Application of Ordinary Space-Time Concepts in Collision Problems and Relation of Classical Theory to Born's Approximation

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QUANTUM mechanics gives not only a complete and accurate account of atomic phenomena, but through the uncertainty principle it also gives a satisfactory interpretation of the status of ordinary classical ideas. It shows where they can be applied and where they must fail. This principle focuses attention on the inability to watch the progress of a phenomenon without disturbing it. This disturbance has a finite minimum, and the failure of the classical picture of the progress of a phenomenon takes place when the events imagined cannot be observed on account of the disturbances produced by any attempted observations. Conversely if the space-time picture can be watched without radically interfering with the phenomenon (in respect of the quantity calculated), then the picture is admissible.

Strong interacting forces between the colliding particles naturally favor the applicability of a classical treatment of collisions. At the other extreme of very weak interacting forces, the general quantum-mechanical treatment takes up the limiting form commonly known as Born's first approximation. In problems of scattering, a very simple picture of this approximation can be formed by virtue of the wave form of the fundamental equations of quantum mechanics. Under the conditions of weak interaction, the collision produces only a small change in the wave function representing the colliding particles. The primary wave, as it were, passes right through the perturbing field without appreciable scattering. Under these conditions the scattered waves originate at every point with an amplitude proportional to the interaction potential and to the *undisturbed* amplitude of the primary wave. The final solution can readily be worked out as a simple problem in interference. The important point of interest which this treatment has in the present connection is that the conditions under which it is valid are, in general, complementary to those which permit the use of the classical orbit treatment. This means that a complete, if approximate, solution of collision problems can often be obtained by a combination of these limiting forms of quantum mechanics.

It is hardly necessary to emphasize that the simple wave picture in the limit of weak interacting forces has not the reality of the classical orbit picture in the limit of strong interacting forces. In the former we imagine a wave motion, but no observation on the system concerned will reveal a wave motion or any associated periodicity. The wave function must be translated into terms of particles before it can be given an observable meaning. In the classical picture, on the other hand, we imagine a particle moving in an orbit, and under the limiting conditions when such a picture is used, observations made at successive intervals during the collision will actually reveal a particle moving in an orbit. In other words what is visualized in the picture can be observed as such by our ordinary senses. This incidentally gives the answer to such questions as, "Is an electron a wave or a particle?." It is, of course, a particle. The wave properties are not properties of the electron but properties of quantum mechanics. It might be noted that the same cannot be said of radiation, where, in the limit of strong fields, periodic effects corresponding to the frequency of the waves can be observed. This difference between particles and radiation corresponds to the appearance of the quantum of action (h) in the *wave* behavior of particles (wave-length $=h/\mathrm{M}v$), and to its appearance in the *particle* behavior of radiation (energy of photon $=h\nu$) but *not* in the wave properties of radiation.

In what follows we shall consider certain collision phenomena from the standpoint of the applicability of classical mechanics and of the

wave treatment. Examples of experimental results are quoted which bear on the main topic of the validity or failure of the classical orbit treatment. In this respect the multiple scattering of electric particles through small angles and the loss of energy by electric particles are very illustrative phenomena. In order to be brief, the detailed qualifying statements which must attend a complete discussion are not always given. Some of the points dealt with here were discussed more fully in an earlier article.[1]

The writer wishes to take this opportunity of stating that his own interest in, and initiation into the theory of collisions, were occasioned by the study (1926) of Professor Bohr's classical papers on "The Passage of Electrical Particles through Matter," which appeared in the Philosophical Magazine in 1913 and 1915.[2] Later (1933–34) the writer had the pleasure of personal discussions with Professor Bohr on such aspects of collision problems as are discussed in this article, and certain points mentioned here are the result of these discussions.

SCATTERING

The initial conditions in a scattering problem are generally a well-defined incident velocity, but a *random* impact parameter—this parameter denoting the perpendicular distance of the undisturbed path of the incident particle from the center of the scattering field. The classical treatment of the scattering, however, cannot proceed without visualizing the conditions in greater detail—the random incidence is broken up into discrete incidences each with a definite impact parameter, the classical deflection is calculated for each discrete orbit, and the statistical scattering actually observed by experiment is obtained by a re-integration of the results. Now this procedure is admissible only if the extra details imagined could be observed without radically disturbing the collision in respect of the quantity calculated, which in this case is the deflection produced by the scattering field.

Let us consider, for instance, a scattering field extending over a limited volume of dimensions of the order of a, and in which the scattering potential is of the order of V. In a classical treatment the orbit of the particle must be well defined in relation to distances of the order of a. This means uncertainties (or disturbances) in the momentum of the particle of the order of at least h/a (or more exactly $h/2\pi a = h/a$). In order that the classical picture should be valid this disturbance, h/a, must, in the first place, be much less than the momentum of the particle, (Mv), i.e.,

$$h/a \ll Mv, \quad \text{or} \quad h/Mv \ll a. \tag{1}$$

h/Mv is the de Broglie wave-length, and the condition may be interpreted as the condition for constructing a wave packet small compared with the size of the scattering field. This, or something equivalent to it, has not infrequently been given as the *sole* condition for a classical treatment.

