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Principles of Micro-Wave Radio *

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INTRODUCTION

BY micro-wave radio is meant the science of electromagnetic radiations in, roughly, the range of wave-lengths from one meter down to one millimeter, that is, of frequencies in the approximate range 3×10^{8} to 3×10^{11} cycle/sec. This region of the spectrum is marked off at its high frequency end by the fact that at higher frequencies the techniques become more "optical" than "electrical." At the lower end it is marked off by the fact that for frequencies below 300 megacycles/sec., the conventional methods of radio engineering based on lumped constant circuit analysis are quite adequate for understanding the phenomena.

The micro-wave field is thus principally characterized by these three features:

(1) Its techniques are essentially electrical rather than optical, particularly in the sense that the sources are man-made oscillators built on a macroscopic scale, rather than the non-coherent superposition of radiations from a large number of atoms or molecules.

(2) The apparatus employed is always at least comparable in size with the wave-length and usually large compared with the wave-length. This fact invalidates, or at least greatly complicates, any attempt to understand the phenomena with the aid of conventional circuit analysis or even usual distributed parameter transmission line theory. It looks as if the engineers will at last really have to learn electrical field theory! At any rate, so far there exists no *technique of evasion* of the use of field theory that corresponds to the use of complex number algebra to avoid the consideration of differential equations in analysis of steady state alternating current circuit problems. No doubt something of this sort will be worked out to correspond with the growing practical needs, but at this stage it seems desirable to consider the subject from the viewpoint of electrical field theory. The pedagogical tricks will come in due course.

(3) The micro-wave electronic apparatus is characterized by the fact that the time of flight of individual electrons is not negligible compared with the time of one cycle. Electrons in ordinary apparatus go with speeds from 0.01 to 0.1 of the velocity of light. Therefore they travel from 0.01 to 0.1 of a wave-length in one cycle. Usually it is impracticable to design tubes for such short paths when the wave-length itself is of the order of centimeters. This fact brought about a

* *Prefatory Note*—The following material is presented, in some respects, particularly in the inclusion of exercises for the reader, more in the style of a textbook than of a review article. That is because it was originally intended for publication as a textbook. Decision to publish it as a paper in the *Reviews of Modern Physics* was based on the fact that pressure of war work is likely greatly to delay completion of the manuscript, and on the fact that it appears desirable to give the completed portion of the manuscript wide circulation now. These circumstances also account for the fact that the bibliographic notes are not as complete as they should be. Nevertheless, it is felt that they afford a reasonably adequate guide to the literature. Plans are made for concluding chapters in a later issue of this journal.

breakdown in the usual modes of thinking about electronic tubes. Recent progress in the field is largely due to the discovery of ways to put finite transit time to good use. In other words, finite transit time is not a limitation on the electronics, but a limitation on the traditional thinking about the subject.

Historically, the earliest work of Hertz, by which electric waves were first intentionally produced by electrical means, was done in what we here call the micro-wave region. But it was characterized by very low radiated power and by the fact that the oscillations were a succession of highly-damped wave trains instead of the much more useful continuous waves which modern technique provides.

From the point of view of application to communication, the principal importance of micro-waves derives from two things: (1) new frequency channels are made available in an already crowded medium, and (2) owing to the fact that for production of very sharply directed beams the antenna must be large compared to the wave-length, this requirement may be satisfied with structures of more convenient size than in the case of longer wave-lengths. The subject is so new that very little work has been done so far on the propagation of these waves over land or sea, or in relation to the ionosphere. Much work needs to be done in this direction.

With the current development of experimental techniques and equipment in the micro-wave field, physicists will have in their hands a tool for investigation of properties of matter, opening up a field that is at present essentially unknown. Yet we do know already that some molecules (e.g., ammonia) have characteristic frequencies in this range that are of the utmost significance for the understanding of molecular structure. Probably much will be learned through the molecular micro-wave spectroscopy of the future. When ferromagnetic conductors are placed in a micro-wave radiation field, the skin depth to which the waves effectively penetrate is of the same order as the size of the ferromagnetic domains. There is, therefore, no question but that the study of ferromagnetics at micro-wave frequencies will contribute to a better understanding of ferromagnetism. Similarly, many dielectric substances show maximum dielectric absorption in the micro-wave frequency range so study of their properties in this range will be essential to a better understanding of the properties of such substances, especially of the modern synthetic resins and rubbers.

All such contributions to a better understanding of dielectric and ferromagnetic materials are, in a larger sense, topics in "applied physics." However, if we look for future fields of application of the micro-wave equipment outside of the research laboratory, it is at once evident that most of the developments now being made will be directly applicable as aids to marine and aerial navigation. Moreover we must not overlook the fact that applicability of micro-waves to medical diathermy is thus far completely unexplored and that they may well prove to have specific therapeutic effects not possessed by the lower frequencies in use at present.

So there is plenty to be done for a long time to come. It is sincerely hoped that the exposition which follows of some parts of the subject will contribute usefully to a vigorous future development of the subject.

CHAPTER I. CAVITY RESONATORS

Instead of the conventional coil and condenser as the basic resonant circuit element, in micro-wave radio the cavity resonator is used. By a cavity resonator is meant a region of space essentially totally enclosed by walls made of good conductors which is used as an oscillating circuit element. It is therefore desirable to begin the study of micro-wave radio by getting a thorough familiarity with the properties of cavity resonators.

1¹. Maxwell's Equations

All electromagnetic field problems are governed by the basic equations of Maxwell which we shall write in the following form:

$$\begin{aligned} &\operatorname{div} \mathbf{D} = 4\pi\rho, && \operatorname{div} \mathbf{B} = 0, \\ &\operatorname{curl} \mathbf{E} = -(1/c)\dot{\mathbf{B}}, && \operatorname{curl} \mathbf{H} = 4\pi \mathbf{i} + (1/c)\dot{\mathbf{D}}, \end{aligned} \tag{1¹1}$$

in which

E is the electric field in statvolt/cm;
D is the electric induction in statvolt/cm;

ρ is the electric charge density in electrostatic units of charge per cm^3;
H is the magnetic field in gauss;
B is the magnetic induction in gauss;
i is the conductive current density in abamp./cm^2.

This is only one of many of the systems of units now competing for public favor, but it is a system which will be found convenient and useful in practical work. Anyway we shall not fall into the common error of becoming slave to a particular unit system and shall not hesitate to change the units whenever it is advantageous to do so. In any ordinary medium we have,

$$\mathbf{B}=\mu\mathbf{H} \quad \text{and} \quad \mathbf{D}=\epsilon\mathbf{E}, \tag{1^{1}2}$$

where μ is the magnetic permeability and ϵ is the dielectric constant of the medium. The coefficients μ and ϵ are here pure numbers and equal to unity for a vacuum. They are thus equal to the values always listed in the tables in the reference books for these quantities.

In a conducting medium, the electric field needed to produce a current density is given by

$$\mathbf{E}=\rho\mathbf{i}, \tag{1^{1}3}$$

in which ρ is the resistivity of the material. A simple check of the dimensions will show that this resistivity is measured in cm. The usual reference tables give ρ in ohm-cm which is the relation between **E** in volt/cm and **i** in amp./cm^2. If ρ' is the resistivity in ohm-cm and ρ is the equivalent quantity in the unit defined here, which is the statvolt-cm/abamp. = cm, the relation is

$$\rho=\rho'/30.$$

Thus for copper at room temperature for which $\rho'=1.7\times10^{-6}$ ohm-cm the resistivity in cm is $\rho=5.7\times10^{-8}$ cm.

At the boundary between two different non-conducting media the conditions are:

Normal components of **D** and **B** continuous,
Tangential components of **E** and **H** continuous. (1¹4)

Continuity of the normal component of **D** implies that there is no surface charge density on the interface. If there is a charge of σ e.s.u./cm^2 on the interface then there is a discontinuity of $4\pi\sigma$ in the normal component of **D** at the interface.

The charge and current density are connected by the relation

$$\operatorname{div}\mathbf{i}+(1/c)\dot{\rho}=0, \tag{1^{1}5}$$

which expresses the fact that a net flow of electric charge out of a region is always accompanied by a corresponding diminution of the charge density there.

In applications **E** is usually expressed in volt/cm and the connection is 1 statvolt = 300 volts. Likewise current is usually measured in amperes, 1 abampere = 10 amperes, and charge in coulombs, 1 coulomb = 3×10^9 e.s.u. Likewise power is expressed in watts or volt-amperes, whereas the unit of our system would be the statvolt-abampere, 1 statvolt-abampere = 3 kilowatts. Although **H** is usually expressed in gauss also in practical work, some people like to express it in amp./cm which is the field in an infinitely long solenoid excited with 1 ampere-turn/cm. The connection is 1 amp./cm = 0.4π gauss.

From Maxwell's equations we may derive the general relation,

$$\operatorname{div}\mathbf{S}+\frac{1}{4\pi}\left(\mathbf{H}\cdot\frac{\partial\mathbf{B}}{\partial t}+\mathbf{E}\cdot\frac{\partial\mathbf{D}}{\partial t}\right)=-c\mathbf{i}\cdot\mathbf{E}, \tag{1^{1}6}$$

in which

$$\mathbf{S}=(c/4\pi)\mathbf{E}\times\mathbf{H}\ \text{erg/cm}^2\ \text{sec}. \tag{1^{1}7}$$

The vector **S** is called Poynting's vector and is interpreted as giving the flow of electromagnetic energy in the field. The exact flow of the field energy is really not known from this or any other consideration, since to **S** could be added any other vector field **S′** whose divergence vanishes everywhere without affecting the validity of (1¹6). However, since electromagnetic energy is only observed by the effects it produces when converted into mechanical or thermal forms, this ambiguity in the flow pattern does not affect any observable results of the calculations.

In ordinary media having constant ϵ and μ the second term in (1¹6) is the time derivative of the quantity

$$W=\frac{1}{8\pi}(\mu\mathbf{H}^2+\epsilon\mathbf{E}^2)\ \text{erg/cm}^3. \tag{1^{1}8}$$

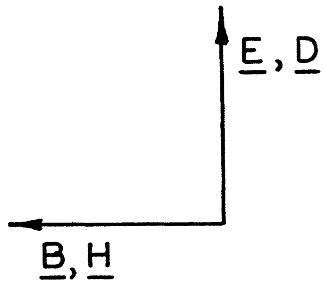


FIG. 1[1]. The relation of vectors in plane wave propagated toward the reader. σ is out from the paper.

This is interpreted as the local density of electrical energy in the field, the first term being the magnetic energy density and the second term being the electric energy density.

One advantage of the system of units we are using is that in a plane electromagnetic wave, **E** in statvolt/cm is equal to **H** in gauss. However for practical work it is handy to have the formula for Poynting's vector expressed in practical units, thus,

$$\mathbf{S} = (1/0.4\pi)\mathbf{E}\times\mathbf{H} \text{ watt/cm}^2, \qquad (1^19)$$

where **E** is in volt/cm and **H** in gauss. In a vacuum in these units H in magnitude is $(1/300)E$ so the magnitude of the Poynting vector is $S=(1/120\pi)E^2$. The numeric $120\pi=377$ is expressed in ohms and in the literature is often dignified by giving it the imposing name, *impedance of free space.*

Referring again to (1^16) we see that in a region where there is no current, so the right side is zero, the equation expresses the conservation of electromagnetic field energy. It also shows that changes in the field energy in any totally enclosed region where there is no outward flow across the boundaries occur only by virtue of flow of electric currents in a direction having a component along the electric field direction. If the current flows with the field the electromagnetic energy diminishes, if the current flows against the field the electromagnetic energy increases.

The whole art and science of micro-wave radio consists in dealing with the generation, transmission, and reception of electromagnetic energy at frequencies so great that the wave-length of the associated waves is not large compared to the apparatus involved. For this reason we have to deal with the distributed fields in accordance with the field equations. In other phases of electrical work, the wave-lengths involved are large compared to the size of the equipment. It is this fact that has made it possible to avoid the use of the field equations in developing the usual lumped constant circuit theory which is the basis of nearly all electrical engineering.

2[1]. Plane Waves

Before taking up the problem of the fields in a cavity resonator it is instructive to get some familiarity with the simpler solutions of the field equations which correspond to progressive and to standing plane waves.

We assume each field vector to be the real part of a constant vector multiplying the factor,

$$e^{2\pi i(\nu t-\boldsymbol{\sigma}\cdot\mathbf{r})}. \qquad (2^11)$$

Here $\boldsymbol{\sigma}$ is the vector whose magnitude indicates the number of waves per unit length and whose direction is normal to the plane wave fronts in the direction of propagation of the phase of the wave, ν is the frequency in cycle/sec. The results which follow could equally well be derived by choosing the opposite sign for the exponent and, in fact, the other choice is more common in the literature of electric waves. But this choice is made to get a positive time factor because that is the custom in other parts of electrical engineering where the vectors in a vector diagram in alternating circuit theory are always regarded as rotating in the counterclockwise sense.

The two equations, div $\mathbf{D}=0$ and div $\mathbf{B}=0$ give

$$\mathbf{D}\cdot\boldsymbol{\sigma}=0 \quad \text{and} \quad \mathbf{B}\cdot\boldsymbol{\sigma}=0,$$

showing that the wave amplitudes must be transverse to the direction of propagation. We shall refer to the direction of **D** or **E** as the direction of polarization of the wave.

The two curl equations in (1^11) give

$$\boldsymbol{\sigma}\times\mathbf{E}=-(\nu/c)\mathbf{B}, \quad \boldsymbol{\sigma}\times\mathbf{H}=+(\nu/c)\mathbf{D}, \qquad (2^12)$$

from which we readily find that

$$\boldsymbol{\sigma}\times(\boldsymbol{\sigma}\times\mathbf{E})=-(\nu/c)^2\epsilon\mu\mathbf{E}, \qquad (2^13)$$

and hence, using $\boldsymbol{\sigma}\cdot\mathbf{E}=0$, that

$$|\boldsymbol{\sigma}|=(\nu/c)(\epsilon\mu)^{\frac{1}{2}}. \qquad (2^14)$$

Therefore the phase velocity of the wave is $c/(\epsilon\mu)^{\frac{1}{2}}$, that is, the refractive index of the medium is $n=(\epsilon\mu)^{\frac{1}{2}}$. From (2^12) it is easy to see the vector directions are related as in Fig. 1[1]. Also $\sqrt{\epsilon}\mathbf{E}=\sqrt{\mu}\mathbf{H}$. In empty space all four vectors, **D**, **E**, **B**, **H**, are numerically equal. For a plane wave the mean energy transport in terms of the amplitude of **E** in volt/cm becomes

$$\left(\frac{\text{watt}}{\text{cm}^2}\right)\mathbf{S}=\frac{1}{2}\,\frac{1}{120\pi(\mu/\epsilon)^{\frac{1}{2}}}E^2. \qquad (2^15)$$

As already remarked in the previous section the coefficient in the denominator is expressed in ohms. Hence we shall say that a medium is characterized by an impedance for plane waves of

$$120\pi(\mu/\epsilon)^{\frac{1}{2}}\ \text{ohms}.$$

The impedance of the plane wave in ohms can also be defined as the ratio of the electric vector (volt/cm) to the magnetic vector (ampere-turns/cm). This definition leads to the same numerical value.

Standing waves arise from the superposition of two progressive plane waves of equal amplitude travelling in opposite directions. Suppose, for example, one has a wave travelling in the $+z$ direction, polarized in the x direction. Then the electric and magnetic vectors are given by

$$\begin{aligned} E_x&=E_1\cos 2\pi(\nu t-\sigma z), & E_y&=E_z=0,\\ H_y&=(\epsilon/\mu)^{\frac{1}{2}}E_1\cos 2\pi(\nu t-\sigma z), & H_x&=H_z=0. \end{aligned} \qquad (2^16)$$

Similarly a wave polarized the same way but travelling in the opposite direction has fields given by,

$$\begin{aligned} E_x&=E_2\cos 2\pi(\nu t+\sigma z),\\ E_y&=E_z=0,\\ H_y&=-(\epsilon/\mu)^{\frac{1}{2}}E_2\cos 2\pi(\nu t+\sigma z),\\ H_x&=H_z=0. \end{aligned} \qquad (2^17)$$

Suppose now the plane $z=0$ is a perfect conductor. At its surface the tangential component of **E** must vanish, and therefore the two waves must be related in such a way that $E_2=-E_1$. The combined fields of the incident

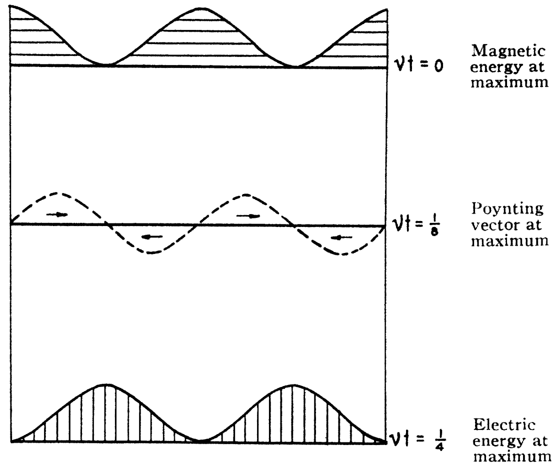


FIG. 2[1]. The pulsation of energy in a standing plane wave.

and reflected waves are then represented by,

$$\begin{aligned} E_x&=2E_1\sin 2\pi\sigma z\sin 2\pi\nu t,\\ H_y&=2(\epsilon/\mu)^{\frac{1}{2}}E_1\cos 2\pi\sigma z\cos 2\pi\nu t. \end{aligned} \qquad (2^18)$$

We observe that in the progressive waves **E** and **H** are in time phase at each place, but that in the standing wave they are in time and space quadrature. With the phases as expressed in (2^18), the energy is all magnetic at $t=0$, and a quarter cycle later it is all electric. The energy in a standing wave does, therefore, not stand entirely still, but pulses back and forth a little as set forth in Fig. 2[1].

The reflection of the plane wave by the perfect conductor comes about by virtue of the flow of induced currents in the conductor. In Chapter IV to be published in a later issue of this journal we shall show how to calculate the radiation from a given current distribution. Here we may anticipate by saying that the induced current sheet flowing in the surface radiates a wave which just cancels the incident wave on the far side of the surface and also radiates the reflected wave on the near side of the surface.

To see what is the magnitude of the induced currents in the reflecting surface one may proceed as follows: Looking at the yz plane near the surface, $z=0$, one has,

$$\begin{aligned} H_y&=2E_1(\epsilon/\mu)^{\frac{1}{2}}\cos 2\pi\nu t \quad &&\text{for}\quad z>0,\\ H_y&=0 &&\text{for}\quad z<0. \end{aligned}$$

Hence the line integral around a path extending for unit length in the y direction just outside the metal and returning just inside the metal does not vanish. By Maxwell's equations it gives the conduction current flowing across the area enclosed, namely, in the surface of the metal, since the displacement current contribution is zero because tangential **E** vanishes at the surface. Therefore the surface current density is

$$i_x = -\frac{2E_1}{4\pi}(\epsilon/\mu)^{\frac{1}{2}} \cos 2\pi\nu t, \qquad (2^19)$$

where i_x is in abamp./cm^2 if E_1 is in statvolt/cm.

3^1. Rectangular cavity resonators[1]

Any region of space totally enclosed by a good metallic conductor may serve as a cavity resonator or "rhumbatron." Any such resonator has an infinite number of resonant frequencies and associated wave fields. First we develop the theory for walls of zero resistivity and later consider the effect of the resistivity of the walls. Also it is more suitable to illustrate by working out the case of a rectangular box since this involves only trigonometric functions.

We have to solve the field equations (1^11) subject to the boundary conditions that **E** must be normal and **H** tangential to the perfectly conducting boundaries. Assume each vector to have a time dependence represented by the factor $e^{2\pi i\nu t}$. Then the equations for the positional dependence, when ϵ and μ are assumed constant throughout the medium, may be written

$$\begin{aligned} &\text{div}\,\sqrt{\epsilon}\mathbf{E} = 0, \quad \text{div}\,\sqrt{\mu}\mathbf{H} = 0, \\ &\text{curl}\,\sqrt{\epsilon}\mathbf{E} = -i(2\pi n\nu/c)\sqrt{\mu}\mathbf{H}, \\ &\text{curl}\,\sqrt{\mu}\mathbf{H} = +i(2\pi n\nu/c)\sqrt{\epsilon}\mathbf{E}. \end{aligned} \qquad (3^11)$$

Here $n = (\epsilon\mu)^{\frac{1}{2}}$, the refractive index as defined in Section 2^1 and the combination $2\pi n\nu/c$ will be denoted by k. From the form of the equations it is evident that $\sqrt{\epsilon}\mathbf{E}$ and $\sqrt{\mu}\mathbf{H}$ satisfy the same equations, with c replaced by c/n, as do **E** and **H** in free space. From this it follows that the theory for a resonator filled with any ordinary medium can be easily derived from the theory for the corresponding shape of empty resonator. For this reason and especially because nearly all the resonators used in practice so far are empty, we shall henceforth suppose ϵ and μ equal to unity.

