Atoms in Variable Magnetic Fields*

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1. INTRODUCTION

IN recent years the investigation of the magnetic, spin, and electrical properties of protons, neutrons, nuclei, and other more complicated systems has made important progress through the study of the change in orientation in a magnetic field which varies appreciably during a Larmor period. It was only through the development of quantum mechanics and the work of Bohr and his school that one could first appreciate how quantization in general and space quantization in particular set in at all, and the appropriate conditions for its study.

In the design and evaluation of experiments of this type there arises the fundamental question: if we have a quantum-mechanical system of total angular momentum[1] j and with magnetic quantum number m with respect to a magnetic field, what will be the final state if the field varies with the time both in magnitude and in direction according to some known vector function $\mathbf{H}(t)$? This problem involves the solution of the time dependent Schroedinger equation and some special cases have been solved rigorously. However, Majorana[2] in a comparatively little-known paper has given some basic general results which are both very useful and greatly deepen our understanding of the process involved. He has demonstrated that the net effect of the varying field $\mathbf{H}(t)$ can be described as a sudden rotation of the angular momentum by an angle α. This angle is obtained from a solution of the dynamical equation and the most important of its properties is that it is independent of the initial magnetic quantum number m but depends only on the gyromagnetic ratio "g" and $\mathbf{H}(t)$. An immediate consequence of this statement can be formulated as follows: after the rotation through the angle α, the system is no longer in a definite state of space quantization with respect to the original field, but is rather to be described by a wave packet or superposition of $2j+1$ states with magnetic quantum numbers m', each with its appropriate probability amplitude. The absolute square of the amplitude for any m' is the probability of finding the system in the state m' after it had initially been in the state m, i.e., it is the transition probability $W(m, m', \alpha)$ for which Majorana has also given the explicit expression.

As an essential feature in the derivation of his formulae, Majorana has shown that the problem of a system with arbitrary angular momentum j can be reduced to the consideration of $2j$ representative points on the unit sphere, each representing the direction of an angular momentum with value $\frac{1}{2}$. Without attempting any interpretation at this point we shall briefly outline his method of reduction: Consider the equation

$$\sum_{r=0}^{2j} a_r \zeta^{2j-r} = 0, \tag{1}$$

and the relation between the $2j+1$ coefficients a_r and the $2j$ roots ζ_s $(s=1, 2\cdots 2j)$ of this equation. Let further C_m denote the probability amplitude of the state with magnetic quantum number m $(m=-j\cdots+j)$ and let

$$a_r = (-1)^r \frac{C_{j-r}}{[(2j-r)!r!]^{\frac{1}{2}}}. \tag{2}$$

Majorana shows then that if the amplitudes $C_m(t)$ are solutions of the time dependent Schroedinger equation, each of the resulting

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[1] We shall throughout measure angular momenta in units $\hbar = h/2\pi$.

[2] E. Majorana, Nuovo Cimento 9, 43 (1932).

roots ζ_s of Eq. (1) can be written as

$$\zeta_s = \beta_s/\alpha_s, \tag{3}$$

where α_s and β_s represent probability amplitudes for a system with angular momentum $\frac{1}{2}$ in a state in which the magnetic quantum number has the values $+\frac{1}{2}$ and $-\frac{1}{2}$ respectively.

Majorana's method, while remarkable in its elegance, has the disadvantage of somewhat obscuring the physical significance of the representative systems with spin $\frac{1}{2}$. It is clear that a simple intuitive understanding of the procedure and of the essential formulae will be very useful to many. In this paper we shall arrive at such an understanding by the application of the familiar vector model where the total spin operator $\mathbf{M}$ is treated as a sum of $2j$ spin operators each representing a system with angular momentum $\frac{1}{2}$ and with the same "g" value as the total system. Mathematically this will be expressed by a different representation of the system in which the variables of the Schroedinger wave function ψ are the spin variables of the constituent systems with spin $\frac{1}{2}$ and which we shall call the "composite" representation in contrast to the usual representation which uses only a single spin variable, referring to the total system of spin j.

This will lead to an elementary discussion of Majorana's reduction and its limitation in the case of fields which are not merely time dependent but vary also in space. We shall also derive the general and explicit solution for a system with spin j in a time dependent magnetic field in terms of that for a system with spin $\frac{1}{2}$. Majorana's expression for the transition probability will appear as a special consequence of this more general formula.