However, the classical calculation of scattering visualizes not only a well defined orbit, but also a well-defined deflection due to the collision. The deflection is determined by the momentum transfer, and the classical value of this in the case under consideration is of the order of V/v. The classical calculation is, therefore, admissible only if the disturbance h/a is also small compared with V/v; i.e.,

$$Va/hv \gg 1. \tag{2}$$

If the collision forces are too weak to satisfy this condition, then the classical picture of the deflection is truly a figment of the imagination. Under these conditions we must proceed in some way which avoids breaking up the random incidence. This cannot possibly be done in classical mechanics, but it can be done in quantum mechanics.[3] In the latter the incident particle with random impact parameter can be represented by a plane wave, and the scattering can then be worked out, without reference to impact parameter, by solving the quantum-mechanical wave equation for the passage of this wave through the scattering field.

The solution of this wave problem shows that

[1] E. J. Williams, Science Progress **121**, 14 (1936).
[2] N. Bohr, Phil. Mag. **25**, 10 (1913); **30**, 58 (1915).
[3] The inability of quantum mechanics to describe the position and momentum of a particle with classical exactness is not infrequently, but falsely of course, regarded as a limitation—to be excused, as it were, by the inability to observe these things accurately. What requires to be emphasized however is the inability of classical mechanics to describe position and momentum ambiguously.

every element of volume, $d\tau$, of the scattering field gives secondary waves of amplitude, at unit distance from $d\tau$, equal to $(2\pi M/h^2)A\,Vd\tau$, where A is the wave amplitude at $d\tau$. If the scattering forces are weak, A may be taken as the amplitude at $d\tau$ of the *undisturbed* incident wave—this is Born's first approximation. Under these conditions (and if $\hbar/Mv \ll a$, Eq. (1)), the resultant secondary amplitude inside the scattering field (i.e., the distortion of the primary wave) may readily be shown to be of the order of $A(Va/\hbar v)$. The condition for the applicability of the above approximation is accordingly

$$Va/\hbar v \ll 1. \tag{3}$$

This condition is just the reverse of condition (2) for the validity of a classical treatment. If $Va/\hbar v \gg 1$ classical orbits are valid, if $Va/\hbar v \ll 1$, the simple wave treatment is valid. The one limiting form merges into the other and gives the same order of scattering cross section when $Va/\hbar v$ is of the order of unity.[4] Between them a complete if approximate solution can be given for all values of the interacting potential. This is an example of the complementary nature of the classical treatment and the simple wave treatment.

Condition (1) for the classical treatment must, however, be remembered, *viz.*, that the de Broglie wave-length must be small compared with the dimensions of the scattering field. If this is not satisfied then, however strong the interaction potential, the orbit picture is never valid. Many important collision problems are in this category, such as the scattering of slow electrons by atoms and the scattering of neutrons by atomic nuclei. In these the interaction potential is too strong for the simple wave treatment, and the de Broglie wave-length of the scattered particle is too large for a classical orbit treatment. The quantum-mechanical treatment in such cases has to proceed along more sophisticated lines, such as the method of Faxen and Holtsmark.

Coulombian Scattering by Atomic Nuclei

The scattering potential in this case is Zze^2/r, where Ze is the nuclear charge, and ze the charge of the scattered particle.

Let us consider the scattering by the Coulombian field at distances of the order of r from the nucleus. Then in terms of the previous section $V=Zze^2/r$, $a=r$, and from (2), the classical orbit picture is valid for this part of the field provided

$$Va/\hbar v=(Zze^2/r)r/\hbar v=Zze^2/\hbar v \gg 1. \tag{4}$$

From (3) the wave treatment (Born's approximation) is valid provided $Zze^2/\hbar v \ll 1$. These conditions are well known. The outstanding point of interest is that the quantity $Zze^2/\hbar v$ is independent of r. In other words a classical calculation is valid for all parts of the field (i.e., all angles of scattering) or not at all, and when it is not valid, the simple wave treatment gives the whole solution.[5]

The classical solution is represented by Rutherford's scattering formula, and as is well known, this formula also represents the wave solution. It may readily be shown that this formal identity of the requirements of these limiting forms of quantum mechanics can only obtain for an inverse square law of force. In Born's approximation the amplitude of the scattered wave is proportional to the scattering potential, and the scattering cross section is therefore proportional to the square of the potential. It follows from dimensional considerations that the scattering cross section S for an inverse-n interaction, $V=k/r^{n-1}$, must in this approximation be of the form

$$S=\text{const. } k^2M^{2n-6}v^{2n-8}h^{4-2n}. \tag{5}$$

Only for an inverse-square interaction, $n=2$, does this expression not involve h, and, therefore, only in this case can the wave scattering be identical with the classical scattering.

It should be noted that though the probability

[4] Under the conditions $Va/\hbar v \ll 1$, if the potential is everywhere increased in a certain ratio, the amplitude of the scattered waves is everywhere increased in the same ratio. Thus the total scattering increases as the square of the potential while its angular distribution remains constant. However, when $Va/\hbar v$ has reached unity then for further increases in the field strength, the total scattering remains practically constant (*all* particles passing through the field are scattered to some extent), but the angular distribution in general changes in favor of large angles. The inverse square field is the only exception which gives the same angular distribution for all field strengths.