From (3^11) we readily find by taking the curl of each curl equation and doing a little reducing that **E** has to satisfy,

$$\nabla^2\mathbf{E} + k^2\mathbf{E} = 0. \qquad (3^12)$$

If an appropriate solution for **E** has been found it is not necessary to solve the corresponding equation for **H** separately, since the associated magnetic field can be calculated from the **E** field by means of the curl **E** equation of (3^11) in the form

$$\mathbf{H} = (i/k)\,\text{curl}\,\mathbf{E}. \qquad (3^13)$$

The magnetic field so calculated will automatically satisfy the correct boundary condition. If we take any path lying in the boundary then the line integral

$$\int \mathbf{E}\cdot d\mathbf{s} = 0$$

since **E** is everywhere normal to the bounding surface. Therefore

$$\int\int \text{curl}\,\mathbf{E}\cdot d\mathbf{S} = 0,$$

where the surface integral extends over any portion of the bounding surface. Hence it follows that the normal component of curl **E** vanishes everywhere on the boundary which is therefore true of the magnetic field calculated from (3^13).

There is no general way of solving (3^12) for cavities of arbitrary shape, and in fact solutions are only known for a very few special shapes. The problem has many points in common with the corresponding acoustical problem of finding the resonant sound waves in a closed cavity. However the electromagnetic problem is more complicated because the wave amplitude is a vector, each component of which must satisfy (3^12) and also satisfy div $\mathbf{E} = 0$; whereas in the

[1] Earliest development of this topic in physics was made in connection with the theory of blackbody radiation. Compare Jeans, *Dynamical Theory of Gases* (Cambridge University Press, London, 1921), third edition, chapter 16; or Fowler, *Statistical Mechanics* (Cambridge University Press, London, 1936), second edition, chapter 4.

acoustic problem there is only a single scalar wave amplitude, for example, the pressure in the wave.

We now consider the solution of the special problem of the rectangular cavity resonator whose walls are at the ends of the ranges,

$$0<x<A, \quad 0<y<B, \quad 0<z<C.$$

If we write

$$E_x = E_1 \overset{\cos}{\underset{\sin}{\cdot}} k_1x \overset{\cos}{\underset{\sin}{\cdot}} k_2y \overset{\cos}{\underset{\sin}{\cdot}} k_3z,$$

$$E_y = E_2 \overset{\cos}{\underset{\sin}{\cdot}} k_1x \overset{\cos}{\underset{\sin}{\cdot}} k_2y \overset{\cos}{\underset{\sin}{\cdot}} k_3z,$$

$$E_z = E_3 \overset{\cos}{\underset{\sin}{\cdot}} k_1x \overset{\cos}{\underset{\sin}{\cdot}} k_2y \overset{\cos}{\underset{\sin}{\cdot}} k_3z,$$

then (3·2) is satisfied for any combination of cos or sin provided the three k's are such that

$$k_1^2+k_2^2+k_3^2=k^2.$$

To make $\mathbf{E}$ normal to all walls we have to specialize the cos or sin alternative and restrict the k's to the following discrete set of values:

$$k_1 = l\pi/A, \quad k_2 = m\pi/B, \quad k_3 = n\pi/C \qquad (3\cdot4)$$

in which (l, m, n) are integers. The solution for $\mathbf{E}$ is therefore

$$\begin{aligned} E_x &= E_1 \cos(l\pi x/A) \\ &\qquad \times \sin(m\pi y/B)\sin(n\pi z/C), \\ E_y &= E_2 \sin(l\pi x/A) \\ &\qquad \times \cos(m\pi y/B)\sin(n\pi z/C), \\ E_z &= E_3 \sin(l\pi x/A) \\ &\qquad \times \sin(m\pi y/B)\cos(n\pi z/C). \end{aligned} \qquad (3\cdot5)$$

The three constant amplitudes, E_1, E_2, and E_3, cannot be chosen independently, but the condition div $\mathbf{E}=0$, imposes the restriction,

$$(l\pi/A)E_1 + (m\pi/B)E_2 + (n\pi/C)E_3 = 0. \qquad (3\cdot6)$$

Therefore, for each set of integers there are two linearly independent modes of oscillation: if we think of (k_1, k_2, k_3) as the components of a vector and (E_1, E_2, E_3) as those of another vector, then any vector (E_1, E_2, E_3) perpendicular to k is permissible.

The possible resonant frequencies are given by

$$(\nu/c)^2 = (l/2A)^2 + (m/2B)^2 + (n/2C)^2 \qquad (3\cdot7)$$

where (l, m, n) are integers, at least two of which are not zero.

For the modes in which one of the integers is zero, the electric vector is everywhere parallel to the axis whose integer is zero, and the resonant frequency is independent of the dimension along that axis. For each such set of integers there is only one vector satisfying (3·6), and so only a single solution for such a set, although as already remarked there are in general two linearly independent solutions associated with each set (l, m, n).

The least resonant frequency for the box is that which corresponds to putting the integers associated with the two larger dimensions each equal to unity, and the third one equal to zero. Thus if A and B are the two larger dimensions, the lowest mode will be polarized with the electric vector along the shortest dimension and the wave-length λ will be

$$\lambda = \frac{2}{(A^{-2}+B^{-2})^{\frac{1}{2}}}.$$

In particular for a cubical box the lowest mode has a wave-length equal to the face diagonal of the box, $\lambda=\sqrt{2}A$.

The number of different resonant modes mounts very rapidly as one goes up the frequency scale. For example consider a shallow square box ($B=A$, $C \ll A$) for which the lower frequency modes will all correspond to $n=0$. The values of $2A\sigma$ are given by

$$2A\sigma = [l^2+m^2+n^2(A/C)^2]^{\frac{1}{2}}.$$

One can easily count up and find that there are 33 different sets of the integers giving rise to frequencies less than or equal to five times the lowest frequency. It should also be observed that the frequencies can be arranged in series: the fundamental is accompanied by all its integral multiples forming the series (110), (220), (330), etc.; another series begins with (120) and (210) and includes their integral multiples as (240) and (420), (360) and (630), etc. This occurrence of the integral multiples among the allowed frequencies is, however, a special property of the rectangular box which other shapes do not possess.

It is important to define some terms which will be used in discussing cavity resonators. Each frequency for which there exists a solution of the field equations satisfying the boundary conditions will be called an *allowed frequency* or a *proper frequency*. The least allowed frequency is called the *fundamental*. The higher allowed frequencies are only called *harmonics* if they are integral multiples of the fundamental.

A particular solution for **E** and **H** will be referred to as a *mode* of oscillation: Any frequency for which there is more than one mode is referred to as a *degenerate* frequency.[2] The *order* of degeneracy is the number of linearly independent *modes* associated with the degenerate frequency. Thus in the example just considered the fundamental is not degenerate, but the next higher frequency is, because (1, 2, 0) and (2, 1, 0) are linearly independent solutions each having this same frequency. As we have seen all of the modes for which no one of the integers is zero are twofold degenerate because of the two linearly independent solutions of (3^16) that are possible. This type of degeneracy we shall call *polarization degeneracy*. Degeneracy also arises from symmetry in the shape of the resonator, thus (1, 2, 0) and (2, 1, 0) have the same frequency only because we assumed a square cross section $A=B$. Degeneracy arising in this way we shall call *symmetry degeneracy*.

A slight departure from the condition $A=B$, intentionally or due to some imperfection of manufacture or an unsymmetric location of the coupling device by which the resonator is placed in a circuit, will cause the degenerate frequencies to become slightly separated. We say that such changes remove the degeneracy.

It is important to recognize a lack of uniqueness in the wave fields associated with a degenerate frequency. For example, in the case of polarization degeneracy, one may choose any two (preferably mutually perpendicular) vectors satisfying (3^16) as the basic modes. Any linear combination of them is a possible mode of oscillation associated with that frequency. Similarly, though our particular analysis may present us with certain particular forms for the degenerate modes in the case of symmetry degeneracy, these have really no special standing in the physics of the problem and the actual mode of oscillation may be any linear combination of these.

For example, consider the modes (1, 2, 0) and (2, 1, 0). From (3^15) these have only a z component of **E** which in the two cases is

$$E_{120}=C\sin(\pi x/A)\sin(2\pi y/A),$$
$$E_{210}=D\sin(2\pi x/A)\sin(\pi y/A),$$

where C and D are arbitrary relative amplitudes. At this frequency quite a variety of different modes are possible according to the relative magnitudes of C and D and the phase relation existing between them. Some of the field distributions which can arise from different relative excitations of these two modes are sketched in Fig. 3^1. Since the combination with $C=D$ has a node on the line $y=A-x$ as well as $y=0$ and $x=0$, it obviously satisfies all the conditions to be the fundamental mode for a right triangular prism whose cross section is formed by these three lines. By this means one can often find particular solutions for special shapes which would not otherwise be easy to find. The trick does not work for a right triangular prism of unequal sides for in that case the two modes of the corresponding rectangular prism would not belong to the same frequency and so could not be superposed in this way.

Exercise: Discuss the modes corresponding to $D=\pm iC$, that is, where the two degenerate modes are in time quadrature.

4^1. Resonator Coordinates[3]

By working out in detail the solution for a rectangular cavity resonator in the preceding section, we have learned most of the general properties which are applicable to resonators of any shape. These are, that the fields in the

[2] The terminology is obviously borrowed from that of an analogous mathematical situation in quantum mechanics.

[3] Most of this section can be skipped at a first reading, but it should be scanned to see the main results concerning orthogonality of the wave functions (4^15) and the dynamical equation for a mode amplitude (4^110). The results are an application of the formalism used in quantum electrodynamics. Compare E. Fermi, Rev. Mod. Phys. **4**, 87 (1932), or W. Heitler, *Quantum Theory of Radiation* (Oxford University Press, London, 1936), p. 40.

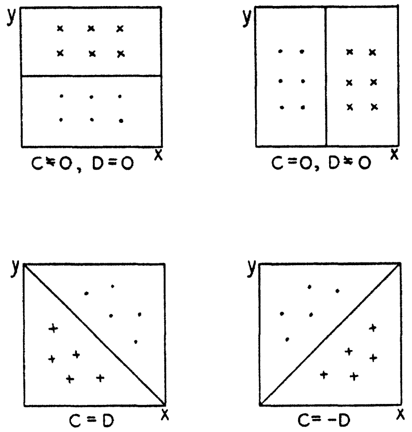


FIG. 3^1. The different types of field distribution resulting from co-existence of the degenerate modes (1, 2, 0) and (2, 1, 0).

resonator can be made to satisfy the boundary conditions only for certain discrete allowed frequencies and associated with each frequency there may be one or more wave patterns. Evidently the most general state of excitation of a resonator would be for all of the possible modes to be simultaneously present, just as many of the different possible modes of vibration of a drum-head are simultaneously present when the drum is struck. To deal mathematically with this situation calls for introduction of a convenient means of describing such general states of oscillation. This we can do by means of resonator coordinates, which are simply the amplitudes of each of the basic wave fields in the actual state of motion.

Vector Potential

Instead of dealing directly with $\mathbf{E}$ and $\mathbf{H}$ it is convenient to derive the electromagnetic fields from the usual scalar and vector potentials, $\mathbf{A}$ and φ, according to the relations

$$\mathbf{E}=-(1/c)\dot{\mathbf{A}}-\text{grad}\,\varphi,\quad \mathbf{H}=\text{curl}\,\mathbf{A}. \qquad (4^11)$$

Here the vector potential is measured in the same units as current, namely abamperes, and the scalar potential is in statvolts. This mode of representation of the field satisfies the field equation for curl $\mathbf{E}$ automatically, as also the equation div $\mathbf{H}=0$. Substituting from (4^11) into the other two field equations we find,

$$\begin{aligned}&-\nabla^2\mathbf{A}+(1/c^2)\ddot{\mathbf{A}}\\&\qquad+\text{grad}\left(\text{div}\,\mathbf{A}+\frac{1}{c}\dot{\varphi}\right)=4\pi\mathbf{i},\\&-\nabla^2\varphi+(1/c^2)\ddot{\varphi}\\&\qquad-\frac{1}{c}\frac{\partial}{\partial t}\left(\text{div}\,\mathbf{A}+\frac{1}{c}\dot{\varphi}\right)=4\pi\rho.\end{aligned} \qquad (4^12)$$

We are at liberty to assume some further relation involving div $\mathbf{A}$ to simplify the equations, and evidently they will be simplified very considerably if we put div $\mathbf{A}+(1/c)\dot{\varphi}=0$, which gives us the following set of equations for the potentials:

$$\begin{aligned}&\frac{1}{c^2}\ddot{\mathbf{A}}-\nabla^2\mathbf{A}=4\pi\mathbf{i},\\&\frac{1}{c^2}\ddot{\varphi}-\nabla^2\varphi=4\pi\rho,\\&\text{div}\,\mathbf{A}+(1/c)\dot{\varphi}=0.\end{aligned} \qquad (4^13)$$

These same equations will play a basic role in Chapter IV when we develop the theory of radiation from a system of moving charges and conductive currents. If we take the divergence of the first equation and take $(1/c)$ times the time derivative of the second and add, we find an equation for the time dependence of [div $\mathbf{A}$ $+(1/c)\dot{\varphi}$], namely,

$$\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2}-\nabla^2\right)\left(\text{div}\,\mathbf{A}+\frac{1}{c}\dot{\varphi}\right)=0.$$

The right side is zero since the charge and current satisfy the conservation equation (1^15). This shows that if we admit only solutions which at $t=0$ satisfy the third equation of (4^13) together with the time derivative of that equation, then the third equation will be satisfied at all times.

Suppose now that the mathematical problem of finding the allowed frequencies and associated wave patterns for the cavity has been solved by some such procedure as that of the preceding section. This means that we know the set of values k_1, k_2, k_3, etc., and associated solutions $\mathbf{A}_1$, $\mathbf{A}_2$, $\mathbf{A}_3$, etc., which satisfy the equations

$$\nabla^2\mathbf{A}+k^2\mathbf{A}=0,\quad \text{div}\,\mathbf{A}=0, \qquad (4^14)$$
$$\mathbf{A}\text{ normal to walls or zero.}$$

As we have seen, it is possible for an allowed value k_a to be degenerate, that is to have more than one linearly independent solution $\mathbf{A}_a$ associated with it. Thus a complete enumeration of the $\mathbf{A}$'s requires another "degeneracy" index, to distinguish the different $\mathbf{A}_a$ belonging to the same k_a. Ordinarily it will not be necessary to write this explicitly: in the formal mathematics we can regard the index a as labelling all of the independent wave functions, then it will happen that the associated k_a are equal for several different values of the index a.

Orthogonality of Wave Functions

The $\mathbf{A}_a$ wave patterns have an important orthogonality property which makes it convenient to use them to represent other functions in a manner similar to Fourier series. Write down the curl curl equation satisfied by $\mathbf{A}_a$ and by $\mathbf{A}_b$, multiply the former by $\mathbf{A}_b$, the latter by $\mathbf{A}_a$ and subtract:

$$\mathbf{A}_b\cdot\text{curl curl }\mathbf{A}_a-\mathbf{A}_a\cdot\text{curl curl }\mathbf{A}_b = (k_a^2-k_b^2)\mathbf{A}_a\cdot\mathbf{A}_b.$$

Now use one of the basic identities of vector analysis,

$$\text{div }(\mathbf{u}\times\mathbf{v})=\mathbf{v}\cdot\text{curl }\mathbf{u}-\mathbf{u}\cdot\text{curl }\mathbf{v}$$

to transform the left side of this equation into

$$\text{div }(\mathbf{A}_a\times\text{curl }\mathbf{A}_b+(\text{curl }\mathbf{A}_a)\times\mathbf{A}_b).$$

Next integrate both sides over the volume of the cavity. The integral of the left side vanishes because it can be transformed to a surface integral over the surface which vanishes because $\mathbf{A}_a$ and $\mathbf{A}_b$ are normal to the walls. Hence,

$$\iiint\mathbf{A}_a\cdot\mathbf{A}_b\,dV=0,\quad\text{if}\quad k_a\neq k_b. \qquad (4{\cdot}5)$$

In the case of a degenerate value of k, it is possible to choose the several $\mathbf{A}$'s belonging to it so they are mutually orthogonal by taking appropriate linear combinations of the original ones if those found in the original solution do not already have this property.

Since a particular solution is still a solution when multiplied by a constant, the constant multiplier of each $\mathbf{A}_a$ may be chosen to suit one's convenience. It turns out that for the work which follows it is convenient to *normalize* the functions in such a way that

$$\iiint\mathbf{A}_a\cdot\mathbf{A}_a{}^*dV=V \qquad (4{\cdot}6)$$

in which $\mathbf{A}_a{}^*$ is the conjugate complex function to $\mathbf{A}_a$. With this choice of normalization, the $\mathbf{A}_a$ functions are physically dimensionless; V is the volume of the cavity.

Let us first consider the case in which the currents in the cavity are distributed in such a way that the charge density is everywhere zero at all times, then $\varphi=0$ and we may try to find a solution of the first of (4·3) by writing

$$\mathbf{A}=\textstyle\sum_a J_a(t)\mathbf{A}_a(x,y,z). \qquad (4{\cdot}7)$$

There is one time-dependent coefficient for each wave pattern. Since the $\mathbf{A}_a$ as normalized by (4·6) are dimensionless, the $J_a(t)$ are measured in the same units as $\mathbf{A}$, namely abamperes. Each J_a gives the amplitude of excitation of its particular mode at a particular instant and is for that reason called a *resonator coordinate.* There is one for each mode and hence an infinite number of them for a particular cavity resonator.

Exciting Current

Similarly we may expand the given current distribution inside the resonator $\mathbf{i}(x,y,z,t)$ in terms of the $\mathbf{A}_a$ functions, denoting the time-dependent coefficients by $I_a(t)$,

$$\mathbf{i}(x,y,z,t)=\textstyle\sum_a I_a(t)\mathbf{A}_a(x,y,z). \qquad (4{\cdot}8)$$

The determination of the coefficients in this expansion is particularly easy formally. Because of the orthogonality property of the $\mathbf{A}$ functions it is just like the method used in Fourier series,

$$I_a(t)=(1/V)\iiint\mathbf{i}\cdot\mathbf{A}_a{}^*dV. \qquad (4{\cdot}9)$$

Thus each $I_a(t)$, like $\mathbf{i}$ is a current density, abampere/cm². We shall call $I_a(t)$ the *exciting current* of the ath mode. It may be remarked in passing that a current distribution $\mathbf{i}$ is most effective in exciting the ath mode if its spatial distribution is like that of the mode it is to excite.

Dynamical Equation

Now substituting (4'8) and (4'7) in the first of (4'3) we may equate coefficients of each $\mathbf{A}_a$ and thus obtain a simple differential equation for the time dependence of each resonator coordinate,

$$\ddot{J}_a(t)+(ck_a)^2J_a(t)=4\pi c^2 I_a(t). \qquad (4'10)$$

This equation is just like that for a simple harmonic oscillator of natural frequency $(ck_a/2\pi)$ driven in forced oscillations by the exciting term on the right side. If the exciting term is zero then the corresponding J_a executes harmonic time variation at its natural frequency with constant amplitude. The free oscillations are undamped because we have supposed the walls to be of zero resistivity: the effect of finite resistivity is considered in Section 8'.

It will make the rest of the discussion easier to follow if we suppose that all of the $\mathbf{A}_a$ are real functions: this is no restriction as it is always possible to choose them in this way.

Expressions for Energy

The electric field energy in the cavity is (in ergs)

$$W_e=\iiint(E^2/8\pi)dV=(V/8\pi c^2)\textstyle\sum_a \dot{J}_a{}^2 \qquad (4'11)$$

as can be found by substituting the expression for $\mathbf{E}$ in terms of $\mathbf{A}$ and using (4'5), (4'6), and (4'7). Similarly the magnetic field energy in the cavity (in ergs) is

$$W_m=\iiint(H^2/8\pi)dV$$
$$=\textstyle\sum_{a,b}\displaystyle\iiint J_a\cdot J_b(\text{curl }\mathbf{A}_a\cdot\text{curl }\mathbf{A}_b)dV.$$

To simplify this we need to evaluate,

$$\iiint(\text{curl }\mathbf{A}_a\cdot\text{curl }\mathbf{A}_b)dV$$
$$=\iiint\text{div }(\mathbf{A}_a\times\text{curl }\mathbf{A}_b)dV$$
$$+k_b{}^2\iiint\mathbf{A}_a\cdot\mathbf{A}_b dV.$$

The integral of the divergence vanishes because it can be transformed to a surface integral of $\mathbf{A}_a\times\text{curl }\mathbf{A}_b$ which has a vanishing normal component at the boundary. Therefore,

$$\iiint(\text{curl }\mathbf{A}_a\cdot\text{curl }\mathbf{A}_b)dV=\begin{cases}0 & b\neq a\\ k_a{}^2V & b=a,\end{cases}$$

so the magnetic energy is

$$W_m=(V/8\pi)\textstyle\sum_a k_a{}^2J_a{}^2. \qquad (4'12)$$

We shall write W_a for the energy associated with the ath mode of oscillation. There is no mutual energy between different modes in consequence of the orthogonality of the $\mathbf{A}$ functions. For W_a we have

$$W_a=(V/8\pi c^2)[\dot{J}_a{}^2+(ck_a)^2J_a{}^2]. \qquad (4'13)$$

We can derive an expression for dW_a/dt from (4'10) in just the same way as is done in obtaining the energy integral in particle dynamics. Multiply through by $(V/4\pi c^2)$ to obtain,

$$(dW_a/dt)=VI_a(t)\dot{J}_a(t). \qquad (4'14)$$

In words: the instantaneous rate of increase of the field energy in the ath mode is equal to the volume times the ath exciting current times the rate of increase of the ath resonator coordinate. This is fully analogous to the expression for power as a force (in this case I_a) multiplied by a velocity, in this case proportional to the rate of increase of J_a.

