The Formation of Absorption Lines in a Moving Atmosphere

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1. INTRODUCTION

THE phenomenon of the scattering of light in a moving atmosphere has considerable interest for astrophysics. It occurs in Novae, Wolf-Rayet stars, planetary nebulae, the solar prominences, and the Corona. And more recently Struve's studies[1] of the spectra of stars like 48 Librae and 17 Leporis have emphasized its importance for stellar spectroscopy in general. But on consideration one soon realizes the unusual difficulties which must confront a rigorous theoretical analysis of these problems. For, in atmospheres in which large scale motions are present, on account of Doppler effect, the radiation scattered in different directions will have different frequencies, and, as a result of this, the radiation field in the different frequencies will interact with each other in a manner which is not always easy to visualize. However, in the astrophysical contexts, two circumstances simplify the problem. First, the velocities which are involved are small compared to the velocity of light, c, and second, the only effects of consequence are those which arise from the sensitive dependence of the scattering coefficient $\sigma(\nu)$ on the frequency ν. This last circumstance in particular allows us to ignore all effects such as aberration etc., and concentrate only on the effects arising from the change of frequency on scattering. The equation of transfer appropriate to these conditions has been written down by W. H. McCrea and K. K. Mitra.[2] But these writers did not succeed in solving any specific problem. However, we shall show how with certain approximations explicit solutions can be found which illustrate the effects which may be expected in the contours of absorption lines formed in an atmosphere in which differential motions exist. On the mathematical side, the novelty of the problem arises from the very unusual type of boundary value problem in hyperbolic equations which it presents.

2. THE EQUATION OF TRANSFER AND ITS APPROXIMATE FORMS

We shall consider an atmosphere stratified in parallel planes and in which all the properties are assumed to be constant over the planes $z = \text{constant}$ (see Fig. 1).

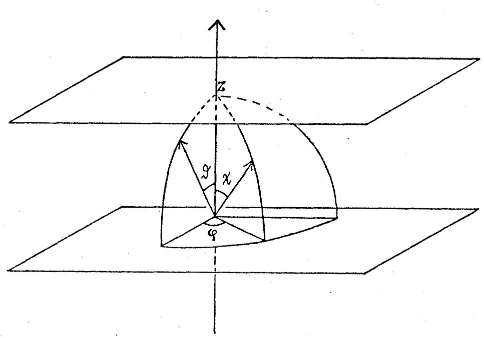

FIG. 1.

Let $\rho(z)$ be the density of the scattering material at height z and $w(z)$ the velocity of the material at the same height assumed parallel to the z direction. Further, let $\sigma(\nu)$ denote the mass scattering coefficient for the frequency ν as judged by an observer at rest with respect to the material. Since our principal interest is in the formation of absorption lines, we shall suppose that $\sigma(\nu)$ differs appreciably from zero only in a small range of ν. However, it is in the essence of the astrophysical problem that the "half-width" of $\sigma(\nu)$ is of the same order as the Doppler shifts in the frequency caused by the differential motions in the atmosphere. Indeed, it is this last

[1] O. Struve, Astrophys. J. **98**, 98 (1943). Also W. Hiltner, Astrophys. J. **99**, 103 (1944); P. W. Merrill and R. Sanford, Astrophys. J. **100**, 14 (1944).
[2] W. H. McCrea and K. K. Mitra, Zeits. f. Astrophys. **11**, 359 (1936).

circumstance which makes the change of frequency on scattering the only optical effect of the motion $w(z)$ which has any importance.

Consider then a pencil of radiation inclined at an angle ϑ to the positive normal and having a frequency ν as judged by a stationary observer. This radiation will appear to an observer at rest with respect to the material at z as having a frequency

$$\nu\left(1-\frac{w}{c}\cos\vartheta\right). \tag{1}$$

It will accordingly be scattered as such in all directions with a scattering coefficient

$$\sigma\left(\nu\left[1-\frac{w}{c}\cos\vartheta\right]\right). \tag{2}$$

We may, therefore, write the equation of transfer for the specific intensity $I(\nu, z, \vartheta)$ in the form

$$\cos\vartheta\frac{\partial I(\nu, z, \vartheta)}{\rho\partial z}=-\sigma\left(\nu\left[1-\frac{w}{c}\cos\vartheta\right]\right)I(\nu, z, \vartheta)+\mathfrak{J}(\nu, z, \vartheta), \tag{3}$$

where $\mathfrak{J}(\nu, z, \vartheta)$ denotes the emission per unit time and per unit solid angle in the frequency ν and in the direction ϑ. It is seen that this emission is given by

$$\mathfrak{J}(\nu, z, \vartheta)=\sigma\left(\nu\left[1-\frac{w}{c}\cos\vartheta\right]\right)\int_0^{2\pi}\int_0^{\pi} I\left(\nu\left[1-\frac{w}{c}\cos\vartheta+\frac{w}{c}\cos\chi\right], z, \chi\right)\sin\chi d\chi\frac{d\varphi}{4\pi}, \tag{4}$$

or in view of the symmetry about the z direction

$$\mathfrak{J}(\nu, z, \vartheta)=\tfrac{1}{2}\sigma\left(\nu\left[1-\frac{w}{c}\cos\vartheta\right]\right)\int_0^{\pi} I\left(\nu\left[1-\frac{w}{c}\cos\vartheta+\frac{w}{c}\cos\chi\right], z, \chi\right)\sin\chi d\chi. \tag{5}$$

To verify the foregoing expression for $\mathfrak{J}(\nu, z, \vartheta)$, we observe that the emission in the direction ϑ arises from the scattering into this direction of radiation from other directions. And, considering the contribution to $\mathfrak{J}$ from the scattering of the radiation in the direction specified by the polar angles χ and φ (see Fig. 1) into the direction $(\vartheta, 0)$, it is evident that the radiation must have the frequency

$$\nu\left(1-\frac{w}{c}\cos\vartheta+\frac{w}{c}\cos\chi\right), \tag{6}$$

as judged by a stationary observer; for, radiation of this frequency in the χ direction will appear to an observer at rest with respect to the material at z as having a frequency

$$\nu\left(1-\frac{w}{c}\cos\vartheta+\frac{w}{c}\cos\chi\right)\left(1-\frac{w}{c}\cos\chi\right)\simeq\nu\left(1-\frac{w}{c}\cos\vartheta\right), \tag{7}$$

which will accordingly be scattered uniformly in all directions with a scattering coefficient

$$\sigma\left(\nu\left[1-\frac{w}{c}\cos\vartheta\right]\right); \tag{8}$$

the radiation scattered into the ϑ-direction will have the same frequency (7) with respect to the material; to a stationary observer, it will appear as having a frequency ν. And summing over the contributions from all directions (χ, φ) we obtain (4).

Combining Eqs. (3) and (5) we have

$$\mu\frac{\partial I(\nu, z, \mu)}{\rho\partial z}=-\sigma\left(\nu\left[1-\frac{w}{c}\mu\right]\right)\left\{I(\nu, z, \mu)-\tfrac{1}{2}\int_{-1}^{+1} I\left(\nu\left[1-\frac{w}{c}\mu+\frac{w}{c}\mu'\right], z, \mu'\right)d\mu'\right\}, \tag{9}$$

where we have written μ and μ' for $\cos\vartheta$ and $\cos\chi$, respectively.

In solving Eq. (9) we shall adopt the method of approximation which has recently been developed in connection with the various problems of radiative transfer in the theory of stellar atmospheres.[3] The essence of this method is to replace the integrals which appear in the equation of transfer by sums according to Gauss's formula for numerical quadratures. Thus, considering Eq. (9) we replace it in the nth approximation by the system of $2n$ equations

$$\mu_i \frac{\partial I_i(\nu, z)}{\rho \partial z} = -\sigma\left(\nu\left[1-\frac{w}{c}\mu_i\right]\right)\left\{I_i(\nu, z) - \tfrac{1}{2}\sum a_j I_j\left(\nu\left[1-\frac{w}{c}\mu_i+\frac{w}{c}\mu_j\right], z\right)\right\}, \quad (i=\pm 1, \cdots, \pm n) \tag{10}$$

where the μ_i's, $(i=\pm 1, \cdots, \pm n)$, are the zeros of the Legendre polynomial $P_{2n}(\mu)$, and the a_i's are the appropriate weights. Further, in Eq. (10) we have written $I_i(\nu, z)$ for $I(\nu, z, \mu_i)$.

