The Influence of the Expansion of Space on the Gravitation Fields Surrounding the Individual Stars

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STATEMENT OF PROBLEM

IN the theory of relativity one is used to representing the gravitation field in the neighborhood of a single star by the centrally symmetric static solution of the field equations, which was first stated by Schwarzschild. This field goes over asymptotically with increasing distance from the generating mass into the Euclidean (or rather, Minkowskian) space. That is to say, it is embedded in a "flat" space. On the other hand, we know that real space is expanding, and that, for the existence of a non-vanishing average density of matter, the field equations will imply such an expansion.

The boundary conditions on which the Schwarzschild solution is based are, therefore, not valid for a real star. In particular the boundary conditions which are valid for the expanding space are dependent on time. One has to expect, therefore, *a priori*, that the field surrounding a single star is essentially dependent on time.

The problem of this time dependence is of particular interest, since such a time-dependent behavior could be of essential importance for the theory of matter. The assumption has been voiced in this connection that there may exist connecting relations between the cosmic and the molecular constants.

The investigation below yields that the expansion of space has no influence on the structure of the field surrounding an individual star, that it is a static field—if only for an exactly delimited neighborhood.

METHOD

As usual for the cosmologic solutions, one starts with a (pressure free) spatially constant density of matter. It is of the form:

$$ds^2 = \frac{-T^2}{(1+zr/2)^2}\delta_{ik}dx_i dx_k + dt^2, \tag{A}$$

where $r=\frac{1}{2}(x_1^2+x_2^2+x_3^2)$. T is a function of t alone. The spherical case corresponds to $z=1$, the pseudo-spherical to $z=-1$, the spatially plane case to $z=0$. The drawing (Fig. 1) is an illustration of the spherical case $z=1$; each of the two circles stands for a three-dimensional spatial section of the four-dimensional continuum. A particle which at the time t_1 is in P_1 and at the time t_2 in P_2 is always on the same radial line in our picture. The spatial coordinates in (A) are chosen so that for a fixed particle they are independent of t ("cosmic coordinates"). The conformally Euclidean representation has an arbitrarily chosen point as the origin of the spatial coordinates.

We now consider a region G cut from the continuum in the following manner: we consider all (two-dimensional) spheres, with a constant radius independent of time (in "cosmic coordinates"), constructed around the origin of each time section. The common interior of all these spheres is the four-dimensional region G. In this region G we consider the metric field as replaced by one whose generating mass (represented by a singularity of the metric field) is localized at the (spatial) origin $x_1=x_2=x_3=0$. Outside the singularity this field shall satisfy the equations $R_{ik}=0$ of empty space. At $r=P$ the field shall pass continuously into the original field (A). At this passage the g_{ik} and their first derivatives shall remain continuous.

The solution of this problem yields a field for the entire continuum, which is generated in the interior of G by a concentrated mass, in the exterior of G by a homogeneous density of matter. Furthermore, it is clear that in other spherical regions outside G one can replace the field by one generated by a point-like mass according to the same method. By continuing this process of replacement, one can obtain a field so that the entire metric is generated by point-like masses rather than by a continuous distribution of matter.

The possibility of obtaining a rigorous solution by this method is gained at the cost that, in order to avoid mathematical complications, we do not allow the cut-out regions to overlap. This implies that we have to introduce infinitely many mass-points with smaller and smaller masses in order to be able to replace the entire continuum by fields generated by discrete mass-points.

This flaw is, however, of little concern to us. We can restrict ourselves to the consideration of the case where we replace only the interior of G by a field generated by a mass-point in its (spatial) center, and connect this solution continuously (for all values of t) to the solution generated by continuously distributed matter at the boundary of G.

