# Field Concepts in Quantum Theory

GUIDO BECK
*Córdoba Observatory, University of Córdoba, Buenos Aires, Argentina*

## 1. INTRODUCTION

THE purpose of the present paper is an attempt to answer the question, whether or not field concepts, introduced by classical electrodynamics, are essential in order to describe the microstructure of matter and radiation.

It is well known that quantum theory has made two independent attempts to describe microscopic phenomena: one by modifying the laws of electrodynamics by the introduction of the photon concept, the other by reformulating the laws of mechanics. Bohr and Rosenfeld have shown, that both of these modifications of classical theory can be derived from one single principle, the uncertainty principle, which limits the simultaneous knowledge of canonic conjugate quantities, indispensable for causal description.

Already a simple dimension consideration shows, that there is room in non-relativistic mechanics for the introduction of a third constant $h$ in addition to the two classical constants $m$ and $e$. Non-relativistic quantum mechanics can be considered, today, to be a definitively established scheme. Making use of the results of a recent paper,[1] we shall denote the physical picture to be associated with non-relativistic quantum mechanics by $P(m, e, h)$.

There is neither any difficulty of principle to introduce quantum concepts into the theory of radiation. Classical electromagnetic radiation theory corresponds to a very simple physical picture, $P(c)$, which, in quantum theory, becomes $P(c, h)$. We have, however, to lay emphasis on the fact, that the pictures $P(m, e, h)$ and $P(c, h)$ use different fundamental concepts and cannot be combined in a trivial way.

We have, finally, to mention a third well-known physical picture, $P(m, c, e)$, which corresponds to relativistic mechanics of a point charge.

While the three mentioned physical pictures can be quantitatively formulated, we meet serious difficulties, as soon as we try to establish an over-determined physical picture $P(m, c, e, h)$ which, sometimes, has been denoted as "relativistic quantum theory." These difficulties arise as soon as we try to generalize either ordinary quantum mechanics in agreement with the theory of relativity, or if we want to introduce into the theory of radiation the concept of a point charge.

We shall show in the next paragraph, that the development of a more general physical picture depends essentially on the choice of a set of appropriate variables.

## 2. KINETIC, CANONICAL, AND FIELD VARIABLES

Let us, first, consider the variables which are implied in classical and relativistic mechanics, or in the physical pictures to be denoted as $P(m, e)$ and $P(m, c, e)$. One realizes immediately, that these variables are respectively

$$\mathbf{v} \quad \text{and} \quad \Phi,$$

and

$$u_i \quad \text{and} \quad A_i.$$

In the following, variables of the type $\mathbf{v}$ and $u_i$ shall be called *kinetic variables*, while variables of the type $\Phi$ and $A_i$ and the derived quantities

$$\mathbf{E}=-\frac{1}{c}\frac{\partial \mathbf{A}}{\partial t}+\text{grad } A_0, \quad \mathbf{H}=\text{rot } \mathbf{A}$$

may be called *field variables*.

The use of these two sets of variables is, however, not the only possible one. Indeed, Hamilton's mechanics and its further development, including the physical picture $P(m, e, h)$ of quantum mechanics, use, in addition to the field variables $A_i$ so-called *canonical variables*,

$$p_i=m\cdot u_i-\frac{e}{c}A_i. \qquad (1)$$

The most important difference between the physical pictures $P(m, c, e)$ (relativistic mechanics) and $P(m, e, h)$ (non-relativistic quantum mechanics) has to be seen in the fact, that the

[1] G. Beck, "Physical picture and mathematical formalism," Phil. Sci. (to be published).

former one contains explicitly the *velocity concept*, while the latter does not use a fundamental quantity of the dimension of a velocity and accounts for the phenomenon of motion in a more abstract way.

In Hamilton's mechanics, the essential difference between the two pictures does not show up fully, because its formalism permits reference fairly easily to the classical picture of motion by the relation

$$\partial H/\partial p = \dot{q}, \tag{2}$$

which, at least in the case of absence of magnetic fields

$$\mathbf{A} = 0, \tag{3}$$

reduces to the simple connection

$$\mathbf{v} = \mathbf{p}/m. \tag{4}$$

In quantum mechanics, the essential difference between the two pictures becomes much more obvious and does not permit more than the use of "correspondence pictures."