At this stage one further simplification of Eq. (9) is possible. In evaluating the Doppler shifts, we need not distinguish between

$$\nu\left(1-\frac{w}{c}\mu\right) \quad \text{and} \quad \nu - \nu_0\frac{w}{c}\mu,$$

where ν_0 denotes the frequency of the center of the line. We may, therefore, replace Eq. (10) by the simpler one

$$\mu_i \frac{\partial I_i(\nu, z)}{\rho \partial z} = -\sigma\left(\nu-\nu_0\frac{w}{c}\mu_i\right)\left\{I_i(\nu, z) - \tfrac{1}{2}\sum a_j I_j\left(\nu-\nu_0\frac{w}{c}\mu_i+\nu_0\frac{w}{c}\mu_j, z\right)\right\}, \quad (i=\pm 1, \cdots, \pm n). \tag{11}$$

The form of Eq. (11) suggests that instead of considering the intensities I_i, $(i=\pm 1, \cdots, \pm n)$, for some fixed frequency ν, we consider them for the frequencies

$$\nu_i = \nu + \nu_0 \frac{w}{c}\mu_i \quad (i=\pm 1, \cdots, \pm n), \tag{12}$$

which are functions of z. In Eq. (12), ν is a "fixed" frequency. If we now let

$$I_i(\nu_i, z) = \psi_i(\nu, z) \quad (i = \pm 1, \cdots, \pm n), \tag{13}$$

we have

$$\frac{\partial \psi_i}{\partial z} = \left[\frac{\partial I_i(\nu, z)}{\partial z}\right]_{\nu=\nu_i} + \frac{\partial I_i(\nu_i, z)}{\partial \nu_i}\frac{\partial \nu_i}{\partial z}, \tag{14}$$

or, according to Eqs. (12) and (13)

$$\frac{\partial \psi_i}{\partial z} = \left[\frac{\partial I_i(\nu, z)}{\partial z}\right]_{\nu=\nu_i} + \mu_i \frac{\nu_0}{c}\frac{dw}{dz}\frac{\partial \psi_i}{\partial \nu}. \tag{15}$$

Substituting for the first term on the right-hand side of the foregoing equation from Eq. (11), we obtain

$$\mu_i \frac{\partial \psi_i}{\partial z} - \mu_i^2 \frac{\nu_0}{c}\frac{dw}{dz}\frac{\partial \psi_i}{\partial \nu} = -\rho\sigma(\nu)(\psi_i - \tfrac{1}{2}\sum a_j \psi_j) \quad (i=\pm 1, \cdots, \pm n), \tag{16}$$

which is clearly the most convenient form in which to study the equation of transfer for a moving atmosphere.

In our subsequent work we shall restrict ourselves to the first approximation. In this approximation

$$\mu_1 = -\mu_{-1} = 1/\sqrt{3} \quad \text{and} \quad a_1 = a_{-1} = 1, \tag{17}$$

and Eqs. (11) and (16) lead to the two pairs of equations

[3] S. Chandrasekhar, Astrophys. J. **100**, 76, 117 (1944) and **101**, 95, 328, 348 (1945).

$$\mu_1 \frac{\partial I_{+1}(\nu, z)}{\rho \partial z} = -\tfrac{1}{2}\sigma\left(\nu - \nu_0 \frac{w}{c}\mu_1\right)\left[I_{+1}(\nu, z) - I_{-1}\left(\nu - 2\nu_0 \frac{w}{c}\mu_1, z\right)\right] \tag{18}$$

$$\mu_1 \frac{\partial I_{-1}(\nu, z)}{\rho \partial z} = +\tfrac{1}{2}\sigma\left(\nu + \nu_0 \frac{w}{c}\mu_1\right)\left[I_{-1}(\nu, z) - I_{+1}\left(\nu + 2\nu_0 \frac{w}{c}\mu_1, z\right)\right], \tag{19}$$

and

$$\mu_1 \frac{\partial \psi_{+1}}{\partial z} - \mu_1^2 \frac{\nu_0}{c} \frac{dw}{dz} \frac{\partial \psi_{+1}}{\partial \nu} = -\tfrac{1}{2}\rho\sigma(\nu)(\psi_{+1} - \psi_{-1}), \tag{20}$$

$$\mu_1 \frac{\partial \psi_{-1}}{\partial z} + \mu_1^2 \frac{\nu_0}{c} \frac{dw}{dz} \frac{\partial \psi_{-1}}{\partial \nu} = -\tfrac{1}{2}\rho\sigma(\nu)(\psi_{+1} - \psi_{-1}), \tag{21}$$

where it may be recalled that

$$\psi_{+1}(\nu, z) = I_{+1}\left(\nu + \mu_1 \nu_0 \frac{w}{c}, z\right), \quad \psi_{-1}(\nu, z) = I_{-1}\left(\nu - \mu_1 \nu_0 \frac{w}{c}, z\right). \tag{22}$$

3. SCHUSTER'S PROBLEM FOR A MOVING ATMOSPHERE

A classical problem first formulated by Schuster[4] provides the simplest model in terms of which the formation of absorption lines in a stellar atmosphere can be analyzed. In this model we consider a plane-stratified scattering atmosphere lying above a plane surface which radiates in a known manner and absorbs all radiation falling on it. The problem is to determine the radiation field in the atmosphere and in particular to relate the distribution in intensity of the emergent radiation with that radiated by the surface below. The appropriateness of this model for a first analysis of stellar absorption lines consists in the suitable idealization which it provides of the notions of a *photospheric surface* and the *reversing layers*. Consequently, when considering moving atmospheres it would seem proper that we retain the essentials of the Schuster model and generalize it only to the extent of admitting large scale motions. More particularly, we shall suppose that the photospheric surface is at $z=0$, and that it radiates uniformly in all outward directions $(0 \leqslant \vartheta < \pi/2)$ and in all frequencies. In other words, we suppose that

$$I(\nu, z, \vartheta) = \text{constant at } z=0 \text{ for } 0 \leqslant \vartheta < \pi/2 \text{ and for all frequencies.} \tag{23}$$

The state of motions in the atmosphere will be specified by the function $w(z)$ giving the velocity (assumed parallel to the z direction) at height z.

Finally, if $z=z_1$ defines the outer boundary of the atmosphere, we must require that here

$$I(\nu, z, \vartheta) \equiv 0, \quad \pi/2 < \vartheta < \pi \text{ at } z = z_1, \tag{24}$$

in accordance with the assumed non-existence of any radiation from the outside being incident on the atmosphere.

Schuster's problem for a moving atmosphere consists then in solving the equation of transfer (9), or the equivalent systems of equations in the various approximations, together with the boundary conditions (23) and (24). In the first approximation, the equivalent boundary conditions are that

$$I_{+1}(\nu, z) = \text{constant independent of } \nu \text{ at } z=0, \tag{25}$$

and

$$I_{-1}(\nu, z) \equiv 0 \text{ at } z = z_1. \tag{26}$$

4. THE REDUCTION TO A BOUNDARY VALUE PROBLEM FOR THE CASE $\sigma(\nu)$ = CONSTANT FOR $\nu_0 - \Delta\nu \leqslant \nu \leqslant \nu_0 + \Delta\nu$ AND ZERO OUTSIDE THIS INTERVAL AND FOR A LINEAR INCREASE OF w WITH THE OPTICAL DEPTH

In this paper we shall consider the solution to Schuster's problem formulated in the preceding

[4] A. Schuster, Astrophys. J. **21**, 1 (1905).

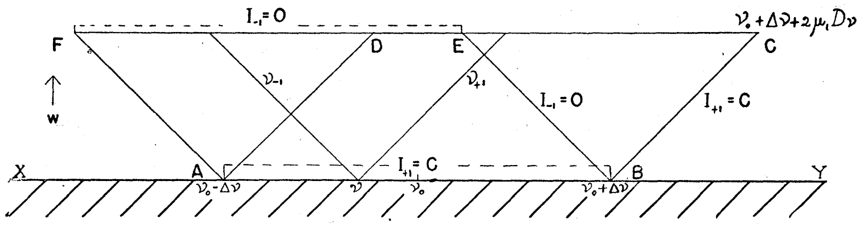

FIG. 2.

section for the case

$$\begin{aligned}\sigma(\nu)&=\text{constant}=\sigma_0 \text{ for } \nu_0-\Delta\nu\leqslant\nu\leqslant\nu_0+\Delta\nu,\\ &=0 \text{ otherwise,}\end{aligned}\Bigg\}\tag{27}$$

and

$$\frac{1}{\rho}\frac{dw}{dz}=\text{constant.}\tag{28}$$

Further, we shall restrict ourselves to the first approximation.

When $\sigma(\nu)$ has the form (27) some care is required in the formulation of the boundary conditions. For, according to Eqs. (18) and (19)

$$\partial I_{+1}/\partial z\neq 0 \text{ only if } \nu_0-\Delta\nu+\mu_1\nu_0\frac{w}{c}\leqslant\nu\leqslant\nu_0+\Delta\nu+\mu_1\nu_0\frac{w}{c},\tag{29}$$

and

$$\partial I_{-1}/\partial z\neq 0 \text{ only if } \nu_0-\Delta\nu-\mu_1\nu_0\frac{w}{c}\leqslant\nu\leqslant\nu_0+\Delta\nu-\mu_1\nu_0\frac{w}{c}.\tag{30}$$

Accordingly, in the (ν, w) plane the lines

$$\nu=\nu_0-\Delta\nu+\mu_1\nu_0\frac{w}{c} \quad\text{and}\quad \nu=\nu_0+\Delta\nu+\mu_1\nu_0\frac{w}{c},\tag{31}$$

demark the regions in which I_{+1} is different from a constant from the regions in which it is a constant for varying z. The situation is further clarified in Fig. 2 where AD and BC represent the lines (31). Similarly, the lines (AF and BE in Fig. 2)

$$\nu=\nu_0-\Delta\nu-\mu_1\nu_0\frac{w}{c} \quad\text{and}\quad \nu=\nu_0+\Delta\nu-\mu_1\nu_0\frac{w}{c},\tag{32}$$

demark the regions in which I_{-1} is different from a constant from the regions in which it is a constant for varying z.

