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Relativistic Wave Equations for the Elementary Particles

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IN this paper we investigate the equations that can be used for describing the behavior of elementary particles of any integral or half-odd integral spin on the assumption that these equations must always be written in the absence of interaction in the form

$$\{p_k\alpha^k+\chi\}\psi=0, \qquad (1)$$

where the p_k are the differential operators $i\partial/\partial x^k$, and the α^k are four matrices describing the spin properties of the particle. χ is an arbitrary constant about the physical meaning of which we make no assumptions for the moment. The α^k satisfy different commutation rules depending on the spin of the particle described and are square matrices of different degrees in each case.

Relativistic wave equations for particles of any integral or half-odd integral spin have been given in several different equivalent forms by Dirac,[1] Fierz,[2] and Pauli[3,4] taking as their guiding principle the assumption that each component of the wave function by itself must satisfy the generalized second-order wave equation in the absence of interaction. Of these the equation for a particle of spin $\frac{1}{2}$, the famous Dirac equation, is already in the form (1), and Kemmer[5] has shown that the pseudoscalar and vector equations for the meson can also be written in the same form. We shall see below that for every case of spin greater than 1 the equations of Dirac, Fierz, and Pauli (abbreviated to D.F.P.) cannot be written in the form (1). The D.F.P. equations connect two irreducible spinors, and by a suitable transformation can be split into two sets, one of which still connects the two irreducible spinors together, while the other set only involves one spinor and is in the nature of subsidiary conditions. We shall see below that the first set can be written in the form (1) but without the second set the first set is not equivalent to the D.F.P. equations. The second set, consisting of the subsidiary conditions, is necessary in order that it should be possible to derive a second-order wave equation for each component. Moreover, the existence of subsidiary conditions has always been a difficulty of the D.F.P. formulation and becomes particularly marked when interaction is introduced. Fierz and Pauli[4] have shown that this can be done in a consistent way by special artifices requiring the introduction of additional subsidiary spinors, involving a certain loss in the elegance of the mathematical formulation. It would, therefore, appear to be more logical to assume that *the fundamental equations of the elementary particles must be first-order equations of the form (1) and that all properties of the particles must be derivable from these without the use of any further subsidiary conditions*. It will be shown below that as a result of this assumption each component of the wave function by itself does not satisfy a second-order wave equation, but one of higher order consisting of products of the usual second-order wave operators. The physical interpretation of this circumstance is that for spins greater than 1 the particle has states of higher rest mass which are simple rational multiples of the lowest value of the rest mass. These states of higher rest mass are an essential feature of the theory and *cannot* be eliminated by an artifice any more than the states of negative mass in the usual formulations of the theory. For example, a particle of spin $\frac{3}{2}$ must appear with two possible rest masses, one three times the other, while a particle of spin 2 has also two rest masses, one being twice the other. The theory put forward in this paper, therefore, predicts that should particles of spin $\frac{3}{2}$ or 2 exist in nature, they would appear each with two possible values of the rest mass, the

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* F.R.S.
[1] Dirac, Proc. Roy. Soc. **A155**, 447 (1936).
[2] Fierz, Helv. Phys. Acta **12**, 3 (1939).
[3] Pauli and Fierz, Helv. Phys. Acta. **12**, 297 (1939).
[4] Fierz and Pauli, Proc. Roy. Soc. **A173**, 211 (1939).
[5] Kemmer, Proc. Roy. Soc. **173**, 91 (1939).

lower values of the rest mass being the stable ones in each case. In general, it can be shown that for a particle of spin n the number of different values of the rest mass is $2n+1$ if n is half an odd integer, each value appearing with its negative, while if n is an integer the number of values is $2n$.

The assumption that all physical and mathematical properties of the particle should be given by Eq. (1) implies that the α-matrices themselves should be capable of generating the nucleus (consisting of the six infinitesimal transformations) of the representation which determines the way the wave function transforms under any given transformation of the Lorentz group (Eq. (8) below), and I have pointed out in a recent note[6] that with this condition the problem of finding all irreducible representations of the matrices in (1) can be connected with the problem of finding the nuclei of all irreducible representations of the Lorentz group in *five* dimensions, the solution of which is completely known. The connection between the wave equation (1) and the representations of the Lorentz group in five dimensions which has been established gives a very powerful method for finding and handling equations of the form (1) and gives a deeper insight into their structure. It is used in this paper to specify all the irreducible equations of the form (1) that satisfy our basic assumption. It is shown that for particles with a maximum spin n, the number of different equations possible is $n+\frac{1}{2}$ if n is half an odd integer, and $n+1$ if n is an integer. These equations are *not* equivalent and describe particles with *different physical properties* with the maximum value n of the spin.* The degree of the representation of the matrices in each case is given. It is shown also that there is only *one* equation for each value of n, such that the particle displays the spin n only in all circumstances. The structure of the equations is analyzed in greater detail in the last section.†

[6] Bhabha, Curr. Sci. **14**, 89 (1945).

* The different physical properties described by each equation manifest themselves strikingly in the non-relativistic approximations of the equation in each state of rest mass. These non-relativistic approximations as also the Lagrangian formulation of the equations and the expressions for the current and energy tensors have been given in a subsequent paper (Proc. Ind. Acad. Sci. A **21**, 241–264 (June, 1945)).

1. GENERAL THEORY

We take the metric tensor to have the form $g_{00}=-g_{11}=-g_{22}=-g_{33}=1$, $g_{kl}=0$ for $k\neq l$. Any general transformation t of the Lorentz group is one whose coefficients $t_l{}^k$ are all real and which leaves the metric form unaltered, that is

$$g_{kl}=g_{mn}t_k{}^m t_l{}^n. \tag{2}$$

We adopt the usual convention of summing from 0 to 3 over any repeated index. Denoting by t^{-1} the inverse transformation of t so that $(t^{-1})_m{}^k t_l{}^m=\delta_l{}^k$ where $\delta_0{}^0=\delta_1{}^1=\delta_2{}^2=\delta_3{}^3=1$ while $\delta_l{}^k=0$, $k\neq l$, the effect of a Lorentz transformation on the Eq. (1) is to transform the p_k and α^k to p_k' and α'^k defined by

$$p_k'=p_l(t^{-1})_k{}^l, \quad \alpha'^k=t_l{}^k\alpha^l. \tag{3}$$

Let the α's be square matrices of some given degree d. The requirement that Eq. (1) shall be invariant for all transformations of the Lorentz group means that a non-singular matrix S of d rows and columns shall exist capable of bringing the α'^m back to their original from α^m by a transformation $S\alpha'^m S^{-1}$. The α^m are, therefore, quantities which transform according to

$$\alpha^m=t_n{}^m(S\alpha^n S^{-1}). \tag{4}$$

With every transformation t, we must be able to associate a matrix S, and it can be seen quite easily that the S's form a representation of the full Lorentz group of degree d. Since the group of transformations t includes improper ones whose determinant is -1, the group of matrices S must include one which reverses the sign of the three matrices α^1, α^2, and α^3, corresponding to an inversion of the space at the origin. Let the infinitesimal transformations of the representation S be the six matrices I^{rs} antisymmetric in r and s. It is usual to refer to these as the nucleus of the representation S. As is well known, the infinitesimal transformations of any representation of the Lorentz group must satisfy the

† The connection between the Eqs. (1) and the Lorentz group in a free-dimensional space is also discussed by J. K. Lubenski, Physica **9**, 310 and 325 (1942). This paper is not accessible in India at present.

commutation relations

$$[I^{mn}, I^{rs}] \equiv I^{mn}I^{rs} - I^{rs}I^{mn} = -g^{mr}I^{ns} + g^{ms}I^{nr} + g^{nr}I^{ms} - g^{ns}I^{mr}, \quad (5)$$

and conversely any collection of six matrices satisfying (5) can be used to build up a representation of the restricted Lorentz group without reflections. From (4) it follows, as usual, that

$$[\alpha^m, I^{rs}] = g^{mr}\alpha^s - g^{ms}\alpha^r. \quad (6)$$

We need concern ourselves first only with an irreducible set of matrices α^m and S satisfying (4) since every other representation can be made up of a direct sum of these. The representation S by itself is not irreducible, and, therefore, its nucleus consisting of the six matrices I^{rs} is not either. But in this particular problem, the six I^{rs} have also to satisfy Eqs. (6), and the collection of the ten matrices I^{rs} and α^m is irreducible.

Remembering that for any three quantities α, β and γ

$$[\alpha\beta, \gamma] = \alpha[\beta, \gamma] + [\alpha, \gamma]\beta,$$

it follows from (6) that

$$[[\alpha^m, \alpha^n], I^{rs}] = -g^{mr}[\alpha^m, \alpha^s] + g^{ms}[\alpha^n, \alpha^r] + g^{nr}[\alpha^n, \alpha^s] - g^{ns}[\alpha^m, \alpha^r] \quad (7)$$

showing that the substitution

$$[\alpha^m, \alpha^n] = cI^{mn},$$

where c is a numerical constant, would be consistent with Eqs. (5) and (6). Any finite constant c can always be removed from this equation by introducing a new set of α's equal to the old one divided by $(c)^{\frac{1}{2}}$. The effect of this change in Eq (1) is to divide χ by $(c)^{\frac{1}{2}}$, but since the physical meaning of χ has not been found yet we may consider this factor absorbed into χ. There are, therefore, two alternatives. Either the last equation holds with a finite c, in which case we can always write it in the more convenient form

$$[\alpha^m, \alpha^n] = I^{mn}, \quad (8)$$

or it does not hold.

The most general form for the α's satisfying Eq. (6) will be investigated in the last section. It will be shown there that (8) does not hold in general. Indeed, it does not hold for any of the alternative forms of the equations for spin greater than 1 given by Dirac, Fierz, and Pauli. It holds, however, for the three important cases where we are fairly certain that the theory is correct, namely for the Dirac equation, and the scalar and vector meson equations considered in the form (1) by Kemmer.

In this section we consider all possible equations of the form (1) for which the condition (8) is fulfilled. The resulting theories for particles of higher spin will be along the lines of the electron and meson theories, and it is possible to carry over many of the features of the latter to the general case. There are no subsidiary conditions, and, therefore, interactions can be introduced without difficulty or resort to any special artifice, exactly as in the case of the electron or meson. There are, therefore, reasons for believing that the equations of the form (1) for which the condition (8) is satisfied are the correct ones for describing particles of higher spin.