We shall show, now, that there exist important arguments which throw doubts on the validity of relations (3) and (4) even in the case of free space.

Already quantum theory of the radiation field leads to the unexpected result, that no field free state of the vacuum can exist. The difficulties which flow from this result for the description of motion have been frequently discussed in literature. It has, however, never been mentioned, that Dirac's theory leads, in an independent way, to the same result.

If we write Dirac's equation under the form

$$p_0 + \boldsymbol{\alpha}\mathbf{p} + \beta mc = 0, \tag{5}$$

we meet a characteristic ambiguity of possible interpretations. We may either accept $p_i$ to be canonical variables and consider $\boldsymbol{\alpha}$ and $\beta$ as formal means to express a linear system of differential equations, which generalize Hamilton's and Schrödinger's equations. In this case, we have to introduce into Eq. (5) independent field variables. This procedure requires, however, the independent introduction of the concepts $e$ and $h$ and leads to all the difficulties implied in an over-determined physical picture $P(m, c, e, h)$.

In order to solve these difficulties along the lines which have so successfully been followed in non-relativistic quantum mechanics, we would have to eliminate from the fundamental concepts of the theory any reference to the velocity concept, represented by the constant $c$. This would mean, first of all, that we would have to abandon Maxwell's picture of electromagnetic radiation, $P(c)$, and that we would have to replace it by a much more abstract description $P(e, h)$ before combining it with the picture of quantum mechanics. No attempt in this direction has ever been made.

The attempts to solve the difficulties which we find in literature are dominated by the tendency to conserve the physical picture $P(c, h)$ of radiation theory, rather than to maintain the picture of quantum mechanics. This means, as long as we want to refer explicitly to the constant $h$ and to the uncertainty principle, that we have to eliminate the constant $e$, the charge concept, and, therefore, any reference to field concepts.

In radiation theory the problem has been solved. The characteristic feature of this theory is the use of particle pictures, appropriate to deal with phenomena showing up in experiments with cosmic rays. Difficulties which have not yet been overcome arise, however, as soon as we want to eliminate field concepts from the description of problems which, usually, are expressed in terms of "static fields."

If the development of the theory along these lines is successful, the question of the role of field concepts becomes a secondary one and we may leave it for later to study what field concepts become and how they degenerate as soon as we leave the domain of applicability of classical theory. Still, even in this case, the question is of importance. Professor Bohr has frequently pointed out that our language and our way of thinking is essentially determined by the accustomed description of macroscopic phenomena by classical theory. No understanding of phenomena is possible unless we know when and where classical field concepts can be used.

The present state of quantum theory does not exclude, however, the possibility of developing other physical pictures. If we find an independent way to study characteristic features of field concepts in microphysics, this study may even furnish new heuristic points of view to the theory.

### 3. DOES DIRAC'S THEORY CONTAIN THE FIELD CONCEPTS?

Let us, now, admit the velocity concept and the constant $c$ explicitly in Dirac's Eq. (5),

$$\left[-\frac{1}{c}\frac{\partial}{\partial t}+\boldsymbol{\alpha}\,\mathrm{grad}+\frac{i\beta}{\Lambda}\right]\Psi=0. \qquad (6)$$

In this case, we can consider the operators $\boldsymbol{\alpha}$ and $\beta$ to be kinematical variables and, in particular to be kinetic variables (velocities) which correspond to the four vector $u_i$ of the theory of relativity and which are measured in units $c$.

This point of view has been frequently discussed in literature. It leads, indeed, to the concepts of spin and trembling movement of an electron in free space. We have, however, to note, that this point of view, once we decide to accept it definitively as a basis for the development of the theory, implies a very fundamental change of description of physical phenomena. Indeed, if we consider $p_i$ as canonical variables and $\boldsymbol{\alpha}$, $\beta$ as kinetic quantities, relation (1) shows immediately, that Dirac's equation does not permit any arbitrary introduction of field variables, but that it contains already implicitly the existence of finite electromagnetic field quantities, even in the case of the vacuum. If we want field variables to show up explicitly, we have either to eliminate from the theory kinetic variables $\boldsymbol{\alpha}$, $\beta$ or the explicit use of canonical variables. In the following we shall adopt the second way.