Now, since the outward intensity I_{+1} is a constant independent of ν on the photospheric surface (represented by the line $XABY$ in Fig. 2), it is clear that, we must, in accordance with our foregoing remarks, require that

$$I_{+1}(\nu, z)=\text{constant along } AB \text{ and } BC.\tag{33}$$

Similarly, the non-existence of any radiation incident on the atmosphere from the outside requires that

$$I_{-1}(\nu, z)=0 \text{ along } BE \text{ and } EF.\tag{34}$$

When we pass to the intensities ψ_{+1} and ψ_{-1} defined as in Eq. (22), the boundary conditions (33) and (34) are equivalent to (cf. Fig. 3):

$$\begin{aligned}\psi_{+1}&=C=\text{constant on } AB\text{:}\quad z=0 \text{ and } \nu_0-\Delta\nu\leqslant\nu\leqslant\nu_0+\Delta\nu,\\ &=\text{the same constant on } BC\text{:}\quad \nu=\nu_0+\Delta\nu \text{ and } 0\leqslant z\leqslant z_1,\end{aligned}\Bigg\}\tag{35}$$

and

$$\begin{aligned}\psi_{-1}&=0 \text{ on } CD\text{:}\quad z=z_1 \text{ and } \nu_0-\Delta\nu\leqslant\nu\leqslant\nu_0+\Delta\nu,\\ &=0 \text{ on } BC\text{:}\quad \nu=\nu_0+\Delta\nu \text{ and } 0\leqslant z\leqslant z_1.\end{aligned}\Bigg\}\tag{36}$$

We now transform Eqs. (20) and (21) to forms which are more convenient for their solution:

Let t denote the optical depth of the atmosphere measured from the boundary inward in terms of σ_0. Then

$$\rho\sigma_0 dz = -dt. \tag{37}$$

In transforming Eqs. (20) and (21) it is, however, more convenient to use instead of the optical depth t the variable

$$x = \frac{1}{2\mu_1}t = \frac{\sqrt{3}}{2}t. \tag{38}$$

In terms of x Eqs. (20) and (21) are

$$\frac{\partial\psi_{+1}}{\partial x} - \mu_1\frac{\nu_0}{c}\frac{dw}{dx}\frac{\partial\psi_{+1}}{\partial\nu} = \psi_{+1} - \psi_{-1}, \tag{39}$$

and

$$\frac{\partial\psi_{-1}}{\partial x} + \mu_1\frac{\nu_0}{c}\frac{dw}{dx}\frac{\partial\psi_{-1}}{\partial\nu} = \psi_{+1} - \psi_{-1}. \tag{40}$$

Now the assumption (28) concerning the variation of w clearly implies that the velocity is a linear function of x. And as it entails no loss of generality, we shall suppose that $w=0$ at the base of the atmosphere. Further, let

$$w = w_1 \text{ at } t=0 \text{ and } x=0. \tag{41}$$

Under these circumstances we can write

$$w = w_1(1 - x/x_1) = w_1(1 - t/t_1), \tag{42}$$

where t_1 denotes the optical thickness in σ_0 of the entire atmosphere lying above the radiating surface. According to Eqs. (41) and (42) w_1 denotes the difference in velocity between the top and the bottom of the atmosphere. This velocity can be expressed in terms of a *Doppler width* $D\nu$ according to

$$D\nu = \tfrac{1}{2}\nu_0 w_1/c. \tag{43}$$

With these definitions

$$\mu_1\frac{\nu_0}{c}\frac{dw}{dx} = -\mu_1\frac{\nu_0}{c}\frac{w_1}{x_1} = -\frac{2}{\sqrt{3}}\frac{D\nu}{x_1} = -\frac{4}{3}\frac{D\nu}{t_1}, \tag{44}$$

and Eqs. (39) and (40) can be rewritten as

$$\frac{\partial\psi_{+1}}{\partial x} + 2\mu_1\frac{D\nu}{x_1}\frac{\partial\psi_{+1}}{\partial\nu} = \psi_{+1} - \psi_{-1}, \tag{45}$$

and

$$\frac{\partial\psi_{-1}}{\partial x} - 2\mu_1\frac{D\nu}{x_1}\frac{\partial\psi_{-1}}{\partial\nu} = \psi_{+1} - \psi_{-1}. \tag{46}$$

We now introduce the variable y defined by

$$(\nu_0 + \Delta\nu) - \nu = 2\mu_1\frac{D\nu}{x_1}y. \tag{47}$$

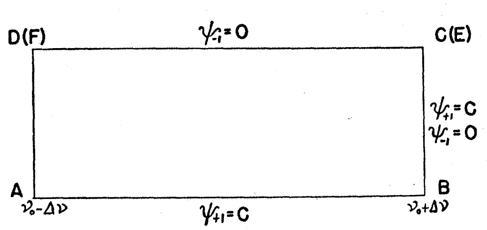

FIG. 3.

In other words, y measures the frequency shifts from the *violet edge* of $\sigma(\nu)$ in units of

$$2\mu_1 D\nu/x_1 \quad \text{(unit of frequency)}. \tag{48}$$

Equations (45) and (46) simplify to the forms

$$\partial\psi_{+1}/\partial x - \partial\psi_{+1}/\partial y = \psi_{+1} - \psi_{-1}, \tag{49}$$

and

$$\partial\psi_{-1}/\partial x + \partial\psi_{-1}/\partial y = \psi_{+1} - \psi_{-1}. \tag{50}$$

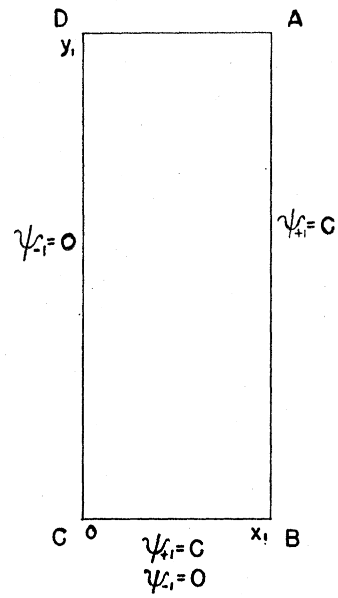

FIG. 4.

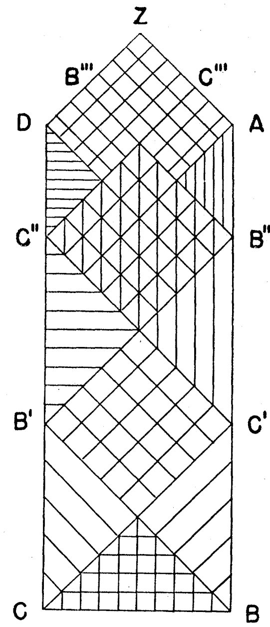

FIG. 5.

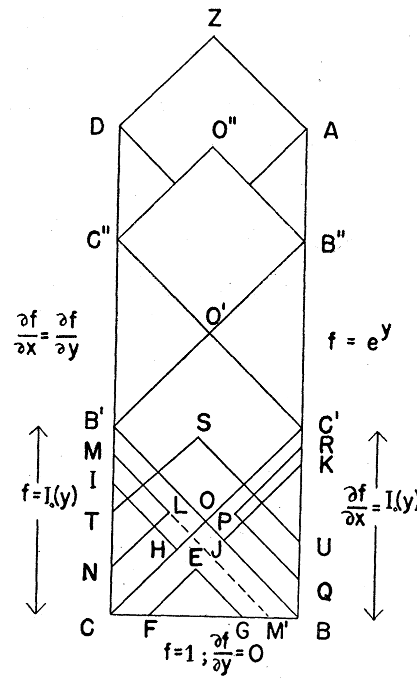

FIG. 6.

The range of the variables x and y in which the solution has to be sought is (Cf. Eq. [47])

$$0 \leqslant x \leqslant x_1 \quad \text{and} \quad 0 \leqslant y \leqslant y_1 = \frac{1}{\mu_1}\frac{\Delta\nu}{D\nu}x_1. \tag{51}$$

And the boundary conditions with respect to which Eqs. (49) and (50) have to be solved in the ranges (51) are (see Fig. 4):

$$\left.\begin{aligned} \psi_{+1} &= C \text{ on } BA: \quad x=x_1 \text{ and } 0\leqslant y\leqslant y_1, \\ &= C \text{ on } CB: \quad y=0 \text{ and } 0\leqslant x\leqslant x_1. \end{aligned}\right\} \tag{52}$$

and

$$\left.\begin{aligned} \psi_{-1} &= 0 \text{ on } CD: \quad x=0 \text{ and } 0\leqslant y\leqslant y_1, \\ &= 0 \text{ on } CB: \quad y=0 \text{ and } 0\leqslant x\leqslant x_1. \end{aligned}\right\} \tag{53}$$

Since the Eqs. (49) and (50) are linear and homogeneous, there is no loss of generality if we set

$$C=1. \tag{54}$$

We shall assume this normalization in our further work. Finally, we may note that in terms of the variables x and y Eq. (12) allowing the passage from the ψ's to the I's becomes

$$y_{\pm 1} = y \mp (x_1 - x). \tag{55}$$

It is convenient to introduce one further transformation of the variables. Let

$$\psi_{+1} = e^{-y}f \quad \text{and} \quad \psi_{-1} = e^{-y}g. \tag{56}$$

Equations (49) and (50) reduce to

$$\partial f/\partial x - \partial f/\partial y = -g, \tag{57}$$

and

$$\partial g/\partial x + \partial g/\partial y = +f. \tag{58}$$

Eliminating g between Eqs. (57) and (58), we obtain

$$\partial^2 f/\partial x^2 - \partial^2 f/\partial y^2 + f = 0. \tag{59}$$

We require to solve this hyperbolic equation with the boundary conditions (cf. Eqs. (52) and (53) and Fig. 5)

$$\left.\begin{aligned} f = e^y \text{ on } AB:\quad & x = x_1 \text{ and } 0 \leqslant y \leqslant y_1,\\ f = 1 \text{ and } \partial f/\partial x = \partial f/\partial y = 0 \text{ on } BC:\quad & y = 0 \text{ and } 0 \leqslant x \leqslant x_1,\\ \partial f/\partial x = \partial f/\partial y \text{ on } CD:\quad & x = 0 \text{ and } 0 \leqslant y \leqslant y_1, \end{aligned}\right\} \tag{60}$$

since $\psi_{-1} = 0$ implies that $g = 0$ and according to Eq. (57) this, in turn, implies that $\partial f/\partial x = \partial f/\partial y$.