[5] It should be noted that the classical condition $\hbar/Mv \ll r$ is automatically met if $Zze^2/\hbar v \gg 1$. For under the latter condition the interaction is so strong that the cross section for large angle scattering is very much greater than $(\hbar/Mv)^2$, and impact parameters of the order of $\hbar/Mv$, therefore, play an insignificant part.

of scattering through an angle θ is formally the same in the wave treatment as it is in the classical treatment, the parts of the field responsible for the scattering through θ are not the same. In the wave treatment, scattering through θ is due to the Coulombian field at distances of the order of $\hbar/Mv\theta$ from the nucleus, while in the classical treatment such scattering is due to the field at a distance of the order of $Zze^2/Mv^2\theta$ from the nucleus. The ratio of these distances is $Zze^2/\hbar v$.

Substituting for $\hbar c/e^2$ its numerical value, *viz.*, 137, the condition (4) for the validity of the classical treatment of nuclear scattering becomes

$$Zz/137\beta \gg 1, \tag{6}$$

where $\beta = v/c$. Rutherford's original application of his formula referred to the scattering of α-particles ($z=2$, $\beta \sim 0.05$) by elements of medium and high atomic numbers ($Z \sim 50$, say). For such cases $Zz/137\beta \sim 20$, so that Rutherford's classical orbit treatment was, in fact, quite valid. Classical mechanics has thus by no means been ousted from all sub-atomic phenomena.

Comparison with Experiment—Multiple Scattering

Since the classical scattering formula is the same as the 'wave' formula, it would appear that an unambiguous demonstration of the validity, under the above conditions, of the classical treatment of nuclear scattering and of the breakdown of the wave treatment is not possible. However, the Coulombian field of atomic nuclei is limited by the atomic electrons. The effect of this shielding is to cut out the Rutherford scattering through very small angles, and from the remarks in the preceding paragraph but one, it is evident that the angle of cut-off is different in the classical treatment from what it is in the wave treatment. Its value in the wave treatment is approximately

$$\theta_m{}^w = \hbar/Mva, \tag{7}$$

where a is the effective shielding radius (roughly equal to the hydrogen radius $\times Z^{-\frac{1}{3}}$). The cut-off in the classical treatment takes place in the region of

$$\theta_m{}^{cl} = Zze^2/Mv^2a = (Zz/137\beta)\theta_m{}^w. \tag{8}$$

For the scattering of α-particles by heavy nuclei, we have seen that $(Zz/137\beta)$ is about 20, so that the classical cut-off takes place in the region of much larger angles than the wave cut-off. For the scattering of α-particles by gold, $\theta_m{}^{cl}$ is of the order of 0.20°, while $\theta_m{}^w$ is of the order of 0.01°. The cut-off is, of course, gradual and begins at about 10 times these angles. Experiments by Rose[6] on the scattering of α-particles by gold show a falling-off below the Rutherford scattering already at about 1°, thus bearing out the classical treatment. However, though Rose experimented with very thin foils, the conditions were not sufficiently those of single scattering, nor were the observations comprehensive enough to make a reliable comparison with the classical shielding effect.

A more quantitative comparison with past experiments is possible if we consider the scattering of α-particles by thicker foils, the conditions for which are those of multiple scattering. This multiple scattering occurs in the region of very small angles for which the Rutherford formula for single scattering can be written

$$P(\theta)d\theta = (4\pi NtZ^2z^2e^4/M^2v^4)d\theta/\theta^3 = ktd\theta/\theta^3. \tag{9}$$

t = thickness of scattering foil, and N = number of atoms per cc. The most probable angle of scattering and also the arithmetic mean angle of scattering are determined by the integral

$$S = \int_{\theta_m}^{\theta_1} \theta^2 \cdot ktd\theta/\theta^3 = kt \log(\theta_1/\theta_m). \tag{10}$$

θ_1 is a property of the statistics of multiple scattering and may be defined such that on the average the α-particle in traversing the foil suffers one collision in which it is deflected through more than θ_1. Hence $\theta_1 = (kt/2)^{\frac{1}{2}}$.

θ_m in the above integral is the angle of cut-off caused by shielding, and its value depends on whether we use classical mechanics or the wave treatment. Since θ_m occurs inside a log term, the multiple scattering is not very sensitive to its value. For the cases investigated experimentally, the difference in the wave and the classical requirements is, however, not inappreciable, being about 50 percent. The following table gives the results of observations by Geiger in

[6] D. C. Rose, Proc. Roy. Soc. 111, 677 (1926).

TABLE I. Multiple scattering of α-particles ($Zze^2/\hbar v > 1$).

Scattering element	$Zze^2/\hbar v$	Most probable angle of scattering, Θ_p (degrees) Observed	Classical	Wave treatment
Gold	20	2.1	1.74	3.07
Tin	13	1.5	1.43	2.20
Silver	12	1.5	1.37	2.11
Copper	7	1.1	1.04	1.46
Aluminium	3	0.6	0.69	0.85
$\Sigma\Theta_p$		6.8	6.3	9.7

1910,[7] and also the calculated values required by classical mechanics and by the wave treatment respectively.[8] (Table I.)