I. Field Equations for the Interior of the Region G and Boundary Conditions for the Transition into the Remaining Space with Homogeneous Density of Matter

A general centrally-symmetric field can be brought into the (conformally Euclidean, not necessarily static) form:

$$ds^2 = -e^{\mu}\delta_{ik}dx_i dx_k + e^{\nu}dt^2 \quad i, k = 1, 2, 3,^1 \quad (1)$$

where μ and ν are functions of r and t. The field equations $R_{ik}=0$ $(i, k=1, \cdots, 4)$ now become:

$$\mu_{rr}+\nu_{rr}-\tfrac{1}{2}(\mu_r{}^2-\nu_r{}^2)-\mu_r\nu_r=0; \quad (2.1)$$

$$r(\mu_{rr}+\tfrac{1}{2}\mu_r{}^2+\tfrac{1}{2}\mu_r\nu_r)+(2\mu_r+\tfrac{1}{2}\nu_r)$$
$$-\tfrac{1}{2}e^{\mu-\nu}(\mu_{tt}+\tfrac{3}{2}\mu_t{}^2-\tfrac{1}{2}\mu_t\nu_t)=0; \quad (2.2)$$

$$2\mu_{rt}-\mu_t\nu_r=0; \quad (2.3)$$

$$r(\nu_{rr}+\tfrac{1}{2}\nu_r{}^2+\tfrac{1}{2}\mu_r\nu_r)+\tfrac{3}{2}\nu_r$$
$$-\tfrac{3}{2}e^{\mu-\nu}(\mu_{tt}+\tfrac{1}{2}\mu_t{}^2-\tfrac{1}{2}\mu_t\nu_t)=0; \quad (2.4)$$

where the subscripts stand for differentiation.

As stated above, the remaining space with homogeneous distribution of matter is a field of

[1] In the following, indices will always refer to 1, 2, 3.

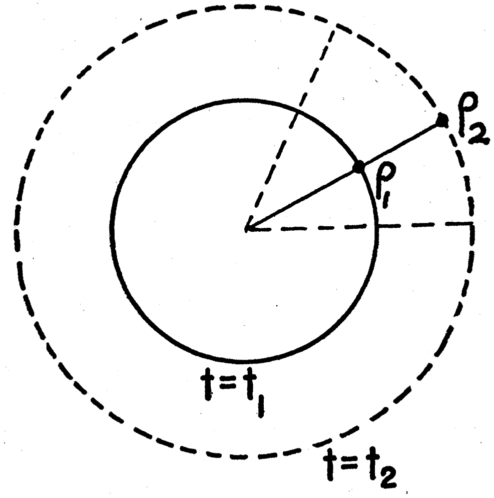

Fig. 1.

constant spatial curvature which conformally Euclidean is given by

$$ds^2 = \frac{-T^2}{(1+zr/2)^2}\delta_{ik}dx_i dx_k + dt^2, \quad (3)$$

where z and T are as above.

Our boundary conditions are now that the field (1) shall go over continuously up to the first derivative into the field (3) for $r=P$,[2] i.e., for $r=P$.

$$e^{\mu}=T^2/(1+zr/2)^2, \quad (4.1)$$

$$\mu_r e^{\mu}=-zT^2/(1+zr/2)^3, \quad (4.2)$$

$$e^{\nu}=1, \quad (4.3)$$

$$\nu_r e^{\nu}=0. \quad (4.4)$$

From the equations we can determine μ, ν, μ_r, ν_r, μ_t, ν_t, μ_{rt}, μ_{tt} for $r=P$. If we substitute these values in the Eqs. (2) we get for $r=P$

$$\mu_{rr}+\nu_{rr}-\frac{z^2}{2c^2}=0, \quad (2.1.1)$$

$$r\left(\mu_{rr}+\frac{z^2}{2c^2}\right)-\frac{2z}{c}-\frac{1}{c^2}(TT''+2T'^2)=0, \quad (2.2.1)$$

$$r\nu_{rr}-(3/c^2)TT''=0, \quad (2.4.1)$$

[2] These boundary conditions are always sufficient, but not always *necessary*, i.e., it could happen that a discontinuity of the g_{ik} or their first derivatives should be caused by a discontinuity of the respective systems of coordinates and not by a discontinuity of the fields. In our case such a possibility is avoided by the conformally Euclidean representation of both fields.