Since, in the problem of the formation of absorption lines, our principal interest is on the ratio of the emergent intensity $I_{+1}(\nu, t)$ at $t = 0$ to the constant outward intensity on the radiating surface, we are most interested in the value of f on CD and DA. We may recall in this connection that $\Delta\nu$, $D\nu$ and x_1 are to be regarded as the parameters of the problem. Given these, y_1 is determined according to the relation (cf. Eq. (51))

$$y_1 = \sqrt{3}x_1 \frac{\Delta\nu}{D\nu} = \frac{3}{2} \text{ optical depth } \frac{\text{line width}}{\text{Doppler width}}. \tag{61}$$

Further, according to Eqs. (48) and (61), y measures the frequency as it enters in ψ_{+1} and ψ_{-1}, from the violet edge of $\sigma(\nu)$ and in the units

$$\frac{2}{y_1}\Delta\nu = \frac{1}{y_1} \text{ line width}. \tag{62}$$

Accordingly, the line contour will have a width

$$\frac{2}{y_1}(x_1 + y_1)\Delta\nu = \left(1 + \frac{x_1}{y_1}\right) \text{ line width}. \tag{63}$$

This is in agreement with what can be inferred directly from Fig. 2. As can be seen from this figure, the contour (on our present first approximation) must extend from

$$\nu_0 - \Delta\nu \text{ to } \nu_0 + \Delta\nu + \mu_1\nu_0 \frac{w_1}{c}, \tag{64}$$

and must, therefore, have the width

$$2\Delta\nu + \mu_1\nu_0 \frac{w_1}{c}, \tag{65}$$

or, according to Eqs. (43) and (61):

$$2\Delta\nu + 2\mu_1 D\nu = 2\Delta\nu\left(1 + \mu_1 \frac{D\nu}{\Delta\nu}\right) = 2\Delta\nu\left(1 + \frac{x_1}{y_1}\right). \tag{66}$$

It is evident that the line contour will itself be given by

$$r = e^{-y} f, \quad x = 0, \quad 0 \leqslant y \leqslant y_1, \tag{67}$$

and

$$r = e^{-y_1} f, \quad y = y_1, \quad 0 \leqslant x \leqslant x_1. \tag{68}$$

Equation (67) refers to the part of the contour which extends from

$$\nu = \nu_0 - \Delta\nu + \mu_1\nu_0 \frac{w_1}{c} \text{ to } \nu_0 + \Delta\nu + \mu_1\nu_0 \frac{w_1}{c} \tag{69}$$

while Eq. (68) refers to the part

$$\nu_0 - \Delta\nu + \mu_1\nu_0 \frac{w_1}{c} \geqslant \nu \geqslant \nu_0 - \Delta\nu. \tag{70}$$

Finally, it may be noted that according to Eq. (55) the scale of frequency is the same for both the x and the y axis.

5. THE SOLUTION OF THE BOUNDARY VALUE PROBLEM

In the preceding section we have seen how the determination of the radiation field in a scattering atmosphere in which differential motions are present can be reduced to a boundary value problem in partial differential equations of the hyperbolic type. Under the conditions (27) and (28), the hyperbolic equation is one with constant coefficients and is of the simplest kind; indeed it is of the same form as the well-known equation of telegraphy.[5] But where our problem differs from the standard ones is in the boundary conditions. And it is the nature of our boundary conditions which prevents a direct application of the methods of Cauchy or Riemann.[6] For in these latter methods, only those situations in which the function and its derivatives are assigned along curves which do not intersect any characteristic more than once are contemplated. Our boundary conditions (60) are not as simple, the "supporting curve" $DCBA$ in fact intersecting every characteristic through a point inside the fundamental rectangle twice. Moreover, the function and its derivatives are assigned only on a part of the contour namely CB, while on the rest of the contour either the function alone, or a relation between its derivatives is specified. We shall, however, show how the boundary conditions (60) just suffice to determine f uniquely in the region $ZDCBA$. The method of solution we are going to describe is an adaptation of Riemann's method and is based on Green's theorem.

Now Green's theorem as applied to Eq. (59) is that the integral

$$\oint Pdy-Qdx \tag{71}$$

where

$$P=v\frac{\partial f}{\partial x}-f\frac{\partial v}{\partial x} \quad \text{and} \quad Q=f\frac{\partial v}{\partial y}-v\frac{\partial f}{\partial y}, \tag{72}$$

around a closed contour vanishes if f and v are any two functions which satisfy Eq. (59) on and inside the contour.

As in Riemann's method we shall apply Green's theorem to contours which in parts are the characteristics $x-\xi=\pm(y-\eta)$ passing through some selected point (ξ, η) and choose for v a solution which is constant along the characteristics through (ξ, η). For Eq. (59) such a "Riemann function" $v(x, y; \xi, \eta)$ is known and depending on the quadrant in which the contour lies is

$$v(x, y; \xi, \eta)=I_0([(y-\eta)^2-(x-\xi)^2]^{\frac{1}{2}}), \tag{73}$$

or

$$v(x, y; \xi, \eta)=J_0([(x-\xi)^2-(y-\eta)^2]^{\frac{1}{2}}), \tag{74}$$

where J_0 and I_0 are the Bessel functions of order zero for real and imaginary arguments, respectively.

With the choice of the Riemann function for v, it is readily verified that

$$\int_{x-\xi=\pm(y-\eta)} Pdy-Qdx=\pm f \tag{75}$$

if the integral on the right-hand side is a line integral along the characteristic $x-\xi=\pm(y-\eta)$.

In solving Eq. (59) consistent with the boundary conditions (60), we shall find it necessary to treat the various regions distinguished in Fig. 5 separately.

[5] Cf. A. G. Webster, *Partial differential equations of mathematical physics* (1933), Section 46, p. 173.

[6] For a general exposition of these classical methods see Webster, reference 5, pp. 160–188 and 239–255; or P. Frank and R. von Mises, *Die Differential und Integralgleichungen der Mechanik und Physik* (Rosenberg, New York, 1943), Vol. I, pp. 779–817.

(a) The Solution in the Region *OCB*

Let the characteristic $x=y$ through C intersect AB at C' and the characteristic $x_1-x=y$ through B intersect CD at B'. Further, let CC' and BB' intersect at O. (See Fig. 6.)

Now, since the function and its derivatives are specified along CB, the solution inside the region OCB (including the sides OC and OB) can be found directly by Riemann's method. Thus applying Green's theorem to a contour such as $EFGE$ where EF and EG are the characteristics through $E=(\xi,\eta)$, and using Eq. (75) to evaluate the integrals along the characteristics we readily find that

$$f(\xi,\eta)=1-\tfrac{1}{2}\int_{\xi-\eta}^{\xi+\eta}\left(f\frac{\partial v}{\partial y}-v\frac{\partial f}{\partial y}\right)_{y=0}dx, \tag{76}$$

or, remembering that along CB

$$f=1 \quad\text{and}\quad \frac{\partial f}{\partial x}=\frac{\partial f}{\partial y}=0, \tag{77}$$

we have

$$f(\xi,\eta)=1-\tfrac{1}{2}\int_{\xi-\eta}^{\xi+\eta}\left(\frac{\partial v}{\partial y}\right)_{y=0}dx. \tag{78}$$

Now the Riemann function appropriate to our present contour is (73). Accordingly,

$$\begin{aligned}\left(\frac{\partial v}{\partial y}\right)_{y=0}&=\left[I_1([(y-\eta)^2-(x-\xi)^2]^{\frac{1}{2}})\frac{y-\eta}{[(y-\eta)^2-(x-\xi)^2]^{\frac{1}{2}}}\right]_{y=0}\\&=-I_1([\eta^2-(x-\xi)^2]^{\frac{1}{2}})\frac{\eta}{[\eta^2-(x-\xi)^2]^{\frac{1}{2}}}.\end{aligned} \tag{79}$$

Hence,

$$f(\xi,\eta)=1+\tfrac{1}{2}\eta\int_{\xi-\eta}^{\xi+\eta}I_1([\eta^2-(x-\xi)^2]^{\frac{1}{2}})\frac{dx}{[\eta^2-(x-\xi)^2]^{\frac{1}{2}}}. \tag{80}$$

Equation (80) determines f in the region OCB.