2. COMPOSITE REPRESENTATION OF A SYSTEM WITH SPIN j AND ITS CONNECTION WITH MAJORANA'S METHOD

Let ψ be the wave function describing a system with angular momentum j and let $\mathbf{M}$ be the vector operator, describing its angular momentum. The components of $\mathbf{M}$ have to satisfy the commutation rule

$$[M_x, M_y] = iM_z, \tag{4}$$

and its cyclical permutations and the existence of a spin j is expressed by

$$M^2\psi = j(j+1)\psi. \tag{5}$$

The usual representation of the operator $\mathbf{M}$ is that in which M_z with the eigenvalues $m = -j \cdots +j$ is diagonalized. In this case ψ has to be understood as consisting of $2j+1$ components C_m and the operator $\mathbf{M}$ as a matrix $\mathbf{M}_{mm'}$ so that

$$(\mathbf{M}\psi)_m = \sum_{m'=-j}^{+j} \mathbf{M}_{mm'} C_{m'}.$$

The vector model, however, suggests another representation in which the angular momentum $\mathbf{M}$ is expressed as a sum of $2j$ angular momenta $\mathbf{M}_k$ of spin $\frac{1}{2}$. The commutation rules (4) shall be satisfied for the components of each operator $\mathbf{M}_k$ and the components of two different operators shall commute with each other.

It is then obvious that the components of

$$\mathbf{M} = \sum_{k=1}^{2j} \mathbf{M}_k \tag{6}$$

satisfy likewise the commutation rules (4). Since they commute with each other we can now use a representation in which the z components and the squares of all $2j$ operators $\mathbf{M}_k$ are simultaneously diagonal; the components of each operator $\mathbf{M}_k$ can then be understood as one half of Pauli's familiar spin matrices σ_x, σ_y, σ_z. Corresponding to an angular momentum $\frac{1}{2}$, each of the M_k^2 will have the eigenvalue $\frac{3}{4}$ and $(\mathbf{M}_k)_z$ will have the two eigenvalues $m_k = \pm\frac{1}{2}$.

Whereas in the usual representation the condition (5) is satisfied by having M^2 diagonal with the eigenvalue $j(j+1)$, it becomes here an auxiliary condition upon the wave function ψ, expressing the fact that the angular momenta $\mathbf{M}_k$ of value $\frac{1}{2}$ shall be coupled such that they all add. The general wave function will evidently have 2^{2j} components since each of its arguments m_k can independently assume either of the two values $+\frac{1}{2}$ or $-\frac{1}{2}$.[3] The condition (5) is then satisfied if ψ is symmetrical in its arguments, i.e., if it does not change its value as any two of them are interchanged.[4]

[3] For a given set of values m_k we shall write the wave function in the form $\psi(m_1, m_2, \cdots m_{2j}) = \psi(m_k)$ and we shall speak about "functions" and "arguments" although we have to deal with discrete quantities.

[4] This can be verified through Dirac's (Proc. Roy. Soc. **123**, 725 (1929)) relation

$$\tfrac{1}{2}[1 + (\boldsymbol{\sigma}_k \cdot \boldsymbol{\sigma}_l)] = \tfrac{1}{2} + 2(\mathbf{M}_k \cdot \mathbf{M}_l) = P_{kl},$$

expressing the fact that this operator, acting upon ψ is equivalent to a commutation of its arguments m_k and m_l.

It shall further be observed that the vector operator (6) is symmetrical in the quantities $\mathbf{M}_k$ so that any function of its components represents an operator which, acting upon a symmetrical function with the arguments m_k produces another symmetrical function. The symmetrical functions ψ form, therefore, a closed subset to which we shall from now on restrict our attention.

Through the condition of symmetry a wave function ψ is defined by giving merely $2j+1$ of its values instead of the general 2^{2j}, necessary for its definition. Let

$$\psi(m_k)=\left[\frac{r!(2j-r)!}{(2j)!}\right]^{\frac{1}{2}}C_m, \tag{7}$$

whenever

$$r=j-m \tag{8}$$

of its arguments m_k have the value $-\frac{1}{2}$ and, therefore,

$$2j-r=j+m \tag{9}$$

have the value $+\frac{1}{2}$. Evidently the $2j+1$ numbers C_m, obtained by letting r vary from $2j$ to zero or m from $-j$ to $+j$, are sufficient to define completely a symmetrical ψ, since its value (7) will be the same no matter which of its $2j$ arguments m_k have the value $-\frac{1}{2}$.