Basing our considerations on expression (6), in which no explicit reference to canonical variables is made, we have to consider Dirac's equation as a kinematical relation, describing a space-time continuum, suitable for the description of motions and fields.

The physical consequences become immediately clear. We arrive at the concept of a space-time continuum which depends on the quantities $\boldsymbol{\alpha}$ and $\beta$ and the microstructure of which accounts for the existence and for the behavior of electromagnetic fields in a way similar to that in which the macrostructure of the space-time continuum accounts for the existence and for the behavior of gravitational fields in general relativity. No simple inertial movement exists in this space-time continuum. Spin and trembling movement become intrinsic properties of space-time structure and, in particular, the spin phenomenon becomes a directly observable evidence for the existence of strong magnetic fields in vacuum.

### 4. GENERAL FEATURES OF MICROKINEMATICS

Before entering into the detailed discussion of the quantities with which we have to deal in a kinematical scheme based upon relation (6), we shall briefly examine its general features and we shall show that it has to deal with magnitudes which are in two respects more general than the ones used in ordinary kinematics.

The physical picture defined by relation (6) is of the type $P(c, \Lambda)$ where $\Lambda$ denotes Compton's wave-length. It means that we consider movements which depend on a critical velocity and on a critical distance. The physical sense of the velocity $c$ is well known from the theory of relativity; the meaning of the distance $\Lambda$ is, as we shall find below, that our kinematics implies characteristic space-time fluctuations which show up in distances smaller than $\Lambda$.

The fact, that the operators $\boldsymbol{\alpha}$ and $\beta$ can be represented by matrices of four rows and columns means, that even in the case when no electron is present, we have to consider four possible states of the space-time continuum and possible transitions between them. In the ordinary treatment of an electron, these four states are well known and are attributed to two different signs of charge and spin direction. We have, therefore, to replace the ordinary space-time continuum of the theory of relativity by the concept of a four-fold space-time continuum.

Observable (diagonal) quantities, e.g., the $z$ component of the spin or of the intrinsic magnetic field of the vacuum, correspond to one single state of space-time. Our kinematical scheme contains, however, beside these diagonal quantities other, non-diagonal ones, which correspond to fluctuating quantities and belong to transitions between two different space-time states.

The possibility of including in kinematics transitions between different space-time states represents the first qualitative generalization implied in the new scheme. We shall see below, that the field quantities to be attributed to the vacuum depend essentially on the possibility of transitions between different space-time states.

A closer mathematical investigation, the bases for which have been given by E. Cartan's theory of semi-vectors, shows that in a continuum with non-positive definite metrics, not all of the possible transitions between the considered states are mathematically equivalent. In kinematics, this means in current terminology, that transitions between different spin states (spin precession) show a different kinematical behavior under a Lorentz transformation than transitions between different charge states (pair production and annihilation). From the mathematical point of view this fact corresponds to the formal differences between a spatial rotation and a Lorentz transformation in ordinary space-time.[2] From the physical point of view, it means that kinematical quantities obey, in our scheme, transformation laws which are of a more general type than the ones used in restricted relativity.

The second essential generalization which has to be introduced is closely related with the behavior of a spin and of a static magnetic field. Even diagonal quantities, such as the $z$ component of the spin and of a constant magnetic field, can only be defined in one system of reference. We cannot define, therefore, a magnetic field in space and time, which is responsible for the spin phenomenon of an electron of arbitrary momentum. As we shall see below, our kinematical scheme leads, indeed, to field quantities which are defined not in space and time, but in the seven-dimensional phase space $(x, y, z; t; p_x, p_y, p_z)$.