To evaluate the integral on the right-hand side of Eq. (80) we let

$$x-\xi=\eta\cos\vartheta, \tag{81}$$

and obtain

$$f(\xi,\eta)=1+\tfrac{1}{2}\eta\int_0^{\pi}I_1(\eta\sin\vartheta)d\vartheta. \tag{82}$$

Replacing I_1 in the foregoing equation by its equivalent series expansion and integrating term by term we find

$$\begin{aligned}f(\xi,\eta)&=1+\tfrac{1}{2}\eta\sum_{m=0}^{\infty}\int_0^{\pi}\frac{(\frac{1}{2}\eta\sin\vartheta)^{2m+1}}{m!\Gamma(m+2)}d\vartheta,\\&=1+\eta\sum_{m=0}^{\infty}(\tfrac{1}{2}\eta)^{2m+1}\frac{1}{m!\Gamma(m+2)}\int_0^{\pi/2}\sin^{2m+1}\vartheta d\vartheta,\\&=1+\eta\sum_{m=0}^{\infty}(\tfrac{1}{2}\eta)^{2m+1}\frac{1}{m!\Gamma(m+2)}\frac{2^{2m}(m!)^2}{(2m+1)!}\\&=1+\sum_{m=0}^{\infty}\frac{\eta^{2m+2}}{(2m+2)!}\\&=\sum_{m=0}^{\infty}\frac{\eta^{2m}}{(2m)!}.\end{aligned} \tag{83}$$

Thus, $$f(\xi, \eta) = \cosh \eta \tag{84}$$

inside and on the triangular contour OCB.

(b) The Integral Equation which Ensures the Continuity of the Solution Along OC

We have seen how the boundary conditions along CB determine the solution in the region OCB and on the sides OC and OB. We shall now show how this knowledge of the function along OC and OB together with the boundary conditions on CB' and BC' enables us to continue the solution into the region $O'B'COBC'O'$ (including the sides $B'O'$ and $O'C'$).

Thus, applying Green's theorem to contours such as $ICHI$ and $KJBK$ we shall obtain integral equations relating the values which the function takes along CO and OB with the values which the function and its derivatives take along CB' and BC'. And, as we shall see presently, these integral equations suffice to determine f along CB' and $\partial f/\partial x$ along BC' uniquely and secure at the same time the continuity of the solutions along OC and OB.

Considering first the condition which ensures the continuity of the solution along CO apply Green's theorem to a contour such as $ICHI$ where $H = (\eta, \eta)$ is a point on CO and HI is the characteristic $\eta - x = y - \eta$ through H. Using Eq. (75) to evaluate the integrals along the characteristics HI and CH and remembering that f takes the values 1 and $\cosh \eta$ at C and H, respectively, we find that

$$2 \cosh \eta - 1 = f(2\eta, 0) + \int_0^{2\eta} \left(v\frac{\partial f}{\partial x} - f\frac{\partial v}{\partial x} \right)_{x=0} dy, \tag{85}$$

or, since $\partial f/\partial x = \partial f/\partial y$ along CB', we have

$$2 \cosh \eta - 1 = f(0, 2\eta) + \int_0^{2\eta} \left(v\frac{\partial f}{\partial y} \right)_{x=0} dy - \int_0^{2\eta} \left(f\frac{\partial v}{\partial x} \right)_{x=0} dy. \tag{86}$$

Integrating by parts the first of the two integrals on the right-hand side of Eq. (86) we obtain

$$2 \cosh \eta = 2f(0, 2\eta) - \int_0^{2\eta} f(0, y) \left(\frac{\partial v}{\partial x} + \frac{\partial v}{\partial y} \right)_{x=0} dy. \tag{87}$$

The Riemann function appropriate to our present contour is (74) with $\xi = \eta$. With this choice of v we find after some minor reductions that

$$\cosh \eta = f(0, 2\eta) - \tfrac{1}{2} \int_0^{2\eta} f(0, y) J_1([\eta^2 - (y-\eta)^2]^{\frac{1}{2}}) \frac{y dy}{[\eta^2 - (y-\eta)^2]^{\frac{1}{2}}}, \tag{88}$$

which is seen to be an integral equation for f along CB'. It is seen that Eq. (88) is equivalent to an integral equation of Volterra's type. For, by differentiating the equation

$$\sinh \eta = \tfrac{1}{2} \int_0^{2\eta} f(0, y) J_0([\eta^2 - (y-\eta)^2]^{\frac{1}{2}}) dy \tag{89}$$

with respect to η we may readily verify that we recover Eq. (88).

(c) The Solution of the Integral Eq. (89)

To solve Eq. (89) we apply a Laplace transformation to this equation. Thus multiplying both sides of Eq. (89) by $e^{-s\eta}$ and integrating over η from 0 to ∞ we obtain

$$\frac{2}{s^2 - 1} = \int_0^\infty d\eta e^{-s\eta} \int_0^{2\eta} dy f(0, y) J_0([\eta^2 - (y-\eta)^2]^{\frac{1}{2}}), \tag{90}$$

or inverting the order of the integration on the right-hand side we have

$$\frac{2}{s^2-1}=\int_0^\infty dyf(0,y)\int_{y/2}^\infty d\eta e^{-s\eta}J_0([2\eta y-y^2]^{\frac{1}{2}}). \tag{91}$$

Introducing the variable

$$t=(2\eta y-y^2)^{\frac{1}{2}} \tag{92}$$

instead of η we find

$$\frac{2}{s^2-1}=\int_0^\infty \frac{dy}{y}e^{-sy/2}f(0,y)\int_0^\infty dt\, t\exp[-st^2/2y]J_0(t). \tag{93}$$

In Eq. (93) the integral over t is equivalent to the so-called Weber's first exponential integral in the theory of Bessel functions[7] and its value is given by

$$\int_0^\infty \exp[-st^2/2y]J_0(t)t\,dt=\frac{y}{s}e^{-y/2s}. \tag{94}$$

Using this result Eq. (93) reduces to

$$\frac{2s}{s^2-1}=\int_0^\infty f(0,y)\exp[-y(s+s^{-1})/2]dy. \tag{95}$$

If we now let

$$s+s^{-1}=2u, \tag{96}$$

Eq. (95) becomes

$$\frac{1}{(u^2-1)^{\frac{1}{2}}}=\int_0^\infty f(0,y)e^{-yu}dy. \tag{97}$$

In other words, we have shown that the Laplace transform of $f(0, y)$ is $(u^2-1)^{\frac{1}{2}}$. But it is known that the Laplace transform of $I_0(y)$ is exactly this. Hence

$$f(0,y)=I_0(y) \quad (0<y\leqslant x_1). \tag{98}$$

Thus, the requirement of the continuity of the solution along CO has determined f along CB'. Its derivatives along CB' are also deducible. We have

$$\left(\frac{\partial f}{dx}\right)_{x=0}=\left(\frac{\partial f}{\partial y}\right)_{x=0}=I_1(y) \quad (0<y\leqslant x_1). \tag{99}$$

(d) The Integral Equation Ensuring the Continuity of the Solution Along OB and Its Solution

Along BA we know f and its derivative with respect to y. But we do not know $\partial f/\partial x$ along this line. However, as the solution along OB is known the requirement that the solution be continuous on this line will determine $\partial f/\partial x$ along BC'. Thus, applying Green's theorem to a contour such as $JKBJ$ where $J=(x_1-\eta,\eta)$ is a point on OB and JK the characteristic $x-x_1+\eta=y-\eta$ through J we find in the usual manner that

$$2\cosh\eta=1+e^{2\eta}-\int_0^{2\eta}\left(v\frac{\partial f}{\partial x}-e^y\frac{\partial v}{\partial x}\right)_{x=x_1}dy. \tag{100}$$

The Riemann function appropriate to our present contour is

$$v=J_0([(x-x_1+\eta)^2-(y-\eta)^2]^{\frac{1}{2}}). \tag{101}$$

With v given by Eq. (101), Eq. (100) becomes

$$2\cosh\eta=1+e^{2\eta}-\int_0^{2\eta}e^yJ_1([\eta^2-(y-\eta)^2]^{\frac{1}{2}})\frac{\eta dy}{[\eta^2-(y-\eta)^2]^{\frac{1}{2}}}-\int_0^{2\eta}J_0([\eta^2-(y-\eta)^2]^{\frac{1}{2}})\left(\frac{\partial f}{\partial x}\right)_{x=x_1}dy. \tag{102}$$

[7] Cf. G. N. Watson, *Theory of Bessel Functions* (Cambridge University Press, England, 1944), p. 393.