Through the definition (7) C_m is the probability amplitude of finding the value m for the z component of the angular momentum, if $\psi(m_k)$ is the probability of finding the values m_k of the constituent components. Indeed,

$$M_z=\sum_{k=1}^{2j}(\mathbf{M}_k)_z$$

has the eigenvalue $m=j-r$ whenever any r of the $2j$ operators $(\mathbf{M}_k)_z$ have the eigenvalues $-\frac{1}{2}$ and the remaining $2j-r$ the eigenvalues $+\frac{1}{2}$. The probability to find the value m is therefore given by

$$W_m=\sum_{m_k}{}'|\psi(m_k)|^2, \tag{10}$$

where the symbol $\sum'$ stands for the summation over all those sets of values m_k of which r have the value $-\frac{1}{2}$ and $2j-r$ the values $+\frac{1}{2}$. Since there are altogether

$$\binom{2j}{r}=\frac{(2j)!}{r!(2j-r)!}$$

such sets and each contributes according to (7) the same value $|C_m|^2r!(2j-r)!/(2j)!$ to the sum (10), we have

$$W_m=|C_m|^2, \tag{11}$$

which proves the significance of C_m as probability amplitude to find the value m.

The relation of our procedure with that of Majorana is now obtained by showing that without loss of generality a symmetrical wave function can be constructed as a symmetrized product of $2j$ functions φ_s, each depending only upon one of the arguments m_k; i.e., that we can always write

$$\psi=\frac{1}{[(2j)!]^{\frac{1}{2}}}\sum^P\prod_{s=1}^{2j}\varphi_s(m_k), \tag{12}$$

where $\sum^P$ stands for the sum over all terms obtained by permutation of the arguments m_k. While a given set of functions φ_s leads through (12) obviously to a symmetrical ψ and therefore through (7) to a given set of values C_m, the converse is also true, that, given the $2j+1$ values C_m, the functions φ_s are essentially determined. To prove this, we shall write

$$\varphi_s(\tfrac{1}{2})=\alpha_s,\quad \varphi_s(-\tfrac{1}{2})=\beta_s, \tag{13}$$

thus defining the functions φ_s by $2j$ pairs of numbers α_s and β_s. Consider now the equation for ζ

$$\prod_{s=1}^{2j}(\alpha_s\zeta-\beta_s)=0, \tag{14}$$

with the $2j$ roots $\zeta_s=\beta_s/\alpha_s$. It can also be written

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It is now

$$M^2=(\sum_{k=1}^{2j}\mathbf{M}_k)^2=\sum_{k=1}^{2j}M_k^2+\sum_{k\neq l}(\mathbf{M}_k\cdot\mathbf{M}_l)$$
$$=\tfrac{3}{4}2j+\tfrac{1}{2}\sum_{k\neq l}[\tfrac{1}{2}+2(\mathbf{M}_k\cdot\mathbf{M}_l)]-\tfrac{1}{4}2j(2j-1)$$
$$=\tfrac{1}{2}\sum_{k\neq l}P_{kl}+2j-j^2.$$

If ψ is symmetrical, i.e., if $P_{kl}\psi=\psi$ for any $k\neq l$, we have therefore $M^2\psi=[\frac{1}{2}2j(2j-1)+2j-j^2]\psi=j(j+1)\psi$ in agreement with (5).

Different coupling schemes and symmetry properties of ψ can of course be introduced if the resultant vector $\mathbf{M}$ is composed of more than $2j$ vectors $\mathbf{M}_k$. For our purposes, however, nothing is gained by this complication and it shall therefore be omitted.

in the form

$$\sum_{r=0}^{2j} a_r \zeta^{2j-r}=0, \qquad (15)$$

with

$$a_0=\alpha_1\alpha_2\cdots\alpha_{2j},$$

$$a_1=-(\beta_1\alpha_2\cdots\alpha_{2j}+\alpha_1\beta_2\cdots\alpha_{2j}+\cdots+\alpha_1\alpha_2\cdots\beta_{2j}) \text{ etc.},$$

$$a_{2j}=(-1)^{2j}\beta_1\beta_2\cdots\beta_{2j};$$

or generally

$$a_r=(-1)^r \sum_{\mu,\nu}{}^{(r)} \prod_{\mu} \alpha_\mu \prod_{\nu} \beta_\nu \qquad (16)$$

where $\sum_{\mu,\nu}^{(r)}$ stands for the sum of all those products which can be formed by letting r of the factors be of the form β_ν, with ν assuming any r different values of s and by letting $2j-r$ more factors be of the form α_μ with μ assuming the remaining $2j-r$ different values of s.