Effective Inductance and Capacity

Those who are accustomed to thinking in terms of resonant circuits in terms of their inductances and capacities will grope for a definition of some sort of effective inductance and capacity which is applicable to the cavity resonator. This can be done as soon as one has fixed on a proper current coordinate by means of which to measure the amplitude of excitation. In an ordinary inductance the magnetic energy is $W_m=Li^2/2$ where W is in ergs if L is in cm and i in abamperes. Here we measure the amplitude of the ath mode by giving the value of its resonator coordinate J_a which is a current, hence we may properly identify the coefficient of $J_a{}^2$ in the expression for the magnetic energy as half the *effective inductance* L_a of the ath mode. This gives, for L_a in cm

$$L_a = Vk_a^2/4\pi = \pi V/\lambda_a^2. \quad (4^115)$$

For conversion to practical units, note that 1 cm of inductance is equal to 10^{-9} henry. To get a correspondingly appropriate definition of the capacity of the ath mode we must choose C_a in such a way that the product L_aC_a gives the correct resonant frequency in accordance with the equation,

$$\lambda_a = 2\pi(L_aC_a)^{\frac{1}{2}}.$$

In this way it is easily found that the electrostatic capacity in cm to be associated with the ath mode is

$$C_a = 4\pi/Vk_a^4. \quad (4^116)$$

Exercise: Show that the normalized $\mathbf{A}_a$ for the (110) mode of a cubical resonator of edge A is

$$\mathbf{A}_{110} = 2\mathbf{k} \sin(\pi x/A) \sin(\pi y/A),$$

where $\mathbf{k}$ is a unit vector parallel to the z axis. Also show that the maximum value of the electric vector occurs at points along the line $x = A/2$, $y = A/2$ and that when the resonator coordinate $J_{110} = 1$ abampere, this maximum electric vector equals $(4\pi/\lambda)$ statvolt/cm where $\lambda = \sqrt{2}A$ as worked out in the preceding section.

Effect of Charge in Cavity

Let us now turn to the more general case in which there is charge density as well as current in the resonator. The expansion (4^17) is no longer adequate, since it gives div $\mathbf{A} = 0$ which is no longer true. A similar remark holds for (4^18). The necessary generalization runs as follows:

Suppose the scalar boundary value problem,

$$\nabla^2\varphi_b + k_b^2\varphi_b = 0, \quad \varphi_b = 0 \text{ on walls,} \quad (4^117)$$

has been solved so its allowed functions and allowed values are known. The functions φ_b may be proved to be orthogonal:

$$\varphi_c\nabla^2\varphi_b - \varphi_b\nabla^2\varphi_c + (k_b^2 - k_c^2)\varphi_b\varphi_c = 0,$$

$$\varphi_c\nabla^2\varphi_b - \varphi_b\nabla^2\varphi_c = \text{div}\,(\varphi_c \text{ grad } \varphi_b - \varphi_b \text{ grad } \varphi_c).$$

Integrate over the volume. The volume integral of the divergence may be seen to vanish by transforming to a surface integral. Therefore $\iiint \varphi_b\varphi_c dV = 0$ if $k_b \neq k_c$. In the case of degeneracy we may choose orthogonal linear combinations of the degenerate wave functions so all φ's are orthogonal. We shall consider the φ's to be normalized as the $\mathbf{A}$'s [compare (4^16)]

$$\iiint \varphi_b^2 dV = V. \quad (4^118)$$

Next we assume that $\rho(x, y, z, t)$ is expanded in terms of the φ_b, so, analogous to (4^18),

$$\rho = \textstyle\sum_b R_b(t)\varphi_b(x, y, z), \quad (4^119)$$

and likewise that the scalar potential can be so expanded,

$$\varphi = \textstyle\sum_b \Phi_b(t)\varphi_b(x, y, z). \quad (4^120)$$

Substitution of these in the equation for φ in (4^13) leads to equations of motion for the coefficients in (4^120), analogous to (4^110),

$$\ddot{\Phi}_b + (ck_b)^2\Phi_b = 4\pi c^2 R_b. \quad (4^121)$$

In addition the expansions for $\mathbf{A}$ and $\mathbf{i}$ have to be extended by bringing in additional terms for which the divergence does not vanish. The functions

$$\mathbf{B}_b = (1/k_b) \text{ grad } \varphi_b, \quad \text{curl } \mathbf{B}_b = 0 \quad (4^122)$$

are appropriate for this purpose. They are all orthogonal to each other and to the $\mathbf{A}_a$ functions. To prove $\mathbf{B}_b$ orthogonal to A_a apply the general formula

$$\iiint (\text{curl } \mathbf{u} \cdot \text{curl } \mathbf{v} + \text{div } \mathbf{u} \text{ div } \mathbf{v} + \mathbf{u} \cdot \Delta\mathbf{v}) dV$$

$$= \iint \mathbf{u} \times \text{curl } \mathbf{v} \cdot \mathbf{dS} + \iint \text{div } \mathbf{v}\, \mathbf{u} \cdot \mathbf{dS}.$$

Identify $\mathbf{v}$ with $\mathbf{A}_a$ and $\mathbf{u}$ with $\mathbf{B}_b$. Then on the left the first integral vanishes since curl $\mathbf{B}_b = 0$, the second since div $\mathbf{A}_a = 0$, and the third reduces to $-k_a^2\iiint \mathbf{A}_a \cdot \mathbf{B}_b dV$. On the right the first integral vanishes because $\mathbf{B}_b$ is normal to the surface and the second because div $\mathbf{A}_a = 0$. Hence the functions are orthogonal.

The factor $(1/k_b)$ is inserted in $(4^1 22)$ to take care that the B_b are normalized like the A's:

$$\iiint \mathbf{B}_b\cdot\mathbf{B}_c dV=\begin{cases}0 & b\neq c\\ V & b=c.\end{cases} \qquad (4^1 23)$$

This follows from the relation,

$$\iiint \operatorname{grad}\varphi_b\cdot\operatorname{grad}\varphi_c\, dV$$

$$=\iiint \operatorname{div}(\varphi_b \operatorname{grad}\varphi_c)dV-\iiint \varphi_b\Delta\varphi_c dV,$$

both as regards orthogonality and normalization.

We may now asume $(4^1 7)$ and $(4^1 8)$ extended as follows:

$$\begin{aligned}\mathbf{A}&=\textstyle\sum_a J_a(t)\mathbf{A}_a+\sum_b K_b(t)\mathbf{B}_b,\\ \mathbf{i}&=\textstyle\sum_a I_a(t)\mathbf{A}_a+\sum_b H_b(t)\mathbf{B}_b.\end{aligned} \qquad (4^1 24)$$

Substituting these into the equation of motion for $\mathbf{A}$ in $(4^1 3)$ we find

$$\ddot{K}_b+(ck_b)^2K_b=4\pi c^2 H_b \qquad (4^1 25)$$

which, together with $(4^1 21)$ and $(4^1 10)$, gives the equations of motion of all of the resonator coordinates. A considerable complication has resulted from the introduction of charge into the cavity: not only was it necessary to introduce the scalar potential, but to extend the expansion for $\mathbf{A}$ as well. The unnumbered equation following $(4^1 3)$ gives a proof that if

$$\dot{\Phi}_b-ck_bK_b=0$$

and

$$\frac{d}{dt}(\dot{\Phi}_b-ck_bK_b)=0$$

initially, then they will remain zero at all times. Hence the solutions of $(4^1 10)$, $(4^1 21)$, and $(4^1 25)$ have to be chosen so as to satisfy these conditions as part of the initial conditions.

If now we compute the electric and magnetic energy expressions, we find no change in the magnetic energy since the curl of the additional terms in $\mathbf{A}$ is zero. But there are added terms in the electric energy and the complete expression to replace $(4^1 11)$ is

$$W_e=(V/8\pi c^2)\textstyle\sum_a \dot{J}_a{}^2+(V/8\pi c^2)\sum_b \dot{K}_b{}^2 \\ +(V/8\pi)\sum_b k_b{}^2\Phi_b{}^2. \qquad (4^1 26)$$

5^1. Cylindrical Resonators[4]

By a cylindrical resonator is meant one which is bounded by the planes $z=0$ and $z=C$ and whose cross section in any plane $z=$constant is the same curve. The three-dimensional problem can be reduced quite generally to a two-dimensional problem for such resonators. We start with Eqs. $(3^1 1)$ in which $\epsilon=\mu=1$,

$$\begin{aligned}&\operatorname{div}\mathbf{E}=0 &&\operatorname{div}\mathbf{H}=0\\ &\operatorname{curl}\mathbf{E}=-ik\mathbf{H} &&\operatorname{curl}\mathbf{H}=+ik\mathbf{E}.\end{aligned} \qquad (5^1 1)$$

It is natural to expect the solutions to depend on z through a $\cos k_3 z$ or $\sin k_3 z$ factor as in the case of the rectangular box (which is a special case of the class of cylindrical resonators).

The modes can be classified into types as follows:

$$\begin{aligned}&E \text{ type, for which } E_z\neq 0, \text{ but } H_z=0,\\ &H \text{ type, for which } H_z\neq 0, \text{ but } E_z=0.\end{aligned} \qquad (5^1 2)$$

E (Electric) Modes

Let us consider first the modes of E type. Since $H_z=0$ we have, from the curl $\mathbf{H}$ equation,

$$\begin{aligned}ikE_x&=-(\partial H_y/\partial z)\\ ikE_y&=+(\partial H_x/\partial z)\\ ikE_z&=(\partial H_y/\partial x)-(\partial H_x/\partial y),\end{aligned} \qquad (5^1 3)$$

and, similarly, the curl $\mathbf{E}$ equation gives

$$\begin{aligned}-ikH_x&=(\partial E_z/\partial y)-(\partial E_y/\partial z),\\ -ikH_y&=(\partial E_x/\partial z)-(\partial E_z/\partial x),\\ -ikH_z&=(\partial E_y/\partial x)-(\partial E_x/\partial y).\end{aligned} \qquad (5^1 4)$$

Using the x and y components of these equations we can express E_x and E_y in terms of derivatives of E_z:

$$\frac{\partial^2 E_x}{\partial z^2}+k^2E_x=\frac{\partial}{\partial x}\left(\frac{\partial E_z}{\partial z}\right),$$

$$\frac{\partial^2 E_y}{\partial z^2}+k^2E_y=\frac{\partial}{\partial y}\left(\frac{\partial E_z}{\partial z}\right).$$

[4] The literature dealing with special properties of resonators of particular shapes is becoming quite extensive. Some useful general references are: Stratton, *Electromagnetic Theory* (McGraw-Hill, New York, 1941), chapters 6 and 7; Bateman, *Electrical and Optical Wave Motion* (Cambridge University Press, London, 1915); Borgnis, Ann. d. Physik **35**, 359 (1939).

The left-hand side of these becomes $(k^2-k_3^2)$ times E_x or E_y no matter whether the z dependence contains a cos or sin factor, so we can write for the component of $\mathbf{E}$ in the cross section

$$\mathbf{E}_s=E_x\mathbf{i}+E_y\mathbf{j},$$
$$(k^2-k_3^2)\mathbf{E}_s=\mathrm{grad}_s\,(\partial E_z/\partial z), \tag{5·5}$$

where grad_s means gradient in the section and is the usual gradient with the z component omitted. Next use (5·4) to eliminate the H components from the z component of (5·3) to obtain the basic differential equation which governs the variation of E_z over the cross section

$$\nabla_s^2E_z+(k^2-k_3^2)E_z=0, \tag{5·6}$$

where ∇_s^2 is the sectional Laplacian, obtained by omitting the z component from the three-dimensional Laplacian.

Finally we may use (5·4) to express $\mathbf{H}$ in terms of $\mathbf{E}$:

$$H_x=\frac{ik}{k^2-k_3^2}\frac{\partial E_z}{\partial y},\quad H_y=\frac{-ik}{k^2-k_3^2}\frac{\partial E_z}{\partial x}, \tag{5·7}$$

or, in vector form,

$$\mathbf{H}=\frac{-ik}{k^2-k_3^2}\mathbf{k}\times\mathrm{grad}_s\,E_z, \tag{5·8}$$

in which $\mathbf{k}$ is the unit vector in the z direction.

The boundary conditions on $\mathbf{E}$ require that $\mathbf{E}$ be normal to all bounding surfaces. This calls for the choice of the factor $\cos k_3z$ instead of $\sin k_3z$ in the expression for E_z in order that the sectional component $\mathbf{E}_s$ shall vanish at the ends. The condition on the cylindrical walls requires that only solutions of (5·6) which vanish at the boundary be admitted.

Suppose we denote by $\psi_a(x,y)$ and k_a the associated functions and proper values which satisfy the two-dimensional boundary value problem

$$\nabla_s^2\psi_a(k,y)+k_a^2\psi_a(x,y)=0,$$
$$\psi_a(x,y)=0,\ \text{on boundary.} \tag{5·9}$$

This two-dimensional problem defines a sequence of proper functions and proper values. They are known in the mathematical literature for a variety of shapes since they occur in other branches of mathematical physics, for instance, the vibrations of a drumhead of the same shape as the resonator section.

Summarizing the results for the E type modes we have,

E type:

$$E_z=A\psi_a(x,y)\cos k_3z,$$

$$k^2=k_a^2+k_3^2,$$

$$\mathbf{E}_s=-(k_3/k_a^2)A\left(\frac{\partial\psi_a}{\partial x}\mathbf{i}+\frac{\partial\psi_a}{\partial y}\mathbf{j}\right)\sin k_3z, \tag{5·10}$$

$$\mathbf{H}_s=-i(k/k_a^2)A\left(-\frac{\partial\psi_a}{\partial y}\mathbf{i}+\frac{\partial\psi_a}{\partial x}\mathbf{j}\right)\cos k_3z.$$

In terms of the normalized vector potentials of Section 4·1 we have for the vector potential $\mathbf{A}$

$$\mathbf{A}=B\{\psi_a(x,y)\cos k_3z\mathbf{k}$$
$$-(k_3/k_a^2)\,\mathrm{grad}_s\,\psi_a\sin k_3z\}, \tag{5·11}$$

where B is the normalizing factor to be chosen to satisfy (4·6). We have

$$\iiint\mathbf{A}^2dV=B^2(C/2)$$
$$\times\left[\iint\psi_a^2dxdy+\frac{k_3^2}{k_a^4}\iint(\mathrm{grad}\,\psi_a)^2dxdy\right].$$

Since $(\mathrm{grad}\,\psi)^2=\mathrm{div}\,(\psi\,\mathrm{grad}\,\psi)-\psi\nabla^2\psi$ this reduces to

$$\iiint\mathbf{A}^2dV=B^2(C/2)$$
$$\times[1+(k_3^2/k_a^2)]\iint\psi_a^2dxdy,$$

so if V is the volume of the resonator, the normalizing factor is

$$B^2=\frac{2Vk_a^2}{Ck^2\iint\psi_a^2dxdy}. \tag{5·12}$$

H (Magnetic) Modes

The theory for the modes of H type is quite similar. In place of (5^13) and (5^14) we have

$$ikE_x=\frac{\partial H_z}{\partial y}-\frac{\partial H_y}{\partial z},\qquad ikE_y=\frac{\partial H_x}{\partial z}-\frac{\partial H_z}{\partial x},$$
$$ikE_z=\frac{\partial H_y}{\partial x}-\frac{\partial H_x}{\partial y},$$
$$-ikH_x=-\frac{\partial E_y}{\partial z},\qquad -ikH_y=\frac{\partial E_x}{\partial z},$$
$$-ikH_z=\frac{\partial E_y}{\partial x}-\frac{\partial E_x}{\partial y}.$$

These permit us to express H_x and H_y in terms of H_z, and yield a relation analogous to (5^15)

$$(k^2-k_3^2)\mathbf{H}_s=\mathrm{grad}_s\left(\frac{\partial H_z}{\partial z}\right). \qquad (5^113)$$

Similarly we find in analogy with (5^16)

$$\nabla_s^2H_z+(k^2-k_3^2)H_z=0, \qquad (5^114)$$

and in analogy with (5^18)

$$\mathbf{E}_s=\frac{ik}{k^2-k_3^2}\mathbf{k}\times\mathrm{grad}_s H_z. \qquad (5^115)$$

Since the boundary conditions require that $\mathbf{H}$ be tangential at the walls, this calls for the $\sin k_3z$ factor in H_z which will also make $\mathbf{E}_s$ vanish at the two ends. From (5^113) we see that, in order to make $\mathbf{H}_s$ be tangential at the cylindrical surfaces, the normal gradient of H_z must vanish at the walls.

Suppose we denote by $\varphi_b(x, y)$ and k_b the proper functions and proper values which satisfy the two-dimensional problem

$$\begin{aligned}&\nabla_s^2\varphi_b+k_b^2\varphi_b=0,\\ &\partial\varphi_b/\partial n=0 \text{ on boundary,}\end{aligned} \qquad (5^116)$$

where $\partial/\partial n$ means differentiation in a direction normal to the boundary. The difference in boundary conditions between (5^19) and (5^116) gives rise to a different set of proper functions and proper values in the two cases.

Summarizing the results for the H type modes we have,

H type:

$$H_z=A\,\varphi_b(x, y)\sin k_3z,$$
$$k^2=k_b^2+k_3^2,$$
$$\mathbf{H}_s=(k_3/k_b^2)A\ \mathrm{grad}_s\ \varphi_b\cdot\cos k_3z, \qquad (5^117)$$
$$\mathbf{E}_s=i(k/k_b^2)A\left(-\frac{\partial\varphi_b}{\partial y}\mathbf{i}+\frac{\partial\varphi_b}{\partial x}\mathbf{j}\right)\sin k_3z.$$

For the vector potential describing these modes we may take

$$\mathbf{A}=B\left(-\frac{\partial\varphi_b}{\partial y}\mathbf{i}+\frac{\partial\varphi_b}{\partial x}\mathbf{j}\right)\sin k_3z,$$

where B is the normalizing factor whose value is readily calculated to be,

$$B^2=\frac{2V}{Ck_b^2\int\int\varphi_b^2dxdy}. \qquad (5^118)$$

Note that there are modes of E type for which $k_3=0$, and that for these the resonant frequencies are independent of the height of the cylinder, but that the H type modes require $k_3\neq 0$. The allowed values of k_3 are, of course,

$$k_3=n\pi/C\quad (n,\text{ an integer}). \qquad (5^119)$$

We shall need a notation to designate a particular mode in a cylindrical resonator. A convenient notation is $E(n, a)$ and $H(n, b)$ to denote an E type or H type mode, respectively, built on the use of $k_3=n\pi/C$ and the scalar functions ψ_a or φ_b, respectively. When we deal with cylindrical resonators of particular cross section the general notations a and b are replaced by more specific designations referring to special properties of the functions ψ_a and φ_b.

Double-Walled Resonators

If the section of the cylindrical resonator consists of the region of space external to curve C_1 and internal to curve C_2 as in Fig. 4^1, then the interior of the cavity resonator is not a singly-connected region (which means simply that an arbitrary

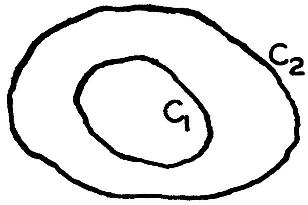


FIG. 4[1]. Sketch of double-walled resonator section.

closed path in the region cannot be shrunk continuously to zero while staying entirely in the region). This gives rise to some special electromagnetic properties which are important. In practice the curves C_1 and C_2 are usually concentric circles but we shall see that the general properties to be discussed are independent of this particular shape.

The two most important properties to be developed are, first, that there exist zero frequency modes giving rise to an internal magnetic field not associated with an electric field, and secondly, that there exist modes for which *both* E_z and H_z vanish, whose frequency depends only on the length, not on the cross section of the cylinder. These will be called coaxial cable modes.

If E_z as well as H_z vanishes then (5[1]3) and (5[1]4) become

$$ikE_x=-\partial H_y/\partial z, \qquad ikE_y=+\partial H_x/\partial z,$$
$$0=(\partial H_y/\partial x)-(\partial H_x/\partial y),$$
$$-ikH_x=-\partial E_y/\partial z, \qquad -ikH_y=+\partial E_x/\partial z,$$
$$0=(\partial E_y/\partial x)-(\partial E_x/\partial y).$$

The z component of these shows that $\mathbf{E}$ or $\mathbf{H}$ may be expressed as the gradient of a scalar function $U(x, y)$. Write

then
$$\begin{aligned}\mathbf{E}_s&=-\mathrm{grad}_s\, U(x,y,z)\,;\\ ik\mathbf{H}_s&=\mathbf{k}\times\mathrm{grad}\,(\partial U/\partial z).\end{aligned} \qquad (5^120)$$

The first two equations require

$$\frac{\partial^2}{\partial z^2}\left(\frac{\partial U}{\partial x}\right)+k^2\left(\frac{\partial U}{\partial x}\right)=0,$$

$$\frac{\partial^2}{\partial z^2}\left(\frac{\partial U}{\partial y}\right)+k^2\left(\frac{\partial U}{\partial y}\right)=0,$$

and the third requires that

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)\left(\frac{\partial U}{\partial z}\right)=0.$$

The boundary conditions require that $\mathbf{E}_s$ vanish at $z=0$ and $z=C$ so $U(x, y, z)$ must contain the factor $\sin n\pi z/C$. Write

$$U(x,y,z)=u(x,y)\sin n\pi z/C,$$

where

$$\nabla^2 u(x,y)=0.$$

To satisfy the boundary conditions on the cylindrical walls we must have $u=$constant on the boundary curves C_1 and C_2.

