma_{\bar{n}}$'s are:

$$\begin{gathered}\gamma_0=\gamma_1=\gamma_2=\gamma_3=100,\quad \gamma_4=84,\\ \gamma_5=50,\quad \gamma_6=30,\quad\cdots.\end{gathered} \tag{32}$$

The number of primaries found at a depth x is then

$$F(x)=\sum_{\bar{n}=0}^\infty\gamma_{\bar{n}}\omega_{\bar{n}}. \tag{33}$$

We have calculated $F(x)$ from these data. (Table IV.) This is an absorption corresponding to an effective mean free path of 2.8 cm Hg, i.e., twice the actual mean free path l. Schein, Iona, and Tabin[14] mentioned that they found that the number of primaries decreased to 45 percent at 6 cm Hg and to 13 percent and 17 cm Hg, which is a much slower decrease with an effective mean free path more than twice that indicated by Table IV. In spite of the admitted crudeness of our theory we do not think that this difference is to be accounted for by its inaccuracy. In the first place the theoretical cross section for meson production was found to agree very well indeed with the direct experiments. Secondly, it is very unlikely that the absorption of the primaries takes place with a cross section smaller than the geometrical cross section of nitrogen. (According to these experiments the cross section would be 2×10^{-25} cm^2, which is certainly too small for nitrogen.) Fast meson producing nucleons, though in small numbers, are also found at sea level and low heights and these can certainly not be primaries. Regener[15] also found that the absorption of these fast nucleons in air is much slower than one would expect from their efficiency in producing mesons.

The explanation is very simple: There are numerous processes by which fast *secondary nucleons* are produced. It has been stated in Section 1 that fast recoil nucleons are also produced whenever a meson is emitted by contribution I. Roughly, these recoil nucleons are produced at a depth of $x=l-2l$. Their number is comparable with the number of the primaries and they will also travel a distance of l or $2l$ (see Table IV). It is clear that these recoil nucleons increase the apparent mean free path by a factor 2 or so, as it is actually found to be the case.

There are also other processes by which fast nucleons are produced throughout the atmosphere and these must be responsible for the fast nucleons producing penetrating showers[16] at sea level and mountain heights. For instance: (i) scattering of a meson by a nucleon with large energy transfer to the nucleon, and (ii) scattering of a light quantum by nucleon.

We hope to consider the production of these penetrating showers more in detail in a later paper together with a revised account of the production of the soft component through the decay of the transverse mesons.

SUMMARY

The theory of the production of mesons given in an earlier paper[1] is revised in accordance with an improvement in the application of the Weizsäcker-Williams method.[2] The energy spectrum and the total number of mesons at sea level, and their variation with height and latitude is worked out and found to be in good agreement with the relevant facts, i.e., within the accuracy of the calculations (which for several reasons is not greater than a factor 2 or so). The positive excess is an effect depending on the finer details of the theory which, for this purpose is considered rather crude. The theoretical value is found to be 2 percent in contrast to the experimental 6–10 percent. In passing through a nucleus a fast proton produces in general several mesons.[4] The total cross section of a nucleus for production of at least one meson and the relative probabilities for the production of n mesons are worked out. In light elements about 50 percent of the total cross section is due to production of only 1–3 mesons and 50 percent to $n=4\cdots A$ (atomic weight). The total cross section is in good agreement with the direct experiments in paraffin.[14] The absorption of the primary protons is found to take place according to a cross section slightly larger than the geometrical cross section. The form of the meson theory used is the charge symmetrical mixture of pseudoscalar and vector mesons.

[15] V. H. Regener, Phys. Rev. **64**, 250 (1943).

[16] Cf. for instance: L. Janossy, Proc. Roy. Soc. **A183**, 190 (1944) and previous papers by the same author quoted there; Wataghin, Santos, and Pompeia, Phys. Rev. **57**, 61 (1940).