On the other hand it follows from (12) that whenever r of the arguments m_k have the value $-\frac{1}{2}$, ψ assumes because of (13) the value

$$\psi=\frac{r!(2j-r)!}{[(2j)!]^{\frac{1}{2}}} \sum_{\mu,\nu}{}^{(r)} \prod_{\mu} \alpha_\mu \prod_{\nu} \beta_\nu \qquad (17)$$

with $\sum_{\mu,\nu}^{(r)}$ having the same significance as in (16). Since, according to (7) and (8) the value (17) of ψ is also given by $\left[\frac{r!(2j-r)!}{(2j)!}\right]^{\frac{1}{2}} C_{j-r}$ we obtain by comparison of (16) and (17)

$$\left[\frac{r!(2j-r)!}{(2j)!}\right]^{\frac{1}{2}} C_{j-r}=(-1)^r\frac{r!(2j-r)!}{[(2j)!]^{\frac{1}{2}}}a_r; \qquad (18)$$

or

$$a_r=(-1)^r\frac{C_{j-r}}{[r!(2j-r)!]^{\frac{1}{2}}}$$

which agrees with (2) and proves our Eq. (15) to be identical with the fundamental Eq. (1) of Majorana.

3. SOLUTIONS IN A TIME DEPENDENT MAGNETIC FIELD

We shall now consider a system with angular momentum j and with a magnetic moment $g\hbar j$ which is to be represented by the vector operator

$$\boldsymbol{\mu}=g\hbar\mathbf{M}. \qquad (19)$$

If the system is exposed to a magnetic field $\mathbf{H}$, its Hamiltonian will be $\mathcal{H}=-(\mathbf{H}\cdot\boldsymbol{\mu})$ and the time dependent Schroedinger equation $-(\hbar/i)(\partial\psi/\partial t)=\mathcal{H}\psi$ takes the form

$$\frac{\partial\psi}{\partial t}=ig(\mathbf{H}\cdot\mathbf{M})\psi. \qquad (20)$$

It can now readily be shown that the solution of this equation with $\mathbf{H}(t)$ as an arbitrary function of time can be reduced to that of the solution for a system with angular momentum $\frac{1}{2}$.

Using the composite representation of the previous paragraph we have seen that ψ can always be written in the form (12) as a symmetrized product of $2j$ functions φ_s. If, besides depending upon the arguments m_k, ψ will also depend upon the time t, the functions φ_s will likewise have to depend upon t. We can write (12) in the form

$$\psi(m_k, t)=\frac{1}{[(2j)!]^{\frac{1}{2}}}\sum{}^{P}\prod_{s=1}^{2j}\varphi_s(m_k, t); \qquad (21)$$

and we have

$$\frac{\partial\psi}{\partial t}=\frac{1}{[(2j)!]^{\frac{1}{2}}}\sum{}^{P}\left(\frac{\partial\varphi_1}{\partial t}\varphi_2\cdots\varphi_{2j}+\varphi_1\frac{\partial\varphi_2}{\partial t}\cdots\varphi_{2j}+\cdots+\varphi_1\varphi_2\cdots\frac{\partial\varphi_{2j}}{\partial t}\right). \qquad (22)$$

Suppose now that each of the functions $\varphi_s(m_k, t)$ is a solution of the equation

$$\frac{\partial\varphi_s(m_k, t)}{\partial t}=ig(\mathbf{H}\cdot\mathbf{M}_k)\varphi_s(m_k, t), \qquad (23)$$

with which one has to deal for a system with angular momentum $\frac{1}{2}$, represented by the operator $\mathbf{M}_k$, provided that it has the *same g-value* as our total system and is exposed to the *same field* $\mathbf{H}=\mathbf{H}(t)$. Since the time derivatives on the right side of (22) apply successively to functions φ the arguments of which assume all values m_k from m_1 to m_{2j}, it follows immediately from (22) that

$$\frac{\partial\psi}{\partial t}=ig(\mathbf{H}\cdot\sum_{k=1}^{2j}\mathbf{M}_k)\frac{1}{[(2j)!]^{\frac{1}{2}}}\sum{}^{P}\varphi_1\cdots\varphi_{2j},$$

and therefore with (6) that (21) is a solution of (20) if the functions φ_s are determined as solutions of (23).

It is also clear that this provides the general solution of (20) in the sense that it permits ψ to have any prescribed value at an initial time $t=0$. This value determines, according to Section 2, merely the initial values of the functions φ_s which are used in (21) to construct ψ and the problem of finding a solution of (20) with arbitrary initial conditions is therefore just as general as that of finding such a solution of Eq. (23) for a system with spin $\frac{1}{2}$.