From potential theory it is known that if the boundary consists of the single curve C_2 so the region is the entire region interior to C_2, then if $u=$constant on the boundary it is constant throughout the inside. Such a solution for u gives vanishing electric and magnetic fields inside the resonator which shows that for such a resonator there are no modes with E_z and H_z both zero. But if the region is bounded by two curves C_1 and C_2 we may satisfy the boundary conditions by putting $u=u_1$ on C_1 and $u=u_2$ on C_2 where u_1 and u_2 are two different constants. This gives rise to a non-constant solution $u(x, y)$, which in fact is the same function of position as the electrostatic potential distribution between the two cylinders if C_1 is at potential u_1 and C_2 at potential u_2.

Since k is not involved in the boundary value problem in the section it follows that k is determined entirely by the length, so $k=n\pi/C$. Hence the result: For any shape of the bounding curves C_1 and C_2 the double-walled cylindrical resonator possesses modes whose frequencies are such that $C=n\lambda/2$ where n is an integer.

If we pass to the limit $k\to 0$ in $U=u(x, y)\sin kz$ we get $\mathbf{E}_s=0$, but

$$\mathbf{H}_s=\mathbf{k}\times\mathrm{grad}\ u.$$

Hence the equations are satisfied by a steady magnetic field, produced by circulation of steady current up the inner cylinder and down the outer cylinder. Such zero-frequency modes always occur if the interior of the resonator is a multiply-connected region.

Use of Function Theory

Since u satisfies Laplace's equation in two dimensions, many results obtainable from the theory of functions of a complex variable are applicable here. For the calculations which follow let $z=x+iy$ (not to be confused with previous use of z as the coordinate along the length of the cylinder). Also let $w=u+iv$. Then if $w=f(z)$ is an analytic function of z we have,

$$w=u(x, y)+iv(x, y). \qquad (5\cdot 21)$$

The Cauchy-Riemann conditions for the existence of a unique derivative $f'(z)$ are:

$$\partial u/\partial x=\partial v/\partial y, \quad \partial u/\partial y=-\partial v/\partial x, \qquad (5\cdot 22)$$

from which it follows that u and v each satisfy Laplace's equation.

From (5·19) it follows that, except for the fact that $\mathbf{E}$ contains a sin factor and $\mathbf{H}$ a cos factor in its dependence on the coordinate along the length of the cylinder, we have:

$$E_x=-(\partial u/\partial x), \quad E_y=-(\partial u/\partial y),$$
$$iH_x=-(\partial v/\partial x), \quad iH_y=-(\partial v/\partial y),$$

which can be summarized in the vector formula

$$\mathbf{E}-i\mathbf{H}=-\operatorname{grad} w. \qquad (5\cdot 23)$$

Consider now any function, $w=f(z)$ such that the equation $u(x, y)=\text{constant}$, defines a family of closed curves, successive members of which enclose the preceding members. Any two of these curves may be chosen as the bounding curves C_1 and C_2 of a double-walled cylindrical cavity resonator, and therefore each such function provides the solution for the fields in a whole family of such cavity resonators.

Circular Coaxial Cable

The simplest application of this general method is the solution which applies to the circular coaxial cable. This is given by the function,

$$w=\log z, \qquad (5\cdot 24)$$

or

$$e^{u+iv}=z,$$

from which we find,

$$e^u=|z|=r, \quad \text{that is,} \quad u=\log r,$$

and

$$v=\operatorname{arc} z=\varphi,$$

so

$$\log z=\log r+i\varphi.$$

Therefore the lines of constant u are the circles $r=\text{constant}$ and the lines of constant v are the radial lines of constant φ. The electric and magnetic fields are given by

$$\mathbf{E}-i\mathbf{H}=-\operatorname{grad} w=-(1/r)\mathbf{r}_0-i(1/r)\varphi_0. \qquad (5\cdot 25)$$

The electric field is directed radially and the magnetic field is directed circumferentially and each varies as the inverse first power of the radius.

If the inner radius is a and the outer radius is b, then the current flowing axially in the inner conductor is $1/4\pi a$ abamp./cm and on the outer conductor is $1/4\pi b$ abamp./cm. The total current flowing on either is the same and is equal to $\frac{1}{2}$ abamp. The line integral of $\mathbf{E}$ from $r=a$ to $r=b$ is $\log(b/a)$ statvolt.

Therefore if the amplitude of excitation of the resonator is such that the maximum current amplitude is 1 ampere at $z=0$ or any other place where $\cos n\pi z/C$ equals ± 1, the maximum voltage amplitude, which occurs at places where $\sin n\pi z/C$ equals ± 1, is equal to $60\log(b/a)$ volts. This relation is expressed by saying that the impedance of the circular coaxial cable resonator is $60\log(b/a)$ ohms.

More general shapes may be treated by remarking that the positional coordinates (x, y) must be periodic functions of v. There is no loss of generality in assuming the period to be 2π, and the most general complex function having this property is the Fourier series,

$$z=\sum_{m=-\infty}^{+\infty} A_m e^{m(u+iv)}. \qquad (5\cdot 26)$$

No extra generality arises from the inclusion of the A_0 term since this simply provides for a shift of origin in the (x, y) plane. The circular coaxial cable, already discussed, is obtained by putting $A_1=1$, and $A_m=0$ for $m\neq 1$.

Elliptic Coaxial Cable

An important simple interesting case is obtained by using (5·26), and by putting

$$A_1=A_{-1}=f/2, \quad A_m=0 \quad \text{for} \quad m\neq\pm 1,$$

which gives,

$$z=f(\cosh u \cos v+i \sinh u \sin v).$$

From this it follows that the curves of constant u are the ellipses

$$\left(\frac{x}{f \cosh u}\right)^2+\left(\frac{y}{f \sinh u}\right)^2=1,$$

that is, confocal ellipses whose foci are at the points $(x, y)=(\pm f, 0)$. Hence this special case is appropriate to the case of a double-walled resonator where the bounding curves are confocal ellipses. Thus if the inner and outer ellipses have semi-major axes a and b, respectively, (both greater than f) then the inner and outer walls are given by u_1 and u_2 where

$$\cosh u_1=a/f \quad \text{and} \quad \cosh u_2=b/f.$$

The line integral from inner to outer wall is

$$u_2-u_1=\cosh^{-1}(b/f)-\cosh^{-1}(a/f).$$

The magnetic field at the point (u, v) is $-\operatorname{grad} v$, so the axial current per unit length on either wall is $(1/4\pi) \operatorname{grad} v$ abamp./cm. Therefore the total current on either conductor is $\frac{1}{2}$ abampere since the integral of grad v around either cylinder is 2π. Hence, using the definition of impedance introduced in discussing the circular coaxial cable, we find

$$60[\cosh^{-1}(b/f)-\cosh^{-1}(a/f)] = 60 \log \frac{b+(b^2-f^2)^{\frac{1}{2}}}{a+(a^2-f^2)^{\frac{1}{2}}} \text{ ohms} \quad (5^1 27)$$

for the impedance of the cylindrical resonator whose walls are confocal elliptic cylinders.

We can give a more general result for the impedance of the resonator formed by two cylinders of arbitrary shape. No matter what the coefficients in ($5^1 26$), the total current flowing in inner or outer conductor is $\frac{1}{2}$ abampere,

TABLE I. Values of X_{mp} for which $J_m(X_{mp})=0$.

	$m=0$	1	2	3
$p=1$	2.405	3.832	5.135	6.379
2	5.520	7.016	8.417	9 760
3	8.654	10.173	11.620	13.017
4	11.792	13.323	14.796	16.229

and the line integral of the electric vector from inner to outer conductor is u_2-u_1 statvolt if the conductors are given by $u=u_1$ and $u=u_2$, respectively. Therefore the impedance of such a resonator is given by

$$60(u_2-u_1) \text{ ohms} \quad (5^1 28)$$

for any shape whatever.

6^1. Circular Cylinder

The general results of the preceding section may be illustrated and useful practical results obtained by specializing to the case of a circular cylinder of radius R. In place of the coordinates (x, y) it is convenient to use polar coordinates (r, φ).

Equations ($5^1 9$) and ($5^1 16$) are satisfied by

$$J_m(k_a r)e^{im\varphi}, \quad (6^1 1)$$

where $J_m(x)$ is the Bessel function usually denoted this way, and m is an integer. The boundary conditions for modes of E type are satisfied by choosing k_a such that

$$J_m(k_a R)=0. \quad (6^1 2)$$

This leads us naturally to replace the general label "a" by two integers m and p, where m is the order of Bessel function used and p is the ordinal number of the root when they are numbered in order of increasing magnitude.

Some of the roots are given in Table I. Thus we denote a particular E mode by $E(n, m, p)$ and the frequencies are given in terms of the dimensions R and C by

$$k^2_{Enmp}=X^2_{mp}/R^2+n^2\pi^2/C^2. \quad (6^1 3)$$

The E mode of lowest frequency is $E(0, 0, 1)$ for which

$$k_{E001}=2.405/R \quad \text{or} \quad \lambda_{E001}=2.61R. \quad (6^1 4)$$

The next higher mode in the symmetric $m=0$ series is $E(0, 0, 2)$ for which

$$k_{E002}=5.520/R \quad \text{or} \quad \lambda_{E002}=1.14R. \quad (6^1 5)$$

Notice that the frequency of $E(0, 0, 2)$ is considerably greater than twice that of $E(0, 0, 1)$.

Similarly for the modes of H type the boundary conditions require that k_b be such that

$$J_m'(k_b R)=0. \qquad (6^1 6)$$

Hence for H type waves we need a table of roots of the equation $J_m'(x)=0$. (See Table II.) The frequencies of the H modes are therefore given by,

$$k^2_{Hnmp}=Y^2_{mp}/R^2+n^2\pi^2/C^2. \qquad (6^1 7)$$

From Table II we see that Y_{11} is smaller than any of the X_{mp}. However since $n=0$ is not allowed with an H wave we find that $E(001)$ has a lower frequency than $H(111)$ for $C<1.15R$ but the order is reversed for $C>1.15R$.

The modes for a resonator whose shape is that of a sector of a circular cylinder are obtained by an easy generalization of the foregoing. Suppose the sector is bounded by the planes $\varphi=0$ and $\varphi=\alpha$ where $\alpha<2\pi$.

For the E modes we must have $E_z=0$ on these bounding planes, which will be satisfied by using for $\psi(r, \varphi)$

$$\psi_a(r, \varphi)=(J m\pi/\alpha)(k_a r)\sin(m\pi\varphi/\alpha) \qquad (6^1 8)$$

in place of $(6^1 1)$. The boundary condition at $r=R$ will require the Bessel function to vanish, and therefore the determination of the allowed frequencies calls for a knowledge of the roots of Bessel functions of fractional order for which the designation $X_{m\pi/\alpha,\, p}$ is a natural notation.

Similarly for the H modes we must use,

$$\varphi_b(r, \varphi)=(J m\pi/\alpha)(k_b r)\cos(m\pi\varphi/\alpha), \qquad (6^1 9)$$

and the allowed values of k_b will be determined by requiring the radial derivative of the Bessel function to vanish at $r=R$. This calls for a knowledge of the roots $Y_{m\pi/\alpha,\, p}$ in an obvious way.

Exercise: If the cross section is a sector of opening α bounded by two circular radii, $A<r<B$, show that the appropriate Bessel junction for the E modes is

$$\psi=[CJ(k_a r)+DN(k_a r)]\sin m\pi\varphi/\alpha,$$

where J and N are two associated Bessel functions of order $m\pi/\alpha$. Discuss the dependence of the fundamental frequency on α and on A/B.

7^1. Figure of Revolution

In practice, resonators in the form of figures of revolution are often useful. In discussing them we use cylindrical polar coordinates (r, φ, z) whose axis is the axis of symmetry of the resonator. Such resonators possess symmetrical modes in which $E_\varphi=0$ and H_φ is independent of φ. This section will develop the theory for modes of this class.

In cylindrical coordinates the curl equations of $(5^1 1)$ become:

$$\left.\begin{aligned}
ikE_r&=\frac{1}{r}\frac{\partial H_z}{\partial\varphi}-\frac{\partial H_\varphi}{\partial z},\\
ikE_\varphi&=\frac{\partial H_r}{\partial z}-\frac{\partial H_z}{\partial r},\\
ikE_z&=\frac{1}{r}\frac{\partial}{\partial r}(rH_\varphi)-\frac{1}{r}\frac{\partial H_r}{\partial\varphi},\\
-ikH_r&=\frac{1}{r}\frac{\partial E_z}{\partial\varphi}-\frac{\partial E_\varphi}{\partial z},\\
-ikH_\varphi&=\frac{\partial E_r}{\partial z}-\frac{\partial E_z}{\partial r},\\
-ikH_z&=\frac{1}{r}\frac{\partial}{\partial r}(rE_\varphi)-\frac{1}{r}\frac{\partial E_r}{\partial\varphi}.
\end{aligned}\right\} \qquad (7^1 1)$$

Now assume that $H_r=H_z=0$ and that H_φ is independent of φ. These reduce to

$$\begin{aligned}
&\text{(a)} & ikE_r&=-\frac{\partial H_\varphi}{\partial z},\\
&\text{(b)} & ikE_\varphi&=0,\\
&\text{(c)} & ikE_z&=\frac{1}{r}\frac{\partial}{\partial r}(rH_\varphi),\\
&\text{(d)} & 0&=\frac{1}{r}\frac{\partial E_z}{\partial\varphi},
\end{aligned} \qquad (7^1 2)$$

TABLE II. Values of Y_{mp} for which $J_m'(Y_{mp})=0$.

	$m=0$	1
$p=1$	3.832	1.840
2	7.016	5.335
3	10.173	8.535
4	13.323	11.705

$$\text{(e)} \qquad -ikH_\varphi = \frac{\partial E_r}{\partial z} - \frac{\partial E_z}{\partial r},$$

$$\text{(f)} \qquad 0 = -\frac{1}{r}\frac{\partial E_r}{\partial \varphi}.$$

Of these (d) and (f) are satisfied because by (c) and (a), E_z and E_r are expressed in terms of H_φ which is independent of φ. A differential equation for H_φ is obtained from (e), by the use of (a) and (c):

$$\frac{\partial^2 H_\varphi}{\partial r^2} + \frac{1}{r}\frac{\partial H_\varphi}{\partial r} + \frac{\partial^2 H_\varphi}{\partial z^2} + \left(k^2 - \frac{1}{r^2}\right) H_\varphi = 0. \qquad (7^13)$$

The boundary condition on **H** is satisfied without restriction on solutions of 7^13, but these must be restricted to fit the boundary conditions on **E**. From (a) and (c) of (7^12) we have

$$\mathbf{E}_s = E_r \mathbf{r}_0 + E_\varphi \varphi_0 = (i/kr)\varphi_0 \times \text{grad } (rH_\varphi). \qquad (7^14)$$

Suppose the boundary of the figure of revolution is the curve $f(r, z) = 0$, so the normal to it is given by

$$\mathbf{n} = \text{grad} f.$$

The boundary condition on **E** requires that **E** have a vanishing tangential component, that is $\mathbf{n} \times \mathbf{E}_s = 0$ which gives the boundary condition in the form

$$\frac{\partial f}{\partial r}\frac{\partial}{\partial r}(rH_\varphi) + \frac{\partial f}{\partial z}\frac{\partial}{\partial z}(rH_\varphi) = 0 \text{ on } f(r, z) = 0. \qquad (7^15)$$

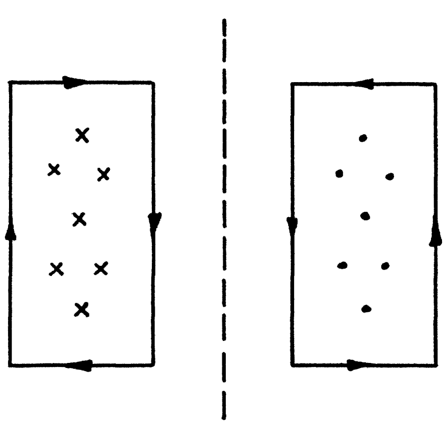


FIG. 5^1. Sketch of currents and magnetic field in zero frequency mode of coaxial cable resonator.

The form of this suggests the convenience of introducing the quantity

$$u = rH_\varphi \qquad (7^16)$$

as the basic scalar function from which solutions are to be derived. The differential equation for u follows from (7^13):

$$\frac{\partial^2 u}{\partial r^2} - \frac{1}{r}\frac{\partial u}{\partial r} + \frac{\partial^2 u}{\partial z^2} + k^2 u = 0,$$

$$u = 0 \quad \text{at} \quad r = 0, \quad \frac{\partial u}{\partial r} = 0 \quad \text{at} \quad f(r, z) = 0. \qquad (7^17)$$

As a simple example let us consider a length $0 < z < C$ of coaxial cable bounded by the radii $r = A$ and $r = B$. Write

$$u(r, z) = v(r) \cdot w(z);$$

then (7^17) is satisfied if

$$v'' - (1/r)v' + k_a^2 v = 0, \quad w'' + k_3^2 w = 0,$$
$$k^2 = k_a^2 + k_3^2. \qquad (7^18)$$

The equation for w together with its boundary conditions is satisfied by writing

$$w(z) = \cos k_3 z \quad \text{with} \quad k_3 = n\pi/C.$$

The simplest solution for v is obtained by putting $v = 1$ and $k_a = 0$. The lowest frequency mode is that corresponding to this solution for v, and to $n = 0$. Its frequency is zero and it corresponds to a steady magnetic field, unaccompanied by an electric field due to circulating steady current as shown in Fig. 5^1. The first mode of non-zero frequency corresponds to $n = 1$. Its frequency is independent of the radii A and B and is such that the length C is half a wavelength. The series of higher modes going with $v = 1$ are the harmonic series such that $C = n\lambda/2$.

The equation for $v(r)$ for $k_a \neq 0$ is satisfied by

$$v(r) = rZ_1(k_a r), \qquad (7^19)$$

in which Z_1 is written for the general Bessel function of the first order. We have

$$Z_1(x) = aJ_1(x) + bN_1(x). \qquad (7^110)$$

The ratio $a : b$ and the parameter k_a now have to be chosen in such a way that $v'(r) = 0$ at

$r=A$ and $r=B$. In this way the frequencies of the higher modes can be calculated if necessary.[5] This boundary value problem defines a sequence of values of k_a each of which can be associated with any value of n to give a mode of symmetric type.

Quarter-Wave Coaxial Resonator

A form which has found much practical application is the quarter-wave coaxial resonator. It is generated by revolving the figure shown in Fig. 6^1 about the z axis. It is not susceptible of exact calculation. An approximate treatment runs as follows.[6] Consider separately the three regions I, II, and III. In I, especially near $z=0$ we expect the fields to be quite accurately represented by the coaxial cable solution, so we assume,

$$u_I = a \cos kz. \quad (0<z<C) \qquad (7^1 11)$$

Here $k=k_3$ since $k_a=0$. By analogy with lumped-constant circuit ideas we might expect a voltage node at the end $z=C$ or D, provided the capacity formed by the region II is great enough to store the charge carried by the current in the walls without developing an appreciable voltage across the region II. That would call for infinite capacity in II: since it is actually finite there will actually be voltage across II.

In the region II we assume

$$u_{II} = brJ_1(kr). \qquad (7^1 12)$$

($k=k_a$ since here $k_3=0$.) In most practical cases B is small compared with a quarter wave-length so $kB \ll \pi/2$. In this region $xJ_1(x)$ is practically equal to the first term in its power series $x^2/2$ so to a good approximation

$$u_{II} = bkr^2/2,$$

which represents a uniform axial field

$$E_z = -ib$$

[5] This problem is treated in explicit detail by Borgnis, Zeits. f. Hochfrequenztechnik **56**, 47 (1940).

[6] A more accurate analysis of this problem was given by W. W. Hansen, J. App. Phys. **10**, 38 (1939). Some interesting experimental results are given by Barrow and Mieher, Proc. I. R. E. **28**, 184 (1940).

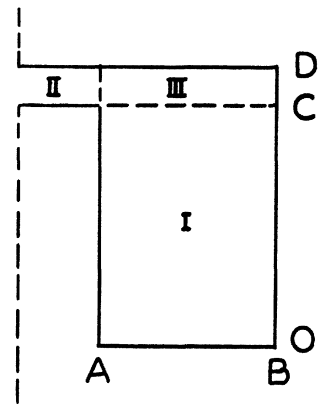


FIG. 6^1. Sketch of cross section of quarter-wave coaxial resonator.

in the region II; hence the line integral from $z=C$ to $z=D$ on $r=A$ is $-ib(D-C)$.

Along the line $z=C$, from $r=A$ to $r=B$, we have approximately

$$E_r = -(ia/r) \sin kC,$$

so the line integral of E on this path is

$$-ia \sin kC \ln (B/A).$$

If we neglect the flux which goes through the region III then the line integral of E around III must vanish and therefore

$$a \sin kC \ln (B/A) = b(D-C). \qquad (7^1 13)$$

We must also have continuity of the magnetic field in the two regions, so equating them at the point $r=A$, $z=C$ we have

$$a \cos kC = bkA^2/2. \qquad (7^1 14)$$

Lacking a more accurate analysis we might just as well equate the magnetic fields at any other point in III. This point is simply preferred because u_I and u_{II} are probably better approximations to the true function there than deeper in region III.

Dividing ($7^1 13$) by ($7^1 14$) we get

$$\tan kC = \frac{2(D-C)/A^2}{k \ln B/A} \qquad (7^1 15)$$

which determines the value of k in terms of the given dimensions.