where $c=1+\frac{1}{2}zr$. (The Eq. (2.3) is satisfied identically.) If we eliminate μ_{rr} and ν_{rr} from these equations we get

$$TT''+\tfrac{1}{2}T'^2=-z/2. \tag{5}$$

By differentiation we get

$$2T'T''+TT'''=0. \tag{5.1}$$

Integrating this we get

$$T^2T''=-k/2. \tag{5.2}$$

Substituting this in (5)

$$T'^2=(k-zT)/T. \tag{5.3}$$

This result is in agreement with that obtained by the solution of the field equations in the case of spatially constant density of matter (cosmologic problem).

II. Approximate Solution of the Field Equations and the Boundary Conditions for a Region Sufficiently near to the Boundary of *G*

We set as first approximation for a region in the neighborhood of the boundary of G:

$$e^{\mu}=-T^{*2}+\sigma,$$
$$e^{\nu}=1+\tau,$$

where $\sigma=a_1r^{-\frac{1}{2}}+a_2r$; $\tau=b_1r^{-\frac{1}{2}}+b_2r$ the a_i, b_i and T^* being functions of t. Here σ and τ are small of the first order. We further assume that differentiation by t increases the order of smallness by $\frac{1}{2}$.

This form contains, besides the term which does not depend on r, a term proportional to $r^{-\frac{1}{2}}$, which corresponds to the field of a mass-point embedded in the Euclidean space. Moreover, it contains a term proportional to r, which corresponds to a regular part of the field. This term appears due to the fact that in the present case we have no embedding in the Euclidean space.

The field Eqs. (2) now become, if we neglect terms of higher order,

$$\sigma_{rr}-T^{*2}\tau_{rr}=0, \tag{2.1.2}$$

$$2r\sigma_{rr}+4\sigma_r-T^{*2}\tau_r+2T^{*2}(2T^{*\prime 2}+T^*T^{*\prime\prime})=0, \tag{2.2.2}$$

$$\frac{T^{*\prime}}{T^*}(T^{*2}\tau_r-2\sigma_r)+\sigma_{rt}=0, \tag{2.3.2}$$

$$r\tau_{rr}-\tfrac{3}{2}\tau_r-3T^*T^{*\prime\prime}=0. \tag{2.4.2}$$

If we separate these equations according to powers of r, we get

$$a_1=T^{*2}b_1, \tag{6.1}$$

$$4a_2-T^*b_2+2T^{*2}(2T^{*\prime 2}+T^*T^{*\prime\prime})=0, \tag{6.2}$$

$$\frac{T^{*\prime}}{T^*}(T^{*2}b_1-2a_1)+a_1'=0, \tag{6.3}$$

$$\frac{T^{*\prime}}{T^*}(T^{*2}b_2-2a_2)+a_2'=0, \tag{6.4}$$

$$b_2-2T^*T^{*\prime\prime}=0; \tag{6.5}$$

and hence,

$$\left.\begin{aligned} a_1&=k_1T^*,\\ a_2&=-T^{*2}T^{*\prime 2},\\ b_1&=k_1T^{*-1},\\ b_2&=2T^*T^{*\prime\prime}. \end{aligned}\right\} \tag{6.6}$$

The boundary conditions (4) become, if we neglect higher powers of P

$$T^{*2}-k_1T^*P^{-\frac{1}{2}}+T^{*2}T^{*\prime 2}P=T^2-zT^2P, \tag{4.1.1}$$

$$\tfrac{1}{2}k_1T^*P^{-\frac{3}{2}}+T^{*2}T^{*\prime 2}=-zT^2, \tag{4.2.1}$$

$$k_1T^{*-1}P^{-\frac{1}{2}}+2T^*T^{*\prime\prime}P=0, \tag{4.3.1}$$

$$-\tfrac{1}{2}k_1T^{*-1}P^{-\frac{3}{2}}+2T^*T^{*\prime\prime}=0. \tag{4.3.2}$$

From this follows that T^* differs from T only by terms of first order and that the constant k in (5.3) is given in the first approximation as: $k=(-k_1/2)P^{-\frac{3}{2}}$. In that case the Eqs. (4) are satisfied.