To derive the general solution of (20) from that of (23) the procedure can thus in principle be described as follows: The initial state for $t=0$ is characterized by $2j+1$ probability amplitudes $C_m{}^0$ $(m=-j\cdots+j)$. Upon substitution of these values into (2) one obtains a set of coefficients $a_r{}^0$ and by solving Majorana's Eq. (1) the corresponding $2j$ roots $\zeta_s{}^0$ $(s=1\cdots 2j)$. These roots determine by virtue of (14) the ratios $\beta_s{}^0/\alpha_s{}^0=\zeta_s{}^0$ and therefore except for a normalization factor the initial values $\varphi_s{}^0$ of the functions φ_s. The solution of (23) gives then the functions φ_s at any later time t in terms of their initial values $\varphi_s{}^0$; inserted in (21) these functions lead to the symmetrical wave function $\psi(m_k, t)$ and finally through (7) to the probability amplitudes $C_m(t)$ in terms of their initial values $C_m{}^0$.

To carry out this procedure for higher spins would prove very troublesome, particularly since it involves the solution of Majorana's Eq. (1) of the $2j$th degree. The reduction to the problem with spin $\frac{1}{2}$ is fortunately greatly simplified because of the linear relation between initial and final probability amplitudes and actually does not require the knowledge of the functions φ_s but merely their transformation properties; it shall be carried out explicitly and generally in the next paragraph.

To characterize the linear relation which connects the initial values $\varphi_s{}^0$ with the functions φ_s at a later time t we use again the notation (13) and write $\alpha_s=\alpha_s(t)$ and $\beta_s=\beta_s(t)$ for the values which $\varphi_s(m_k, t)$ assumes for $m_k=+\frac{1}{2}$ and $m_k=-\frac{1}{2}$, respectively. If $\alpha_s{}^0$ and $\beta_s{}^0$ are the corresponding initial values at the time $t=0$ the result of integration of Eq. (23) can always be expressed as a linear transformation in the form

$$\alpha_s=A\alpha_s{}^0+B\beta_s{}^0, \quad \beta_s=C\alpha_s{}^0+D\beta_s{}^0, \qquad (24)$$

where the coefficients A, B, C, D are functions of the time t. For $t=0$ they assume the values $A=D=1$; $B=C=0$ and they are otherwise uniquely determined through the magnetic field vector $\mathbf{H}(t)$; particularly their values do not depend upon the index s. In the next section we shall investigate the transformation which leads from the initial values $C_m{}^0$ of the probability amplitudes to their values C_m at a later time t in consequence of the relation (24).

4. LINEAR TRANSFORMATIONS OF COMPONENT AND RESULTANT WAVE FUNCTIONS

We shall assume that a set $\varphi_s{}^0(s=1\cdots 2j)$ of wave functions for spin $\frac{1}{2}$, characterized by the $2j$ pairs of numbers $\alpha_s{}^0$ and $\beta_s{}^0$, is related to another set φ_s, characterized by α_s and β_s, through the linear transformation (24) with coefficients A, B, C, D, independent of s. Because of the results of Section 2 each of them leads to a corresponding set of probability amplitudes $C^0_{j-r}(r=0, 1\cdots 2j)$ and C_{j-r}, respectively, for a system with spin j and we shall here investigate the consequent transformation which is established between these two sets of probability amplitudes.

It was pointed out in the previous section that this transformation leads to the solution for a system in a time dependent magnetic field if the transformation coefficients A, B, C, D are obtained as functions of time by integration of (23). The same transformation, however, leads also to the result of a rotation of the coordinate system since, with a different significance of the transformation coefficients such a rotation is likewise expressed by a relation of the type (24); applying the general formula to this special case, we shall thus directly obtain Majorana's formula[5] for the probability $W(m, m')$ of finding the value m of the z component in the new coordinate system if it is known to have the value m' in the original one.

To obtain the desired transformation of the probability amplitudes, we use Eqs. (7) and (17) and write the probability amplitude C_{j-r}, corresponding to the pairs α_s, β_s in the form

$$C_{j-r}=[r!(2j-r)!]^{\frac{1}{2}} \sum_{\mu,\nu}{}^{(r)} \prod_{\mu} \alpha_\mu \prod_{\nu} \beta_\nu, \qquad (25)$$

where the sum $\sum_{\mu,\nu}^{(r)}$ contains all the products with r different factors β_ν and $2j-r$ remaining factors α_μ to be obtained from the $2j$ pairs of numbers

[5] Equation (4), reference 2.