It will be seen that the observed scattering bears out the classical orbit treatment in accordance with the general requirements of quantum mechanics and the uncertainty principle. Although experimental evidence for the validity of quantum mechanics is hardly required at this stage, these results have been given in some detail as they offer the only example, to the writer's knowledge, of distinctive quantitative evidence for the accuracy under appropriate conditions of the classical orbit treatment in atomic phenomena.

For the scattering of fast electrons ($z=1$, $\beta\sim 1$) the quantity $Zze^2/\hbar v$ equals $Z/137$ and is less than unity even for heavy elements. Thus the wave treatment is the appropriate form of quantum mechanics, especially for light elements. Table II gives actual results for the multiple scattering of fast electrons. The experimental values are due to Kultichitsky and Latyshev[9] (1942) and refer to the multiple scattering of $2\frac{1}{4}$-million volt electrons. The theoretical values required by the wave treatment are those calculated by these authors from the writer's theory of multiple scattering[10] and not from the later theory of Goudsmit and Saunderson[11] which, however, agrees numerically with the former within a few percent.

The experimental results for the light elements clearly bear out the wave treatment, rather than the classical treatment which gives about 25 percent too much scattering. Tables I and II between them thus demonstrate the change-over from classical orbits to 'waves,' as the quantity $Zze^2/\hbar v$ decreases from values appreciably greater than unity to values appreciably less than unity.

The quantity $Zze^2/\hbar v$ has the same value for fast cosmic-ray mesotrons ($z=1$, $\beta\sim 1$) as for fast electrons, *viz.*, $Z/137$. Their multiple scat-

TABLE II. Multiple scattering of fast electrons ($Zze^2/\hbar v < 1$).

Scattering element	$Zze^2/\hbar v$	Half-width of Gaussian distribution (degrees) Observed	Wave treatment	Classical
Aluminium	0.10	9.5	9.8	12.1
Iron	0.20	9.6	9.9	11.9
Tin	0.37	10.6	10.9	11.9
Lead	0.60	9.6	10.8	10.6

[7] H. Geiger, Proc. Roy. Soc. **83**, 492 (1910).

[8] The exact relation of the multiple scattering to the integral S, and other details of the calculations, have been given elsewhere by the writer (Phys. Rev. **58**, 292 (1940)). The following are some points not previously mentioned which may be of interest to those who wish to consider the more detailed side of the problem.

(i) The argument already given does not indicate what is the best exact value of the quantity $Zze^2/\hbar v$ to take as an effective demarcation between the wave treatment and the classical treatment. We know it is of the order of unity, but unity itself may be appreciably different from the best value. The following argument provides some guidance in this respect. The gross failure of the classical calculation of atomic scattering, when $Zze^2/\hbar v \ll 1$, is in the direction of too much scattering. When $Zze^2/\hbar v \gg 1$ the gross failure of the wave treatment is in the same direction. If we assume that the partial failure of both treatments when $Zze^2/\hbar v \sim 1$, is also in the direction of too much scattering, then evidently the more correct treatment is the one which gives the *less* scattering. The best value of $Zze^2/\hbar v$ to take as the transition point for atomic scattering is thus the one for which the classical and wave scattering are equal, or most nearly equal. From Fig. (2) of the above paper, it is found that this is about 0.6, quite close to unity. A corollary to this argument is that even the more accurate treatment still gives too much scattering, and this may not be inappreciable for $Zze^2/\hbar v$ close to the transition point (i.e., 0.6), for which neither treatment is as accurate as one or the other is for greater or less values of $Zze^2/\hbar v$. This inference may be related to the discrepancy of about 10 percent for lead in Table II.

(ii) As $Zze^2/\hbar v$ is increased, the transition from the wave treatment to a classical treatment does not take place simultaneously for all parts of the atomic field. The transition is delayed for the weaker fields near the periphery. However the spread in the transition caused by this effect corresponds in the main only to a factor of about 2 in $Zze^2/\hbar v$.

(iii) The effect of polarization of the atomic electrons by the moving particle is probably quite negligible in all the cases quoted in Tables I and II. Even for α-particles, though their velocity is less than the orbital velocities of the inner atomic electrons of the elements concerned, the nuclear field acting on these inner electrons far exceeds the average field that will be exerted on them by the α-particle during a collision.

[9] L. A. Kultichitsky and G. D. Latyshev, Phys. Rev. **61**, 254 (1942).

[10] E. J. Williams, Phys. Rev. **58**, 292 (1940).

[11] S. Goudsmit and J. L. Saunderson, Phys. Rev. **57**, 24 (1940).

tering should, therefore, obey the wave treatment, and this in fact has been found to be the case, the agreement being very close.[12,13] It is, however, of interest to note that in this case the classical treatment gives practically the same formula as the wave treatment. This is so because the finite size of the nucleus cuts out the Rutherford scattering above a certain angle θ_x, which is small. The upper limit to the integral S for the multiple scattering (Eq. (10)) is this θ_x, and not θ_1 determined by statistics.[12] Denoting nuclear dimensions by d, then in the classical treatment θ_x is of the order of Zze^2/Mv^2d, in the wave treatment it is of the order of $\hbar/Mvd$. The lower limit θ_m is of the order of Zze^2/Mv^2a and $\hbar/Mvd$, in the respective treatments, where a represents atomic dimensions. The scattering integral depends on the ratio θ_x/θ_m, which is therefore the same in the two treatments, approximately equal to a/d, i.e., the ratio of the size of the atom to the size of the nucleus.