Putting

$$y-\eta=\eta \cos \vartheta \tag{103}$$

in the first of the two integrals on the right-hand side of Eq. (102), it can be expressed in the form

$$2 \cosh \eta=1+e^{2\eta}-e^{\eta}G(\eta)-\int_0^{2\eta} J_0([\eta^2-(y-\eta)^2]^{\frac{1}{2}})\left(\frac{\partial f}{\partial x}\right)_{x=x_1} dy, \tag{104}$$

where

$$G(\eta)=\eta\int_0^{\pi} e^{\eta \cos \vartheta}J_1(\eta \sin \vartheta)d\vartheta. \tag{105}$$

To evaluate $G(\eta)$ we replace $e^{\eta \cos \vartheta}$ and $J_1(\eta \sin \vartheta)$ by their respective series expansions and integrate term by term. In this manner we find that

$$\begin{aligned}
G(\eta)&=\eta\int_0^{\pi} \sum_{n=0}^{\infty} \frac{(\eta \cos \vartheta)^n}{n!} \sum_{m=0}^{\infty} (-1)^m\frac{(\frac{1}{2}\eta \sin \vartheta)^{2m+1}}{m!\Gamma(m+2)}d\vartheta,\\
&=\sum_{n=0}^{\infty} \frac{\eta^{2n+1}}{(2n)!}\Gamma(n+\tfrac{1}{2}) \sum_{m=0}^{\infty} (-1)^m\frac{(\frac{1}{2}\eta)^{2m+1}}{\Gamma(m+2)\Gamma(m+n+\frac{3}{2})},\\
&=-\sum_{n=0}^{\infty} \frac{(2\eta)^{n+\frac{1}{2}}\Gamma(n+\frac{1}{2})}{(2n)!} \sum_{m=1}^{\infty} (-1)^m\frac{(\frac{1}{2}\eta)^{2m+n-\frac{1}{2}}}{\Gamma(m+1)\Gamma(m+n+\frac{1}{2})},\\
&=\sum_{n=0}^{\infty} \frac{(2\eta)^{n+\frac{1}{2}}\Gamma(n+\frac{1}{2})}{(2n)!}\left[\frac{(\frac{1}{2}\eta)^{n-\frac{1}{2}}}{\Gamma(n+\frac{1}{2})}-J_{n-\frac{1}{2}}(\eta)\right],\\
&=2\sum_{n=0}^{\infty} \frac{\eta^{2n}}{(2n)!}-\sum_{n=0}^{\infty} \frac{(2\eta)^{n+\frac{1}{2}}}{(2n)!}\Gamma(n+\tfrac{1}{2})J_{n-\frac{1}{2}}(\eta),\\
&=2 \cosh \eta-(2\pi\eta)^{\frac{1}{2}} \sum_{n=0}^{\infty} \left(\frac{\eta}{2}\right)^n \frac{1}{n!}J_{n-\frac{1}{2}}(\eta).
\end{aligned} \tag{106}$$

But according to a formula of Lommel[8]

$$\sum_{n=0}^{\infty} (\tfrac{1}{2}\eta)^n \frac{1}{n!}J_{n-\frac{1}{2}}(\eta)=\left(\frac{2}{\pi\eta}\right)^{\frac{1}{2}}. \tag{107}$$

Hence

$$G(\eta)=2(\cosh \eta-1). \tag{108}$$

Substituting this result in Eq. (104) we find

$$\sinh \eta=\tfrac{1}{2}\int_0^{2\eta} J_0([\eta^2-(y-\eta)^2]^{\frac{1}{2}})\left(\frac{\partial f}{\partial x}\right)_{x=x_1} dy, \tag{109}$$

which is a Volterra integral equation for $(\partial f/\partial x)_{x=x_1}$.

It is seen that Eq. (109) is of the same form as Eq. (89). Accordingly

$$(\partial f/\partial x)_{x=x_1}=I_0(y) \quad (0\leqslant y\leqslant x_1). \tag{110}$$

(e) The Solution in the Region $O'B'COBC'O'$

With the determination of f along CB' and of $\partial f/\partial x$ along BC' our knowledge of the function and its derivatives along $B'CBC'$ is complete, and in the region $O'B'COBC'O'$ the solution becomes determinate. Thus, as in Riemann's method, applying Green's theorem to contours such as $LMNL$, $PQRP$, and $STCBUS$ we find that we can express f in the regions $OB'C$, OBC', and $O'B'OC'$ as follows:[9]

[8] See G. N. Watson, *Theory of Bessel Functions* (Cambridge University Press, England, 1944), p. 141, Eq. (7).
[9] For a point such as L, Green's theorem can be applied to either of the two contours $LMNL$ and $LNCM'L$. But clearly no ambiguity is implied as the continuity of the solution along OC (and OB) has already been ensured.

$$f(\xi,\eta)=I_0(\xi+\eta)-\tfrac{1}{2}\xi\int_0^{\pi}I_0(\eta+\xi\cos\vartheta)J_1(\xi\sin\vartheta)(1+\cos\vartheta)d\vartheta,\quad [(\xi,\eta) \text{ in } OB'C], \tag{111}$$

$$f(\xi,\eta)=e^{\eta}-\tfrac{1}{2}(x_1-\xi)\int_0^{\pi}I_0(\eta+[x_1-\xi]\cos\vartheta)J_0([x_1-\xi]\sin\vartheta)\sin\vartheta d\vartheta,\quad [(\xi,\eta) \text{ in } OBC']; \tag{112}$$

and

$$\begin{aligned} f(\xi,\eta)=&I_0([\eta^2-\xi^2]^{\frac{1}{2}})+e^{\eta-x_1+\xi}+\eta\int_{\cos^{-1}(x_1-\xi)/\eta}^{\cos^{-1}-\xi/\eta}I_1(\eta\sin\vartheta)d\vartheta\\ &-\xi\int_0^{\cosh^{-1}\eta/\xi}I_0(\eta-\xi\cosh\vartheta)I_1(\xi\sinh\vartheta)(\cosh\vartheta-1)d\vartheta\\ &+(x_1-\xi)\int_0^{\cosh^{-1}\eta/(x_1-\xi)}I_0([x_1-\xi]\sinh\vartheta)I_0(\eta-[x_1-\xi]\cosh\vartheta)\sinh\vartheta d\vartheta\\ &+(x_1-\xi)\int_0^{\cosh^{-1}\eta/(x_1-\xi)}e^{\eta-(x_1-\xi)\cosh\vartheta}I_1([x_1-\xi]\sinh\vartheta)d\vartheta,\quad [(\xi,\eta) \text{ in } O'B'OC']. \end{aligned} \tag{113}$$

In particular we may note that Eq. (113) will enable us to determine the solution along the sides $B'O'$ and $O'C'$.

(f) Further Continuation of the Solution

In the preceding paragraphs we have seen how the knowledge of the function along COB, together with the boundary conditions on CB' and BC' enables us to determine f in the region $O'B'COBC'$ including the sides $B'O'$ and $O'C'$. It is now apparent that in the same way we can utilize our present knowledge of the function along $B'O'C'$ to extend the solution further into the region $O'B'C''O''B''C'$. And this process can be continued until the solution inside the entire rectangle $DCBA$ is completed. However, in this paper we shall not consider these further extensions but content ourselves with the solution which has been completed in the first square $B'CBC'$. According to Eq. (61) this will suffice to determine the radiation field in all cases in which the ratio $D\nu:\Delta\nu$ exceeds $\sqrt{3}$.

6. THE CONTOURS OF THE ABSORPTION LINES FORMED. NUMERICAL ILLUSTRATIONS

In the preceding section we have seen how the boundary value problem formulated in Section 4 can, in principle, be solved. In terms of the solution thus found, we can specify the radiation field in an atmosphere with differential motions and under the conditions prescribed in Sections 3 and 4. While the determination of the radiation field in the entire atmosphere is necessary to answer all questions relating to the formation of the absorption lines (see Section 7 below), greatest interest is, however, attached to the contour of the resulting line. In the first approximation in which we have studied the problem, this is given in terms of the emergent value of the outward intensity $I_{+1}(\nu)$. More specifically the form of the line is given by Eqs. (67) and (68) where f is the solution of the boundary value problem. We shall now consider in some detail the predicted nature of these contours.

Now along CB' the solution is given by (cf. Eq. (98))

$$f=I_0(y)\quad (x=0,\ 0\leqslant y\leqslant x_1). \tag{114}$$

According to Eq. (68) we may, therefore, write down a formula for the *residual intensity* r which will be valid for a part of the line contour. Thus

$$r=e^{-y}I_0(y), \tag{115}$$

will describe the line in the frequency interval

$$\nu_0+\Delta\nu+2\mu_1 D\nu\geqslant\nu\geqslant\nu_0-\Delta\nu+2\mu_1 D\nu, \tag{116}$$

or

$$\nu_0+\Delta\nu+2\mu_1 D\nu\geqslant\nu\geqslant\nu_0+\Delta\nu, \tag{116'}$$

TABLE I. The function $r=e^{-y}I_0(y)$.

y	$e^{-y}I_0(y)$	y	$e^{-y}I_0(y)$	y	$e^{-y}I_0(y)$	y	$e^{-y}I_0(y)$
0	1.0000	0.7	0.5593	1.40	0.3831	3.0	0.2430
0.1	0.9071	0.8	0.5241	1.50	0.3674	3.5	0.2228
0.2	0.8269	0.9	0.4932	1.75	0.3346	4.0	0.2070
0.3	0.7576	1.0	0.4658	2.00	0.3085	4.5	0.1942
0.4	0.6974	1.1	0.4414	2.25	0.2874	5.0	0.1835
0.5	0.6450	1.2	0.4198	2.50	0.2700	6.0	0.1667
0.6	0.5993	1.3	0.4004	2.75	0.2555	8.0	0.1434
						10.0	0.1278

TABLE II. $f(\xi, \eta)$.