α_s, β_s. If one expresses in (25) these numbers in terms of the numbers $\alpha_s{}^0$ and $\beta_s{}^0$ according to (24), one obtains the quantities C_{j-r} in the form

$$C_{j-r}=\sum_{r'=0}^{2j} S_{rr'}C^0_{j-r'}\,; \tag{26}$$

where

$$C^0_{j-r'}=[r'!(2j-r')]^{\frac{1}{2}}\sum_{\mu,\nu}{}^{(r')}\prod_{\mu}\alpha_\mu{}^0\prod_{\nu}\beta_\nu{}^0 \tag{27}$$

is the probability amplitude corresponding to the pairs $\alpha_s{}^0$, $\beta_s{}^0$ and where $\sum_{\mu,\nu}{}^{(r')}$ has the same significance as $\sum_{\mu,\nu}{}^{(r)}$ in (25) except for the replacement of r by r'.

To obtain the transformation matrix $S_{rr'}$ is now simply a problem of combinations. One has to observe that it will contain the product $A^{2j-r-r'+\rho}B^{r'-\rho}C^{r-\rho}D^{\rho}$ as many times as there appear of the products $\prod_{\mu}\alpha_\mu{}^0\prod_{\nu}\beta_\nu{}^0$ those for which among r' chosen values of the $2j$ indices s there are ρ factors of the form $\beta_\nu{}^0$ and among the remaining $2j-r'$ values of the indices s there are $r-\rho$ factors of the form $\beta_\nu{}^0$. For every value of ρ for which none of the four numbers ρ, $r-\rho$, $r'-\rho$ and $2j-r-r'+\rho$ is negative, one obtains therefore from the sum $\sum_{\mu,\nu}{}^{(r)}$ on the right side of (25) the sum $\sum_{\mu,\nu}{}^{(r')}$ of (27) multiplied with

$$\frac{r'!}{\rho!(r'-\rho)!}\,\frac{(2j-r')!}{(r-\rho)!(2j-r-r'+\rho)!}\times A^{2j-r-r'+\rho}B^{r'-\rho}C^{r-\rho}D^{\rho}.$$

The final result for $S_{rr'}$ is obtained by summation over all possible values of ρ and through the relations (25) and (27) between C_{j-r}, $C^0_{j-r'}$ and $\sum_{\mu,\nu}{}^{(r)}$, $\sum_{\mu,\nu}{}^{(r')}$ respectively. It is namely

$$S_{rr'}=[r!(2j-r)!r'!(2j-r')!]^{\frac{1}{2}}\times\sum_{\rho}\frac{A^{2j-r-r'+\rho}B^{r'-\rho}C^{r-\rho}D^{\rho}}{\rho!(r-\rho)!(r'-\rho)!(2j-r-r'+\rho)!}. \tag{28}$$

As pointed out before, the summation over ρ shall contain all those terms for which ρ, $r-\rho$, $r'-\rho$ and $2j-r-r'+\rho$ are not negative. Actually the summation $\sum_{\rho}$ in (28) can be considered as extending over all positive integers, including zero if the factorials in the denominator of (28) are understood in the usual generalized sense which makes them infinite for negative integer arguments and thus automatically suppresses all terms in $\sum_{\rho}$ for which any one of the four numbers ρ, $r-\rho$, $r'-\rho$, $2j-r-r'+\rho$ is negative.

We shall finally rewrite the formulae (26) and (28) by using instead of r and r' the values m and m' of the z component of the angular momentum which, according to (8) are given by $m=j-r$ and $m'=j-r'$, respectively. We have then:

$$C_m=\sum_{m'=-j}^{+j} T_{mm'}C_{m'}{}^0, \tag{29}$$

with

$$T_{mm'}=S_{j-m,\,j-m'}$$

or from (28)

$$T_{mm'}=[(j+m)!(j-m)!(j+m')!(j-m')!]^{\frac{1}{2}}\times\sum_{\rho}\frac{A^{m+m'+\rho}B^{j-m'-\rho}C^{j-m-\rho}D^{\rho}}{(m+m'+\rho)!(j-m'-\rho)!(j-m-\rho)!\rho!}. \tag{30}$$

The special case $j=\frac{1}{2}$ gives of course from (30) $T_{\frac{1}{2}\frac{1}{2}}=A$, $T_{\frac{1}{2}-\frac{1}{2}}=B$, $T_{-\frac{1}{2}\frac{1}{2}}=C$, $T_{-\frac{1}{2}-\frac{1}{2}}=D$, i.e., Eq. (29) becomes identical with the original transformation (24) of the probability amplitudes for spin $\frac{1}{2}$ upon which our derivation of (30) was based.