We have to study the algebra of ten matrices, the four α's and the six I's satisfying Eqs. (5), (6), and (8). Equation (5) merely states that if all the four indices m, n, r, j on the left are different, the matrices commute, while if one index of the two matrices on the left is common, then we always get a general commutation relation of the form

$$[I^{mn}, I^{ns}] = g^{nn}I^{ms}. \quad \text{(no summation over } n) \quad (9)$$

Hence if we take any three matrices I^{mn}, I^{ns}, I^{sm}, they satisfy the three cyclic commutation relations

$$\begin{aligned} [I^{mn}, I^{ns}] &= g^{nn}I^{ms} = -g^{nn}I^{sm}, \\ [I^{ns}, I^{sm}] &= -g^{ss}I^{mn}, \\ [I^{sm}, I^{mn}] &= -g^{mm}I^{ns}. \end{aligned} \quad (10)$$

The g^{nn} are here simple coefficients having the value 1 if $n=0$ and -1 otherwise. Consider first the case when m, n, l are equal to 1, 2, 3 respectively. Writing $iI^{23}=X$, $iI^{31}=Y$, $iI^{12}=Z$, the three equations (10) become

$$[X, Y] = iZ, \quad [Y, Z] = iX, \quad [Z, X] = iY. \quad (11)$$

Equations (11) are the well-known commutation rules for the three components of angular momentum and all representations by finite matrices of three quantities X, Y, Z satisfying (11) are known. As is well known, any irreducible representation can be labelled by a number λ which is an integer or a half-odd integer, and in this representation all the three matrices have the eigenvalues $\lambda, \lambda-1, \lambda-2, \cdots, -\lambda+1, -\lambda$, and X, Y and Z satisfy the characteristic equation

$$\{X^2-\lambda^2\}\{X^2-(\lambda-1)^2\} \times\{X^2-(\lambda-2)^2\}\cdots=0, \quad (12)$$

the last factor in the product being X or $\{X^2-(\frac{1}{2})^2\}$ depending on whether λ is an integer or half an odd integer. Now the representation of the I's with which we have to deal are certainly reducible as far as any three of them satisfying the relations (11) are concerned and since they are the direct sum of irreducible representations, can be characterized by the numbers $\lambda', \lambda'', \cdots$ belonging to each irreducible component. It will be found in the next section that the numbers $\lambda', \lambda'', \cdots$ must either be *all* integers or all half-odd integers. Since the characteristic equation for an X labelled by a given integer or half-odd integer is a factor of the equation for an X labelled by a larger integer or half-odd integer respectively, it follows that every such *reducible* X satisfies the characteristic Eq. (12) where λ is the largest positive number of the set $\lambda', \lambda'', \cdots$.

Suppose now that one of the indices in (10), say m, is zero. We then have to put $I^{0n}=X$, $I^{ns}=iY$, $I^{s0}=Z$ to reduce the corresponding Eqs. (10) to the form (12).

Lastly, consider the set of three matrices α^m, α^n, and I^{mn}. From (6) and (8), it follows that these satisfy the three commutation relations

$$[\alpha^m, \alpha^n]=I^{mn}, \quad [\alpha^n, I^{mn}]=-g^{nn}\alpha^m, \quad [I^{mn}, \alpha^m]=-g^{mm}\alpha^n. \quad (13)$$

Putting, for example, $i\alpha^m=X$, $i\alpha^n=Y$, $iI^{mn}=Z$ if $m, n\neq 0$, these can again be brought to the form (11). Taking three of our ten α's and I's at a time and proceeding in this way, it can be proved that α^0, $i\alpha^1$, $i\alpha^2$, $i\alpha^3$, I^{0k}, iI^{23}, iI^{31}, iI^{12} all satisfy the *same* characteristic Eq. (12) with the same value of λ, which is, therefore, a number that (together with others) can be used for labelling an irreducible representation of the ten matrices, furnishing a possible equation for a spinning particle. That the α-matrices, multiplied in some cases by i, satisfy the same characteristic equation as the I's as a result of Eqs. (6) and (8), was deduced after relatively lengthy calculations by Madhava Rao[7] in the case of spin $\frac{3}{2}$ and 2. The above argument shows without any calculation that it is generally true for all spins.

The similarity of (10) and (13) immediately suggests the next step. We introduce an additional index 4, and define

$$I^{m4}\equiv -I^{4m}\equiv\alpha^m, \quad g^{44}=-1, \quad g^{m4}=0 \quad \text{for } m\neq 4. \quad (14)$$

Then, as I have already pointed out in a previous note,[6] Eqs. (5), (6), and (8) are all combined into one, Eq. (5), if we let the indices in the latter run from 0 to 4 instead of from 0 to 3. We make the convention that a small capital letter used as an index stands for any of the numbers 0 to 4. Then ten matrices I^{KL} satisfying (5) with the indices running from 0 to 4 now satisfy the same commutation rules as the ten infinitesimal transformations of the Lorentz group in five dimensions. The problem of finding all relativistically invariant equations of the form (1) under the assumption (8) is reduced to the one of finding all irreducible representations of the Lorentz group in five dimensions, the solution of which is already known. It will, therefore, suffice to state in the next section some of the results we need concerning the representation of the Lorentz group in five dimensions and refer the reader to books[8,9] on the subject for further details and proofs.

2. THE IRREDUCIBLE REPRESENTATIONS OF THE WAVE EQUATIONS (1)

From every irreducible representation of the real orthogonal group in n dimensions, we can obtain an irreducible representation of the Lorentz group by a suitable change in the reality conditions governing the parameters and the elements of the group. Every representation of the Lorentz group in five dimensions can be labelled by two numbers λ_1 and λ_2 such that $\lambda_1\geqslant\lambda_2\geqslant 0$ where λ_1 and λ_2 are both integers or zero, or both half-odd integers (and, therefore, neither is zero). In the former case we get the one-valued representations expressible in terms

[7] Madhava Rao, Proc. Ind. Acad. Sci. **A15**, 139 (1942).
[8] Weyl, *The Classical Groups* (Princeton University Press, New Jersey, 1939).
[9] Murnaghan, *The Theory of Group Representations* (Johns Hopkins, 1938).

of tensors, in the latter the two-valued representations expressible in terms of spinors only.

The degree d of a representation $R_5(\lambda_1, \lambda_2)$ of the five-dimensional real orthogonal group is given by the formula

$$d_5(\lambda_1, \lambda_2) = \tfrac{2}{3}(\lambda_1+\tfrac{3}{2})(\lambda_2+\tfrac{1}{2}) \times(\lambda_1-\lambda_2+1)(\lambda_1+\lambda_2+2). \quad (15)$$

On the other hand, the degree of a representation $R_4(\lambda_1, \lambda_2)$ of the four-dimensional real orthogonal group is given by

$$\left.\begin{array}{l} d_4(\lambda_1, \lambda_2) = 2(\lambda_1-\lambda_2+1)(\lambda_1+\lambda_2+1) \\ \qquad\qquad\qquad \text{provided } \lambda_2 \neq 0 \\ \text{and} \\ d_4(\lambda_1, \lambda_2) = (\lambda_1-\lambda_2+1)(\lambda_1+\lambda_2+1) \\ \qquad\qquad\qquad \text{if } \lambda_2 = 0 \end{array}\right\}. \quad (16)$$

Every transformation of determinant one of the real orthogonal group in four dimensions belongs to a class, the typical element T of which consists of a rotation of any two axes through an angle θ_1 and a rotation of the two other axes through an angle θ_2. In five dimensions, the typical element of a class is of the same type, but leaves in addition the fifth axis unchanged. Consider the expression

$$\rho(m_1, m_2) = \sum \exp(im_1\theta_1 + im_2\theta_2) \quad (17)$$

with fixed positive numbers m_1 and m_2, the sum being over all permutations and reversals of sign of θ_1 and θ_2. Since ρ is symmetric in m_1 and m_2, we make the convention of writing the larger number first $m_1 \geqslant m_2 \geqslant 0$. A certain $\rho(m_1', m_2')$ will be said to precede another ρ when $m_1' > m_1$ or when $m_2' > m_2$ if $m_1' = m_1$. Then the spur of the matrix representing the transformations belonging to the class of T, we have described above in the representation $R_4(\lambda_1, \lambda_4)$ is

$$\sum_{m_1, m_2} \rho(m_1, m_2), \quad (18)$$

the sum being over certain integral values of m_1 and m_2 if λ_1 is an integer and over half-odd integral values if it is half an odd integer. The important point is that the 'leading term' in the sum (18), consisting of the ρ which precedes all the others, is

$$\rho(\lambda_1, \lambda_2) = \sum \exp(i\lambda_1\theta_1 + i\lambda_2\theta_2). \quad (19)$$

This circumstance enables us to identify the spin of the particle described by a given representation and set up the connection with the more usual notation.

In the usual rotation used by physicists, an irreducible representation of the *proper* Lorentz group $D(k, l)$ is labelled by two positive numbers k, l which can take on any integral or half-odd integral values independently. This representation is of degree $(2k+1)(2l+1)$ and its basic vectors $v_{m,m'}$ may be labelled by two numbers m, m' of which m takes on the values $k, k-1, \cdots -k+1, -k$, while m' takes on the values $l, l-1, \cdots -l+1, -l$ (cf., for example, van der Waerden[10]). The typical transformation T corresponding to a rotation of the axes 1, 2 through an angle θ_1, and the axes 0, 3 through the hyperbolic angle θ_2 is represented with a suitable choice of the basic vectors $v_{mm'}$ by a matrix which simply multiplies the vector $v_{mm'}$ by $\exp\{i(m+m')\theta_1 + (m-m')\theta_2\}$. A reflection turns the representation $D(k, l)$ into the representation $D(l, k)$, and in consequence, both must appear together in a representation of the full Lorentz group if $k \neq l$. The degree of this irreducible representation of the full Lorentz group is, therefore, $2(2k+1) \times(2l+1)$ if $k \neq l$, and $(2k+1)(2l+1)$ if $k = l$. The spur of the matrix representing T is, therefore,

$$\sum_{m, m'} \{\exp[i(m+m')\theta_1 + (m-m')\theta_2] + \exp[i(m+m')\theta_1 + (m'-m)\theta_2]\}, \quad (20)$$

the sum being over all values of m from $-k$ to k and over all values of m' from $-l$ to l. This expression can be identified with (17) if we write $i\theta_2$ in place of θ_2 as is required in passing from the Lorentz group to the real orthogonal group. The biggest positive integer which multiplies θ_1 in (20) is $k+l$, and in this term θ_2 is multiplied by $\pm|k-l|$. We therefore have the identification

$$\lambda_1 = k+l, \quad \lambda_2 = |k-l|. \quad (21)$$

The infinitesimal transformations iI^{23}, iI^{31}, iI^{12} can also be interpreted as the spin operators of the particle, and the highest eigenvalue of these in the representation we have just considered is $k+l=\lambda_1$. Hence we can interpret Eq. (1) as describing a particle of spin λ_1, when the α's in it are the four infinitesimal transformations I^{K4}

[10] van der Waerden, *Die gruppentheoretische Methode in der Quantenmechanik* (Verlagsbuchhandlung, Julius Springer, Berlin, 1932).

of the representation $R_5(\lambda_1, \lambda_2)$ of the Lorentz group in five dimensions. That λ_1 should have a value equal to the spin of the particle we wish to describe is clear. We leave the question open for the moment as to whether this condition is sufficient and discuss it again at the end of the section.