### 5. KINEMATICAL QUANTITIES

We shall call a solution of Eq. (6) a complete and normalized set of orthogonal functions, depending on space and time coordinates, on the matrices $\boldsymbol{\alpha}$ and $\beta$ and on three parameters, $p_1, p_2, p_3$,

$$\Psi_p(x, y, z, t, \boldsymbol{\alpha}, \beta). \tag{7}$$

The solution (7) represents a matrix of four rows and columns.[3] Comparing the solution in cartesian coordinates,

$$\Psi_p = \frac{(k_0+1)-\mathbf{k}\boldsymbol{\gamma}}{[L^3 2k_0(k_0+1)]^{\frac{1}{2}}} \exp\left[i(\beta k_0 ct+\mathbf{kr})\right], \tag{8}$$

with

$$k_0^2 - k^2 = 1/\Lambda^2,$$

with the ordinary way of writing the solutions of Dirac's equation, we find that each column of the matrix (8) represents an ordinary four-component spinor, corresponding respectively to plane waves of different spin direction and charge sign.

Dirac's matrices $\boldsymbol{\alpha}$ and $\beta$ represent a fundamental system which permits, in particular, one to form a linear basis system of 16 independent matrices and to express any given matrix of four rows and columns by a linear combination of them. We shall choose the following linear basis system:

$$\begin{array}{cc} & \beta \\ 1, & \boldsymbol{\alpha} \\ i\boldsymbol{\gamma}, & \beta\boldsymbol{\sigma} \\ \tau, & \boldsymbol{\sigma} \\ & i\beta\tau \end{array} \tag{9}$$

with

$$\boldsymbol{\gamma}=\beta\boldsymbol{\alpha}; \quad \boldsymbol{\sigma}=\frac{1}{2i}(\boldsymbol{\alpha}\times\boldsymbol{\alpha}); \quad \tau=\tfrac{1}{3}(\boldsymbol{\sigma}\cdot\boldsymbol{\alpha}).$$

We shall call

$${}^1M=\tfrac{1}{2}(M+\beta\cdot M\cdot\beta); \quad {}^2M=\tfrac{1}{2}(M-\beta\cdot M\cdot\beta)$$
$$M={}^1M+{}^2M,$$

respectively, the parts of *first* and *second* kind of the matrix $M$. 1, $\beta$, $\boldsymbol{\sigma}$, $\beta\boldsymbol{\sigma}$, are matrices of first kind, $\tau$, $\beta\tau$, $\boldsymbol{\alpha}$, $\boldsymbol{\gamma}$ are matrices of second kind.

We pretend, that the kinematical quantities of our scheme are given by the bilinear forms

$$\{M\}=\tilde{\Psi}_p M \Psi_{p'}, \tag{10}$$

where $M$ denotes any one of the 16 matrices (9) and where $\tilde{\Psi}$ is the transposed conjugate of $\Psi$.

The physical interpretation we can give to the quantities (10) depends on their transformation character and on the differential relations which we may find them to obey.

[2] See A. J. Fernandes De Sa, "Sur le comportement relativiste des grandeurs quantiques," Thèse, Porto, 1943. G. Beck, "El espacio fisico," Ciencia y Técnica **102**, No. 501 (1944).

[3] Solutions of Dirac's equations depending explicitly on $\boldsymbol{\alpha}$ and $\beta$ have been considered, for the first time, by F. Sauter. Their formal properties have been pointed out to the author in some more detail by Mr. Bogdanovich in Kiew.

### 6. TRANSFORMATION LAWS

We refer in this paragraph to the results obtained in a previous paper mentioned above.[4]

Under a spatial rotation the transformation character of the bilinear forms belonging to (9) is given by:

$$\begin{array}{ll} \{1\},\ \{\beta\} & \text{scalars,} \\ \{\boldsymbol{\alpha}\},\ \{i\boldsymbol{\gamma}\} & \text{polar vectors,} \\ \{\boldsymbol{\sigma}\},\ \{\beta\boldsymbol{\sigma}\} & \text{axial vectors,} \\ \{\tau\},\ \{i\beta\tau\} & \text{pseudoscalars.} \end{array}$$