Our field now has the form,

$$\begin{aligned} ds^2&=(-T^{*2}+k_1T^*r^{-\frac{1}{2}}-T^{*2}T^{*\prime 2}r)\delta_{ik}dx_idx_k+(1+k_1T^{*-1}r^{-\frac{1}{2}}+2T^*T^{*\prime\prime}r)dt^2\\ &=[-T^{*2}+k_1T^*r^{-\frac{1}{2}}-T^*(k-zT^*)r]\delta_{ik}dx_idx_k+[1+k_1T^{*-1}r^{-\frac{1}{2}}-kT^{*-1}r]dt^2. \end{aligned} \tag{7}$$

We consider this solution for a small interval of time around $t=t_0$ and transform it so that it remains conformally Euclidean and that the coefficient of $\delta_{ik}dx_idx_k$ becomes equal to 1 except for infinitely small terms ("local coordinates"). We then get the form (neglecting smaller terms):

$$ds^2=(-1+k_1r'^{-\frac{1}{2}})\delta_{ik}dx_idx_k+(1+k_1r'^{-\frac{1}{2}})dt'^2. \quad (7.1)$$

This result is most remarkable since it represents an entirely static field which in the first approximation is identical with the Schwarzschild solution.

This result suggests that the field in the interior of G is rigorously equal to a Schwarzschild field.

III. Proof that the Solution in the Interior of G can be Transformed into a Static Schwarzschild Field

The static Schwarzschild field in its conformally Euclidean form is given by

$$ds^2=-a^4\delta_{ik}dx_i'dx_k'+\frac{b^2}{a^2}dt'^2, \quad (8)$$

where

$$a=1+\frac{m}{r'^{\frac{1}{2}}};\quad b=1-\frac{m}{r'^{\frac{1}{2}}}.$$

The general transformation which will leave this field centrally symmetric is given by

$$x_i'=Ux_i;\quad t'=V \quad (9)$$

where U, V are functions of r and t.

The field (8) now gets the form:

$$ds^2=-a^4U^2\delta_{ik}dx_idx_k+\left[-2a^4U_r(U+rU_r)+\frac{b^2}{a^2}V_r^2\right]x_ix_kdx_idx_k$$
$$+\left[-a^4U_t(U+2rU_r)+\frac{b^2}{a^2}V_rV_t\right]x_idx_idt+\left[-2ra^4U_t^2+\frac{b^2}{a^2}V_t^2\right]dt^2, \quad (8.1)$$

where now

$$a=1+\frac{m}{r^{\frac{1}{2}}U};\quad b=1-\frac{m}{r^{\frac{1}{2}}U}.$$

It is now our aim to show that we can choose U and V so that (8.1) is of the form (1) and that the boundary conditions (4) are satisfied. When we have proven that, we know that the Schwarzschild field can be transformed into a solution of our problem as far as the inside of G is concerned, and since the boundary conditions imply the uniqueness of the solution, our theorem will then be proven.

In order that our field shall be of the form (1) we must have:

$$-2a^4U_r(U+rU_r)+\frac{b^2}{a^2}V_r^2=0, \quad (10.1)$$

$$-a^4U_t(U+2rU_r)+\frac{b^2}{a^2}V_rV_t=0. \quad (10.2)$$

The Schwarzschild field then has the form

$$ds^2=-a^4U^2\delta_{ik}dx_idx_k+\left(-2ra^4U_t^2+\frac{b^2}{a^2}V_t^2\right)dt^2. \quad (8.2)$$

The boundary conditions (4) for $r=P$ are:

$$a^4U^2=T^2/c^2, \quad (4.1.2)$$

$$\frac{\partial}{\partial r}(a^2U)=(\partial/\partial r)(T/c), \quad (4.2.2)$$

$$-2ra^4U_t^2+\frac{b^2}{a^2}V_t^2=1, \quad (4.3.2)$$

$$\frac{\partial}{\partial r}\left(-2ra^4U_t^2+\frac{b^2}{a^2}V_t^2\right)=0, \quad (4.4.2)$$

where we abbreviate: $c=1+zr/2$; $d=1-zr/2$.