The existence of the number λ_2 shows that there is more than one such equation for any given value of λ_1. Since $\lambda_1 \geqslant \lambda_2 \geqslant 0$, it follows that the number of equations for a given value of λ_1 must be equal to λ_1+1 if λ_1 is an integer not equal to zero, or $\lambda_1+\frac{1}{2}$ if λ_1 is half an odd integer. For example, for $\lambda_1=\frac{1}{2}$, there is only one possibility $\lambda_2=\frac{1}{2}$, and we simply get the Dirac equation. For $\lambda_1=1$ there are two possibilities $\lambda_2=0$ and $\lambda_2=1$. The former gives the scalar meson theory with a representation of degree 5, the latter the vector meson theory with the representation of degree 10, both studied by Kemmer.

In the first column of Table I the possible values of λ_1 and λ_2 for equations describing particles of any spin up to 2 have been given. The respective degrees of the representations concerned, as calculated by formula (15), are given in the second column. The same values of λ_1 and λ_2 also label an irreducible representation of the Lorentz group in four dimensions which, as we have seen, is identical with the usual representation $D(k, l)+D(l, k)$ if $k\neq l$ and $D(k, k)$ otherwise. The values of k and l given by (21) are shown in the third column, and the degrees of the representations, as calculated by (16), in the fourth. In the fifth column are given the types of spinors corresponding to the irreducible representations of the four-dimensional group in the usual notation.

TABLE I.

λ_1	λ_2	d_5	k	l	d_4		
0	0	1	0	0	1	a	
$\frac{1}{2}$	$\frac{1}{2}$	4	$\frac{1}{2}$	0	4	a^{μ},	$a_{\dot{\mu}}$
1	0	5	$\frac{1}{2}$	$\frac{1}{2}$	4	$a^{\lambda\dot{\mu}}$	
1	1	10	1	0	6	$a^{\lambda\mu}$,	$a_{\dot{\lambda}\dot{\mu}}$
$\frac{3}{2}$	$\frac{1}{2}$	16	1	$\frac{1}{2}$	12	$a_{\dot{\nu}}{}^{\lambda\mu}$,	$a_{\dot{\lambda}\dot{\mu}}{}^{\nu}$
$\frac{3}{2}$	$\frac{3}{2}$	20	$\frac{3}{2}$	0	8	$a^{\lambda\mu\nu}$,	$a_{\dot{\lambda}\dot{\mu}\dot{\nu}}$
2	0	14	1	1	9	$a_{\dot{\nu}\dot{\rho}}{}^{\lambda\mu}$	
2	1	35	$\frac{3}{2}$	$\frac{1}{2}$	16	$a_{\dot{\rho}}{}^{\lambda\mu\nu}$,	$a_{\dot{\lambda}\dot{\mu}\dot{\nu}}{}^{\rho}$
2	2	35	2	0	10	$a^{\lambda\mu\nu\rho}$,	$a_{\dot{\lambda}\dot{\mu}\dot{\nu}\dot{\rho}}$

Every irreducible representation of the Lorentz group in five dimensions provides a representation, usually a reducible one, of the Lorentz group in four dimensions. For example, the antisymmetric tensor F_{KL} with ten components breaks up into two irreducible tensors, an antisymmetric one F_{kl} with six components and a vector F_{k4} with four components, if we consider only rotations keeping the axis 4 fixed. It corresponds to the decomposition

$$R_5(1, 1)\rightarrow R_4(1, 1)+R_4(1, 0). \tag{22}$$

This shows that the ten-rowed representation of a particle of spin 1 must involve a wave function which is made up of an antisymmetric tensor with two indices, and a vector. Similarly a five-dimensional vector F_K breaks into a vector F_k and a scalar F_4 under restriction to the four-dimensional group, corresponding to

$$R_5(1, 0)\rightarrow R_4(1, 0)+R_4(0, 0). \tag{23}$$

These two examples show in a new light the structure of the meson equations as studied by Kemmer.

Another example is provided by the simple rule (Murnaghan, page 287)

$$R_5(\lambda_1, 0)\rightarrow R_4(\lambda_1, 0)+R_4(\lambda_1-1, 0) +R_4(\lambda_1-2, 0)+\cdots+R_4(0, 0), \tag{24}$$

for λ_1, an integer. As a particular case of this rule, $\lambda_1=2$, we have one of the possible equations of a particle of spin 2 whose wave function consists of a symmetric tensor G_{KL} satisfying $G_K{}^K=0$, which breaks up under restriction to the Lorentz group in four dimensions into a symmetric tensor G_{kl}, a vector G_{k4} and scalar G_{44}. The degrees of the representations furnish a check on this decomposition: $14=9+4+1$.

For half-odd integral values of the λ's we have, as an example, which may be of physical interest,

$$R_5(\tfrac{3}{2}, \tfrac{3}{2})\rightarrow R_4(\tfrac{3}{2}, \tfrac{3}{2})+R_4(\tfrac{3}{2}, \tfrac{1}{2}) \tag{25}$$

with the check by degrees

$$20=8+12.$$

The corresponding Eq. (1) could describe a particle of spin $\frac{3}{2}$, and its wave function would involve the spinors $a^{\lambda\mu\nu}$, $a_{\dot{\lambda}\dot{\mu}\dot{\nu}}$, $a_{\dot{\nu}}{}^{\lambda\mu}$ and $a_{\dot{\lambda}\dot{\mu}}{}^{\nu}$.

The only other equation possible for describing a particle of spin $\frac{3}{2}$ is that given by the nucleus of the representation $R_5(\frac{3}{2}, \frac{1}{2})$. The wave function decomposes under restriction to the four-dimensional rotation group according to

$$R_5(\tfrac{3}{2}, \tfrac{1}{2}) \rightarrow R_4(\tfrac{3}{2}, \tfrac{1}{2}) + R_4(\tfrac{1}{2}, \tfrac{1}{2}), \tag{26}$$

the check according to degrees being $16 = 12 + 4$. The wave function consists of the spinors $a_{\dot{\nu}}{}^{\lambda\mu}$, $a_{\dot{\lambda}\dot{\mu}}{}^{\nu}$, a^{μ} and $a_{\dot{\mu}}$. We shall consider the structure of both the equations given above in more detail in Section 5.

We now return to the question as to whether the representations for a given value of λ_1 but different values of λ_2 all describe particles of spin λ_1. The scalar wave equation $R_5(1, 0)$ is generally thought to represent a particle without spin, and Kemmer has shown that the expectation value of the spin operator is zero in this case. On the other hand, as Kemmer has pointed out, the particle would still show a magnetic moment in the relativistic region. There is, therefore, some justification for regarding this particle as one possessing a spin which manifests itself in the relativistic region. According to a possible classification it would be considered as a particle of spin 1, even though it manifests no spin in the non-relativistic approximation. There are several equations of this type, as for example those given by $R_5(2, 0)$ and $R_5(\frac{3}{2}, \frac{1}{2})$ mentioned above. In all cases of the above type, the wave function falls into parts which transform according to different representations $D(k, l)$ of the four-dimensional Lorentz group with $k+1$ equal to *and less than* λ_1. We may consider the particle described by equations of this type to manifest different spins under different circumstances, the highest value being always λ_1. If, however, we require that all the different parts should correspond to representations for which $k+l$ is constant, then there is only *one* possible representation for each value of the spin λ_1 namely $R_5(\lambda_1, \lambda_1)$ with the degree $\frac{1}{6}(2\lambda_1+1)(2\lambda_1+2)(2\lambda_1+3)$ given by (15). This equation would certainly describe a particle of spin λ_1 only.

3. DETERMINATION OF THE VALUES OF THE REST MASS

The constant χ is obviously connected with the rest mass of the particle. To find the actual values of the mass it is necessary to eliminate the spin matrices α^k from Eq. (1) and in so doing derive an equation of higher order containing only the differential operators p_k and the constant χ. This can be done quite easily by a direct method already given in a previous note.[6]

Let $P \equiv p_k \alpha^k$. For the argument which follows we may regard the p_k as simple numbers; they become this in fact if the wave function is a plane wave. Since the α's are finite matrices of degree d, P must satisfy a characteristic equation whose degree cannot exceed d^2. This characteristic equation is a polynomial in P with the p_k as coefficients. Further, it can be proved immediately by using (4) that this equation must be invariant for all transformations of the Lorentz group; that is, it must be unchanged when the p_k are replaced by the p_k' to which they are connected by a transformation (3). The p_k can, therefore, only appear in the combination $p^2 \equiv p_k p^k$. Now consider the characteristic equation of P in the Lorentz system in which p_k has the special form $p_1 = p_2 = p_3 = 0$. P then becomes equal to $p_0\alpha^0$ where p_0 is just a number. For a particle of spin λ the operator α^0 satisfies the characteristic Eq. (12), and hence P satisfies the equation obtained from this by multiplying λ^2, $(\lambda-1)^2$, etc., by p^2. Written in a relativistically invariant way, $p_0{}^2 = p_k p^k$ and hence the characteristic equation of P must be

$$\{P^2 - p^2\lambda^2\}\{P^2 - p^2(\lambda-1)^2\} \times \{P^2 - p^2(\lambda-2)^2\} \cdots = 0 \tag{27}$$

the last term being $\{P^2 - p^2\}P$ or $\{P^2 - \frac{1}{4}p^2\}$ depending on whether λ is an integer or half an odd integer. Operating on ψ with the left-hand side of this equation we get, replacing every P by χ by a repeated use of (1),

$$\{\chi^2 - p^2\lambda^2\}\{\chi^2 - p^2(\lambda-1)^2\} \cdots \times \{\chi^2 - \lambda^2\} x\psi = 0 \tag{28a}$$

if λ is an integer, or

$$\{\chi^2 - p^2\lambda^2\}\{\chi^2 - p^2(\lambda-1)^2\} \cdots \times \{\chi^2 - \tfrac{9}{4}p^2\}\{\chi^2 - \tfrac{1}{4}p^2\}\psi = 0 \tag{28b}$$

if λ is half an odd integer. We may now replace p^2 by the second-order differential operator $-\partial^2/\partial x_k \partial x^k$. Each bracket is then just the usual second-order differential operator for a particle of finite mass. Equations (28), therefore, show that a particle of integral spin λ has 2λ possible values of the rest mass, namely $\pm\chi$, $\pm\chi/2$,