Under a Lorentz transformation, a simple tensor character can be attributed to the parts of first and second kind of bilinear forms only. The parts of first kind, which include in particular the observable (diagonal) terms represent anti-symmetrical tensors of the rank indicated in (11)

$$\begin{array}{ll} {}^1\{\beta\} & \text{rank 0 (scalar),} \\ {}^1\{1\},\ {}^1\{\boldsymbol{\alpha}\} & \text{rank 1 (four vector),} \\ {}^1\{i\boldsymbol{\gamma}\},\ {}^1\{\beta\boldsymbol{\sigma}\} & \text{rank 2 (tensor),} \\ {}^1\{\tau\},\ {}^1\{\boldsymbol{\sigma}\} & \text{rank 3 (dual vector),} \\ {}^1\{i\beta\tau\} & \text{rank 4 (pseudoscalar).} \end{array} \tag{11}$$

The parts of second kind of our bilinear forms have anti-symmetric tensor character too, but according to the scheme (12)

$$\begin{array}{ll} {}^2\{1\} & \text{rank 0,} \\ {}^2\{\beta\},\ {}^2\{\boldsymbol{\gamma}\} & \text{rank 1,} \\ {}^2\{i\boldsymbol{\alpha}\},\ {}^2\{\boldsymbol{\sigma}\} & \text{rank 2,} \\ {}^2\{\beta\tau\},\ {}^2\{\beta\boldsymbol{\sigma}\} & \text{rank 3,} \\ {}^2\{i\tau\} & \text{rank 4,} \end{array} \tag{12}$$

and they are no longer exclusively of hermitian type, part of them being anti-hermitian.

No unambiguous co- and contra-variant transformation character can be attributed to the tensors (11) and (12) unless we consider each of their four parts

$$\begin{array}{l} \tfrac{1}{4}(\{M\}-\beta\{M\}-\{M\}\beta+\beta\{M\}\beta), \\ \tfrac{1}{4}(\{M\}+\beta\{M\}+\{M\}\beta+\beta\{M\}\beta), \\ \tfrac{1}{4}(\{M\}-\beta\{M\}+\{M\}\beta-\beta\{M\}\beta), \\ \tfrac{1}{4}(\{M\}+\beta\{M\}-\{M\}\beta-\beta\{M\}\beta), \end{array}$$

separately.

[4] See Section 4. The origin of the quoted modifications of relativistic transformation rules is found in the fact, that the definition of kinetic variables $\boldsymbol{\alpha}$, $\beta$ requires a plane wave (8) to correspond to a velocity

$$\mathbf{v}=-\beta\mathbf{k}/k_0,$$

instead of the usual value

$$\mathbf{v}=\mathbf{k}/k_0.$$

To touch at transformation properties of physical quantities is, indeed, a very serious matter. We have, therefore, to appeal to experience itself, before accepting a generalization of Lorentz' hundred-fold confirmed transformation rules. The examples to be chosen have to refer both to mechanics and to electrodynamics.

It is an experimental, rather than a theoretically derived fact, that the motion of a macroscopic body can be characterized by an energy-momentum four vector. Energy and momentum of the fields in the interior of the body have to be included (mass defects), though according to current theories energy and momentum of fields do not form, in general a four vector. Only in the restricted case of a free radiation field theory leads to a four vector and permits the unambiguous use of a particle picture (photons). Still, there can be no doubt, that observable field quantities obey always Lorentz' transformation rules.

The second point we want to mention concerns the concepts of angular momentum and magnetic moment. In electrodynamics there is no difficulty in considering the density of magnetic moment as part of an anti-symmetric tensor $u_{ik}$ which accounts simultaneously for the density of electric dipole moment. Difficulties arise, however, as soon as we want to interpret these quantities by a particle model in restricted relativity.