Inverse 'n' Field

It is instructive to consider this general case from the standpoint of the limiting forms of quantum mechanics—the classical orbit treatment for strong interacting forces and the wave treatment for weak forces.

Denoting the scattering potential by k/r^{n-1}, the classical momentum transfer for orbits approaching within distances of the order of r of the scattering center is in general of the order $(k/r^{n-2})r/v=k/vr^{n-1}$. To make the orbit picture valid, this momentum transfer must be large compared with the disturbance, $\hbar/r$, produced in observing r, i.e.,

$$k/vr^{n-1}\gg\hbar/r, \quad \text{or} \quad k/\hbar vr^{n-2}\gg 1. \tag{11}$$

This result also follows directly from conditions (2) if we put $V=k/r^{n-1}$ and put $a=r$.

The condition (3) for the wave treatment by a field V over dimensions a is $Va/\hbar v\ll 1$, giving for an inverse n field,

$$k/\hbar vr^{n-2}\ll 1 \tag{12}$$

viz., the reverse of (11).

The conditions (11) and (12) do not involve r for $n=2$, which means again that the orbit treatment is valid for *all* regions of a Coulombian field, or not at all. For the general case we must distinguish between the fields inside and outside a radius of the order of b, where

$$b=(k/\hbar v)^{1/(n-2)}. \tag{13}$$

If $n>2$ the field well inside $r=b$ is strong enough to scatter classically, while the field well outside $r=b$ is weak enough to scatter according to the wave treatment. In general a combination of the two limiting treatments will, therefore, give a complete though approximate solution. If $n<2$ the field *outside* $r=b$ scatters classically. These broad results require qualifications, but they provide a basis for a general understanding of the quantum-mechanical scattering by an inverse $-n$ field. The following points may be specially noted for fields with $n>2$.

(i) If the demarcation radius b is less than the de Broglie wave-length, λ, then there is, of course, no region amenable to the orbit treatment. Under these conditions, it may be shown that the core of strong field inside b does not make a significant addition to the total scattering, and the whole scattering is approximately equal to the wave scattering by the field outside $r=b$. If $n<3$ then the field inside b does produce a significant effect, which cannot be dealt with by the orbit treatment or the simple wave treatment.

(ii) If $b\gg\lambda$, and if $n<3$, then the classical scattering by the field inside b, and the wave scattering by the field outside b, both contribute significantly to the total scattering, the angle of demarcation between the two types of scattering being of the order of λ/b. If, however, $n>3$ then the total wave scattering by the field outside b is not important. In fact fields with $n>3$ fall off so rapidly that no sooner are the distant parts too weak to scatter classically than they are almost too weak to produce any scattering at all.

(iii) The most interesting case of all is probably that of attractive fields with $n\geqslant 3$. In this case the general rule that, for $b\gg\lambda$, the field inside b scatters classically requires qualification. These high power attractive fields produce classical orbits which pass through the center of force. These orbits are defined by an impact parameter less than a certain critical value of the order of $(k/mv^2)^{1/(n-1)}$. In the general quantum-mechanical theory of scattering (e.g., method of Faxen and Holtsmark), there are serious difficulties in the treatment of attractive fields with $n\geqslant 3$. It is shown, for example, in the treatment of $n=3$ in Mott and Massey's *Theory of Collisions*, page 30, that no general solution of the scattering exists for such a case. Not only is the wave function infinite at the origin, but the phases of the spherical harmonies which determine the scattering are indeterminate. It follows from Mott and Massey's treatment that this applies only to those harmonics which

[12] E. J. Williams, Proc. Roy. Soc. **169**, 531 (1939).
[13] P. M. S. Blackett and T. G. Wilson, Proc. Roy. Soc. **160**, 304 (1937).

correspond to an angular momentum less than $(2k/mv^2)^{\frac{1}{2}}$, and this is precisely the critical angular momentum for the classical orbits which pass through the center of force. There can, therefore, be little doubt that the classical orbits through the center of force correspond to the indeterminate solutions in the general quantum-mechanical treatment.

Now although in general the classical orbits inside the radius b can be observed without significantly disturbing the final deflection, this is not true of those orbits which pass through the center. The passage of the particle into and just through the center can be observed since, as r decreases, the momentum of the particle increases faster than $1/r$. However the disturbance produced by such observation will radically modify the ultimate motion away from the center and hence, the ultimate deflection. Thus the whole classical orbit cannot be observed however strong the center of force. There is, therefore, no proof along those lines that there must exist a complete solution to the scattering problem by attractive fields with $n \geqslant 3$. It should, however, be noted that the classical orbits which pass through the field inside the radius b, but do not pass through the center, give a small but observable deflection, and the scattering through *small* angles by fields with $n \geqslant 3$ has therefore a definite solution.