$\pm\chi/3\cdots\pm\chi/\lambda$ while a particle of half-odd integral spin has $2\lambda+1$ values of the rest mass $\pm 2\chi$, $\pm 2\chi/3$, $\pm 2\chi/5\cdots\pm\chi/\lambda$. The negative values of the mass only give the 'anti-particles' corresponding to each positive value of the mass, that is the particles of opposite charge exactly as in Dirac's theory of the electron. In the special case of spin $\frac{3}{2}$, we have two possible values of the rest mass, namely $m=2\chi/3$ and $3m=2x$. The higher value of the mass is, therefore, three times the lower. In the case of spin 2, we also have two possible values of the rest mass, namely $m=\chi/2$ and $2m=\chi$. The higher value is here just double the lower.

The above derivation shows that the number of values of the rest mass depends only on the maximum spin of the particle, or more precisely, on the magnitude of the constant λ_1 and is independent of the value of λ_2. Moreover, these states of higher rest mass are an intrinsic and necessary part of our theory and cannot be eliminated any more than the states of negative energy in Dirac's theory of the electron. According to our theory, if particles of spin greater than 1 exist in nature, then they must certainly manifest the phenomenon of appearing sometimes with one, sometimes with another value of the rest mass. The states of different rest mass being merely different states of the same particle, transitions from one mass to another would always be possible under the influence of interaction if sufficient energy were available for the purpose.

4. COMMUTATION RULES. REDUCIBLE REPRESENTATIONS

Equations (5), (6), and (8) or, if we adopt the five-dimensional notation, Eq. (5) alone with the indices running from 0 to 4, are commutation rules which must be valid for all representations and hence for all values of the spin. From these we can exclude all but a finite number of representations by prescribing a maximum value for the eigenvalues of any one of the α's or I's. It has been shown in Section 1 that all the X's and I's, multiplied in some cases by i, satisfy the same characteristic Eq. (12). We can derive particular commutation rules for each value λ_1 of the spin by prescribing the value of λ in (12) and then using this equation in conjunction with (5), (6), and (8). The method has been used by Madhava Rao[7,11] to derive the commutation rules of the α's for spins $\frac{3}{2}$ and 2. It is clearly quite general and can be used for any other value of the spin. From the fact that these commutation rules are derived by making use of (12), we can see at once that for a spin λ they must contain terms consisting of products of $2\lambda+1$ different α's on the one hand and connect these with terms containing products of $2\lambda-1$, $2\lambda-3$, $\cdots$, etc., λ's. On putting all the α's equal in these commutation rules, one must get back Eq. (12). The contribution of this paper to the solution of this particular problem is that it allows one to write down all the irreducible representations without knowing the commutation rules themselves, and it also tells us how many inequivalent representations there are for each value of the spin.

In this connection one important point should be noticed. Any matrix which satisfies Eq. (12) for a given value of λ *ipso facto* satisfies the same equation for a value λ' greater than λ by a positive integer. Hence matrices satisfying the commutation rules for some spin λ automatically satisfy the commutation rules for a spin greater than λ by an integer. In other words, if only the commutation rules corresponding to some value of the spin are given, there are included among the possible representations of the matrices those corresponding to particles of lower spin. For example, the commutation rules for spin $\frac{3}{2}$ are also satisfied by the Dirac matrices[12] besides the two new representations for this case which have been given above.

It may now be asked whether it is possible to specify an abstract algebra in such a way that it only has the one representation that we wish to select. For example, in the case of spin 1, can we so specify the algebra, besides giving the commutation rules of Duffin, that it possesses only one of the two possible representations that are known? The answer to this question must be in the affirmative. It depends on a well-known theorem in algebra (cf. van der Waerden[13])

[11] Madhava Rao, J. Mysore Univ. **3**, 59 (1942).

[12] This particular fact was pointed out to me in a conversation by Prof. D. D. Kosambi, and I am grateful to him for it.

[13] van der Waerden, *Moderne Algebra* (Verlagsbuchhandlung, Julius Springer, Berlin, 1931), Vol. II.

which states: Every irreducible representation of an abstract finite semi-simple algebra is a faithful representation of a simple sub-algebra, the other sub-algebras being represented by zero matrices in this representation. Corresponding to each irreducible representation for a given value of the spin, therefore, there must be an abstract simple sub-algebra, and the specification of it automatically restricts one to just this one representation.

It was first discovered by Cartan that the orthogonal group in 2ν or $2\nu+1$ dimensions has a spinor representation of degree 2^ν. As shown by Brauer and Weyl[14] spinors in n dimensions can be introduced exactly as in four dimensions. We start with n matrices γ^K satisfying the relations

$$\gamma^K\gamma^L+\gamma^L\gamma^K=2g^{KL}, \tag{29}$$

the g^{KL} being just a generalization to n dimensions of the g^{kl} given previously. There are only 2^ν independent elements of this algebra. The last element $\gamma^1\gamma^2\cdots\gamma^n$ commutes with all the elements, if n is odd and since its square is ± 1 it can only be represented by plus or minus a certain multiple of i. Corresponding to these two possibilities it can be proved that there are only two inequivalent irreducible representations of degree 2^ν. One of these is obtained from the other by merely reversing the sign of all the γ's. Spinors can then be introduced by considering the transformations of the form $p_K\gamma^K$ and connecting each orthogonal transformation in n dimensions with a certain S, exactly as in (4). The particular spinor representation so obtained can be regarded as the basic one, since every other representation of the n-dimensional orthogonal group is contained in the reducible ones obtained by forming its direct product with itself a sufficient number of times. In five dimensions the degree of the basic spinor representation is four. It is the representation denoted by $R_5(\frac{1}{2},\frac{1}{2})$ in Table I. This summary suffices for the purposes of this paper, and the reader is referred to the paper by Brauer and Weyl for further details.

Now consider the Kronecker direct product $R_5(\lambda_1,\lambda_2)\times R_5(\lambda_1',\lambda_2')$ of two irreducible representations of the five-dimensional orthogonal group of degrees d and d'. As it is well known, this representation is reducible, and its nucleus consists of the infinitesimal transformations

$$I^{KL}(\lambda_1,\lambda_2)\times E_{d'}+E_d\times I^{KL}(\lambda_1',\lambda_2') \tag{30}$$

E_d and $E_{d'}$ standing for the unit matrices of degrees d and d', respectively. Since the eigenvalues of $I^{KL}(\lambda_1,\lambda_2)$ are $\lambda_1,\ \lambda_1-1,\ \cdots-\lambda_1$ and those of $I^{KL}(\lambda_1',\lambda_2')$ run from λ_1' to $-\lambda_1'$ it follows that the eigenvalues of the nucleus of the product representation must range from plus to minus $\lambda_1+\lambda_1'$ and an irreducible representation corresponding to this value of the spin must appear in the reduction of the product representation. Further, the matrices (30) must satisfy the characteristic Eq. (12) with $\lambda=\lambda_1+\lambda_1'$. We can, therefore, write down immediately matrices which satisfy the characteristic Eq. (12) for any value of λ. They are in fact

$$\begin{aligned}I^{MN}\equiv I^{(1)MN}&\times E^{(2)}\times E^{(3)}\cdots\times E^{(2\lambda)}\\ &+E^{(1)}\times I^{(2)MN}\times E^{(3)}\times\cdots\times E^{(2\lambda)}\cdots\\ &+E^{(1)}\times E^{(2)}\times\cdots\times E^{(2\lambda-1)}\times I^{(2\lambda)MN},\end{aligned} \tag{31}$$

where the numbers in parentheses denote different independent sets of matrices of degree 4 each satisfying (29). Putting the four matrices I^{M4} of this set into (1) in place of the α^M, in accordance with (14), we get an equation whose wave function is a direct product of 2λ wave functions of the Dirac equation. The reduction of the set (31) into the sum of irreducible representations leads to a direct sum of all possible equations for particles of spin λ and those of lower spin. The reduction can be effected by repeated use of the formula (cf. Murnaghan[8])

$$\begin{aligned}R(\tfrac{1}{2},\tfrac{1}{2})\times R(\lambda_1,\lambda_2)&=R(\lambda_1+\tfrac{1}{2},\lambda_2+\tfrac{1}{2})\\ &+R(\lambda_1+\tfrac{1}{2},\lambda_2-\tfrac{1}{2})+R(\lambda_1-\tfrac{1}{2},\lambda_2+\tfrac{1}{2})\\ &+R(\lambda_1-\tfrac{1}{2},\lambda_2-\tfrac{1}{2}),\end{aligned} \tag{32}$$

it being understood that the third term on the right is to be dropped if $\lambda_1-\frac{1}{2}<\lambda_2+\frac{1}{2}$. The reduction indicated by Kemmer of the equation considered by de Broglie in which the α^k in (1) have the form $\alpha^k+\alpha'^k$ is but a particular case of (32) with $\lambda_1=\lambda_2=\frac{1}{2}$. We get on the right in this case $R(1,1)+R(1,0)+R(0,0)$, the first two representations being the two irreducible ones of the vector and scalar meson theory.

[14] Brauer and Weyl, Am. J. Math. 57, 425 (1935).