The preceding analysis is quite crude, and should leave the reader dissatisfied. Nevertheless, it corresponds to the result obtained by applying the standard engineering form of transmission line theory, as will be shown in Section 8^2. This mode of derivation has the merit that it shows more vividly what approximations have been made. The ratio of a to b can be obtained from either (7^113) or (7^114), after k has been calculated.

$$\frac{a}{b}=\frac{(D-C)}{\sin kC\cdot\ln B/A}=\frac{kA^2}{2\cos kC}. \qquad (7^116)$$

The foregoing theory will now be illustrated by a numerical design example. Suppose the frequency is 150 mc so the wave-length is 200 cm. Suppose we wish to make a resonator with $D-C=3$ inches, $A=6$ inches, $B=10$ inches, what is the proper value of C? From (7^115), since $k=1/12.5$ in.$^{-1}$,

$$\tan kC=4.07 \quad \text{so} \quad kC=76.2°$$

and therefore the proper value of C is

$$C=(76.2/90)(\lambda/4)=16.6 \text{ inches.}$$

Therefore the resonator in this case is about 15 percent shorter than a quarter wave-length. We find from (7^116)

$$b/a=0.0647.$$

Hence, if the excitation is such that the magnetic field amplitude is 1 gauss in the corner, $z=0$, $r=A$, the amplitude of the electric field on the axis in region II is b statvolts with $a=15.26$ (since $A=15.26$ cm); therefore $b=0.986$ so the line integral of electric vector from $z=C$ to $z=D$ on $r=0$ is 2260 volts.

Another method of deriving the symmetric modes of a figure of revolution is sometimes useful. If ψ is any function satisfying the scalar wave equation

$$\nabla^2\psi+k^2\psi=0, \qquad (7^117)$$

then it is easily verified that if $\mathbf{C}$ is any constant vector, the vector,

$$\mathbf{A}=\mathbf{C}\times\text{grad }\psi \qquad (7^118)$$

satisfies the vector wave equation

$$\text{curl curl }\mathbf{A}=k^2\mathbf{A} \qquad (7^119)$$

as does also $\mathbf{B}=\text{curl }\mathbf{A}$.

Suppose that we use particular solutions of (7^117) that are independent of φ and that we choose for $\mathbf{C}$ the unit vector $\mathbf{k}$ in the direction of the axis of revolution. Then $\mathbf{A}$ is entirely in the direction of φ_0 and so is suitable to represent the magnetic field in the symmetric modes of a figure of revolution. Writing

$$\mathbf{H}=\mathbf{k}\times\text{grad }\psi=\frac{\partial\psi}{\partial r}\varphi_0 \qquad (7^120)$$

we have for the associated electric field

$$ik\mathbf{E}=\text{curl }\mathbf{H}=-\frac{\partial^2\psi}{\partial z\partial r}\mathbf{r}_0+\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\psi}{\partial r}\right)\mathbf{k}. \qquad (7^121)$$

The boundary conditions, when we compare with (7^15), are that the normal derivative of $u=r(\partial\psi/\partial r)$ be zero on the boundary and that $u=0$ at $r=0$. The solutions of (7^117) that are finite on the axis and independent of φ are of the form,

$$\psi=J_0(\alpha r)e^{i\beta z} \quad \alpha^2+\beta^2=k^2;$$

hence a general solution will be

$$\psi(z,r)=\int_{-k}^{+k}g(\beta)J_0(k^2-\beta^2r)^{\frac{1}{2}}e^{i\beta z}d\beta,$$

where $g(\beta)$ is an arbitrary function. Since $J_0'(x)=-J_1(x)$,

$$u=r\frac{\partial\psi}{\partial r}=r\int_{-k}^{+k}h(\beta)J_1(k^2-\beta^2r)^{\frac{1}{2}}e^{i\beta z}d\beta,$$

where $h(\beta)$ is an arbitrary function.

8^1. Skin Effect

In preparation for developing the theory of losses in a cavity resonator due to the finite conductivity of the metal walls let us consider the propagation of an electromagnetic wave in a good conductor. From the field equations (1^11) for a medium with constants ϵ, μ, and ρ we can find the wave equations that are satisfied by the space dependence of the field quantities whose time dependence is represented by the factor $e^{i\omega t}$ to be

$$\begin{aligned}\text{curl curl }\mathbf{E}&=k^2\mu(\epsilon-2i\lambda/\rho)\mathbf{E},\\ \text{curl curl }\mathbf{H}&=k^2\mu(\epsilon-2i\lambda/\rho)\mathbf{H},\end{aligned} \qquad (8^11)$$

where, as usual, $k=\omega/c$. These equations are of the same form as those governing propagation in a non-conducting medium except that the medium is characterized by a complex index of refraction,

$$n^2=\mu(\epsilon-2i\lambda/\rho). \qquad (8^12)$$

Since for metals ρ is of the order of 10^{-8} cm it follows that the pure imaginary component of n^2 is very large compared to the real part. In fact this is true even at optical frequencies. Hence we may neglect ϵ in comparison with λ/ρ, which amounts physically to neglecting the displacement current relative to the conduction current, which gives as an entirely adequate approximation to the index of refraction,

$$n=(\mu\lambda/\rho)^{\frac{1}{2}}(1-i). \qquad (8^13)$$

The general plane wave solution thus appears in the form

$$\mathbf{E}=\mathbf{E}_0e^{i\omega t-knx}=E_0e^{-x/\delta}\cos(\omega t-x/\delta), \qquad (8^14)$$

in which the quantity δ is called the *skin depth* and is given by

$$\delta=\frac{1}{2\pi}(\rho\lambda/\mu)^{\frac{1}{2}}. \qquad (8^15)$$

Evidently the length δ in cm, if ρ and λ are in cm, gives a measure of the depth of penetration of the rapidly damped wave in the metal. For copper the values of δ at some representative wave-lengths are given in Table III. At 60 cycle/sec. the value of δ in copper is 0.85 cm.

If we have a wave propagated into a metal in the z direction with its electric vector along the x axis when its magnetic vector is along the y axis so,

$$H_y=H_{y0}e^{-z/\delta}\cos(\omega t-z/\delta), \qquad (8^16)$$

then the associated set of conduction currents is obtained by calculating $\mathbf{i}=(1/4\pi)$ curl $\mathbf{H}$ which gives $i_y=i_z=0$ and

$$i_x=-(H_{y0}/4\pi\delta)e^{-z/\delta}[\cos(\omega t-z/\delta)-\sin(\omega t-z/\delta)], \qquad (8^17)$$

where, of course, i_x is in abamp./cm^2 if H is in gauss.

TABLE III. Depth of penetration (δ) in copper.

λ cm	δ cm in copper
1	0.368×10^{-4}
3	0.670×10^{-4}
10	1.22×10^{-4}
30	2.11×10^{-4}
100	3.86×10^{-4}
1000	12.2×10^{-4}

The power instantaneously converted into heat in unit volume is ρci^2 erg/cm^3 sec., so the power loss below unit area at all depths and averaged over a cycle is

$$\rho c\frac{H_{y0}^2}{4\pi\delta^2}\int_0^\infty e^{-2z/\delta}dz=\frac{\pi\delta}{\lambda}\cdot c\frac{\mu H_{y0}^2}{8\pi}. \qquad (8^18)$$

In this expression, $\mu H_{y0}^2/8\pi$ represents magnetic energy density at the surface (erg/cm^3) which multiplied by c gives power per unit area (erg/cm^2 sec.), a small fraction of which, $\pi\delta/\lambda$, represents the power per unit area that is absorbed in the walls.

In view of the extreme smallness of δ, we may neglect the curvature of the walls in all practical work (except very fine wires) and suppose the actual losses in unit area of a wall to be given by (8^18) even where the walls are curved. If this assumption is made, the whole power loss in the walls of a cavity resonator is given by

$$\frac{c\delta}{8\lambda}\int\int\mu\mathbf{H}^2dS, \qquad (8^19)$$

where the integral is extended over the whole bounding surface and $\mathbf{H}$ is the tangential magnetic vector at the surface.

Because of the finite conductivity of the walls, the electric vector is not strictly normal to a metal wall. From the expression for i_x evaluated at $z=0$, we find

$$E_{x0}=\rho i_x=\frac{\sqrt{2}\rho H_{y0}}{4\pi\delta}\sin\left(\omega t-\frac{\pi}{4}\right). \qquad (8^110)$$

Therefore the actual tangential electric field at the surface is small compared with the tangential magnetic field in the ratio of $\sqrt{2}\rho/4\pi\delta$.

As will be seen in later sections, the losses in one cycle in a cavity resonator are small compared to the energy stored in the resonator. For this reason the approximation procedure by which we first find the fields which would exist in case of infinite conductivity and then calculate the losses gives a very good approximation.

9[1]. Resonator Losses

In radio engineering, the losses of an oscillatory system are conventionally measured by giving the Q value, a kind of figure of merit which is an inverse measure of the damping.[7] The quantity Q can be defined by saying that the damping of a free oscillation is such that the amplitude of free oscillation contains a factor

$$e^{-\omega t/2Q}, \tag{9¹1}$$

so the total field energy in the oscillator during free oscillation is

$$W = W_0 e^{-\omega t/Q}, \tag{9¹2}$$

which amounts to saying that

$$Q = 2\pi \frac{\text{Energy stored in oscillating system}}{\text{Energy dissipated in a cycle}}. \tag{9¹3}$$

Using the formula of the preceding section for the losses in a cavity resonator we have, therefore,

$$Q = (2/\delta\mu)\frac{\iiint H^2 dV}{\iint H^2 dS}. \tag{9¹4}$$

For rough order of magnitude ideas, we observe that since $\mathbf{H}$ has a loop at the surface, the mean value of $\mathbf{H}^2$ on the surface will be roughly twice the mean value in the volume, and therefore, very roughly,

$$Q = V/\delta\mu S, \tag{9¹5}$$

where V is the volume and S the bounding area of the resonator. Hence for a resonator whose linear dimensions are large compared with δ we may expect that Q will be of the order of a linear dimension divided by the skin depth. Since the ratio of integrals in (9¹4) has the dimensions of length and δ varies as $\sqrt{\lambda}$, it follows that the Q value of geometrically similar resonators varies as the square root of a linear dimension and hence as the square root of the wave-length of any particular resonant mode.

In practical cavity resonators in the micro-wave region one may expect $Q > 1000$ and therefore the actually existing fields in the cavity are only very slightly different from those calculated on the assumption of perfect conductivity of the walls. In applying (9¹4) one uses therefore the fields as calculated by assuming perfect conductivity. This is the basis of all the calculations of Q that have thus far been made.

As an illustrative example, consider the Q value for the $(0, m, n)$ mode of a rectangular resonator of edges A, B, and C as discussed in Section 3¹. It is easily calculated to be

$$Q = \frac{1}{\delta\mu}\frac{ABC}{BC + 2AC\dfrac{(m/B)^2}{(m/B)^2+(n/C)^2} + 2AB\dfrac{(n/C)^2}{(m/B)^2+(n/C)^2}}. \tag{9¹6}$$

If the prism is square, $B = C$, this reduces to

$$Q = (B/\delta)\frac{A/B}{1+2(A/B)},$$

and for a cube this reduces to $Q = A/3\delta$. For the mode of lowest frequency $\lambda = \sqrt{2}A$ and, therefore,

$$Q = (\lambda/3\sqrt{2}\delta\mu).$$

The Q value of a copper cube filled with material of unit permeability is, therefore,

$$Q = 5920\sqrt{\lambda} = 7040\sqrt{A},$$

so the Q value for a copper cube resonator designed for $\lambda = 10$ cm is $Q = 18{,}800$.

Next we consider how the losses affect the equations of motion of the resonator coordinates $J_a(t)$ introduced in Section 4¹. When we take account of the losses it becomes necessary to introduce a damping term in (4¹10). It is easily verified that the damping is correctly represented if we replace (4¹10) by

$$\ddot{J}_a + (\omega_a/Q_a)\dot{J}_a + \omega^2 J_a = 4\pi c^2 I_a(t). \tag{9¹7}$$

[7] Compare Terman, *Radio Engineering* (McGraw-Hill, New York, 1937), p. 37 *et seq.* and chapter 3.

Multiplying this through by $(V/4\pi c^2)\dot{J}_a$ we find the equation for $\dot{W}_a$ which replaces (4'14):

$$\dot{W}_a = VI_a(t)\dot{J}_a(t) - (V\omega_a/4\pi c^2 Q_a)\dot{J}_a^2. \quad (9'8)$$

The second term on the right, which represents the losses, is essentially negative, since it contains the square of the speed of the resonator coordinate.

In the steady state in which $J_a(t)$ executes a harmonic time variation, the mean rate of conversion of energy into heat due to losses in the cavity walls is, therefore,

$$P = (V\omega_a/4\pi c^2 Q_a)(\dot{J}_a^2/2) = (2\pi^2 Vc/\lambda_a^3 Q_a)(J_a^2/2),$$

in which J_a stands for the amplitude of the sinusoidal variation of $J_a(t)$. If P is to be expressed in watts and J_a in amperes, then we need to write $c = 30$ ohms. The coefficient of $J_a^2/2$ in this expression will be called the resonator resistance R_a of the ath mode. Hence we have,

$$R_a = 60\pi^2 V/Q_a\lambda_a^3 \text{ ohms}. \quad (9'9)$$

Thus, for a cube resonator of edge A with copper walls, we have for the (011) mode, $R_a = 0.0112$ ohms.

Shunt Resistance

Another measure of the losses which is often more convenient, is called the shunt resistance of the resonator. In some calculations of the theory of electronic oscillator tubes involving cavity resonators we like to specify the amplitude, not by J_a, but by the magnitude of the line integral in volts of the electric vector along some particular path through the resonator. We have

$$\mathbf{E} = -(1/c)\dot{J}_a\mathbf{A}_a,$$

so the amplitude of the electric vector is $k_a J_a \mathbf{A}_a$ statvolt/cm if J_a is in abamperes or $30k_a J_a \mathbf{A}_a$ if $\mathbf{E}$ is in volt/cm and J_a is in amperes, so the line integral of $\mathbf{E}$ in volts is

$$V_a = 30k_a J_a \int A_a \cdot \mathbf{ds}. \quad (9'10)$$

We can now calculate the resistance which when shunted across a voltage of this magnitude will dissipate power at the same rate as the actual dissipation in the resonator. This is called the *shunt resistance* and is dependent not only on the resonator and the particular mode of oscillation in question, but also on the particular path in the resonator along which V_a is calculated. Calling the shunt resistance S_a, we have

$$V_a^2/2S_a = (60\pi^2 V/Q_a\lambda_a^3)(J_a^2/2),$$

and, therefore,

$$S_a = 60Q_a\lambda_a \frac{\left(\int \mathbf{A}_a \cdot \mathbf{ds}\right)^2}{\iiint \mathbf{A}_a^2 dV}. \quad (9'11)$$

On substituting for Q_a its expression by (9'4) and using the vector formula in the equation just preceding (4'12) we may write the following expression for the shunt resistance:

$$S_a = \frac{480\pi^2}{\lambda_a\delta\mu} \frac{\left(\int \mathbf{A}_a \cdot \mathbf{ds}\right)^2}{\iint \mathbf{H}^2 dS}. \quad (9'12)$$

By taking the path to be from $z=0$ to $z=A$ on the line $x=A/2$, $y=A/2$, one readily finds, for the (110) mode of a copper cube of edge A, the shunt resistance to be $S = 105600A^{5/2}$ ohms where A is in cm.

E(00p) Modes of Circular Cylinder

It is, of course, not necessary nor even desirable to refer the calculation back to normalized vector potentials when one is seeking explicit results for a particular mode. To illustrate, let us derive the formulas for the $E(00p)$ modes of a circular cylinder.

From (6') we know that

$$E_z = AJ_0(kr) \text{ statvolt/cm},$$

where $kR = X_{0p}$, the pth root of $J_0(x) = 0$. Hence the statvoltage amplitude for a path along the axis from $z=0$ to $z=C$ is

$$V = AC \text{ statvolt}.$$

From (5'8), the magnetic field, except for the time phase which does not enter this calculation, is

$$\mathbf{H} = AJ_1(kr)\boldsymbol{\phi}_0 \text{ gauss}.$$

From $(8^{1}9)$ the power loss on each end is

$$P_E=A^2(\pi c\delta\mu/4\lambda)\int_0^R J_1^2(kr)rdr$$
$$=A^2(\pi c\delta\mu/4\lambda)\frac{R^2}{2}J_1^2(X_{0p}),$$

and the power loss on the cylindrical wall is

$$P_C=A^2(\pi c\delta\mu/4\lambda)RCJ_1^2(X_{0p}),$$

so the total power loss is

$$(P_C+2P_E)=A^2(\pi c\delta\mu/4\lambda)(R^2+RC)J_1^2(X_{0p}).$$

The energy stored is

$$W=\iiint \mathbf{H}^2/8\pi dV=(A^2/8)CR^2J_1^2(X_{0p})\,;$$

therefore from the definition of Q in $(9^{1}3)$ we have

$$Q_p=(2\pi R/\rho\mu)^{\frac{1}{2}}\frac{C}{R+C}(X_{0p})^{\frac{1}{2}}. \qquad (9^{1}13)$$

It is interesting to note the effect of varying C with a fixed value of R. This does not affect the frequencies of the modes, which depend solely on R. For C small compared to R, the Q value is small because the end losses remain as a "fixed charge" although there is very little volume for stored energy. As the height C is increased, Q increases, but approaches a limit as C becomes large compared to R for then the gain in extra field energy stored is offset by the corresponding increase in losses in the extra length of side wall.

The shunt resistance S_p is now obtained by equating $V^2/2S_p$ with V in volts to the total power loss expressed in watts. This gives,

$$S_p=120(2\pi R/\rho\mu)^{\frac{1}{2}}$$
$$\times\frac{C^2}{R^2+RC}\,\frac{1}{(X_{0p})^{\frac{1}{2}}J_1^2(X_{0p})}\text{ ohm.} \qquad (9^{1}14)$$

TABLE IV. Values useful in calculating the shunt resistance S_p.

p	X_{0p}	$J_1(X_{0p})$	$\sqrt{X_{0p}}J_1^2(X_{0p})$
1	2.4048	+0.5191	0.417
2	5.5207	− .3403	.272
3	8.6537	+ .2705	.216
4	11.7915	− .2325	.186
5	14.9309	+ .2065	.165

Comparing the formula for S_p with the one for Q_p we find that they are simply related,

$$S_p=120(C/R)[1/X_{0p}J_1^2(X_{0p})]Q_p\text{ ohm.} \qquad (9^{1}15)$$

In making numerical applications of $(9^{1}14)$ Table IV is useful. Here X_{0p} is the pth root of $J_0(x)=0$ as in Section 6^{1}.

Coaxial Cable Modes

Another example that is important in practice is the circular coaxial cable of inner radius $r=a$ and outer radius $r=b$, terminated by $z=0$ and $z=C$. The fields have been discussed in (5^{1}) and are given by $(5^{1}25)$. From $(5^{1}25)$, inserting the dependence on z which is not explicitly written there, we have,

$$\mathbf{H}=(A/r)\cos n\pi z/C\phi_0\text{ gauss.}$$

By applying $(8^{1}9)$ we find the losses in either end are

$$P_E=A^2(\pi c\delta\mu/4\lambda)\log b/a,$$

and the losses on the inner and outer walls are, respectively.

$$P_I=A^2(\pi c\delta\mu/4\lambda)C/2a,\quad P_O=A^2(\pi c\delta\mu/4\lambda)C/2b.$$

Hence the total power loss is

$$(P_I+P_O+2P_E)=A^2(\pi c\delta\mu/4\lambda)$$
$$\times(2\log b/a+C/2a+C/2b).$$

The energy stored is

$$W=(A^2/8)C\log b/a,$$

and, therefore, from the definition of Q we have

$$Q_n=(2C/\rho\mu)^{\frac{1}{2}}\frac{2\pi\log b/a}{4\log b/a+C/a+C/b}\sqrt{n}. \qquad (9^{1}16)$$

The most natural path with respect to which we may define the shunt resistance is from $r=a$ to $r=b$ in a plane of constant z for which $\sin n\pi z/C=\pm1$, that is, at a voltage loop. On any such path

$$V=A\log b/a\text{ statvolt.}$$

Calculating the shunt resistance S_n from this expression and that for the power loss, we get, in ohms,

$$S_n = 120(2C/\rho\mu)^{\frac{1}{2}} \frac{2(\log b/a)^2}{4 \log b/a + C/a + C/b} \frac{1}{\sqrt{n}}. \quad (9^1 17)$$

Hence the relation between S_n and Q_n which is the analogue of ($9^1 15$) is

$$S_n = (120 Q_n/\pi n) \log b/a \text{ ohms.} \quad (9^1 18)$$

Exercise: Consider a resonator for which $C = 150$ cm, and hence the wave-length of the lowest resonant frequency is 3 meters. Suppose $a = 30$ cm and $b = 45$ cm, what power will be required in a copper resonator to get a voltage amplitude, at a voltage loop, of one million volts? *Answer:* 1720 kilowatts.