ENERGY LOSS IN COLLISIONS WITH ATOMS

The classical theory of the rate at which α- and β-particles lose energy in passing through matter was first fully worked out by Bohr in 1913.[2] Relativistic and other effects were considered in a later paper by Bohr in 1915. The ideas underlying the treatment given in these papers were in many respects more general than classical theory, and they still serve as a basis for understanding most of the features of energy loss. Prior to the application by Bethe in 1931[14] of Born's quantum-mechanical theory of collisions, it did in fact appear that Bohr's classical formula for energy loss might well be quite generally true, *statistically*. We shall briefly consider the status of the classical treatment and also give an approximate derivation of Bethe's formula for energy loss.

The moving α- or β-particle loses its energy through the excitation and ionization of the atoms traversed, and the problem is to calculate the perturbation of an atomic electron by the moving particle. If this perturbation was uninfluenced by the binding force of the nucleus, the dynamical problem would be identical with that of Coulombian scattering. In that case the scattering of the moving particle (mass M, charge ze) due to interaction with the atomic electrons (NZ per unit volume) would be approximately given by (cf. Eq. (9)):

$$P(\theta)d\theta = (8\pi NtZz^2e^4/M^2v^4)d\theta/\theta^3 = k^1td\theta/\theta^3. \quad (14)$$

A deflection through an angle θ corresponds to a momentum transfer to the atomic electron approximately equal to $Mv\theta$, and hence an energy transfer equal to $(Mv\theta)^2/2m$ (m = electron mass). The maximum value of θ is of the order of m/M. Thus the energy loss per cm caused by collisions with θ greater than a certain minimum, ϕm, say, is

$$dT/dx = \int_{\phi m}^{m/M} k^1 d\theta/\theta^3 \cdot (Mv\theta)^2/2m$$
$$= (4\pi NZ^2e^4/mv^2) \log (m/M\phi_m). \quad (15)$$

The binding forces determine the value of ϕm. The essentially new point introduced into the classical theory by Bohr, in his 1913 paper, was that binding forces restrict the energy transfer only when the time of the collision is comparable with or greater than the natural period of the atomic electrons. The time of collision is of the order of p/v where p is the impact parameter. It follows that the electron may be treated as free from impact parameters less than ρ where

$$\rho \sim v/\nu \sim (v/u)d. \quad (16)$$

ν denotes the natural frequency of the atomic electron, u its orbital velocity, and d the orbital or atomic dimensions. In nearly all practical cases for α- and β-particles, their velocity v is much greater than u, so that $\rho \gg d$, i.e., the critical impact parameter at which the atomic electron ceases to behave as if it were free is much greater than atomic dimensions.

How general is this argument? In the first place it rests on the conception of time of collision and impact parameter. Since $\rho \gg d$ the essential argument regarding the effect of binding forces requires only the concept of impact parameter in *distant* collisions. The impact parameter in such collisions can be defined as the distance of the path of the moving particle from the atom as a whole or from the atomic nucleus. This impact parameter can therefore be determined experimentally by observations only on the

[14] H. Bethe, Ann. d. Phys. **5**, 325 (1930).

moving particle and the atomic nucleus. For fast particles ($v \gg u$) the disturbances produced by such observations produce a negligible effect on the space-time picture of the particle traveling along a well defined path with respect to the atom, and, therefore, such a picture is perfectly admissible for calculating the potential perturbing the atomic electrons.[15]

The concept of an impact parameter ρ (Eq. (16)), inside which the time of collision is small compared with the natural period or periods of the atomic electron, is therefore quite valid in quantum mechanics. The general theorem that the sum of the strengths of the virtual oscillators corresponding to one atomic electron is equal to that of *one* free electron furthermore means that for these collisions the energy transfer is statistically equal to the transfer to a free electron.[16] The exact energy transfers in individual collisions are, however, outside the scope of a space-time treatment. For the duration of the collisions concerned is much less than $1/\nu$ and hence, any energy measurements during the collision would involve uncertainties large compared with $h\nu$, i.e., large compared with the excitation and ionization potentials of the atomic electrons concerned.

The classical Rutherford formula represents quite generally the Coulombian interaction of two free particles. It would, therefore, appear that Bohr's classical formula for the *average* energy loss should also be generally true, since we have seen that this energy loss is statistically the same as the energy loss to free electrons.

The fallacy, of course, arises from the fact that if the quantity, $ze^2/\hbar v$, $=z/137\beta$, is much less than unity, the Rutherford classical formula is only accidentally valid, and the classical treatment does not give correctly the effect of any deviation from Coulombian interaction, such as a limitation of the interaction to distances less than ρ of Eq. (16). The position with respect to energy loss is, in fact, closely analogous to that in respect of multiple scattering. Comparison of (10) and (15) shows that both are determined by the integral of the squares of the scattering angle as given by the Rutherford scattering formula. The lower limit to the integral in the case of multiple scattering is the cut-off of the Rutherford scattering due to the shielding of the nucleus by the atomic electrons. In the case of energy loss, the lower limit, ϕ_m, is the angle of cut-off corresponding to a Coulombian field limited to the radius, ρ, which marks the onset of adiabatic conditions. As in the case of nuclear scattering, this angle ϕ_m is or is not given by classical mechanics according as the interaction between the moving particle (ze) and the atomic electron (e) is or is not strong enough to give scattering according to the classical orbit treatment. In exact analogy with Eqs. (7) and (8), if $ze^2/\hbar v \ll 1$ then the Born approximation is valid, and