5. GENERAL ANALYSIS OF THE STRUCTURE OF EQ. (1)

We now proceed to a general analysis of the structure of Eq. (1) without requiring Eq. (8) to be fulfilled. For this purpose we shall have to make use of certain well-known formulae[4,10,15] connecting tensors with spinors which are summarized below.[16]

Spinor indices are denoted by Greek letters and only take on the values 1 and 2. The antisymmetric spinors $\epsilon_{\mu\nu}$ and $\epsilon^{\mu\nu}$ defined by $\epsilon_{12}=-\epsilon_{21}=1$, $\epsilon_{11}=\epsilon_{22}=0$ and $\epsilon^{12}=-\epsilon^{21}=1$, $\epsilon^{11}=\epsilon^{22}=0$ can be used for lowering and raising indices according to the formulae

$$b_\mu=\epsilon_{\mu\nu}b^\nu, \quad b^\nu=b_\mu\epsilon^{\mu\nu}.$$

Antisymmetric spinors $\epsilon_{\dot\mu\dot\nu}$ and $\epsilon^{\dot\mu\dot\nu}$ for raising and lowering dotted indices are defined similarly. Now consider the four matrices σ_k defined by

$$\sigma_0=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}, \quad \sigma_1=\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}, \quad \sigma_2=\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}, \quad \sigma_3=\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}, \tag{33}$$

or an equivalent representation. They satisfy the well-known relations

$$\sigma_k\sigma_l+\sigma_l\sigma_k=0 \quad \text{for } k\neq l\neq 0,\ \sigma_k^2=1 \text{ all } k \text{ and } \sigma_1\sigma_2=i\sigma_3 \tag{34}$$

together with the equations obtained from the last one by a cyclic permutation of the indices. The rows and columns of the matrices (33) can be labelled by an upper undotted and dotted index respectively thus giving the quantities $\sigma_k{}^{\mu\dot\nu}$ which have one tensor and two spinor indices. They retain the same form (33) after undergoing a transformation of the Lorentz group on the index k and corresponding spinor transformations of the indices μ, $\dot\nu$. The tensor index of these can be raised and lowered as usual by the tensor g_{kl}, the spinor indices by using the ϵ's. On doing this we find that $\sigma_{0\dot\mu\nu}$, and $-\sigma_{k\dot\mu\nu}(k\neq 0)$ have again the form (33) if the lower dotted index is now used to label the rows and the lower undotted one the columns. The first two equations of (7) can then be written in the spinor form

$$\begin{aligned}\sigma_k{}^{\nu\dot\mu}\sigma_{l\dot\mu\rho}+\sigma_l{}^{\nu\dot\mu}\sigma_{k\dot\mu\rho}&=2g_{kl}\delta_\rho{}^\nu\\ \sigma_{k\dot\rho\mu}\sigma_l{}^{\mu\dot\nu}+\sigma_{l\dot\rho\mu}\sigma_k{}^{\mu\dot\nu}&=2g_{kl}\delta_{\dot\rho}{}^{\dot\nu}.\end{aligned} \tag{35}$$

The third set of equations in (34) can be written

$$\tfrac{1}{2}(\sigma_k{}^{\mu\dot\nu}\sigma_{l\dot\nu\rho}-\sigma_l{}^{\mu\dot\nu}\sigma_{k\dot\nu\rho})=\sigma_k{}^{\mu\dot\nu}\sigma_{l\dot\nu\rho}=\frac{i}{2}\epsilon_{klmn}\sigma^{m\mu\dot\nu}\sigma_{\dot\nu\rho}{}^n \tag{36a}$$

and

$$\tfrac{1}{2}(\sigma_{k\dot\lambda\rho}\sigma_l{}^{\rho\dot\nu}-\sigma_{l\dot\lambda\rho}\sigma_k{}^{\rho\dot\nu})=\sigma_{k\dot\lambda\rho}\sigma_l{}^{\rho\dot\nu}=-\frac{i}{2}\epsilon_{klmn}\sigma_{\dot\lambda\rho}{}^m\sigma^{n\rho\dot\nu}. \tag{36b}$$

Finally there are the equations

$$\sigma^{m\mu\dot\nu}\sigma_{n\dot\nu\mu}=2\delta_n{}^m \tag{37}$$

$$\sigma_m{}^{\mu\dot\lambda}\sigma^{m\rho\dot\nu}=2\epsilon^{\mu\rho}\epsilon^{\dot\lambda\dot\nu}. \tag{38}$$

With every tensor α^k we can connect a spinor $A^{\lambda\mu}$ and *vice versa* through the equations

$$A^{\lambda\dot\mu}=\alpha^m\sigma_m{}^{\lambda\dot\mu}, \quad \alpha^m=\tfrac{1}{2}\sigma_{\dot\mu\lambda}{}^m A^{\lambda\dot\mu}. \tag{39}$$

Similarly, an antisymmetric spinor I^{kl} can be connected with two *symmetric* spinors $K^{\mu\nu}$ and $L^{\dot\mu\dot\nu}$ by the equations

$$4K_\nu{}^\mu=-I_{mn}\sigma^{m\mu\dot\lambda}\sigma_{\dot\lambda\nu}{}^n \tag{40a}$$

$$4L_{\dot\mu}{}^{\dot\lambda}=I_{mn}\sigma_{\dot\mu\nu}{}^m\sigma^{n\nu\dot\lambda} \tag{40b}$$

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[15] Uhlenbeck and Laporte, Phys. Rev. **37**, 1380 (1931).

[16] The definitions given here do not follow those of the authors mentioned but those given in my essay for the Adams Prize (Cambridge, 1942) on "The theory of the elementary physical particles and their interactions" to be published shortly in enlarged form by the Clarendon Press, Oxford.

and the converse

$$I_{mn} = -\tfrac{1}{2}\sigma_m{}^{\lambda\dot{\mu}}\sigma_n{}^{\nu\dot{\rho}}(\epsilon_{\dot{\mu}\dot{\rho}}K_{\lambda\nu} + \epsilon_{\lambda\nu}L_{\dot{\mu}\dot{\rho}}). \tag{41}$$

(40a) is equivalent to the three equations

$$K_1{}^1 = -K_2{}^2 = K_z, \quad K_2{}^1 = K_x - iK_y, \quad K_1{}^2 = K_x + iK_y \tag{42}$$

where

$$K_x = \tfrac{1}{2}(iI^{23} + I^{01}), \quad K_y = \tfrac{1}{2}(iI^{31} + I^{02}), \quad K_z = \tfrac{1}{2}(iI^{12} + I^{03}). \tag{43}$$

Similarly (40b) is equivalent to

$$L_{\dot{1}}{}^{\dot{1}} = -L_{\dot{2}}{}^{\dot{2}} = -L_z, \quad L_{\dot{2}}{}^{\dot{1}} = -L_x - iL_y, \quad L_{\dot{1}}{}^{\dot{2}} = -L_x + iL_y \tag{44}$$

where

$$L_x = \tfrac{1}{2}(iI^{23} - I^{01}), \quad L_y = \tfrac{1}{2}(iI^{31} - I^{02}), \quad L_z = \tfrac{1}{2}(iI^{12} - I^{03}). \tag{45}$$

The formulae (39) to (45) hold even when the α's and I's are matrices as in the previous sections. When the I's are the infinitesimal transformations of a representation of the Lorentz group satisfying (5), the K's and L's form two sets of matrices which commute with each other while the three matrices of each set satisfy the commutation rules (11) which follow from (5). In fact, $K^2 \equiv K_x^2 + K_y^2 + K_z^2$ commutes with every one of the six I's and if the representation concerned is the irreducible one we have labelled $I^{mn}(k, l)$, its value is $k(k+1)$. We denote this particular representation of the K's by $K(k)$. Similarly $L^2 \equiv L_x^2 + L_y^2 + L_z^2$ commutes with all the I's and has the value $l(l+1)$ in this representation. The representation of the K's is of degree $2k+1$, that of the L's of degree $2l+1$. The eigenvalues of all the K's are $k, k-1, \cdots -k+1, -k$. We may as usual take K_z to be in diagonal form. Labelling the corresponding rows and columns of the matrices by m, where m runs from k to $-k$ the matrix elements of the usual representation of the K's are given by

$$\begin{aligned}(m|K_z|m) &= m \\ (m+1|K_x + iK_z|m) &= [(k-m)(k+m+1)]^{\frac{1}{2}} \\ (m-1|K_x - iK_z|m) &= [(k+m)(k-m+1)]^{\frac{1}{2}}.\end{aligned} \tag{46}$$

All the other matrix elements vanish. The representation of L_x, L_y, and L_z is similar with l replacing k.

Finally we shall need the two matrices $u^\rho(k)$ of $2k+1$ rows and $2k$ columns and the two matrices $v^\rho(k)$ of $2k$ rows and $2k+1$ columns introduced by Dirac[1] and treated by Fierz[2] which satisfy the equations

$$-u_\mu(k+\tfrac{1}{2})v^\mu(k+\tfrac{1}{2}) = v_\mu(k)u^\mu(k) = 2k+1, \tag{47a}$$

$$v_\mu(k)v^\mu(k+\tfrac{1}{2}) = u_\mu(k+\tfrac{1}{2})u^\mu(k) = 0, \tag{47b}$$

$$-v^\mu(k+\tfrac{1}{2})u_\nu(k+\tfrac{1}{2}) = K_\nu{}^\mu(k) + (k+1)\delta_\nu{}^\mu, \tag{48a}$$

$$-u^\mu(k)v_\nu(k) = K_\nu{}^\mu(k) - k\delta_\nu{}^\mu. \tag{48b}$$

Equations (47) and (48) uniquely define the u's and v's. There are corresponding matrices $u^{\dot{\mu}}(l)$ and $v^{\dot{\mu}}(l)$ satisfying the same equations with all the indices dotted and the $L(l)$ written in place of the $K(k)$.

Writing $k-\frac{1}{2}$ in place of k in (48a), multiplying it from the left by $u^\rho(k)$ and then subtracting from it (48b) multiplied by $u^\rho(k)$ from the right we get after using (47) and (48)

$$u^\rho(k)K^{\mu\nu}(k-\tfrac{1}{2}) - K^{\mu\nu}(k)u^\rho(k) = \tfrac{1}{2}\epsilon^{\rho\mu}u^\nu(k) + \tfrac{1}{2}\epsilon^{\rho\nu}u^\mu(k). \tag{49}$$

Similarly we can deduce the equation

$$v^\rho(k)K^{\mu\nu}(k) - K^{\mu\nu}(k-\tfrac{1}{2})u^\rho(k) = \tfrac{1}{2}\epsilon^{\rho\mu}v^\nu(k) + \tfrac{1}{2}\epsilon^{\rho\nu}v^\mu(k). \tag{50}$$

Equations (49) and (50) uniquely determine the u's and v's except for an arbitrary multiplying constant if the representations of $K^{\mu\nu}(k)$ and $K^{\mu\nu}(k-\frac{1}{2})$ are given.