In non-relativistic electron mechanics both magnetic moment and angular momentum depend on the expression

$$\mathbf{r}\times\mathbf{v}; \tag{13}$$

they do not represent fundamental kinematical quantities, but derived ones. Expression (13) refers to an arbitrarily chosen point in space which we may consider to be the origin of our coordinate system. The relativistic generalization of (13) is the expression

$$x_iu_k-x_ku_i \tag{14}$$

which does not refer to a point in space but to a point-event in space-time. Expression (14) is, in general, no integral of motion even if angular momentum remains constant. Only the first term of the three components

$$x_iu_0-x_0u_i \tag{15}$$

corresponds to known physical quantities: mass moment and electric moment. Only in the restricted case of stationary movement, $u_i=0$, $i=1, 2, 3$, does (15) represent physical quantities. In the general case, (15) not only depends on time, but even on the arbitrary zero point from which we decide to count the variable $x_0$. No physical sense can, therefore, be attributed to the components (15) of the tensor (14).

On the other hand, we know from quantum theory and from experience, that angular momentum represents a very important characteristic of a mechanical system of arbitrarily rapid movement and is even related to the universal constant $h$. This raises the question, whether (14) is the correct relativistic generalization of the quantity (13) and whether the kinematical scheme to which it belongs is sufficiently general to account for the underlying physical phenomena.

Both examples show, that Lorentz' transformation laws do not exclude the possibility of generalization. In both of the mentioned problems the particular kinematics implied in Dirac's theory opens new possibilities of interpretation.

While observable (diagonal) kinematical quantities (10), including field quantities, obey well accustomed tensor transformation laws, the connection between the transformation character of quantities in non-linear relations becomes more complicated, because of the intervention of the fluctuating parts of second kind of physical quantities. This applies, in particular, to the quadratic energy expressions of field theory.

On the other hand, (10) contains angular momentum, $\{\boldsymbol{\sigma}\}$, not as a derived quantity, but already as a fundamental kinematical term. According to (11), the observable part of $\{\boldsymbol{\sigma}\}$ represents an angular momentum density and angular momentum

$$\int\{\boldsymbol{\sigma}\}d\tau \tag{16}$$

is no longer connected with quantities of the type (15), to which no physical sense can be attributed.

### 7. FERNANDES DE SA'S RELATIONS

We have, now, to show that the 16 quantities (10) obey 32 differential relations which Fernandes De Sa has obtained by adding and subtracting the two equations arising from (6)

$$\begin{aligned}&\tilde{\Psi}M\cdot\left(-\frac{1}{c}\frac{\partial\Psi}{\partial t}+\boldsymbol{\alpha}\cdot\operatorname{grad}\Psi+\frac{i\beta}{\Lambda}\Psi\right)=0,\\&\pm\left(-\frac{1}{c}\frac{\partial\tilde{\Psi}}{\partial t}+\operatorname{grad}\tilde{\Psi}\cdot\boldsymbol{\alpha}-\tilde{\Psi}\frac{i\beta}{\Lambda}\right)\cdot M\Psi=0.\end{aligned}\tag{17}$$

Letting $M$ be each one of the 16 operators of (9) and simplifying the resulting relations by introducing two differentiations, respectively denoted by latin and greek symbols,

$$\begin{aligned}\left\{\frac{\partial M}{\partial\xi}\right\}&=\frac{\partial\tilde{\Psi}M}{\partial\xi}\Psi+\tilde{\Psi}\frac{\partial M\Psi}{\partial\xi}=\frac{\partial\{M\}}{\partial\xi},\\\left\{\frac{\delta M}{\delta\xi}\right\}&=\frac{\partial\tilde{\Psi}M}{\partial\xi}\Psi-\tilde{\Psi}\frac{\partial M\Psi}{\partial\xi}=2\frac{\partial\tilde{\Psi}M}{\partial\xi}\Psi-\frac{\partial\{M\}}{\partial\xi}\\&=\frac{\partial\{M\}}{\partial\xi}-2\tilde{\Psi}\frac{\partial M\Psi}{\partial\xi},\end{aligned}\tag{18}$$

we obtain:

$$\left\{\frac{1}{c}\frac{\partial 1}{\partial t}\right\}-\{\operatorname{div}\boldsymbol{\alpha}\}=0,\tag{I}$$