$$\phi_m{}^w \sim \hbar/Mv\rho \sim \hbar\nu/Mv^2. \tag{17}$$

If $ze^2/\hbar v \gg 1$ the classical treatment is valid and

$$\phi_m{}^{cl} \sim ze^2/Mv^2\rho \sim (z/137\beta)\phi_m{}^w. \tag{18}$$

Substituting in the formula (15) for energy loss, we have

$$dT/dx = (4\pi Nz^2e^4/mv^2) \log (g_1 mv^2/h\nu) \tag{19}$$

if $ze^2/\hbar v \ll 1$; and

$$dT/dx = (4\pi Nz^2e^4/mv^2) \log (g_2 mv^3/ze^2\nu) \tag{20}$$

if $ze^2/\hbar v \gg 1$.

Equation (19) is Bethe's formula obtained by applying Born's approximation;[14] (20) is Bohr's classical formula.[2] The numerical coefficients g_1 and g_2 inside the log terms are left undetermined in the present approximate treatment. The log terms are, however, insensitive to their values. Detailed calculations have been made which give accurate values of g_1 and g_2 for hydrogen-like atoms[14] and for helium.[17]

The conditions deduced above for the validity of the classical formula and the Born approximation formula respectively were first derived by Bloch[18] from a more detailed application of quantum mechanics. The equivalence of the method of impact parameter for calculating the energy loss, to Born's method, was first demonstrated by Mott[19] by a detailed mathematical

[15] It is important to notice here that we are not trying to study the reaction of the collision on the moving particle. To do that would require the observational disturbances to be small compared with the calculated reaction.

[16] E. J. Williams, Proc. Roy. Soc. **139**, 163 (1933).

[17] E. J. Williams, Proc. Camb. Phil. Soc. **33**, 179 (1937).

[18] F. Bloch, Ann. d. Physik **5**, 285 (1933).

[19] N. F. Mott, Proc. Camb. Phil. Soc. **27**, 553 (1931).

TABLE III. Energy loss.[20]

Moving particle	Gas traversed	Initial and final velocity ($\times 10^{-9}$)	$ze^2/\hbar v$	Distance travelled (cm) Observed	Wave treatment (Bethe)	Classical (Bohr)
Alpha	Hydrogen	2.054→1.709	0.23	19.0	18.9	16.3
Alpha	Hydrogen	1.709→1.802	0.25	15.8	16.2	13.7
Alpha	Helium	2.054→1.709	0.23	22.6	22.3	18.4
Beta	Hydrogen	5.11 →0	0.06	0.76	0.77	0.52
Beta	Hydrogen	4.08 →0	0.07	0.37	0.34	0.23

proof of the identity of the requirements of the two methods. It is important to note that Mott's proof and the present more general argument based on the uncertainty principle, both refer to an impact parameter defined with reference to the atomic nucleus and not to the perturbed atomic electron. Furthermore, the method is fully applicable to *all* collisions, close as well as distant, provided only the moving particle is sufficiently heavy (α-particle e.g.) so as not to be appreciably disturbed from rectilinear motion even in the closest collisions.

As a parallel to Tables I and II for multiple scattering, we give the above table (Table III) of results for energy loss. Even for α-particles the quantity $ze^2/\hbar v$ is much less than unity, so that it is Bethe's formula and not the classical formula of Bohr that gives the correct representation of quantum mechanics in these cases.

The ranges given by the wave treatment (Born's approximation) are seen to agree very well with the experimental values. The latter, however, very definitely exceed the values given by the classical formula, the failure of which is most pronounced for β-particles. These results are as they should be since $ze^2/\hbar v$ is much less than unity and is much less for β-particles than for α-particles.

GENERAL REMARKS

Several other collision phenomena, besides scattering and energy loss, permit in their treatment the use of ordinary space-time ideas to a greater or lesser extent. The method of Fourier analysis of the perturbing field and comparison of the harmonic components with radiation,[21,22] essentially depends on the ordinary space concept of impact parameter. This Fourier analysis method has been applied to energy loss, emission of radiation in collisions, production of electron pairs by photons in a nuclear field, production of pairs in collision of two electric particles, etc.[21a] In all cases the criterion which determines whether the method is applicable or not is whether the impact parameter can be observed without radically disturbing or confusing the quantity calculated. In the above cases it can.

All the collision phenomena concerned can, however, be treated, and in some respects more completely treated, by the quantum-mechanical method of stationary states (Born), and historically the first quantum-mechanical treatment of most collisions effects was in fact by this method.[23] What then, it may be asked, is the point of introducing classical ideas even though a partial introduction of such ideas is perfectly rigorous in principle? There is a point.