10^1. Spherical Resonators

The theory of the modes of oscillation of a spherical cavity resonator may be developed as a special case of a method which is capable of more general application. Use an orthogonal curvilinear coordinate system x_1, x_2, x_3 such that the line element is

$$ds^2 = e_1^2 dx_1^2 + e_2^2 dx_2^2 + e_3^2 dx_3^2. \quad (10^1 1)$$

For example, for spherical polar coordinates we have

$$ds^2 = dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\varphi^2. \quad (10^1 2)$$

and hence

$$x_1 = r, \quad x_2 = \theta, \quad x_3 = \varphi; \quad e_1 = 1, \quad e_2 = r, \quad e_3 = r \sin \theta.$$

In such a general curvilinear coordinate system the curl equations of ($5^1 1$) become:

$$\begin{aligned}
ike_2e_3E_1 &= (\partial/\partial x_2)(e_3H_3) - (\partial/\partial x_3)(e_2H_2), \\
ike_3e_1E_2 &= (\partial/\partial x_3)(e_1H_1) - (\partial/\partial x_1)(e_3H_3), \\
ike_1e_2E_3 &= (\partial/\partial x_1)(e_2H_2) - (\partial/\partial x_2)(e_1H_1), \\
-ike_2e_3H_1 &= (\partial/\partial x_2)(e_3E_3) - (\partial/\partial x_3)(e_2E_2), \\
-ike_3e_1H_2 &= (\partial/\partial x_3)(e_1E_1) - (\partial/\partial x_1)(e_3E_3), \\
-ike_1e_2H_3 &= (\partial/\partial x_1)(e_2E_2) - (\partial/\partial x_2)(e_1E_1).
\end{aligned} \quad (10^1 3)$$

Suppose further that the choice of x_1 is such that $e_1 = 1$, as is the case with ordinary spherical polar coordinates, and, moreover, that the coordinate system is such that e_2/e_3 is independent of x_1. We find now that the modes fall into two types, each derivable from a scalar wave function. The two modes will be called E type and H type, corresponding to the classification already introduced in (5^1).

For the E type modes $H_1 = 0$. The fourth of ($10^1 3$) is satisfied if we write,

$$e_2E_2 = (\partial P/\partial x_2), \quad e_3E_3 = (\partial P/\partial x_3).$$

If we write $P = (\partial U/\partial x_1)$ then the second and third of ($10^1 3$) give

$$H_2 = (ik/e_3)(\partial U/\partial x_3), \quad H_3 = -(ik/e_2)(\partial U/\partial x_2).$$

The first and fifth of ($10^1 3$) give two different expressions for E_1 in terms of U. The condition that these be consistent leads to an equation for U:

$$\frac{\partial^2 U}{\partial x_1^2} + \frac{1}{e_2e_3}\left(\frac{\partial}{\partial x_2}\frac{e_3}{e_2}\frac{\partial U}{\partial x_2} + \frac{\partial}{\partial x_3}\frac{e_2}{e_3}\frac{\partial U}{\partial x_3}\right) + k^2 U = 0, \quad (10^1 4)$$

the most convenient expression for E_1 in terms of U being

$$E_1 = k^2 U + (\partial^2 U/\partial x_1^2).$$

With these choices the sixth of ($10^1 3$) is satisfied identically.

In a similar way the H type modes are given by putting $E_1 = 0$. This leads to a similar scheme of equations for deriving field components from a scalar function U which satisfies the same wave equation ($10^1 4$). To summarize, the equations for the field components in terms of U are:

E type:

$$\begin{aligned}
E_1 &= k^2 U + \frac{\partial^2 U}{\partial x_1^2}, & H_1 &= 0, \\
E_2 &= \frac{1}{e_2}\frac{\partial^2 U}{\partial x_1 \partial x_2}, & H_2 &= \frac{ik}{e_3}\frac{\partial U}{\partial x_3}, \\
E_3 &= \frac{1}{e_3}\frac{\partial^2 U}{\partial x_1 \partial x_3}, & H_3 &= -\frac{ik}{e_2}\frac{\partial U}{\partial x_2}.
\end{aligned} \quad (10^1 5)$$

H type:

$$\begin{aligned}
E_1 &= 0, & H_1 &= k^2 U + \frac{\partial^2 U}{\partial x_1^2}, \\
E_2 &= -\frac{ik}{e_3}\frac{\partial U}{\partial x_3}, & H_2 &= \frac{1}{e_2}\frac{\partial^2 U}{\partial x_1 \partial x_2}, \\
E_3 &= +\frac{ik}{e_2}\frac{\partial U}{\partial x_2}, & H_3 &= \frac{1}{e_3}\frac{\partial^2 U}{\partial x_1 \partial x_3}.
\end{aligned} \quad (10^1 6)$$

Now we may specialize this general method to the case of a sphere, using spherical polar coordinates as in (10·2). Equation (10·4) for U becomes,

$$\frac{\partial^2 U}{\partial r^2}+\frac{1}{r^2}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial U}{\partial\theta}\right)+\frac{1}{\sin^2\theta}\frac{\partial^2 U}{\partial\varphi^2}\right)+k^2U=0. \tag{10·7}$$

The general solution of this is of the form

$$U=R(r)\Theta(\theta,\varphi),$$

where

$$\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Theta}{\partial\theta}\right)+\frac{1}{\sin^2\theta}\frac{\partial^2\Theta}{\partial\varphi^2}+l(l+1)\Theta=0 \tag{10·8}$$

and

$$\frac{d^2R}{dr^2}+\left(k^2-\frac{l(l+1)}{r^2}\right)R=0. \tag{10·9}$$

TABLE V. Values of $j_l(x)$.

l	$j_l(x)$
0	$\sin x/x$
1	$\sin x/x^2-\cos x/x$
2	$(3/x^3-1/x)\sin x-(3/x^2)\cos x$
3	$(15/x^4-6/x^2)\sin x-(15/x^3-1/x)\cos x$

The quantity l assumes integral values for solutions of (10·8) that are finite and single-valued in all directions. Any solution of (10·8) is known as a spherical harmonic and a great deal of information can be found about them in books on harmonic analysis. The solutions of (10·9) will be called spherical Bessel functions, from a terminology introduced by Morse.

$$R(r)=(kr)z_l(kr), \tag{10·10}$$

where

$$z_l(x)=(\pi/2x)^{\frac{1}{2}}Z_{l+\frac{1}{2}}(x).$$

For any spherical Bessel function we have,

$$z_{n-1}+z_{n+1}=(2n+1/x)z_n,$$

$$(d/dx)z_n(x)=[nz_{n-1}-(n+1)z_{n+1}]/(2n+1), \tag{10·11}$$

$$(d/dx)x^{n+1}z_n=x^{n+1}z_{n-1},$$

$$(d/dx)x^{-n}z_n=-x^{-n}z_{n+1}.$$

It is necessary to specialize to the particular Bessel function which is finite at $r=0$. These are denoted by $j_l(x)$. The functions j_lx are given in Table V.

The spherical harmonics will be used in the following notation:

$$\Theta(\theta,\varphi)=\Theta(l,m)e^{im\varphi}, \tag{10·12}$$

where

$$m\geq 0\begin{cases}\Theta(l,m)=(-1)^m\left[(2l+1)\dfrac{(l-m)!}{(l+m)!}\right]^{\frac{1}{2}}\\ \qquad\qquad\times\sin^m\theta\dfrac{d^m}{d(\cos\theta)^m}P_l(\cos\theta)\\ \Theta(l,-m)=+\text{same expression}\end{cases}$$

in which $P_l(\cos\theta)$ is the lth Legendre polynomial.

A list of explicit expressions for some of the spherical harmonics is given below:

$$\Theta(0,0)=1,$$

$$\Theta(1,0)=\sqrt{3}\cos\theta,$$

$$\Theta(2,0)=\sqrt{5}/2(3\cos^2\theta-1),$$

$$\Theta(3,0)=\sqrt{7}/2(2\cos^3\theta-3\cos\theta\sin^2\theta).$$

The coefficients appearing here are chosen to normalize in such a way that

$$\int\int|\Theta|^2\sin\theta d\theta d\varphi=4\pi,$$

the integral extending over all directions in space.

In calculations involving the spherical harmonics the following properties of them are useful,

$$\frac{\partial}{\partial\theta}\Theta(l,m)=\tfrac{1}{2}[(l-m)(l+m+1)]^{\frac{1}{2}}\Theta(l,m+1)-\tfrac{1}{2}[(l+m)(l-m+1)]^{\frac{1}{2}}\Theta(l,m-1),$$

$$\cos\theta\,\Theta(l,m)=\Theta(l+1,m)\left[\frac{(l+1-m)(l+1+m)}{(2l+1)(2l+3)}\right]^{\frac{1}{2}}+\Theta(l-1,m)\left[\frac{(l-m)(l+m)}{(2l-1)(2l+1)}\right]^{\frac{1}{2}},\tag{10'13}$$

$$\sin\theta\,\Theta(l,m)=-\Theta(l+1,m+1)\left[\frac{(l+m+1)(l+m+2)}{(2l+1)(2l+3)}\right]^{\frac{1}{2}}+\Theta(l-1,m+1)\left[\frac{(l-m)(l-m-1)}{(2l-1)(2l+1)}\right]^{\frac{1}{2}}$$

$$=\Theta(l+1,m-1)-\Theta(l-1,m-1).$$

The final result is that the E and H modes of the sphere may be derived from the following expression for U:

$$U=kr\cdot j_l(kr)\Theta(l,m)e^{im\varphi},\tag{10'14}$$

with $|m|\leqslant l$, and $l=0, 1, 2, 3\cdots$.

The modes of *electric type* are given by applying 10'5:

$$E_r=(k^2/r^2)l(l+1)U,$$

$$E_\theta=-\frac{k}{r}\frac{\partial}{\partial(kr)}[krj_l(kr)]\cdot\frac{\partial}{\partial\theta}\Theta(l,m)\cdot e^{im\varphi},$$

$$E_\varphi=\frac{ikm}{r\sin\theta}\frac{\partial}{\partial(kr)}[krj_l(kr)]\Theta(l,m)e^{im\varphi},\tag{10'15}$$

$$H_r=0,$$

$$H_\theta=\frac{-km}{r\sin\theta}U,$$

$$H_\varphi=-\frac{ik}{r}[krj_l(kr)]\frac{\partial}{\partial\theta}\Theta(l,m)e^{im\varphi}.$$

The boundary conditions require that $E_\theta=E_\varphi=0$ at $r=R$, if R is the radius of the sphere. Hence for the E type modes we must have

$$kR=S_{nl},\tag{10'16}$$

where S_{nl} is a root of the equation

$$(d/dx)(xj_l(x))=0.$$

Similarly for modes of *magnetic type:*

$$E_r=0,$$

$$E_\theta=(km/r\sin\theta)U,$$

$$E_\varphi=\frac{ik}{r}krj_l(kr)\frac{\partial}{\partial\theta}\Theta(l,m)e^{im\varphi},$$

$$H_r=\frac{k^2}{r^2}l(l+1)U,\tag{10'17}$$

$$H_\theta=-\frac{k}{r}\frac{\partial}{\partial(kr)}[krj_l(kr)]\frac{\partial}{\partial\theta}\Theta(l,m)e^{im\varphi},$$

$$H_\varphi=\frac{ikm}{r\sin\theta}\frac{\partial}{\partial(kr)}[krj_l(kr)]\Theta(l,m)e^{im\varphi}.$$

For the modes of magnetic type the boundary conditions require that

$$kR=T_{nl},\tag{10'18}$$

where T_{nl} is a root of the equation $j_l(x)=0$.

The modes will be designated by the notation $E(n,l,m)$ and $H(n,l,m)$. Inspection of (10'15) and (10'17) shows that there is no solution corresponding to $l=0$, $m=0$, hence the least value of l is unity. Since the roots S_{nl} and T_{nl} are independent of m, it follows that there are $(2l+1)$ modes of either E or H type going with a particular (n,l) all of which have the same frequency. This degeneracy arises from the spherical symmetry.

The values of the roots for the fundamental modes of each type are

$$S_{11}=2.74,\quad T_{11}=4.49.\tag{10'19}$$

Therefore, the resonant wave-length for the $E(11)$ modes is $2.29R$.

The spherical harmonics introduced in (10'12) are appropriate for problems involving the complete sphere for they are finite at $\theta=0$ and $\theta=\pi$, the singular points of (10'8). Spherical coordinates may also be used to discuss the fields in a conical resonator bounded by $\theta=\theta_0$ as well as $r=R$, or in a region consisting of a sphere with two conical dimples cut out of it, that is the region $0<r<R$ and $\theta_0<\theta<\theta_1$. To deal with such cases it is necessary to introduce more general solutions of (10^18) which have singularities at the excluded poles 0 or π for θ.

Sphere with Conical Dimples

As a specific example consider the fundamental electric mode for the sphere with conical dimples.[8] For this the appropriate U function is

$$U=\log\tan(\theta/2)\cdot(\sin kr/kr) \qquad (10^120)$$

which would not be admissible for a complete sphere because of its logarithmic singularities. Since this corresponds to an $l=0$ solution, it follows that $E_r=0$, as well as $H_r=0$. Applying (10^115) we find that the only non-vanishing field components are

$$\begin{aligned} E_\theta &= k^2(1/\sin\theta)(\cos kr)/kr, \\ H_\varphi &= -ik^2(1/\sin\theta)(\sin kr)/kr. \end{aligned} \qquad (10^121)$$

Since E_r and E_φ vanish everywhere the only boundary condition to be imposed is that $E_\theta=0$ at $r=R$, which requires that

$$kR=(n+\tfrac{1}{2})\pi$$

independently of the location of the angular boundaries. The lowest mode is that for which $n=0$ and for this the wave-length is exactly equal to four times the radius of the sphere.

For this solution E_θ becomes infinite as r approaches zero, but in such a way that the line integral of E along a path of constant r from one dimple to the other is finite. The solution is therefore appropriate for representation of the fields which exist when the apices of the two conical dimples are not quite in electrical contact.

Exercises: 1. Show that the Q value for this mode of the dimpled spherical resonator is

$$Q=\left(\frac{4R}{\rho\mu}\right)^{\frac{1}{2}}\frac{\log\tan\theta_1/2-\log\tan\theta_0/2}{\log\tan\theta_1/2-\log\tan\theta_0/2+I(\csc\theta_1+\csc\theta_0)}$$

in which

$$I=\int_0^{\pi/2}\frac{\sin^2 x}{x}dx=0.825.$$

Show that if $\theta_1=\pi-\theta_0$, the Q value as a function of θ_0 has a maximum at about $\theta_0=34°$.

2. Show that the shunt resistance for this mode, with voltage measured from apex to apex is

$$S=120\left(\frac{4R}{\rho\mu}\right)^{\frac{1}{2}}\frac{\log^2(\tan\theta_1/2/\tan\theta_0/2)}{\log\dfrac{\tan\theta_1/2}{\tan\theta_0/2}+I(\csc\theta_1+\csc\theta_0)}.$$

Show that for $\theta_1=\pi-\theta_0$, the shunt resistance as a function of θ_0 has a maximum at about 9°.

CHAPTER II. TRANSMISSION LINES[1]

In low frequency radio work, power is transmitted from one place to another by transmission lines consisting of two conductors, such as parallel wire lines, or coaxial cable. Such lines also play a great role in micro-wave radio. In addition it becomes practical at the shorter wave-lengths to transmit power through hollow pipes. In the literature it has been customary to call two-conductor lines transmission lines and to call hollow pipes wave guides. This chapter deals with two-conductor lines while the properties of hollow wave guides will be developed in Chapter III.

1². Two-Conductor Transmission Lines

The commonest form of two-conductor transmission line is the coaxial cable, consisting of an inner circular conductor of radius $r=a$, and an outer circular conductor of radius $r=b$. The theory is very closely related to that of coaxial cavity resonators discussed in (5^1) under the subhead, double-walled resonators.

Let z be the coordinate along the length of the line and suppose any section by a plane $z=$ constant, gives a region bounded by two curves C_1 and C_2, the latter enclosing the former, as in Fig. 4^1.

[8] W. W. Hansen and R. D. Richtmyer, J. App. Phys. 10, 189 (1939).

[1] The general literature on electrical transmission lines is very extensive since this topic is important for long power transmission lines as well as in telegraphy and telephony. In this chapter a brief account of the subject is given from the point of view of micro-wave applications.

We seek solutions of ($5^1 1$) in which the dependence on z is given by a factor $\exp(-ik_3 z)$, which when combined with the time factor $e^{i\omega t}$, represents a progressive wave moving in the $+z$ direction. The phase velocity v_p is given by

$$v_p = \omega / k_3. \qquad (1^2 1)$$

If one writes $k = \omega/c$ and assumes $E_z = H_z = 0$, then Eqs. ($5^1 1$) are found to give the following for the factors which represent the dependence of the field components on x and y,

$$\begin{aligned} kE_x &= k_3 H_y, \qquad kE_y = -k_3 H_x, \\ & 0 = (\partial H_y/\partial x) - (\partial H_x/\partial y), \\ -kH_x &= k_3 E_y, \qquad -kH_y = -k_3 E_x, \\ & 0 = (\partial E_y/\partial x) - (\partial E_x/\partial y). \end{aligned}$$

The z component of these shows that $\mathbf{E}$ or $\mathbf{H}$ may be expressed as the gradient of a scalar function, $u(x, y)$. Write

$$\begin{aligned} \mathbf{E}_s &= -\text{grad } u(x, y), \\ -k\mathbf{H}_s &= k_3 \mathbf{k} \times \text{grad } u(x, y), \end{aligned} \qquad (1^2 2)$$

which are the transmission line analogues of ($5^1 19$). These equations imply that $k_3 = \pm k$, hence, the phase velocity of the waves is $\pm c$. The z component of the equation for curl $\mathbf{H}$ requires that $u(x, y)$ satisfy Laplace's equation in the cross section

$$\nabla^2 u(x, y) = 0. \qquad (1^2 3)$$

Since this is the same as ($5^1 20$) with the same boundary conditions, namely u = constant on C_1 and C_2 it follows that the discussion following ($5^1 20$) is applicable here.

Suppose $u(x, y)$ is a solution of the boundary value problem such that the coordinates (x, y) are periodic functions with period 2π in the conjugate harmonic function $v(x, y)$, as in ($5^1 26$). Let the values of u corresponding to the inner and outer conductors be u_1 and u_2, respectively. Then for a wave propagated in the $+z$ direction we have

$$\begin{aligned} \mathbf{E}_s &= -\text{grad } u(x, y) \cos(\omega t - kz), \\ \mathbf{H}_s &= -\mathbf{k} \times \text{grad } u(x, y) \cos(\omega t - kz), \end{aligned} \qquad (1^2 4)$$

and for a wave propagated in the $-z$ direction,

$$\begin{aligned} \mathbf{E}_s &= -\text{grad } u(x, y) \cos(\omega t + kz), \\ \mathbf{H}_s &= +\mathbf{k} \times \text{grad } u(x, y) \cos(\omega t + kz). \end{aligned} \qquad (1^2 5)$$

Characteristic Impedance

As remarked just before Eq. ($5^1 28$), the solution in which (x, y) have the period 2π in v, corresponds to a current amplitude of $\frac{1}{2}$ abampere in the inner and outer conductors, and to a statvoltage amplitude $(u_2 - u_1)$ in the line integral of $\mathbf{E}$ from one conductor to the other in a plane of constant z. Hence the ratio of voltage amplitude to current amplitude in amperes is

$$Z = 60(u_2 - u_1) \text{ ohm}. \qquad (1^2 6)$$

This quantity is called the characteristic impedance, or surge impedance of the transmission line. The surge impedance of the line, as thus defined, is therefore the same as the impedance of the double-walled cylindrical cavity resonator as introduced in Section 5^1.

Another way of looking at the surge impedance may help to bring out more clearly its physical significance. Let C_1 be the capacity per unit length of the condenser formed by the two conductors of the line. Suppose one of them is at potential O and the other at V statvolt. Then the charge per unit length is $C_1 V$ e.s.u./cm. If now the line is to be fed in such a way as to set up on it a wave travelling from left to right with speed c, then at the input end one must supply current which will keep the charge on each conductor at its requisite amount. This is a current $cC_1 V$ e.s.u./sec. or $C_1 V$ abamp. Hence the input impedance in ohms, the ratio of voltage to current in amperes, is

$$Z = (300 V / 10 C_1 V) = 30 / C_1 \text{ ohms}. \qquad (1^2 7)$$

If the space between the conductors is filled with a medium whose constants are (ϵ, μ) it is easy to see that the input impedance of the line is

$$Z = (\mu/\epsilon)^{\frac{1}{2}} \cdot 30 / C_1 \text{ ohm}, \qquad (1^2 8)$$

in which C_1 is the geometrical capacity per unit length in the absence of the medium.

For circular coaxial cable, the capacity per unit length is

$$C_1 = 1/(2 \log b/a) \qquad (1^2 9)$$

and, therefore, the surge impedance of such a transmission line is

$$Z=(\mu/\epsilon)^{\frac{1}{2}}\,60\log b/a. \tag{1²10}$$

Likewise for coaxial confocal elliptic cylinders of focal length f and inner and outer semi-major axes a and b, as in (5¹27), the surge impedance is

$$Z=(\mu/\epsilon)^{\frac{1}{2}}60[\cosh^{-1}(b/f)-\cosh^{-1}(a/f)]. \tag{1²11}$$

Transmission lines in which one conductor completely surrounds the other are to be preferred to "open" lines like a pair of parallel wires, because they are self-shielding and do not interact with nearby conductors. The theory developed in this section is, however, equally applicable to open lines in which the curves C_1 and C_2, which bound the conductors, lie external to each other. The result in (1²8) is applicable in this case as well, it being supposed that the line is "balanced to ground," that is, that the potential of C_2 is as much negative with respect to distant points as C_1 is positive.