Formula (30) represents the general and explicit reduction of the problem of general spin to that of spin one-half since it gives the transformation coefficients $T_{mm'}$ of the wave amplitudes for the former in terms of the latter.

As a first application we shall investigate the transformation (29) which the probability amplitudes undergo as a consequence of a rotation of the coordinate system, characterized by the Euler angles α, ϕ, ψ. According to Pauli[6] we have here

$$A=\cos\frac{\alpha}{2}e\left(i\frac{\phi+\psi}{2}\right),\quad B=i\sin\frac{\alpha}{2}e\left(-i\frac{\phi-\psi}{2}\right),$$

$$C=i\sin\frac{\alpha}{2}e\left(i\frac{\phi-\psi}{2}\right),\quad D=\cos\frac{\alpha}{2}e\left(-i\frac{\phi+\psi}{2}\right),$$

and substituting these values in (30)

[6] W. Pauli, Zeits. f. Physik **43**, 601 (1927).

$$T_{mm'}=[(j+m)!(j-m)!(j+m')!(j-m')!]^{\frac{1}{2}}i^{2j-m-m'}e^{i(m\psi+m'\phi)}\sin^{2j}\frac{\alpha}{2}$$
$$\times\sum_{\rho}(-1)^{\rho}\frac{\cot\text{an}^{m+m'+2\rho}\,\alpha/2}{(m+m'+\rho)!(j-m'-\rho)!j-m-\rho)!\rho!}.\quad(31)\dagger$$

The absolute square of (31) gives immediately the probability $W(m, m')$ to find the value m of the z component of the angular momentum in the rotated coordinate system if it had the value m' in the original one; i.e.,

$$W(m,m')=(j+m)!(j-m)!(j+m')!(j-m')!\sin^{4j}\frac{\alpha}{2}$$
$$\times\left|\sum_{\rho}(-1)^{\rho}\frac{\cot\text{an}^{m+m'+2\rho}\,\alpha/2}{(m+m'+\rho)!(j-m'-\rho)!(j-m-\rho)!\rho!}\right|^{2}.\quad(32)$$

This formula becomes identical with Majorana's formula[5] if one introduces instead of ρ a new summation index r by the relation $r=j-m'-\rho$; it has however, the advantage that the symmetry of $W(m, m')$ in m and m' becomes evident which is not the case in the form given by Majorana.

As a second application we shall consider the magnetic resonance where the system is subjected to a magnetic field which has a constant z component H_0 and a projection upon the $x-y$ plane of magnitude H, rotating around the z axis with angular frequency ω. Upon integration of Eq. (23) one obtains here the solution for spin $\frac{1}{2}$ in the form (24) with

$$A=D^*=\cos\frac{\lambda}{2}t-i\frac{\omega-\omega_0}{\lambda}\sin\frac{\lambda}{2}t,$$
$$B=C=i\frac{gH}{\lambda}\sin\frac{\lambda}{2}t,$$

where $\lambda=[(\omega-\omega_0)^2+(gH)^2]^{\frac{1}{2}}$ and $\omega_0=gH_0$. By comparison with the corresponding formulae for a rotation of the coordinate system it is seen that the formulae (31) and (32) can also be used for this case, if, instead of having the significance of Euler angles of rotation of the coordinate system, the angles α, ϕ, and ψ are defined by

$$\alpha=2\arcsin\left(\frac{gH}{\lambda}\sin\frac{\lambda}{2}t\right),\quad(33)$$

$$\phi=\psi=-\arctan\left(\frac{\omega-\omega_0}{\lambda}\tan\frac{\lambda}{2}t\right).\quad(34)$$

† Note added in proof: Professor Pauli has kindly brought our attention to a paper by P. Guettinger (Zeits. f. Physik **73**, 169 (1931), in which this formula has been previously derived [Guettinger's Eq. (59)] by the methods of group theory and where the relation between the sum $\sum_\rho$ in our Eq. (31) and the hypergeometric polynomials has been observed).