We now return to Eq. (6). Multiplying it by $\sigma_m{}^{\beta\dot{\lambda}}\sigma_r{}^{\gamma\dot{\nu}}\sigma_{s\dot{\nu}\rho}$ and remembering (39) and (40), the left-hand side becomes $-4[A^{\beta\dot{\lambda}}, K_\rho{}^\gamma]$ while the first term of the right-hand side gives

$$g^{mr}\sigma_m{}^{\beta\dot{\lambda}}\sigma_r{}^{\gamma\dot{\nu}}A_{\dot{\nu}\rho} = 2\epsilon^{\beta\gamma}\epsilon^{\dot{\lambda}\dot{\nu}}A_{\dot{\nu}\rho} = -2\epsilon^{\beta\gamma}A_\rho{}^{\dot{\lambda}}$$

because of (39). The second term on the right can be transformed similarly. Raising the index ρ we get

$$2[A^{\beta\dot{\lambda}}, K^{\gamma\rho}] = \epsilon^{\beta\gamma}A^{\rho\dot{\lambda}} + \epsilon^{\beta\rho}A^{\gamma\dot{\lambda}}. \tag{51}$$

Similarly, multiplying (6) by $\sigma_m{}^{\beta\dot{\lambda}}\sigma_{r\dot{\nu}\rho}\sigma_s{}^{\rho\dot{\mu}}$ we get

$$2[A^{\beta\dot{\lambda}}, L^{\dot{\mu}\dot{\nu}}] = \epsilon^{\dot{\lambda}\dot{\mu}}A^{\beta\dot{\nu}} + \epsilon^{\dot{\lambda}\dot{\nu}}A^{\beta\dot{\mu}}. \tag{52}$$

The factor 2 at the left of these equations is important, as we shall soon see.

The matrices I^{rs} in (6) must be a reducible representation of the nucleus of the proper Lorentz group, and they can always be written in the form

$$I^{rs} = \left\{ \begin{array}{c|c|c|c} I^{rs}(k_1, l_1) & & & \\ \hline & I^{rs}(k_2, l_2) & & \\ \hline & & I^{rs}(k_3, l_3) & \\ \hline & & & \end{array} \right. \tag{53}$$

where all the empty rectangles are filled with zeros. We leave the number of irreducible elements contained in I^{rs} unspecified. Corresponding to this reduction, both $K^{\gamma\rho}$ and $L^{\dot{\mu}\dot{\nu}}$ take the same form, the former having $K^{\gamma\rho}(k_s)$ in the respective boxes, the latter $L^{\dot{\mu}\dot{\nu}}(l_s)$. We can also divide the matrices $A^{\beta\dot{\lambda}}$ into squares and rectangles corresponding to the above reduction and label the sub matrices in any rectangle $(k_s, l_s|A^{\beta\dot{\lambda}}|k_t, l_t)$. Corresponding to this reduction, (51) and (52) become

$$(k_sl_s|A^{\beta\dot{\lambda}}|k_tl_t)K^{\gamma\rho}(k_t) - K^{\gamma\rho}(k_s)(k_sl_s|A^{\beta\dot{\lambda}}|k_tl_t) = \tfrac{1}{2}\epsilon^{\beta\gamma}(k_sl_s|A^{\rho\dot{\lambda}}|k_tl_t) + \tfrac{1}{2}\epsilon^{\beta\rho}(k_sl_s|A^{\gamma\dot{\lambda}}|k_tl_t), \tag{54}$$

and

$$(k_sl_s|A^{\beta\dot{\lambda}}|k_tl_t)L^{\dot{\mu}\dot{\nu}}(l_t) - L^{\dot{\mu}\dot{\nu}}(l_s)(k_sl_s|A^{\beta\dot{\lambda}}|k_tl_t) = \tfrac{1}{2}\epsilon^{\dot{\lambda}\dot{\mu}}(k_sl_s|A^{\beta\dot{\nu}}|k_tl_t) + \tfrac{1}{2}\epsilon^{\dot{\lambda}\dot{\nu}}(k_sl_s|A^{\beta\dot{\mu}}|k_tl_t). \tag{55}$$

The index λ remains attached to the $A^{\beta\dot{\lambda}}$ unaffected in (54), while the index β remains unaffected in (55). The general solution of these equations is, therefore, clearly of the form

$$(k_sl_s|A^{\beta\dot{\lambda}}|k_tl_t) = (k_s|A^\beta|k_t)(l_s|A^{\dot{\lambda}}|l_t), \tag{56}$$

the expression on the right being really the direct product of two rectangular matrices the first having $2k_s+1$ rows and $2k_t+1$ columns, and the second $2l_s+1$ rows and $2l_t+1$ columns. The first equation, therefore, reduces to

$$(k_s|A^\beta|k_t)K^{\gamma\rho}(k_t) - K^{\gamma\rho}(k_s)(k_s|A^\beta|k_t) = \tfrac{1}{2}\epsilon^{\beta\gamma}(k_s|A^\rho|k_t) + \tfrac{1}{2}\epsilon^{\beta\rho}(k_s|A^\gamma|k_t), \tag{57}$$

with a similar equation for $(l_s|A^{\dot{\lambda}}|l_t)$. Equation (57) is exactly the same as (49) or (50) if $k_t = k_s \pm \frac{1}{2}$. We have to investigate what happens in the other cases. It will be shown that $(k_s|A^\beta|k_t) = 0$ unless $k_t = k_s \pm \frac{1}{2}$.

Putting $\beta = \gamma = 1$, $\rho = 2$ in (57) we get, remembering (42),

$$(k_s|A^1|k_t)K_z(k_t) - K_z(k_s)(k_s|A^1|k_t) = \tfrac{1}{2}(k_s|A^1|k_t). \tag{58}$$

Let us assume the K's to be in the form (46). Using the letters m and m' to label the rows and columns of $K_z(k_s)$ and $K_z(k_t)$, respectively, (58) then becomes

$$(k_s,\, m\,|\,A^1\,|\,k_t,\, m')\{m'-m-\tfrac{1}{2}\}=0. \tag{59}$$

Since m takes on the values $k_s,\, k_s-1,\, \cdots$ and m' the values $k_t,\, k_t-1,\, \cdots$ the factor in curly brackets can only vanish if k_s and k_t differ by half an odd integer, and the only non-vanishing elements of A^1 are then $(k_s,\, m\,|\,A^1\,|\,k_t,\, m+\frac{1}{2})$. If $k_t \neq k_s \pm \frac{1}{2}$, then the corresponding sub matrix of A^1 is zero. Let $k_t=k_s+(n/2)$, n being an odd integer.

If we put $\beta=\gamma=\rho=1$ in (57) we get, remembering (42),

$$(k_s\,|\,A^1\,|\,k_t)\{K_x(k_t)-iK_y(k_t)\}-\{K_x(k_s)-iK_y(k_s)\}(k_s\,|\,A^1\,|\,k_t)=0. \tag{60}$$

Using (46) we obtain

$$(k_s,\, m\,|\,A^1\,|\,k_t,\, m+\tfrac{1}{2})[(k_t+m+\tfrac{3}{2})(k_t-m-\tfrac{1}{2})]^{\frac{1}{2}}-[(k_s+m+1)(k_s-m)]^{\frac{1}{2}}(k_s,\, m+1\,|\,A^1\,|\,k_t,\, m+\tfrac{3}{2})=0,$$

or writing $k_t=k_s+(n/2)$ and dropping the indices s and t

$$\left(k,\, m\,|\,A^1\,|\,k+\frac{n}{2},\, m+\frac{1}{2}\right)\left[\left(k+\frac{n}{2}+m+\frac{3}{2}\right)\left(k+\frac{n}{2}-m-\frac{1}{2}\right)\right]^{\frac{1}{2}}$$
$$=[(k+m+1)(k-m)]^{\frac{1}{2}}\left(k,\, m+1\,|\,A^1\,|\,k+\frac{n}{2},\, m+\frac{3}{2}\right). \tag{61}$$

First consider the case when n is positive, $n \geqslant 1$. The matrix element of A^1 on the left of (61) does not exist if $m=-k-1$, while the one on the right exists, but no contradiction arises since the coefficient of the latter vanishes for $m=-k-1$. For $m=k$ the matrix element of A^1 on the left exists while that on the right does not. Hence (61) would lead to a vanishing of $(k,\, -k\,|\,A^1\,|\,k+n/2,\, -k+\frac{1}{2})$ (and by repeated use of (61) of all other matrix elements of $(k\,|\,A'\,|\,k+n/2))$ unless the coefficient of the one on the left vanished, i.e., if

$$\left(k+\frac{n}{2}+k+\frac{3}{2}\right)\left(k+\frac{n}{2}-k-\frac{1}{2}\right)=0. \tag{62}$$

This can only happen if $n=1$. In this case (57) reduces to (50), and we see that $(k\,|\,A^1\,|\,k+\frac{1}{2})$ is just the matrix $v^1(k+\frac{1}{2})$ multiplied by an arbitrary constant. In fact, from (61) we can derive an explicit expression for the matrix elements of $(k\,|\,A^1\,|\,k+\frac{1}{2})$ or v^1. It is

$$(k,\, m\,|\,A^1\,|\,k+\tfrac{1}{2},\, m+\tfrac{1}{2})=c[(k+m+1)]^{\frac{1}{2}}, \tag{63}$$

c being an arbitrary constant. All other elements vanish.