$$i\left\{\frac{1}{c}\frac{\delta 1}{\delta t}\right\}-i\{\delta\iota\upsilon\;\boldsymbol{\alpha}\}=\frac{2}{\Lambda}\{\beta\}.\tag{II}$$

$$\begin{aligned}\left\{\frac{1}{c}\frac{\partial\beta}{\partial t}\right\}&=i\{\delta\iota\upsilon\;i\boldsymbol{\gamma}\},\\\{\operatorname{grad}\beta\}&=i\left\{\frac{1}{c}\frac{\delta\iota\boldsymbol{\gamma}}{\delta t}\right\}+i\{\rho o\tau\;\beta\boldsymbol{\sigma}\}.\end{aligned}\tag{III}$$

$$\begin{aligned}\{\operatorname{grad}1\}-\left\{\frac{1}{c}\frac{\partial\boldsymbol{\alpha}}{\partial t}\right\}&=\frac{2}{\Lambda}\{i\boldsymbol{\gamma}\}+i\{\rho o\tau\;\boldsymbol{\sigma}\},\\\{\operatorname{rot}\boldsymbol{\alpha}\}&=-\frac{2}{\Lambda}\{\beta\boldsymbol{\sigma}\}-i\{\gamma\rho\alpha\delta\;\tau\}+i\left\{\frac{1}{c}\frac{\delta\boldsymbol{\sigma}}{\delta t}\right\}.\end{aligned}\tag{IV}$$

$$\begin{aligned}\{\operatorname{div}i\boldsymbol{\gamma}\}&=\frac{2}{\Lambda}\{1\}-i\left\{\frac{1}{c}\frac{\delta\beta}{\delta t}\right\},\\-\{\operatorname{rot}\beta\boldsymbol{\sigma}\}-\left\{\frac{1}{c}\frac{\partial i\boldsymbol{\gamma}}{\partial t}\right\}&=-\frac{2}{\Lambda}\{\boldsymbol{\alpha}\}+i\{\gamma\rho\alpha\delta\;\beta\}.\end{aligned}\tag{V}$$

$$\begin{aligned}-\{\operatorname{div}\beta\boldsymbol{\sigma}\}&=-i\left\{\frac{1}{c}\frac{\delta i\beta\tau}{\delta t}\right\},\\\{\operatorname{rot}i\boldsymbol{\gamma}\}-\left\{\frac{1}{c}\frac{\partial\beta\boldsymbol{\sigma}}{\partial t}\right\}&=i\{\gamma\rho\alpha\delta\;i\beta\tau\}.\end{aligned}\tag{VI}$$

$$\{\text{grad } \tau\} - \left\{\frac{1}{c}\frac{\partial \boldsymbol{\sigma}}{\partial t}\right\} = i\{\rho o \tau\ \boldsymbol{\alpha}\},$$
$$\{\text{rot } \boldsymbol{\sigma}\} = -i\{\gamma\rho\alpha\delta\ 1\} + i\left\{\frac{1}{c}\frac{\delta\boldsymbol{\alpha}}{\delta t}\right\}. \tag{VII}$$

$$\left\{\frac{1}{c}\frac{\partial i\beta\tau}{\partial t}\right\} = \frac{2}{\Lambda}\{\tau\} - i\{\delta\iota\upsilon\ \beta\boldsymbol{\sigma}\},$$
$$\{\text{grad } i\beta\tau\} = \frac{2}{\Lambda}\{\boldsymbol{\sigma}\} - i\left\{\frac{1}{c}\frac{\delta\beta\boldsymbol{\sigma}}{\delta t}\right\} + i\{\partial o\tau\ i\boldsymbol{\gamma}\}. \tag{VIII}$$

$$\left\{\frac{1}{c}\frac{\partial\tau}{\partial t}\right\} - \{\text{div } \boldsymbol{\sigma}\} = -\frac{2}{\Lambda}\{i\beta\tau\}, \tag{IX}$$

$$i\left\{\frac{1}{c}\frac{\delta\tau}{\delta t}\right\} - i\{\delta\iota\upsilon\ \boldsymbol{\sigma}\} = 0. \tag{X}$$