ξ \ η	1.0	2.0	2.5	3.0	4.0	5.0
0	1.2661	2.2796	3.2898	4.8808	11.302	27.240
0.5	1.4762	2.9697	4.4134	6.6792	15.847	39.601
1.0	1.5431	3.4374	5.2449	8.0836	19.628	51.362
1.5	1.5431	3.6897	5.7814	9.0767	23.354	62.659
2.0	1.5431	3.7622	6.0569	9.6891	26.931	73.746
2.5	1.5431	3.7622	6.1323	10.6881	30.579	84.930
3.0	1.5431	3.7622	6.7927	11.8944	34.506	96.502
3.5	1.5431	4.3857	7.7767	13.4600	38.859	108.657
4.0	1.5431	5.2378	9.0554	15.4183	43.693	121.460
4.5	2.0969	6.2657	10.5584	17.6724	48.998	134.802
5.0	2.7183	7.3891	12.1825	20.0855	54.598	148.413

TABLE III. Line contours of absorption lines formed in a moving atmosphere ($x_1=5$; $y_1=1$, 2, 2.5, 3, 4, 5).

$\frac{\nu-\nu_0+\Delta\nu}{2\Delta\nu}$	r	$\frac{\nu-\nu_0+\Delta\nu}{2\Delta\nu}$	r	$\frac{\nu-\nu_0+\Delta\nu}{2\Delta\nu}$	r	$\frac{\nu-\nu_0+\Delta\nu}{2\Delta\nu}$	r	$\frac{\nu-\nu_0+\Delta\nu}{2\Delta\nu}$	r	$\frac{\nu-\nu_0+\Delta\nu}{2\Delta\nu}$	r
0.0	1.0000	0.0	1.0000	0.0	1.0000	0.0	1.0000	0.0	1.0000	0.0	1.0000
0.5	0.7714	0.25	0.8480	0.2	0.8667	$0.1\dot{6}$	0.8799	0.125	0.8974	0.1	0.9083
1.0	0.5677	0.50	0.7089	0.4	0.7433	$0.3\dot{3}$	0.7676	0.250	0.8003	0.2	0.8184
1.5	0.5677	0.75	0.5935	0.6	0.6384	0.50	0.6701	0.375	0.7117	0.3	0.7321
2.0	0.5677	1.00	0.5092	0.8	0.5576	$0.6\dot{6}$	0.5922	0.500	0.6320	0.4	0.6502
2.5	0.5677	1.25	0.5092	1.0	0.5034	$0.8\dot{3}$	0.5321	0.625	0.5601	0.5	0.5723
3.0	0.5677	1.50	0.5092	1.2	0.4972	1.00	0.4824	0.750	0.4933	0.6	0.4969
3.5	0.5677	1.75	0.4993	1.4	0.4746	$1.1\dot{6}$	0.4519	0.875	0.4277	0.7	0.4222
4.0	0.5677	2.00	0.4652	1.6	0.4305	$1.3\dot{3}$	0.4025	1.000	0.3595	0.8	0.3461
4.5	0.5431	2.25	0.4019	1.8	0.3623	1.50	0.3325	1.125	0.2902	0.9	0.2668
5.0	0.4658	2.50	0.3085	2.0	0.2700	$1.6\dot{6}$	0.2430	1.250	0.2070	1.0	0.1835
5.5	0.6450	2.75	0.3674	2.2	0.3085	$1.8\dot{3}$	0.2700	1.375	0.2228	1.1	0.1942
6.0	1.0000	3.00	0.4658	2.4	0.3674	2.00	0.3085	1.500	0.2430	1.2	0.2070
		3.25	0.6450	2.6	0.4658	$2.1\dot{6}$	0.3674	1.625	0.2700	1.3	0.2228
		3.50	1.0000	2.8	0.6450	$2.3\dot{3}$	0.4658	1.750	0.3085	1.4	0.2430
				3.0	1.0000	2.50	0.6450	1.875	0.3674	1.5	0.2700
						$2.6\dot{6}$	1.0000	2.000	0.4658	1.6	0.3085
								2.125	0.6450	1.7	0.3674
								2.250	1.0000	1.8	0.4658
										1.9	0.6450
										2.0	1.0000

depending on whether

$$\mu_1 D\nu \geqslant \Delta\nu \quad \text{or} \quad \mu D\nu \leqslant \Delta\nu. \tag{117}$$

It should be noted in this connection that in our present context y measures the frequency shifts from the violet edge $\nu_0+\Delta\nu+2\mu_1 D\nu$ of the *line contour* in the unit $2\Delta\nu/y_1$.

For convenience we have provided a brief table of the function on the right-hand side of Eq. (115) (see Table I).

Again, according to Eqs. (68) and (84), when

$$D\nu > (2/\mu_1)\Delta\nu = 2\sqrt{3}\Delta\nu, \tag{118}$$

in the frequency interval

$$\nu_0 - 3\Delta\nu + 2\mu_1 D\nu \geqslant \nu \geqslant \nu_0 + \Delta\nu, \tag{119}$$

the contour is flat, the residual intensity having the constant value

$$r = e^{-y_1}\cosh y_1 \quad (y_1 \leqslant x \leqslant x_1 - y_1). \tag{120}$$

This flat portion occupies a fraction

$$(x_1 - 2y_1)/(x_1 + y_1) \tag{121}$$

of the entire contour. As $D\nu/\Delta\nu \rightarrow \infty$ and $y_1 \rightarrow 0$, the fraction (121) tends to unity: the line accordingly becomes very shallow and very broad. More specifically, as $y_1 \rightarrow 0$

$$1 - r \rightarrow y_1 \quad (y_1 \rightarrow 0), \tag{122}$$

and

$$\text{the width of the line contour} \rightarrow 2\Delta\nu x_1/y_1 \quad (y_1 \rightarrow 0). \tag{123}$$

The equivalent width, therefore, tends to the limiting value

$$\text{Equivalent width} \rightarrow 2\Delta\nu x_1 = \sqrt{3}\Delta\nu t_1 = \sqrt{3}\Delta\nu N\sigma_0 m^{-1} \quad (y_1 \rightarrow 0), \tag{124}$$

where N denotes the number of scattering atoms in a column of unit cross section in the atmosphere and m the mass of the atom.

Returning to the general case, it is seen that the specification of the line contour over its entire range

$$\nu_0 - \Delta\nu \leqslant \nu \leqslant \nu_0 + \Delta\nu + 2\mu_1 D\nu, \tag{125}$$

requires a knowledge of the function f along the line $y = y_1$ and for $0 \leqslant x \leqslant x_1$. However, since the solution for f has been found in an explicit form only in the first square, $B'CBC'$, complete contours can be given only for those cases in which $D\nu \geqslant \sqrt{3}\Delta\nu$. And even then, the part of the solution not included in the triangle OBC and the sides CB' and BC' can be found only after several numerical quadratures. For, in these regions the solution is given by the formulae (111)–(113), and it does not seem that the various integrals occurring in these formulae can be evaluated explicitly.

As illustrating the solution found in Section 5 we have considered in detail the case

$$x_1 = 5 \tag{126}$$

and determined the line contours for the following ratios of the Doppler width to the line width:

$$\mu_1 D\nu/\Delta\nu = 5,\ 2.5,\ 2,\ 1\tfrac{2}{3},\ 1.25, \text{ and } 1. \tag{127}$$

According to Eq. (61) the specification of the contours for these ratios of $\mu_1 D\nu : \Delta\nu$, requires the evaluation of f along the lines

$$y = 1,\ 2,\ 2.5,\ 3,\ 4, \text{ and } 5, \tag{128}$$

for $0 \leqslant x \leqslant 5$. The values of f for several points along these lines and intercepted in the region $O'B'COBC'$ were determined according to Eqs. (111)–(113). The various integrals occurring in these equations were evaluated numerically.[10] The results of these calculations are included in Table II. In Table III, the values of f given in Table II are converted into residual intensities according to

[10] The carrying out of the numerical quadratures were immensely facilitated by the *British Association Mathematical Tables*, VI: *Bessel functions of order zero and unity* (Cambridge University Press, England, 1937). I should record here my indebtedness to Mrs. Frances Herman Breen for assistance with these calculations.

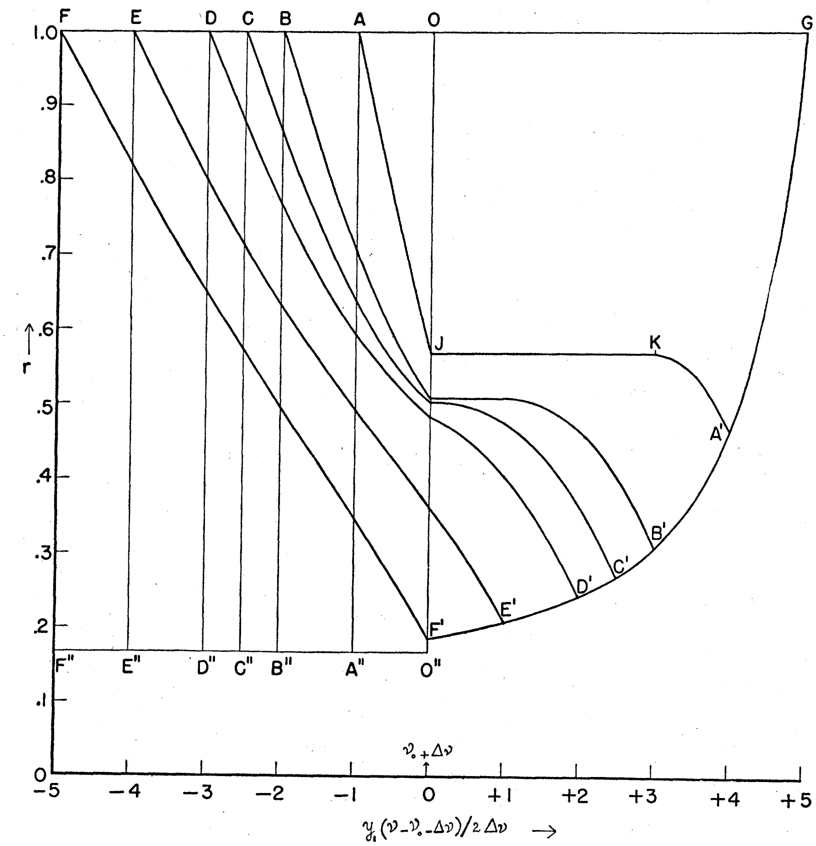

FIG. 7.