These constitute four boundary conditions for the Eqs. (10). From the existence theorems for differential equations we know that the Eqs. (10) together with the boundary conditions (4.1.2) and (4.3.2) have a unique[3] solution. In order to prove the existence of U and V satisfying all the conditions (10) and (4), we must show that the conditions (4.2.2) and (4.4.2) follow from the conditions (10) and (4.1.2), (4.3.2).

[3] Except for the fact that there is an arbitrary constant in the value of V at the boundary.

From (10.2) follows:

$$V_r=\frac{a^6}{b^2}\frac{U_t}{V_t}(U+2rU_r), \tag{10.3}$$

and by substitution in (10.1) we get:

$$-2U_r(U+rU_r)+\frac{a^6}{b^2}\frac{U_t^2}{V_t^2}(U+2rU_r)^2=0. \tag{10.4}$$

The boundary conditions (4) imply for $r=P$

$$T=a^2cU, \tag{11.1}$$

$$U_t=\frac{T'}{abc};\quad U_t^2=\frac{k-za^2cU}{a^4b^2c^3U}, \tag{11.2}$$

$$U_r=\frac{2mr^{-\frac{3}{2}}-zU}{2bc};$$

$$U+rU_r=\frac{2U-zmr^{\frac{1}{2}}}{2bc};\quad U+2rU_r=\frac{adU}{bc}, \tag{11.3}$$

$$V_t^2=\frac{a^2}{b^2}(1+2ra^4U_t^2)$$

$$=\frac{a^2}{b^2}\frac{2rk+cU(b^2c^2-2zra^2)}{b^2c^3U}, \tag{11.4}$$

$$\frac{\partial}{\partial r}(V_t^2)=\frac{\partial}{\partial r}\left[\frac{a^2}{b^2}(1+2ra^4U_t^2)\right]. \tag{11.5}$$

If we substitute the Eqs. (11.2)–(11.4) in (10.4) we get an expression which vanishes identically in U if

$$k=2m\left(1+\frac{zP}{2}\right)^3P^{-\frac{3}{2}}. \tag{12}$$

That is, in this case the Eq. (4.2.2) follows from the Eqs. (10) and (4.1.2), (4.3.2).

If we further substitute the Eqs. (10.3) and (11.5) in the equation:

$$V_r\frac{\partial}{\partial r}(V_t^2)-V_t\frac{\partial}{\partial t}(V_r^2)=0, \tag{13}$$

then considering the value of k in (12) this becomes (for $r=P$)

$$\frac{a^7dU}{b^3c}\frac{U_t}{V_t}\frac{\partial}{\partial r}(V_t^2)-V_t\frac{\partial}{\partial U}(V_r^2)U_t=0, \tag{13.1}$$

or

$$\frac{a^7dU}{b^3c}\frac{\partial}{\partial r}(\log V_t^2)-\frac{\partial}{\partial U}(V_r^2)=0, \tag{13.2}$$

which is again an identity in U. Hence also the condition (4.4.2) is a consequence of (10) and (4.1.2), (4.3.2). Our assumption is thereby confirmed.

CONCLUSION

The field of the mass point in the interior of G, which is imbedded in an expanding space is, considered in "local coordinates," a static field given by the Schwarzschild solution. The time dependence implied by the expansion does not make the solution time-dependent. What becomes time-dependent is the boundary of G where the Schwarzschild field goes over into the field generated by homogeneously distributed matter.