From the mathematical point of view, relations (I)–(X) have a very remarkable property. They have been grouped in order to show their tensor character in first kind. This condition is, however, according to (11) and (12), not sufficient to prove their relativistic invariance. Relations (I)–(X) can, however, easily be regrouped in order to form sets of tensor relations in second kind too. Still, total invariance can no longer be attributed to one single tensor relation, but only to the complete set of our equations. We may call a mathematical structure of the type (I)–(X) an invariant *cycle of tensor relations*. Relativistic invariance imposes, therefore, more restrictive conditions upon kinematical quantities in Dirac's theory, than ordinary Lorentz invariance.

### 8. INTERPRETATION OF KINEMATICAL QUANTITIES

The interpretation of our kinematical quantities (10) has to be based on their transformation character (Section 6) and on the differential relations which they obey (Section 7).

We pretend, that Eqs. (V) and (VI) correspond, in the restricted case of absence of matter and radiation, to Maxwell's equations, i.e., that apart from a constant dimension factor we have to put in vacuum

$$\mathbf{E} \cong \{i\boldsymbol{\gamma}\}, \quad \mathbf{H} \cong -\{\beta\boldsymbol{\sigma}\}. \tag{19}$$

The quantities (19) are defined in phase space-time, as indicated above. For a point at rest, $\mathbf{k}=0$, $H_z$ is an observable, diagonal quantity, suitable to account for the spin movement of an electron, while $\mathbf{E}$ is a fluctuating quantity of second kind and accounts for the unobservable trembling movement.

Vacuum contains, according to (19) strong, fluctuating (not observable) electric and magnetic current densities of second kind, but no electric and magnetic charge densities, as can be easily proved by introducing (8) into Eqs. (V) and (VI). One recognizes, that this picture depends essentially on the new transformation laws found in Section 6 and would be in contradiction with ordinary Lorentz invariance.

The right-hand terms of Eq. (V) remember Dirac's subtraction terms in the hole theory. Here, they appear, however, without any hypothesis *ad hoc*. It can be easily shown, that these terms lead to finite polarization values of the vacuum in an appropriately introduced external field.

Equations (I) and (IV) describe the motion of continuously distributed, conservative matter. In particular (IV) represents an alternative form of generalization of Euler's hydrodynamic equations and depends on the action of forces, including electromagnetic forces. Still, in the vacuum (8), these forces are essentially fluctuating quantities of second kind (trembling movement).

Relations (I), (IV) and (V), (VI) are indispensable for a kinematical scheme which intends to serve for the description of conservative matter and fields. We find, however, additional relations, to which a physical meaning will have to be attributed, since they are indispensable for the mathematical consistency of our scheme.

We could try to interpret Eqs. (VII) and (IX) by properties of angular momentum in space-time. The formal analogy of (VII), (IX) and (I), (IV) suggests, however, that $\{\tau\}$, $\{\boldsymbol{\sigma}\}$ describes non-conservative matter, depending on the action of forces. It is still more difficult, at present, to interpret the quantities $\{\beta\}$ and $\{i\beta\tau\}$. It is not impossible that dynamics, built on the basis of Dirac's kinematics, will be able to use these quantities for the description of highly excited states of space-time (mesotrons, neutrinos).

Let us finally observe, that the great number of finite quantities which we have to attribute, according to the present theory, to vacuum, does not imply infinite energy and momentum values. We can, indeed, derive from (I)–(X) vanishing total energy and momentum expressions, corresponding to dynamical equilibrium.

### 9. CONCLUSIONS

The present study does not intend more than to show that Dirac's theory determines a well defined kinematical scheme, as soon as we decide to consider $\alpha$ and $\beta$ as kinetic variables. This scheme does not permit more than an unambiguous description of vacuum, but it contains the necessary elements to deal with fields and matter. From a general point of view one may conclude, that it should not present more than mathematical difficulties to formulate well-known dynamical laws of phenomena which take place in a given space-time continuum.