Similarly, putting $\beta=1$, $\gamma=\rho=2$ in (57), we get an equation which connects $(k\,|\,A^2\,|\,k+\frac{1}{2})$ with $(k\,|\,A^1\,|\,k+\frac{1}{2})$. It follows from (50) that it must be proportional to v^2. We get, in fact, for its elements

$$(k,\, m\,|\,A^2\,|\,k+\tfrac{1}{2},\, m-\tfrac{1}{2})=c(k-m+1)^{\frac{1}{2}}. \tag{64}$$

If n in (61) is negative, then it can be shown by a similar argument that it must be equal to -1. Equation (57) then takes the form of (49), and we find that

$$(k\,|\,A^\mu\,|\,k-\tfrac{1}{2})=du^\mu(k). \tag{65}$$

We have obtained the final result that two irreducible components in (53) of the Γ's can only be connected together if their k's and l's differ by $\pm\frac{1}{2}$. There are, therefore, two typical types of connection. We call the coupling of $I(k,\, l)$ with $I(k+\frac{1}{2},\, l-\frac{1}{2})$ a connection of the first type and denote it by $(k,\, l)\rightleftharpoons(k+\frac{1}{2},\, l-\frac{1}{2})$. The coupling of $I(k,\, l)$ with $I(k+\frac{1}{2},\, l+\frac{1}{2})$ will be called one of the second type and denoted by $(k,\, l)\rightleftharpoons(k+\frac{1}{2},\, l+\frac{1}{2})$. For the first type of connection the relevant matrix elements of the α's have the form

$$\begin{aligned}(k,\, l\,|\,A^{\beta\dot\lambda}\,|\,k+\tfrac{1}{2},\, l-\tfrac{1}{2})&=cv^\beta(k+\tfrac{1}{2})u^{\dot\lambda}(l)\\(k+\tfrac{1}{2},\, l-\tfrac{1}{2}\,|\,A^{\beta\dot\lambda}\,|\,k,\, l)&=du^\beta(k+\tfrac{1}{2})v^{\dot\lambda}(l).\end{aligned} \tag{66}$$

For the second type they have the form

$$
\begin{aligned}
(k, l|A^{\beta\dot\lambda}|k+\tfrac{1}{2}, l+\tfrac{1}{2}) &= cv^{\beta}(k+\tfrac{1}{2})v^{\dot\lambda}(l+\tfrac{1}{2})\\
(k+\tfrac{1}{2}, l+\tfrac{1}{2}|A^{\beta\dot\lambda}|k, l) &= du^{\beta}(k+\tfrac{1}{2})u^{\dot\lambda}(l+\tfrac{1}{2}).
\end{aligned}
\tag{67}
$$

We note at once that by a transformation of the type $\alpha^k \rightarrow Q\alpha^k Q^{-1}$ which leaves the I's unchanged, $QI^{rs}Q^{-1}=I^{rs}$, the constants c and d in (66) and (67) can be changed at will, leaving only their product cd constant. We can, therefore, always bring them to the form $|c|=|d|$. For a reason which will become obvious further down, it is convenient to write $c=-d=(a)^{\frac{1}{2}}$.

Since Eq. (1) and the I's must be invariant for reflections or improper Lorentz transformations in general if an $I(k, l)$ appears in the decomposition (53), then $I(l, k)$ must also appear in it if $k\neq l$. There are now two possibilities. Either one can pass from any $I(k, l)$ to every other through a chain of successive connections of the types we have described, or there are at most two unconnected sets in each of which one can pass from any $I(k, l)$ to any other through a chain. The two unconnected sets (unconnected, that is, by matrix elements of the α's) pass into each other by a reflection. Any other possibility would lead to reducible representations of the α's and I's. If the second alternative holds, then we shall only consider one of the sets and ignore the other, since the latter can always be obtained from the former by a reflection and has the same structure. Corresponding to the decomposition (53), the wave function of (1) also decomposes into a *direct* sum of component wave functions which we write

$$
\psi=\psi(k_1, l_1)+\psi(k_2, l_2)+\psi(k_3, l_3)+\cdots. \tag{68}
$$

We consider first the simplest case where there is only one connection of the first type, $(k, l) \rightleftharpoons (k+\frac{1}{2}, l-\frac{1}{2})$. Remembering that $p_k\alpha^k=\frac{1}{2}p_{\lambda\dot\mu}\alpha^{\lambda\dot\mu}$ and omitting the factor 2, we find Eq. (1) takes the form

$$
\begin{aligned}
cp_{\beta\dot\lambda}v^{\beta}(k+\tfrac{1}{2})u^{\dot\lambda}(l)\psi(k+\tfrac{1}{2}, l-\tfrac{1}{2})+\chi\psi(k, l)&=0,\\
-cp_{\beta\dot\lambda}u^{\beta}(k+\tfrac{1}{2})v^{\dot\lambda}(l)\psi(k, l)+\chi\psi(k+\tfrac{1}{2}, l-\tfrac{1}{2})&=0.
\end{aligned}
\tag{69}
$$

The equations given by Dirac are equivalent to

$$
\begin{aligned}
p_{\dot\lambda\beta}v^{\beta}(k+\tfrac{1}{2})\psi(k+\tfrac{1}{2}, l-\tfrac{1}{2}) &= \left(\frac{2k+1}{2l}\right)^{\frac{1}{2}}\chi v_{\dot\lambda}(l)\psi(k, l),\\
p^{\beta\dot\lambda}v_{\dot\lambda}(l)\psi(k, l) &= \left(\frac{2l}{2k+1}\right)^{\frac{1}{2}}\chi v^{\beta}(k+\tfrac{1}{2})\psi(k+\tfrac{1}{2}, l-\tfrac{1}{2}).
\end{aligned}
\tag{70}
$$

Multiplying these by $u^{\dot\lambda}(l)$ and $u_{\beta}(k+\frac{1}{2})$, respectively, and using (47a), we get equations of the form (69). Multiplying Eqs. (70) by $v^{\dot\lambda}(l-\frac{1}{2})$ and $v_{\beta}(k)$, respectively, and using (47b), we get

$$
\begin{aligned}
p_{\dot\lambda\beta}v^{\dot\lambda}(l-\tfrac{1}{2})v^{\beta}(k+\tfrac{1}{2})\psi(k+\tfrac{1}{2}, l-\tfrac{1}{2})&=0,\\
p^{\beta\dot\lambda}v_{\dot\lambda}(l)v_{\beta}(k)\psi(k, l)&=0,
\end{aligned}
\tag{71}
$$

which are clearly in the nature of subsidiary conditions involving only one of the component ψ's. Equations (71) alone are *not* equivalent to (70), but (69) and (71) together are. We can only pass back from (69) to (70) by using (71), and it is also necessary to use (71) in order to deduce that both the ψ's satisfy the second-order wave equation. It is, therefore, clear that the D.F.P. equations cannot be written in the form (1) without imposing further subsidiary conditions not contained in (1). The one exception to the preceding statement is the Dirac equation $(\frac{1}{2}, 0)\rightleftharpoons(0, \frac{1}{2})$ where *both* the subsidiary conditions (71) vanish.[17]

We now investigate whether Eq. (8) is satisfied for the case we have just considered. We have to calculate the matrix $[\alpha^m, \alpha^n]$ with the α's given by (66). $[\alpha^m, \alpha^n]$ being an antisymmetric tensor,

[17] The meson equations do *not* strictly come under the D.F.P. scheme, but under one of the general schemes given below. In this one other exceptional case, however, the two are practically equivalent.

we can associate with it a symmetric spinor $B_\nu{}^\mu$ by (40a). This gives, by (39),

$$4B_\nu{}^\mu = -[\alpha_m, \alpha_n]\sigma^{m\mu\dot\lambda}\sigma_{n\dot\lambda\nu} = -[A^{\mu\dot\lambda}, A_{\dot\lambda\nu}]. \tag{72}$$

Similarly we get an antisymmetric spinor $B_{\dot\mu}{}^{\dot\lambda}$ defined by

$$4B_{\dot\mu}{}^{\dot\lambda} = [A_{\dot\mu\nu}, A^{\nu\dot\lambda}]. \tag{73}$$

It is obvious from (66) and (67) that all the non-diagonal elements of $B_\nu{}^\mu$ and $B_{\dot\mu}{}^{\dot\lambda}$, for example, $(k, l|B_\nu{}^\mu|k', l')$ for $k\neq k'$, $l\neq l'$ vanish. For the diagonal elements we get, using (66) and remembering that $-cd=a$,

$$\begin{aligned}4(k, l|B_\rho{}^\mu|k, l) &= a\{v^\mu(k+\tfrac{1}{2})u_\rho(k+\tfrac{1}{2})v^{\dot\nu}(l+\tfrac{1}{2})u_{\dot\nu}(l+\tfrac{1}{2}) - v_\rho(k+\tfrac{1}{2})u^\mu(k+\tfrac{1}{2})v_{\dot\nu}(l+\tfrac{1}{2})u^{\dot\nu}(l+\tfrac{1}{2}),\\ &= -2a(2l)K_\rho{}^\mu(k),\end{aligned} \tag{74}$$

after making use of (47) and (48). Similarly we find that

$$4(k+\tfrac{1}{2}, l-\tfrac{1}{2}|B_\rho{}^\mu|k+\tfrac{1}{2}, l-\tfrac{1}{2}) = 2a(2l+1)K_\rho{}^\mu(k+\tfrac{1}{2}). \tag{75}$$

Hence $B_\rho{}^\mu \neq K_\rho{}^\mu$, and they cannot be made equal by any adjustment of the constants at our disposal. An exception occurs only when $k=0$, for then $K_\rho{}^\mu(0)=0$ and we can choose $a=1$. The additional formulae

$$\begin{aligned}(k, l|B_{\dot\mu}{}^{\dot\lambda}|k, l) &= \tfrac{1}{2}a(2k+2)L_{\dot\mu}{}^{\dot\lambda}(l)\\ (k+\tfrac{1}{2}, l-\tfrac{1}{2}|B_{\dot\mu}{}^{\dot\lambda}|k+\tfrac{1}{2}, l-\tfrac{1}{2}) &= -\tfrac{1}{2}a(2k+1)L_{\dot\mu}{}^{\dot\lambda}(l-\tfrac{1}{2})\end{aligned} \tag{76}$$

can be calculated easily. It is clear that $B_{\dot\mu}{}^{\dot\lambda}\neq L_{\dot\mu}{}^{\dot\lambda}$ except when $l-\frac{1}{2}=0$. Thus the Dirac equation is the only one of the type $(k, l)\rightleftharpoons(k+\frac{1}{2}, l-\frac{1}{2})$ for which Eq. (8) is satisfied.