An important special case is the pair of round wires each of radius a, whose center-to-center distance is d, for which

$$C_1=1/(4\cosh^{-1}d/2a)$$

so

$$Z=(\mu/\epsilon)^{\frac{1}{2}}\,120\cosh^{-1}d/2a. \tag{1²12}$$

Another example is the parallel plate transmission line, made of two plates whose width is b, separated by a distance d which is small compared with the width. For such a line

$$C_1=d/4\pi b \quad \text{so} \quad Z=(\mu/\epsilon)^{\frac{1}{2}}\,120\pi d/b. \tag{1²13}$$

Exercise: What separation between the surfaces of the wires of a parallel wire transmission line is required to make the surge impedance of the line equal to 73 ohms? *Answer*: About 19 percent of the diameter of a wire.

What should be the ratio of the radii of a coaxial cable in order to give a surge impedance of 73 ohms? *Answer*: $b/a=3.36$.

(The point of these questions is that the radiation resistance of a half-wave dipole is approximately 73 ohms.)

Transmission Line Equations

We have developed the theory of transmission lines from the point of view of the field theory.

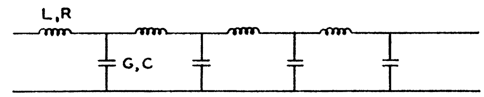


FIG. 1². Equivalent lumped-constant circuit of a transmission line.

In the engineering literature[2] the subject is usually approached as an extension of the theory of networks having lumped constants. This method will now be briefly presented in order to compare it with what has gone before. (See Fig. 1².)

The line is regarded as equivalent to the limiting case of a circuit of the type shown, in which the meshes are assigned smaller parameters and more meshes are put in per unit length in such a way that, for example, the inductance per mesh multiplied by the number of meshes per unit length approaches a definite limit L, the inductance per unit length. A similar situation exists for the resistance R, the conductance G and the capacitance C per unit length.

If $V(z, t)$ is the potential difference (volts) of the upper line with respect to a point on the lower at the same z, and if $I(z, t)$ is the current (amperes) flowing toward the right in the upper line and toward the left in the lower line, then we must have

$$(\partial V/\partial z)=-RI-L(\partial I/\partial t), \tag{1²14}$$

where R and L are resistance (ohms) and inductance (henries) per unit length. Similarly if G and C are conductance (mhos) and capacitance (farads) per unit length then

$$(\partial I/\partial z)=-GV-C(\partial V/\partial t). \tag{1²15}$$

These two equations form the basis of the circuit theory approach to transmission line theory. The field theory treatment given in the first part of this section corresponds to the ideal case in which R and G are negligible.

[2] For a good elementary introduction see Everitt, *Communication Engineering* (McGraw-Hill, New York, 1937), chapters 4 and 5. Also Guillemin, *Communication Networks* (John Wiley, New York, 1935), Vol. 2. Some important recent papers are: Nergaard, RCA Rev. **3**, 156 (1938); Nergaard and Salzberg, Proc. I. R. E. **27**, 579 (1939); Reukema, Elec. Eng. **56**, 1002 (1937); King, Proc. I. R. E. **23**, 885 (1935); Mason and Sykes, Bell Sys. Tech. J. **16**, 275 (1938).

Before going on to discuss solutions of (1^{2}14) and (1^{2}15) it is desirable to connect their derivation with the field theory. In the first place we speak of "potential difference" between the two lines. Yet we know that a rapidly-varying electric field is not derivable from a scalar potential. We can remove this ambiguity by agreeing that $V(z, t)$ means the line integral of $\mathbf{E}(x, y, z, t)$ on a path from one line to the other, in a plane of constant z. Since we have seen that in such a plane the $\mathbf{E}_s$ is derivable from a scalar potential (1^{2}2) there is no need further to specify the path in the plane of constant z.

The connection of "current" with the field quantities is to be understood as follows. We take the line integral of $\mathbf{H}$ around a closed path surrounding either line in a plane of constant z and very close to one of the conductors. From the equation curl $\mathbf{H}=4\pi\mathbf{i}+(1/c)\dot{\mathbf{D}}$ we have

$$\int \mathbf{H}\cdot d\mathbf{s}=\int\int \operatorname{curl} \mathbf{H}\cdot d\mathbf{S}=4\pi\int\int \mathbf{i}\cdot d\mathbf{S}.$$

The line integral is over the path just described; the surface integral is on a plane of constant z bounded by this path. The displacement current makes no contribution to this line integral because it is everywhere normal to the conductor. Hence the total current $I(z, t)$ in a conductor is

$$I(z, t)=(1/4\pi)\int \mathbf{H}\cdot d\mathbf{s}, \qquad (1^216)$$

where I is in abamperes if H is in gauss.

As to inductance per unit length, that is to be understood as follows. The spatial distribution of magnetic field in the space between the conductors is the same for the high frequency case as it is for the direct current. The magnetic field energy stored between z and $z+dz$ can be regarded as $\int\int\int(\mathbf{H}^2/8\pi)dv$ in the space between these planes. Equating this to $(Ldz)I^2/2$ we obtain a suitable precise definition of L, the inductance per unit length. If I is in abamperes and the energy is in ergs then L is a pure number.

In the same way the capacitance per unit length is related to the electric field energy stored between the planes z and $z+dz$ by the relation,

$$(Cdz)\,V^2/2=\int\int\int(E^2/8\pi)dv.$$

If I is in statvolts and the energy is in ergs, then C is a pure number. It can be shown that $LC=1$ at frequencies such that the magnetic flux in the conductors is negligible. The resistance per unit length has to be defined with due regard to the skin effect and is the sum of the resistance per unit length in each of the two lines. The conductance per unit length arises from the dissipative characteristic of the dielectric as discussed further in Section 5^2.

Voltage and Current Distribution

We look now for a solution of (1^{2}14) and (1^{2}15) in which the dependence of V and I on position is that associated with progressive simple harmonic waves, hence,

$$V=Ve^{i(\omega t-kz)}, \quad I=Ie^{i(\omega t-kz)}. \qquad (1^217)$$

Substituted in (1^{2}14) and (1^{2}15) this gives for the voltage and current amplitudes

$$ikV=(R+i\omega L)I, \quad ikI=(G+i\omega C)V. \qquad (1^218)$$

This pair of equations leads to non-vanishing values of V and I only if the propagation constant k have the value

$$\begin{aligned} k^2&=\omega^2(L-iR/\omega)(C-iG/\omega)\\ &=-(R+i\omega L)(G+i\omega C). \end{aligned} \qquad (1^219)$$

In case the line is without loss, so that $R=0$ and $G=0$, this reduces to

$$k=\omega(LC)^{\frac{1}{2}}, \qquad (1^220)$$

and the waves are propagated without attenuation in either direction and with the phase velocity $1/(LC)^{\frac{1}{2}}$. From the definitions of L and C it follows that this is equal to c, the velocity of light.

In the general case of a line with loss, Eq. (1^{2}19) leads to a complex value of k which means simply that the wave is attenuated in being propagated along the line. In general the magnitude of the attenuation (measured by the imaginary part of k) depends on the frequency, and the line introduces distortion in transmitting a signal which is not a monochromatic wave. However, in the special case that $LG=RC$ it is easily seen that the attenuation is independent of ω and the real part of k is proportional to ω, hence, the phase velocity is the same for all frequencies. Such a line is called distortionless.

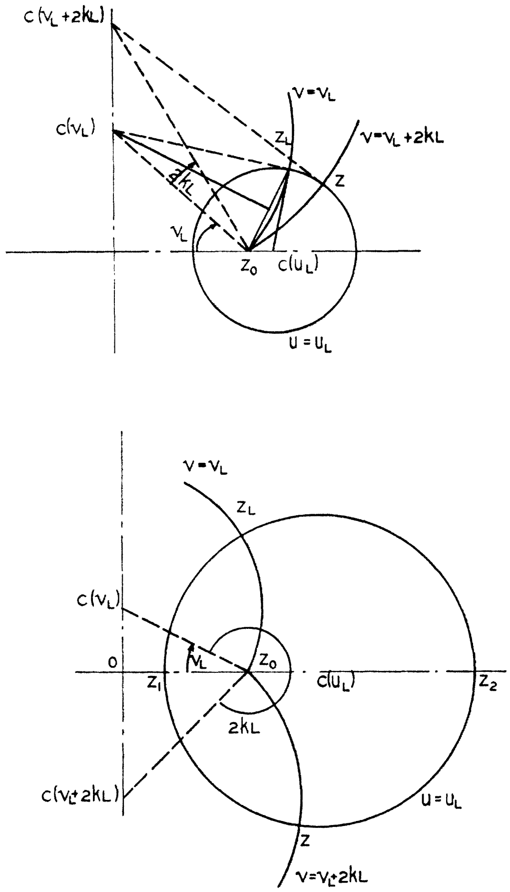


FIG. 2²A. Geometrical construction for determining Z_0 and kL, given Z_0 and Z_L. Detailed construction for one case is shown in the upper figure.

Further developments of the theory along these lines are of the greatest importance in power engineering for long-distance transmission of electric power, and in telephony at audio- or carrier frequencies. For that reason the theory has had a very thorough practical development which can be found in standard textbooks and will not be fully developed here.

2². Transmission Line with Load

Consider a transmission line terminated at $z=L$ by a load of arbitrary impedance Z_L. In general, waves will exist on the line which are travelling both to and from the load. These interfere with each other, producing a standing wave system superposed on a progressive wave. In consequence, the ratio of voltage to current is different at different points on the line. Suppose the line characteristic impedance is Z_0.

If the voltage amplitudes of the waves travelling toward $+z$ and $-z$ are V_1 and V_2, respectively, then the voltage at any point z is the real part of

$$V=(V_1e^{-ikz}+V_2e^{+ikz})e^{i\omega t}, \qquad (2^21)$$

and the total current flowing in the line at z is

$$I=(1/Z_0)(V_1e^{-ikz}-V_2e^{+ikz})e^{i\omega t}. \qquad (2^22)$$

Note that V_1 and V_2 are in general complex numbers.

At $z=L$, where the line is terminated in the impedance Z_L we must have $V=Z_LI$ so

$$Z_0\frac{V_1e^{-ikL}+V_2e^{+ikL}}{V_1e^{-ikL}-V_2e^{+ikL}}=Z_L. \qquad (2^23)$$

At the other end of the line $z=0$, the input impedance Z is the ratio of V to I so

$$Z_0(V_1+V_2)/(V_1-V_2)=Z. \qquad (2^24)$$

Equation (2²3) determines the ratio V_2/V_1 of reflected to incident waves. Solving for this ratio we have,

$$(V_2e^{ikL}/V_1e^{-ikL})=(Z_L-Z_0)/(Z_L+Z_0). \qquad (2^25)$$

Here V_2e^{ikL} is the amplitude of the reflected wave at $z=L$ and V_1e^{-ikL} is the amplitude of the incident wave at the load. From (2²5) we see that $V_2=0$ if $Z_L=Z_0$, that is, the reflected wave vanishes if the load impedance matches that of the line.

It is convenient to introduce an auxiliary quantity ψ by the defining relation

$$V_2/V_1=-e^{-2\psi}, \qquad (2^26)$$

in terms of which we note that (2²3) and (2²4) can be written

$$Z_L=Z_0\tanh(\psi-ikL), \qquad (2^27)$$
$$Z=Z_0\tanh\psi. \qquad (2^28)$$

If we write

$$Z=Z_0\tanh(u+iv),$$
$$Z_L=Z_0\tanh(u_L+iv_L),$$

we see that

$$u+iv=\psi \quad\text{and}\quad u_L+iv_L=\psi-ikL,$$

and, therefore,

$$u = u_L \quad \text{and} \quad V = v_L + kL. \qquad (2^29)$$

Suppose now that on an impedance plane, $Z = R + iX$, we plot the two mutually orthogonal families of curves corresponding to u = constant and v = constant. The load impedance Z_L will correspond to a pair of values u_L, v_L. From (2^29) we see that the impedance transformation produced by putting in an electrical length kL of the transmission line corresponds to a displacement along the curve $u = u_L$ from the point $v = v_L$ to the point $v = v_L + kL$. It is therefore of great importance to learn more about the curves defined by the transformation

$$Z = Z_0 \tanh (u + iv) = Z_0 \frac{\tanh u + i \tan v}{1 + i \tanh u \tan v}.$$

The curve $u = 0$, gives $Z = iZ_0 \tan v$, hence, Z sweeps out the imaginary axis as v increases from 0 to π. For u infinite we have $Z = Z_0$ for all values of v, and the "curve" has shrunk to a point. For $v = 0$, we have $Z = Z_0 \tanh u$ which sweeps over the part of the real axis between 0 and Z_0 as u increases from 0 to infinity. For $v = \pi/2$, we have $Z = Z_0 \coth u$ which sweeps over the real axis from infinity to Z_0 as u increases from 0 to infinity. Therefore the curve u = constant intersects the real axis at two points, namely ($Z_0 \tanh u$, 0) and ($Z_0 \coth u$, 0). The curve u = constant is, in fact, a circle whose center is at

$$(Z_0 \coth 2u,\ 0)$$

and whose radius is $Z_0/\sinh 2u$.

Similarly the curves v = constant form an orthogonal family of circles with center at

$$(0,\ -Z_0 \cot 2v)$$

and with radius equal to $Z_0/\sin 2v$.

Suppose we are given Z_L and wish to determine what kind of line, as regards the value of Z_0, and how much, given by kL, should be introduced in order to transform to a given input impedance Z. In Fig. 2^2A, we draw the perpendicular bisector of the line Z_LZ. Its intersection with the real axis will be the center of the circle $u = u_L$ along which the transformation proceeds as various lengths of line of the as yet unknown Z_0 are introduced. This circle will intersect the real axis in two points, the product of whose abscissas is equal to the square of Z_0. This enables the calculation of Z_0 after which it can be plotted on the diagram. With Z_0 known we can now carry out the construction indicated in detail in the upper part of Fig. 2^2A which permits us to locate $C(v_L)$ and $C(v_L + kL)$, the centers of the circles

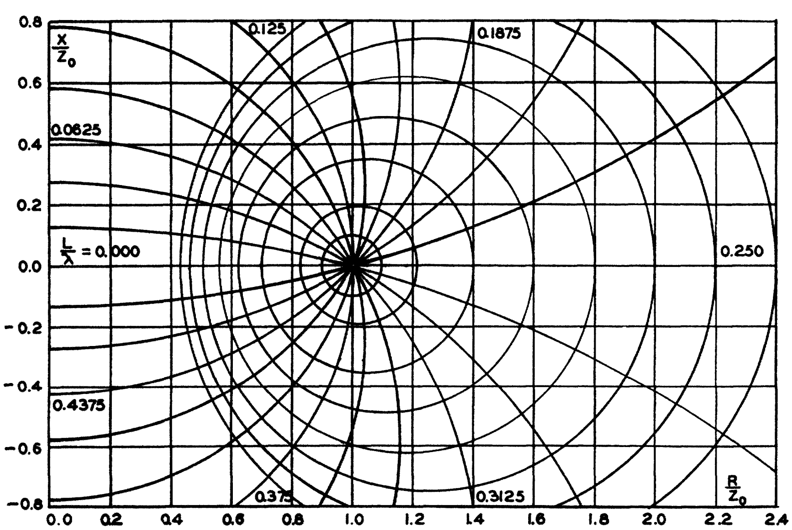
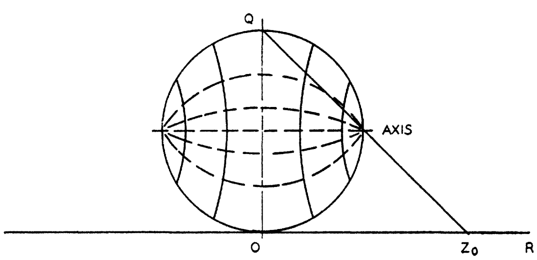


FIG. 2^2B. Special plotting paper for impedance calculations with the circles, u = constant, v = constant superposed on Cartesian scales for R and X.

$v=v_L$ and $v=v_L+kL$. Finally $2kL$ is the angle at Z_0 subtended between the lines drawn out to the centers of the two circles.

It is evident that the frequency dependence of Z arises jointly from any inherent frequency dependence there may be in Z_L and the variation of electrical length of the line due to the variation in k. If $Z_L(k)$ is given one may construct a series of points giving the corresponding values of $Z(k)$ as a means of determining the frequency dependence of line and load. In this connection it is instructive to note that if the line is many wave-lengths long then a small fractional change in k will cause kL to change by several times 2π. This in itself produces a variation of several revolutions around the Z_0 point on the plane when connected with what is usually a rather slow variation with k of Z_L, except in case Z_L exhibits a sharp resonance in the range of frequencies involved.

If very many calculations of this kind are to be made it is convenient to prepare special plotting paper, as in Fig. 2²B, on which the circles u = constant and v = constant are superposed on ordinary Cartesian scales for R and X, for some particular value of Z_0. The values of u and v corresponding to given R and X then can be read at a glance with sufficient accuracy for most purposes.



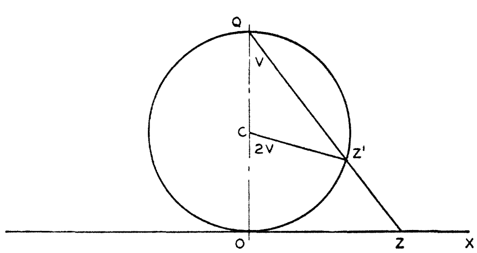


FIG. 2²C. (Upper) Stereographic projection of the RX plane on a sphere with diameter Z_0.

FIG. 2²D. (Lower) A section through the imaginary axis and Q. $u=0$.

These geometrical constructions suffer from the complication that the motion of Z along the circle $u=u_L$ is non-uniform as kL is increased at a constant rate. Another disadvantage is that an infinite half-plane is required on which to carry out the calculations for all possible impedances. This suggests seeking a diagram in which the circles $u=u_L$ are all concentric and the curves v = constant become straight lines running out from the common center, as in ordinary polar coordinates.

We can see that this is possible, and how to construct the new kind of diagram, by making use of the properties of the stereographic projection of a plane on a sphere. Suppose in Fig. 2²C a sphere of diameter Z_0 is tangent to the RX plane at the origin and let Q be the end of the diameter through O that is opposite O. Any point Z in the RX plane is associated with a point Z' on the sphere which is the intersection of the line QZ with the sphere. It is a property of this projection that any circle on the plane transforms into a circle on the sphere, and vice versa.

If now we think of the system of small circles and meridian great circles laid out on the sphere, having as its axis a diameter parallel to the R axis, then we can readily see that the system of circles u = constant on the Z plane corresponds to the circles of constant "latitude" and that the circles v = constant on the Z plane correspond to great circles on constant "longitude" on the sphere.

The situation for $u=0$ is shown in Fig. 2²D, a section through the imaginary axis and Q. Since for $u=0$ we have $Z=iZ_0 \tan v$, it follows that the angle OQZ is v and therefore the angle OCZ' is $2v$. Hence an increase of v by π corresponds to a variation of $2v$ through its entire period 2π. The lines of constant v are therefore the meridian circles for which the longitude is $2v$.

Figure 2²E shows a section through the real axis and Q. The quadrantal arc OP is the locus $v=0$, and the quadrantal arc PQ is the locus $v=\pi/2$. The value of u corresponding to any particular small circle is that for which

$u = \tanh^{-1}(R/Z_0)$, the small circle being drawn through the point Z' on the sphere associated with the point $(R, 0)$ on the plane. In particular, the equator is the circle $u=0$, and the pole is the limiting circle $u=$ infinity.

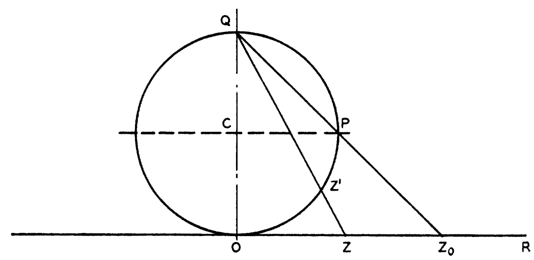
We may also project on the sphere the Cartesian coordinate lines for $R=$constant and $X=$constant. Evidently the locus of points belonging to $R=$constant will be the small circle which is the intersection with the sphere of the plane through Q and the line $R=$constant. This family of circles will have a common tangent at Q. Similarly the lines $X=$constant project into a similar set of circles orthogonal to the first set and also having a common tangent at Q. Thus the appearance of the sphere in the neighborhood of Q with lines of constant R and X drawn on it will be as in Fig. 2²F.

With this system of $R=$constant and $X=$constant circles mapped out on the sphere, one can now dispense with the impedance plane altogether. On the sphere we have two systems of mutually orthogonal circles, one giving the (R, X) coordinates of a point, the other its (u, v) coordinates. If we are given R_L, X_L we locate it on the sphere by using the (R, X) nets. Then the change in impedance due to a length of line kL is obtained by moving along the small circle $u=u_L$ until the longitude has been increased by an amount $2kL$.

It is quite instructive to think this all through but it is not very practical to work out the impedance transformations by reference to curves drawn on a sphere. But we can now go back to a wide variety of diagrams on a plane by reprojecting the sphere from any point Q' on it to a plane tangent at the opposite end of the diameter through Q'. Of all the plane diagrams which might be made in this way one is particularly valuable, namely, that in which Q' is chosen to be the pole opposite the point corresponding to $Z=Z_0$ on the sphere. It is evident in Fig. 2²G that the hemisphere corresponding to positive resistances projects into a circle of radius Z_0, which is the projection of the equator $u=0$, and the center corresponds to the pole $u=$infinity. Other values of u are represented by concentric circles. Likewise the meridian circles, $v=$constant, project into radial lines on the plane. The circles on the sphere for $R=$constant and $X=$constant project into a similarly disposed set of circles on the plane, as indicated in Fig. 2²G.