Eq. (68) and are tabulated together with the values of r for the remaining parts of the contours given by Eq. (115). The arguments in Table III are the frequency shifts measured from the red end $\nu_0-\Delta\nu$ of the contour in the unit $2\Delta\nu$.

The residual intensities tabulated in Table III are further illustrated as line contours in Fig. 7. In this figure the various contours are plotted on different frequency scales, the width $2\Delta\nu$ of $\sigma(\nu)$ always extending from the red end of the contour to O. Thus the contour $BB'G$ corresponds to a case in which the line formed under the same conditions in a static atmosphere would extend from B to O.

From Fig. 7 it is apparent that in all cases in which $D\nu > 2\sqrt{3}\Delta\nu$ the contour consists of four distinct parts, namely,

$$\left.\begin{array}{rr} \nu_0-\Delta\nu \leqslant \nu \leqslant \nu_0+\Delta\nu, & \text{(i)} \\ \nu_0+\Delta\nu \leqslant \nu \leqslant \nu_0-3\Delta\nu+2\mu_1 D\nu, & \text{(ii)} \\ \nu_0-3\Delta\nu+2\mu_1 D\nu \leqslant \nu \leqslant \nu_0-\Delta\nu+2\mu_1 D\nu, & \text{(iii)} \\ \nu_0-\Delta\nu+2\mu_1 D\nu \leqslant \nu \leqslant \nu_0+\Delta\nu+2\mu_1 D\nu. & \text{(iv)} \end{array}\right\} \tag{129}$$

In each of these parts r is given by a different analytical expression. It decreases from 1 in the first interval, remains constant in the second, and decreases some more in the third attaining its minimum at $\nu=\nu_0-\Delta\nu+2\mu_1 D\nu$. In the last interval it increases again to 1. It is in this fourth interval that the line contour is described by Eq. (115). In Fig. 7 we have indicated these four parts on the contour $AA'G$. The parts are respectively, AJ, JK, KA', and $A'G$. The reason for the existence of these four parts can be understood from a reference to Fig. 8. In this figure, which is similar to Fig. 2, the regions in which I_{+1} and I_{-1}, respectively, are different from constants (for varying z) are marked. We have further

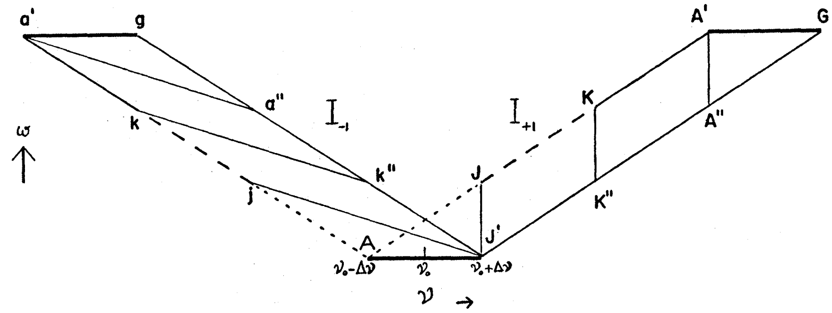

FIG. 8.

indicated the frequency intervals in which the different parts of the contour arise. (The lettering in Figs. 7 and 8 correspond). Now, according to our discussion in Sections 3 and 4, the outward intensity I_{+1} for a frequency ν interacts with the inward intensity I_{-1} for a frequency $\nu-2\mu_1\nu_0(w/c)$. Hence I_{+1} for the frequencies in the intervals AJ, JK, KA', and $A'G$ in their transfer through the atmosphere have interacted with I_{-1} for the frequencies in the regions AjJ', $jkk''J'$, $ka'a''k''$, and $a'ga''$, respectively. The reason for the existence of the four distinct parts in the line contour now becomes apparent. Moreover, this discussion makes it clear why it is that the problem increases in complexity as $\mu_1 D\nu/\Delta\nu$ decreases below unity.

Finally, it is of interest to compare the contours we have obtained with those which would be expected in an atmosphere in which no gradient of velocity exists. To discuss this case we have to go back to Eqs. (39) and (40). Setting $dw/dx=0$ in these equations and solving them with the boundary conditions appropriate to Schuster's problem we readily find that

$$\left.\begin{aligned} r&=\frac{1}{1+x_1} \quad (\nu_0-\Delta\nu\leqslant\nu\leqslant\nu_0+\Delta\nu)\\ &=0 \qquad\qquad\qquad\qquad \text{otherwise.}\end{aligned}\right\} \tag{130}$$

The contours are therefore rectangular. For $x_1=5$, $r=\frac{1}{6}$. These rectangular contours which will be obtained in the limit $D\nu=0$ are also shown in Fig. 7. Thus the contour $BB'G$ should be compared with $BB''O''O$; and similarly for the others.

7. REMARKS ON FUTURE WORK

The successful solution of a specific problem in the theory of moving atmospheres which we have presented in the preceding sections justifies the hope that it will be possible to solve problems more general and less idealized than the one considered in this paper. Indeed, there are several problems in the theory of moving atmospheres which come already within the scope of the methods developed in this paper. For example, there is the problem of the variation of line contours with the angle of emergence from the atmosphere. The solution to this problem will depend on the radiation field in the entire atmosphere. For, the intensity $I(\nu, z_1, \mu)$ of the radiation of frequency ν (as judged by an observer at rest with respect to the radiating surface at $z=0$) emergent in a direction with a direction cosine μ with respect to the positive normal can be expressed as an integral in the form (cf., Eq. (9))

$$I(\nu, z_1, \mu)=\int_0^{z_1} J(\nu, z, \mu)\exp\left\{-\int_z^{z_1}\rho\sigma(\nu-[\nu_0/c]w\mu)/\mu\right\}\frac{dz}{\mu}, \tag{131}$$

where

$$J(\nu, z, \mu)=\tfrac{1}{2}\int_{-1}^{+1} I\left(\nu-\nu_0\frac{w}{c}\mu+\nu_0\frac{w}{c}\mu', z, \mu'\right)d\mu'. \tag{132}$$

In the first approximation we can express the integral on the right-hand side of Eq. (132) as a Gauss sum with two terms. Thus,

$$J(\nu, z, \mu) = \tfrac{1}{2}\left\{ I_{+1}(\nu - \nu_0\frac{w}{c}\mu + \nu_0\frac{w}{c}\mu_1) + I_{-1}\left(\nu - \nu_0\frac{w}{c}\mu - \nu_0\frac{w}{c}\mu_1, z\right)\right\}. \tag{133}$$

The source function J can, therefore, be expressed in terms of the solutions for $I_{+1}(\nu, z)$ and $I_{-1}(\nu, z)$ which we have found in Section 5 and there will be no formal difficulty in solving for $I(\nu, z_1, \mu)$ according to Eq. (131).

Again, the determination of $I(\nu, z_1, \mu)$ by the procedure we have outlined above will be of particular importance for deriving contours comparable to those *observed* in cases in which the photospheric surface is itself moving with a velocity w_0. It will be recalled in this connection that our discussion of the equation of transfer involves no assumption concerning w_0 since everything was referred to an observer at rest with respect to the surface at $z=0$ and $t=t_1$. However, the line contour as seen by an observer outside the star will not be given by $F(\nu, z_1)$, as allowance will have to be made for the fact that the photospheric surface from which the radiation is emerging at an angle ϑ has a motion $w_0 \cos\vartheta$ towards the observer. Accordingly, the contour as judged by an external observer at a great distance from the star will be determined by

$$\mathfrak{F}(\nu) = 2\int_0^1 I\left(\nu + \nu_0\frac{w_0}{c}\mu, z_1, \mu\right)\mu d\mu, \tag{134}$$

where $I(\nu, z_1, \mu)$ has the same meaning as in Eq. (131).

Another problem which can be solved by the methods of the present paper is the radiative equilibrium of a planetary nebula. It is known that large differential motions are present in planetary nebulae and a problem of considerable interest relates to the question of the radiation pressure in the Lyman α-radiation. It can be shown that with the same assumptions (27) and (28) concerning $\sigma(\nu)$ and the variation of w through the atmosphere, the problem can be reduced to a boundary value problem very similar to the one considered in Sections 4 and 5 and the solution can also be found by similar methods.

And finally there is the general problem of line formation in moving atmospheres in which $\sigma(\nu)$ is allowed to be more general than the rectangular form considered in this paper. It can be shown that under these more general conditions the problem can still be reduced to a boundary value problem in hyperbolic equations. If the velocity be assumed to vary linearly with the optical depth, the equation we have to consider differs from (59) only in the occurrence of a factor depending on y in front of f. It does not seem impossible that with suitable simplifications, progress toward the solution of these more difficult problems can be made.