For a coupling of the second type given by (67), the corresponding formulae read

$$\begin{aligned}(k, l|B_\rho{}^\mu|k, l) &= \tfrac{1}{2}a(2l+2)K_\rho{}^\mu(k)\\ (k+\tfrac{1}{2}, l+\tfrac{1}{2}|B_\rho{}^\mu|k+\tfrac{1}{2}, l+\tfrac{1}{2}) &= -\tfrac{1}{2}a(2l+1)K_\rho{}^\mu(k+\tfrac{1}{2}).\end{aligned} \tag{77}$$

The formulae for $B_{\dot\rho}{}^{\dot\mu}$ in this case are the same, with k and l interchanged on the right of (77) and $L_{\dot\rho}{}^{\dot\mu}$ written in place of $K_\rho{}^\mu$. In this case also $B_\rho{}^\mu\neq K_\rho{}^\mu$ unless $k=0$. Also, $B_{\dot\rho}{}^{\dot\mu}=L_{\dot\rho}{}^{\dot\mu}$ can be shown to require $l=0$. The only possible simple connection of the second type for which (8) is satisfied is $(0, 0)\rightleftharpoons(\frac{1}{2}, \frac{1}{2})$, and this gives just the scalar meson equations.

We now proceed to consider more complicated cases involving the coupling of more than two $I(k, l)$'s. It is convenient to distinguish two types. One consists of open chains in which the I's at the two ends of the chain are only connected to one other I, while those in the middle are connected to one on each side. In the other more complicated type there is no I which is connected to *only one* other.

Let us consider first an example belonging to the first type, namely $(k, l)\rightleftharpoons(k+\frac{1}{2}, l-\frac{1}{2})\rightleftharpoons(k+1, l-1)\cdots\rightleftharpoons(k+n+\frac{1}{2}, l-n-\frac{1}{2})$. $A^{\beta\dot\mu}$ is just made up of several matrix elements of the type (66) the ones connecting $(k+r, l-r)$ with $(k+r+\frac{1}{2}, l-r-\frac{1}{2})$ being multiplied by an arbitrary constant which we write as $(a_r)^{\frac{1}{2}}$ instead of $(a)^{\frac{1}{2}}$. Here again the non-diagonal elements of the $B_\rho{}^\mu$ vanish while in the diagonal elements (k, l) we simply have a sum of two terms one of which comes from the connection on each side of the I concerned. Thus, using (74) and (75)

$$(k+r, l-r|B_\rho{}^\mu|k+r, l-r) = \tfrac{1}{2}\{(2l-2r+2)a_{r-\frac{1}{2}} - (2l-2r)a_r\}K_\rho{}^\mu(k+r). \tag{78a}$$

Similarly, using (76) we get

$$(k+r, l-r|B_{\dot\mu}{}^{\dot\lambda}|k+r, l-r) = \tfrac{1}{2}\{-(2k+2r)a_{r-\frac{1}{2}} + (2k+2r+2)a_r\}L_{\dot\mu}{}^{\dot\lambda}(l-r). \tag{78b}$$

$B_\rho{}^\mu = K_\rho{}^\mu$ requires

$$-2la_0 = (2l+1)a_0 - (2l-1)a_{\frac{1}{2}} = \cdots = (2l-2r+2)a_{r-\frac{1}{2}} - (2l-2r)a_r = (2l-2n+1)a_n = 2. \tag{79}$$

Solving these equations successively for a_n, $a_{n-\frac{1}{2}}$, we see that all the coefficients must be positive, which leads to a contradiction with the last equation $-2la_0=2$. Hence in general a solution of this type does not satisfy (8). The contradiction is avoided only if $k=0$ for then $K_\rho^\mu(0)=0$, and the last equation disappears. Similarly it can be shown that $B_{\dot\mu}^{\dot\lambda}=L_{\dot\mu}^{\dot\lambda}$ requires that $(l-n-\frac{1}{2})=0$. The solution of (79) in this case is simply $a_r=1$ for all r. The chain covers all $I(k, l)$ with $k+l$ equal to constant λ_1 say, $I(0, \lambda_1)\rightleftharpoons I(\frac{1}{2}, \lambda_1-\frac{1}{2})\rightleftharpoons\cdots\rightleftharpoons I(\lambda_1, 0)$. It is obvious from the analysis of the representations involved that this solution of the I's and α's just corresponds to the representation $R_5(\lambda_1, \lambda_1)$. There is one equation, (1), of this type for each value of the spin $\lambda_1=k+l$. Written in terms of spinors with the decomposition of ψ given in (68), Eq. (1) with this representation of the α's is equivalent to

$$-p_{\beta\dot\lambda}u^\beta(r)v^{\dot\lambda}(\lambda_1-r+\tfrac{1}{2})\psi(r-\tfrac{1}{2}, \lambda_1-r+\tfrac{1}{2})+p_{\beta\dot\lambda}v^\beta(r+\tfrac{1}{2})u^{\dot\lambda}(\lambda_1-r)\psi(r+\tfrac{1}{2}, \lambda_1-r-\tfrac{1}{2}) +\chi\psi(r_1\lambda_1-r)=0, \quad (80)$$

it being understood that in the first and last equations of the set ($r=0$ or λ_1) any term for which the argument inside brackets becomes negative is to be dropped. The vector meson equation belongs to this type. It is interesting to note that in this case there are only three equations, and while the two end ones for $r=0$ and $r=1$ are of the type (70), the one for $r=\frac{1}{2}$ does not belong to this type since it connects three ψ's; $\psi(0, 1)$, $\psi(\frac{1}{2}, \frac{1}{2})$ and $\psi(1, 0)$. If we take $\lambda_1=\frac{3}{2}$, we get one of the two possible equations describing a particle of spin $\frac{3}{2}$.

Another type of open chain is provided by the scheme $(k, l)\rightleftharpoons(k+\frac{1}{2}, l+\frac{1}{2})\rightleftharpoons\cdots\rightleftharpoons(k+n, l+n)$. The elements of $A^{\beta\dot\lambda}$ are now all of type (67), and it can be shown by an analysis similar to the foregoing that $B_\rho^\mu=K_\rho^\mu$ and $B_{\dot\rho}^{\dot\mu}=L_{\dot\rho}^{\dot\mu}$ only if $k=l=0$. The one possible equation of this type satisfying (8) has the scheme $(0, 0)\rightleftharpoons(\frac{1}{2}, \frac{1}{2})\rightleftharpoons\cdots\rightleftharpoons(k, k)$ it being possible to break off the chain for any value of k. Writing $\lambda_1=2k$ it is clearly seen to be the case described in Section 2 as $R_5(\lambda_1, 0)$ the structure we have just shown corresponding to that given by formula (24). Equations of this type can only describe particles of integral spin. The scalar meson equation is the simplest one of this type.

It will suffice to mention only two other equations in detail, namely two belonging to the second type where the chain is not open. Let us analyze the structure of the other possible equation for a particle of spin $\frac{3}{2}$ besides the one we have considered. Its structure is given by (26). Connecting up all the I's which can possibly be connected we arrive at the scheme

$$\begin{array}{ccc} (\frac{1}{2}, 1) & \rightleftharpoons & (1, \frac{1}{2}) \\ \downarrow\uparrow & & \downarrow\uparrow \\ (0, \frac{1}{2}) & \rightleftharpoons & (\frac{1}{2}, 0). \end{array} \quad (81)$$

It can then be proved by an analysis similar to the previous ones that by a proper and uniquely determined choice of the a's in (66) and (67) the α's can be made to satisfy (8). In fact, calling the constant involved in the left vertical link in (81) a, that in the right vertical link c, the one in the top horizontal link b and the one in the bottom link d it can be shown quite easily that

$$a=c=-\tfrac{5}{4}, \quad b=\tfrac{1}{4}, \quad d=\tfrac{9}{4}.$$

By using these values and (66) and (67), the equations for this case, equivalent to (1), can be written down explicitly without difficulty. Although the maximum spin of the particle described is $\frac{3}{2}$, it has mixed up with it a particle of spin $\frac{1}{2}$. It has some very interesting physical properties which will be dealt with in another paper.**

Finally we give the scheme corresponding to the remaining equation for a particle of spin 2 men-

** Proc. Ind. Acad. Sci. A 21, 241–264 (June 1945). The non-relativistic equation of the state of lowest rest mass describes a particle of spin $\frac{1}{2}$. It can be used to describe the proton, but predicts some hitherto undiscovered properties for it. In general, the constant λ_1 determines the maximum spin of the particle while the constant λ_2 determines the spin in the state of lowest rest mass.

tioned in Table I and described by the representation $R_5(2, 1)$. It is

$$\begin{array}{ccccc} (0,1) & \rightleftharpoons & (\frac{1}{2},\frac{1}{2}) & \rightleftharpoons & (1,0) \\ \upharpoonleft\downharpoonright & & \upharpoonleft\downharpoonright & & \upharpoonleft\downharpoonright \\ (\frac{1}{2},\frac{3}{2}) & \rightleftharpoons & (1,1) & \rightleftharpoons & (\frac{3}{2},\frac{1}{2}). \end{array} \tag{82}$$

The constants of the connecting matrix elements of $A^{\beta\lambda}$ can be determined without difficulty.

Since all the equations considered in this section which satisfy (8) can be written in the form (1) without any subsidiary conditions, it is obvious that interaction with an electromagnetic field can be introduced in all of them, as in (1), by simply writing $p_k - e\phi_k$ in place of p_k, the ϕ_k being the electromagnetic potentials. The Eq. (1) can clearly be derived from a Lagrange function, and hence no inconsistency is introduced in the equations by this procedure.

SUMMARY

The general structure of relativistically invariant wave equations of the form (1) is investigated, it being postulated that all properties of the particle should be derivable from (1) without the help of any auxiliary conditions. It is proved that the equations investigated by Dirac, Fierz, and Pauli for spins greater than 1 do not satisfy this requirement.

It is shown that all irreducible representations of the spin matrices in (1) satisfying the condition (8) can be obtained from the irreducible representations of the orthogonal group in five dimensions. The wave functions of (1) do not in general satisfy a second-order wave equation in the absence of interaction, but one of a higher order. A consequence of the equations is that every particle of spin greater than one must appear with several values of the rest mass which are multiples of the lowest value. For example, a particle of spin $\frac{3}{2}$ must have two values of the mass, one three times the other, and a particle of spin 2 also two values of the mass, the higher double the lower. These higher values of the mass are a necessary feature of the theory and cannot be eliminated.