5. LIMITATION OF THE METHOD

We have seen in Section 2 that the vector model offers a new representation of a system with spin j and that the wave function ψ describing such a system can always be written in the form (12) as a symmetrized product of wave functions φ_s of $2j$ systems with spin $\frac{1}{2}$. While this result is general, the explicit reduction of the problem of spin j to spin $\frac{1}{2}$ in Section 4 was carried out under the special assumption that the physical problem implies a linear transformation of the form (24) with the same transformation coefficients A, B, C, D for all wave functions φ_s. This assumption is satisfied if one deals with rotations of the coordinate system or, as shown in Section 3, with the action of a magnetic field which depends only on the time. In dealing with rays of atoms or neutrons, it is often necessary, however, to consider magnetic fields which depend not only upon the time but also upon the position coordinates of the particle and we shall here discuss the reasons for which the general reduction becomes in this case impossible.

The general wave function ψ, describing a particle with spin j and position vector $\mathbf{r}$ consists of $2j+1$ components C_m $(m=-j\cdots+j)$ each of which depends both upon the time t and the three components of $\mathbf{r}$. According to the results of Section 3 it is still possible for any given set of values of $\mathbf{r}$ and t to find $2j$ wave functions φ_s

for spin $\frac{1}{2}$ such that ψ can be written in the form (12) as a symmetrized product; these functions φ_s are related to the values C_m and will therefore in general likewise depend upon both t and $\mathbf{r}$.

The essential difference between this more general case and the case of a purely time dependent field arises from the fact that no analogue can be found to Eqs. (23) according to which all wave functions φ_s satisfy the same differential equation and therefore undergo in course of time a linear transformation (24) with coefficients, independent of s. The Schroedinger equation for ψ has in this case, instead of (20) the form

$$\frac{\partial\psi}{\partial t}=ig(\mathbf{H}\cdot\mathbf{M})+\frac{i\hbar}{2m_0}\nabla^2\psi, \tag{35}$$

where the second term on the right side has to be included to take into account the kinetic energy of the particle with mass m_0. According to Section 2, Eq. (21) can be generalized in the form

$$\psi(m_k, \mathbf{r}, \mathbf{t})=\frac{1}{[(2j)!]^{\frac{1}{2}}}\sum{}^P\prod_{s=1}^{2j}\varphi_s(m_k, \mathbf{r}, t) \tag{36}$$

but the substitution of this expression into (35) gives from the Laplace operator ∇^2 not only terms of the form $\nabla^2\varphi_s$ but also cross derivatives of the form (grad $\varphi_s\cdot$grad $\varphi_{s'}$) between any two different wave functions φ_s and $\varphi_{s'}$. While the presence of the terms $\nabla^2\varphi_s$ would allow a simple generalization of the results of Section 3 by merely including them on the right side of (23), the terms (grad $\varphi_s\cdot$grad $\varphi_{s'}$) prohibit such a generalization, since they cause the values of any function φ_s to depend upon those of all the other functions $\varphi_{s'}$. This means that the mere solution of the problem with spin $\frac{1}{2}$ is not sufficient to furnish the general solution for arbitrary spin, if the dependence of the wave function upon the position of the particle becomes essential.

The ordinary Stern-Gerlach arrangement for the deflection of atomic beams can be considered as a simple illustration of this situation. The solution for spin $\frac{1}{2}$ gives here wave functions $\varphi_s(m, \mathbf{r})$ of which the component with $m=\frac{1}{2}$ vanishes outside the one, the component with $m=-\frac{1}{2}$ outside the other of the two beams into which the original unpolarized beam will separate. A wave function $\psi(m_k, \mathbf{r})$, constructed according to (36) from products of these functions would therefore have the absurd property that in the region of separation it could differ from zero only for $m_1=m_2=\cdots=m_{2j}=\pm\frac{1}{2}$, i.e., for $m=\sum_{k=1}^{2j} m_k=\pm j$ and then only inside one of the two beams obtained for spin $\frac{1}{2}$. It is essential for the description of $2j+1$ beams, one for each value of m, that the functions φ_s in (36) do not vary independently but are coupled through the terms (grad $\varphi_s\cdot$grad $\varphi_{s'}$) with the result that even one component of a given function φ_s will split into several beams, depending upon the splitting of the others. The method of composite wave functions introduces under these circumstances merely complications and cannot be fruitfully applied. It is obvious, however, that there is a limited use of the method, even in the case of space dependent magnetic fields. If the magnetic field varies appreciably only over regions of space, large compared to the de Broglie wave-length of the particle and if deflections due to its inhomogeneity can be neglected, one can effectively consider a wave packet in uniform motion and a coordinate system, moving together with this packet. In this coordinate system and within the wave packet, the magnetic field will again be only a function of time alone, thus making the results of Section 4 applicable.