In the first place the semiclassical method has proved much more useful for appreciating the basic features of the theory involved and for assessing the expected validity of the formal results. An example of this is provided by the phenomenon of ionization by particles with velocity close to that of light. The application of the method of stationary states to this phenomenon gives a logarithmic rise in the ionization with energy, as does the semiclassical method. However, the latter shows quite clearly that this rise does not involve relativistic quantum mechanics[16,20,24]—the mechanics involved is nonrelativistic, and relativity comes in only in the Lorentz contraction of the field of a particle moving with uniform velocity. This very secure basis of the theory, which is not as apparent in the other type of treatment, has played a part

[20] Results for hydrogen and helium, only, are given since these involve very little computational error. In the case of α-particles, in order to avoid complications due to capture and loss of electrons towards the end of their range, the table gives the difference in ranges between fast α-particles. Details relating to the comparison are given in an earlier paper by the writer (Proc. Roy. Soc. **135**, 108 (1932)). The theoretical value for helium was dealt with in a later paper (see reference 17).

[21] E. J. Williams, (a) Proc. Danish Acad. **13**, 4 (1935); (b) Phys. Rev. **45**, 129 (1934).

[22] C. F. v. Weizsacker, Zeits. f. Physik **88**, 612 (1934).

[23] There are exceptions, for example, the logarithmic rise in ionization by very fast particles after their velocity exceeds about 0.97 c, was first demonstrated on the basis of the method of impact parameter. (Reference 24.)

[24] E. J. Williams, Proc. Roy. Soc. **130**, 328 (1931).

in the study of cosmic rays, and in particular made it possible, before the accumulation of other evidence, to rule out electrons as the average high energy cosmic-ray particles at sea-level (1934).[21b] Another example of the usefulness of the semi-classical method in this respect is provided by the emission of radiation by high energy electrons in nuclear collisions. The semi-classical method readily shows that practically all the theoretical effect involves relativistic quantum mechanics only for quantum energies of the order of mc^2 or less—however high the energy of the electron. This relatively simple feature of the theory was again not apparent in the method of treatment by stationary states. In fact, at the time of the original application of the latter method to this problem, by Bethe and Heitler, these authors suggested a possible breakdown of the theory when the de Broglie wave-length of the incident electron was of the order of or less than electron radius (i.e., energy of electron $\geqslant 137mc^2$). The semiclassical treatment, however, readily shows that the relation of this wave-length to electron radius has no relevance at all.

The above contributions from the semiclassical method, however, do not indicate a fundamental advantage over the method of stationary states. The basic theory involved in different collision phenomena can no doubt also be unraveled in terms of the latter though not so readily. It is desired to emphasize, however, that the semi-classical method does possess a distinctive fundamental merit in the very fact that it makes use of ordinary ideas. We have seen that such ideas are applicable when what is imagined in their application can be observed without significantly disturbing the phenomenon. Thus in applying the semiclassical method to collisions, what we in essence do is to shine a light on one or more of the colliding particles and watch with our ordinary senses what is happening. To forego the possibility of doing this is deliberately to limit our ordinary instinctive understanding of certain aspects of the phenomenon, to accept as it were, an unnecessary black-out on ordinary ideas. Thus the scattering of α-particles by heavy nuclei ($Zze^2/hv \gg 1$) permits full illumination, and the orbits can be seen to obey Newton's law. On the other hand the scattering of fast electrons by light nuclei ($Zze^2/hv \ll 1$) requires a total black-out—the phenomenon is a 100 percent quantum one. Finally as an illustrative intermediate example, let us return to the ionization by fast particles. This permits partial illumination, *viz.*, illumination of the moving particle and of the atomic nucleus, but not of the atomic electron. However, this is enough to show that the failure of the moving particle to ionize at distances beyond the radius ρ (Eq. (16)) is related to the absence, at such distances, of harmonics in the field of the particle of sufficiently high frequency to ionize the atom. The absence of such harmonics has nothing to do with h and follows simply from our ordinary ideas of motion. In the method of stationary states, the limiting distance, ρ, again appears, but as a wave interference effect, *viz.*, the distance over which the change in the de Broglie wave-length due to the ionization leads to a phase difference of the order of π.[25] This gives a simple mathematical picture of ρ, but the real essence of the quantity is veiled. The method of stationary states has to draw this veil because it deals with the whole collision as one system with a fixed total energy. The method gives the reaction on the ionizing particle—which the semiclassical method cannot touch—as well as the perturbation of the atomic electron. This completeness in the statistical treatment is, of course, an essential merit, but it entails a complete black-out on ordinary ideas. The semi-classical method raises this black-out where possible.

[25] Denoting the wave-length of the moving particle before and after ionizing by λ_1 and λ_2, respectively, the secondary waves from a volume of dimensions ρ will, because of the change in wave-length alone, destructively interfere if ρ is of the order of, or exceeds, a value given by $\rho/\lambda_1 - \rho/\lambda_2 \sim 1$. ρ determines the maximum distance from the atom at which the moving particle can ionize. Now $\lambda = h/mv$, so that the condition is $(\rho m/h)(v_1 - v_2) \sim (\rho/hv) \times (E_1 - E_2) \sim \rho(\delta E/hv) \sim 1$, where δE is the ionizing energy. $\delta E = h\nu$ where ν is the corresponding atomic frequency. Thus $\rho\nu/v \sim 1$, i.e., $\rho \sim v/\nu$, as in Eq. (16).