Thus we have achieved the purpose of constructing a diagram on which the $u=$constant circles are concentric and $v=$constant circles are equally spaced radii for equal intervals of v. In Fig. 2²H a diagram of this type is presented to show its general appearance. For practical work one may prepare diagrams of this type on a large scale as a means of making transmission line calculations rapidly to an accuracy quite sufficient for most purposes.

Exercise: Given Z_L and Z_0, find the input impedance graphically for any length of line kL. In Fig. 2²I, the circle $v=v_L$ has its center on the imaginary axis and passes through both Z_L and



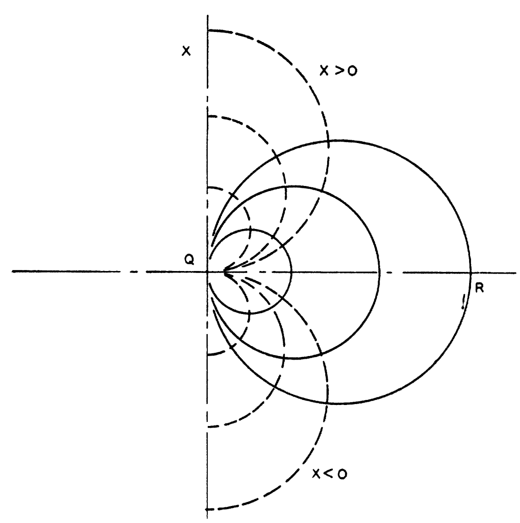


FIG. 2²E. (Upper) A section through the real axis and Q. $v=0$.

FIG. 2²F. (Lower) Projection onto the sphere of the Cartesian coordinate lines, $R=$constant, $X=$constant.

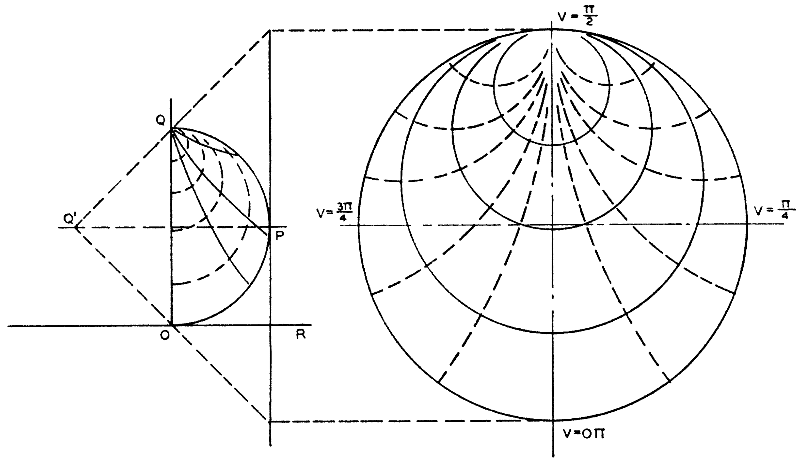


FIG. 2²G. Reprojection of the sphere from any point Q' on it to a plane tangent at the opposite end of the diameter through Q'.

Z_0. Construct the perpendicular bisector of the line Z_0Z_L, its intersection of the imaginary axis is the center of the circle $v=v_L$. To find the center of the circle $u=u_L$ draw a perpendicular to CZ_L at Z_L; the center is at the intersection of this perpendicular with the real axis. Next draw the circle $u=u_L$. The impedance transformation introduced by the line length kL is obtained by adding kl to v_L, finding the new center C', and drawing the new circle $v=v_L+kL$ to its intersection with the circle $u=u_L$.

3². Variable Impedance Transformers

Since there are losses in transmission lines as well as possibility of insulation failure in power lines, it is desirable to lead power into a load in such a way that there is no reflected wave. This requires that the load impedance Z_L be "matched" to the line impedance Z_0, which then raises the question of design of adjustable transformers to be inserted between the line and the load to permit matching the load to the line.

First let us see what can be done by connecting a shorted line of adjustable length L_1 in parallel with the load. Assume the surge impedance of the parallel unit to be the same as that of the line. By Eq. (2²9) the impedance of the unit is $iZ_0 \tan kL_1$. Since it is connected in parallel with the load it is more convenient to carry out the calculations with reciprocal impedances, that is, admittances.

Let $Y_L=G_L-iB_L$ be the admittance of the load and $-iY_0 \cot kL_1$ be that of the parallel shorted line, where $Y_0=1/Z_0$ is the surge admittance of the transmission line. Then the combined admittance of the two in parallel is

$$Y_1=Y_L-iY_0 \cot kL_1. \tag{3²1}$$

By varying L_1 over the range of one-half wavelength the second term can be made to take on any numerical value, hence, the resultant admittance can be made to assume any value on a vertical line through Y_L on the complex admittance plane. Therefore if the real part of Y_L happened to be equal to the characteristic admittance of the line it would be possible to get a perfect match by an appropriate choice of L_1.

Since a complete match involves equating two complex numbers, it is evident that a transformer suitable for all cases must involve at least two adjustable elements. Let us see what can be done by inserting another parallel shorted line of adjustable length L_2 into the line at a distance L_3 away from the load.

The admittance given by (3^21) is transformed by the length L_3 of line to

$$Y = Y_0\frac{1-w}{1+w} \quad \text{where} \quad w = \frac{Y_0 - Y_1}{Y_0 + Y_1}e^{-2i\theta},$$

and

$$\theta = kL_3,$$

and the effect of the second shorted line in parallel with this will be to add $-iY_0 \cot kL_2$. The second shorted line or piston can thus be used to balance out any reactive component there is in Y. The problem thus reduces to a study of the range of values which the real part of Y may be made to assume for various choices of L_1 and L_3. Writing $Y_1/Y_0 = g - ib$ we find

$$\frac{Y}{Y_0} = \frac{g}{(\cos\theta - b\sin\theta)^2 + g^2\sin^2\theta} - i\frac{\sin\theta\cos\theta(1-g^2-b^2) + b(\cos^2\theta - \sin^2\theta)}{(\cos\theta - b\sin\theta)^2 + g^2\sin^2\theta}. \qquad (3^22)$$

Hence, by varying b we can make the real part of Y/Y_0 take on all values from 0 (for b infinite) to $1/g\sin^2\theta$ [for $(\cos\theta - b\sin\theta) = 0$]. Therefore it will be possible to match any load to the line for which $g\sin^2\theta$ is less than unity. Since the real part of Y_1 is the same as that of Y_L it follows that with the two-piston transformer it will be possible to match any load admittance such that

$$G_L\sin^2\theta < Y_0. \qquad (3^23)$$

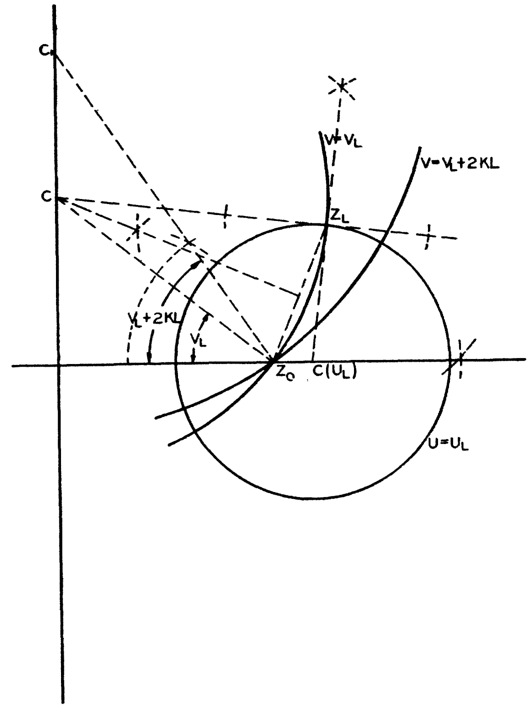


FIG. 2^2I. Construction for determining Z for any KL, given Z_0 and Z_L.

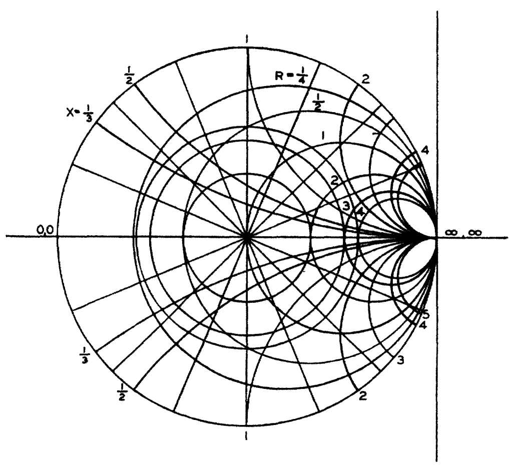


FIG. 2^2H. Nets for impedance calculations, in which the circles u = constant are concentric and the circles v = constant are equally spaced radii.

At first sight it might appear that this restriction could be removed simply by choosing L_3 such that $\theta = n\pi$, so $\sin\theta = 0$. However if this were done the first piston loses control, since its position appears in the combination $b\sin\theta$ in the real part of (3^22). Therefore one is confronted by the need to compromise as follows: In order to make (3^23) as little restrictive as possible one should design for a small value of $\sin\theta$, but in doing this it becomes necessary to be able to make very accurate adjustments of position of the piston L_1.

A reasonable choice of θ is to make $L_3 = \lambda/8$ or $3\lambda/8$ so the sines and cosines are each equal to $1/\sqrt{2}$ in magnitude. This permits matching of all impedances for which $G_L < 2Y_0$ without a very great sacrifice in control by the first piston. If $Z_L = Ae^{ia}$ then $G_L = A^{-1}\cos a$, hence, (3^23) requires that $A^{-1}\cos a$ be less than $2/Z_0$. On the impedance plane for Z_L this means that Z_L must

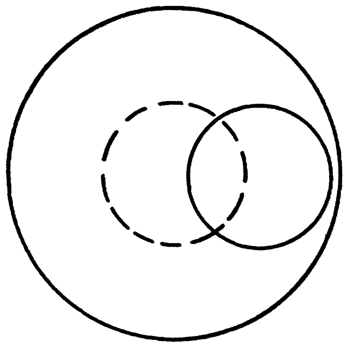


FIG. 3[2]. A quarter wave-length coaxial section with variable surge impedance.

lie outside a circle of radius $Z_0/4$ whose center is at $(Z_0/4, 0)$.

Another useful type of variable element consists of a section one quarter wave-length long whose surge impedance can be continuously varied from a maximum to a minimum value. A suitable construction is indicated in Fig. 3[2]. An inner conductor is eccentrically mounted on an eccentric shaft so on turning it through 180° it varies from the coaxial position (dotted) to one in which it comes very close to the outer conductor. In the coaxial position the surge impedance of the section is a maximum while in the position where the center element is closest to the wall it is a minimum.

If Z_L is the load impedance connected to such a unit then the input impedance from (2^26) is,

$$Z = Z_a{}^2/Z_L.$$

Thus one can use a single piston in parallel with Z_L to cancel out the reactive component of Z_L followed by a quarter-wave unit of the type just described to transform the magnitude of R_L so as to make it match the line impedance Z_0. Alternatively one can use the quarter-wave section first to effect a reciprocal transformation on Z_L followed by a piston in parallel to cancel out the reactive component remaining after the reciprocal transformation.

4[2]. Losses in Transmission Lines

Losses in transmission lines arise from the lack of perfect conductivity of the conductors and from the imperfection of the dielectric which is between them. In coaxial cable, for instance, it is necessary to use dielectric to give mechanical support to the center conductor. If the cable is to be flexible it is almost necessary to use solid (plastic) dielectric filling the whole cable to keep the center conductor in place when the cable is bent.

The losses in the conductors have to be handled as in Section 8[1]. From (8^19) the power loss in unit length of the line is

$$\frac{c\delta\mu}{8\lambda}\int\int H^2 dS, \tag{4^21}$$

where the integration extends over unit length of both line conductors. For coaxial cable, if I is the current amplitude in abamperes in either conductor $[I(t, z) = I\cos(\omega t - kz)]$, then the field at the inner conductor is $(2I/a)\cos(\omega t - kz)$ and at the outer conductor it is $(2I/b)\cos(\omega t - kz)$. Hence the time average of the power loss in unit length is

$$(\pi c\delta\mu/2\lambda)(1/a + 1/b)I^2$$

which means that the line has an effective resistance per unit length of

$$R_1 = 15(\rho\mu/\lambda)^{\frac{1}{2}}(1/a + 1/b) \text{ ohm/cm.} \tag{4^22}$$

The mean power flow down the line at a place where the current amplitude is I is $Z_0I^2/2$ watts if I is in amperes and Z_0 is the surge impedance in ohms, while the mean power loss per unit length is $R_1I^2/2$ in watts/cm if I is in amperes and R_1 in ohm/cm. Hence the power level P is attenuated according to the law,

$$(dP/dx) = -(R_1/Z_0)P,$$

and

$$P(z) = P(0)\exp[-(R_1/Z_0)z]. \tag{4^23}$$

Therefore, the quantity Z_0/R_1 gives the distance along the line in which the power level drops by a factor e^{-1} because of the losses in the conductors. This quantity is large if the losses are low and so is qualitatively analogous to the Q value for a cavity resonator. We shall denote it by L. For coaxial cable we have

$$L = 4b(\lambda/\rho\mu)^{\frac{1}{2}}\frac{\log b/a}{1 + b/a} \text{ cm for } e^{-1} \text{ loss.} \tag{4^24}$$

This factor is quite closely analogous to (9¹16) for the Q value of a coaxial cable resonator of finite length. The principal difference arises from the fact that here there are no end losses to be considered.

In radio engineering power ratios are usually expressed in decibels (db) where 1 db corresponds to a power ratio of $10^{0.1}=1.258$. Since $\log_{10} e = 0.434$, a factor e^{-1} corresponds to a loss of 4.34 db in the power level. Since the losses are not great in copper coaxial cable it is convenient to express L in meter/db loss. Using $\rho=5.7\cdot 10^{-8}$ cm we have the practical formula

$$L=10.72b\sqrt{\lambda}\frac{\log b/a}{0.279(1+b/a)}\frac{\text{meter}}{\text{db}}, \quad (4^25)$$

where b, a, and λ are in cm. The factor depending on b/a has a rather flat maximum at $b/a=3.58$, the maximum value being equal to 1. Since the maximum is so flat it is not necessary to design close to the optimum value to get a good line, as the list of values in Table I² shows. As a specific design example, suppose the outer diameter is $\frac{5}{8}$ inch and the line is used for 15-cm waves, then the maximum value of L is obtained if the inner diameter is 175 mils. For such a line $L=33$ meter/db loss.

Exercise: Show that if the cable is filled with perfect dielectric of dielectric constant ϵ, and that if the inner and outer conductors are made of different metals having resistivity and permeability, $\rho_a\mu_a$ and $\rho_b\mu_b$, respectively, then the appropriate generalization of (4²4) is

$$L=4b\left(\frac{\lambda/\epsilon}{\rho_b\mu_b}\right)^{\frac{1}{2}}\frac{\log b/a}{1+(\rho_a\mu_a/\rho_b\mu_b)^{\frac{1}{2}}(b/a)}. \quad (4^24\text{a})$$

In Section 5¹ we learned how functions of a complex variable of the form (5¹26) can be used to work out the fields in two conductor lines of more general shape. Let us now consider the losses in such lines. If $z=f(w)$ and the inverse function is $w=g(z)$, then

$$\text{grad}^2\, v=(\partial v/\partial x)^2+(\partial v/\partial y)^2=|g'(z)|^2.$$

On a curve of constant u,

$$ds=[(\partial x/\partial v)^2+(\partial y/\partial v)^2]^{\frac{1}{2}}=|f'(w)|dv.$$

Since $|g'(z)|=1/|f'(w)|$, the integral appearing in (4²1) is

$$\int H^2 ds=\int_0^{2\pi}\frac{dv}{|f'(w)|},$$

where the integral is to be evaluated with $u=u_1$ for the inner conductor and $u=u_2$ for the outer conductor.

This gives the losses associated with a current of $\frac{1}{2}$ abampere in either conductor. Therefore, by steps analogous to those used in deriving (4²2), the effective resistance per unit length in ohm/cm is

$$R_1=15\cdot(\rho\mu/\lambda)^{\frac{1}{2}} \cdot\frac{1}{2\pi}\left[\int_0^{2\pi}\frac{dv}{|f'(w)|_{u_1}}+\int_0^{2\pi}\frac{dv}{|f'(w)|_{u_2}}\right], \quad (4^26)$$

and the quantity L is easily obtained from the relation $L=Z_0/R_1$ on using the formula (5¹28) for Z_0.

As a specific example consider the calculation of R_1 for the line consisting of confocal elliptic cylinders whose surge impedance was calculated in (5¹27). We have $z=f\cosh w$, hence, $f'(w) = f\sinh w$ and the integral to be calculated is

$$\frac{1}{2\pi}\int_0^{2\pi}\frac{dv}{f'(w)}=\frac{1}{2\pi f\cosh u}\int_0^{2\pi}\frac{dv}{(1-k^2\sin^2 v)^{\frac{1}{2}}},$$

where $k^2=1/\cosh^2 u$. This is a complete elliptic integral (see Peirce's Tables, No. 524 for definition and p. 121 for tables). Since $\cosh u_1=a/f$ and $\cosh u_2=b/f$ the final result for R_1 is

$$R_1=15(\rho\mu/\lambda)^{\frac{1}{2}}\left[\frac{1}{a}\frac{2}{\pi}K\left(\frac{f}{a}\right)+\frac{1}{b}\frac{2}{\pi}K\left(\frac{f}{b}\right)\right]. \quad (4^27)$$

TABLE I². List of values for Eq. (4²5).

x	$\frac{\log x}{0.279(1+x)}$
1.5	0.58
2.0	.83
2.5	.94
3.0	.98
3.5	1.00
4.0	.99
4.5	.98
5.0	.96

If f/a and f/b are small compared with unity, the two elliptic integrals approach $\pi/2$ and this result reduces to the formula for the effective resistance of circular coaxial cable as it should. The first-order correction to R_1 for small values of f is obtained by using power series expansions for the elliptic integrals to give the result

$$R_1=15(\rho\mu/\lambda)^{\frac{1}{2}}\left[\frac{1}{a}+\frac{1}{b}+\frac{f^2}{4}\left(\frac{1}{a^3}+\frac{1}{b^3}\right)+\cdots\right]. \quad (4^28)$$

Since the first-order correction depends only on f^2 it is evident that the losses are not changed much by moderate flattening of the conductors.

5². Dielectric Losses

Suppose the space between the conductors is filled with dielectric of dielectric constant ϵ. Then, according to (1²8) the surge impedance is changed from its vacuum value by the factor $\epsilon^{-\frac{1}{2}}$. If the dielectric shows loss then it can be described by a complex dielectric constant.

It is worth while to go back to (1¹1) to the equation for curl $\mathbf{H}$. By assuming a time dependence by the factor $e^{i\omega t}$ it becomes

$$\operatorname{curl}\mathbf{H}=4\pi\mathbf{i}+ik\mathbf{D}.$$

If the material has a resistivity ρ and a dielectric constant ϵ then the right side of this can be written

$$ik\mathbf{E}(\epsilon-2i\lambda/\rho),$$

as was already remarked in dealing with skin effect in metals in Section 8¹. In metals ρ is so small that the second term is very much greater than the first. For dielectrics the reverse is true.

The actual phenomena which occur in real dielectrics are much more complicated than is usually admitted in discussions of the formal mathematical field theory.[3] The actual molecular processes involve dissipative energy losses by other mechanisms than those represented by ohmic conduction. Among these, for example, is the dissipation represented by turning of molecules with permanent dipole moments against viscous dragging forces. But all such dissipative processes have this in common, that they give rise to current density in the dielectric, that is, in time phase with $\mathbf{E}$ and which can be taken into account formally by means of an imaginary term in the dielectric constant. In addition there may be ohmic conduction of a sort which would be represented by a resistivity. These losses are dependent upon the frequency and there is no unambiguous way in which dipole turning losses can be separated experimentally from ohmic conduction losses.

It is therefore more satisfactory to discard any attempt at distinction between "true" ohmic conduction and other dissipative mechanisms. Phenomenologically the imperfect dielectric is to be described by means of a complex dielectric constant

$$\epsilon=\epsilon'-i\epsilon'' \quad (5^21)$$

in which the quantities ϵ' and ϵ'' are frequency dependent quantities characteristic of the material. Sometimes the losses are measured by giving the magnitude and phase angle of the complex dielectric constant

$$\epsilon=\epsilon_0 e^{-i\delta}. \quad (5^22)$$

Before considering losses in lines due to imperfect dielectric it will pay to reconsider the work of Section 3¹ to see how a cavity resonator is affected by being filled with leaky dielectric. Referring to Eqs. (3¹1), let us agree to work with $\sqrt{\epsilon}\mathbf{E}$ and $\sqrt{\mu}\mathbf{H}$ as the basic field vectors. In (3¹1) introduction of a complex dielectric constant results in a complex index of refraction $n=(\epsilon\mu)^{\frac{1}{2}}$. The index of refraction appears in the combination $k=n\omega/c$. Most of the theory of Chapter I consisted in devising ways to find allowed values of k which would give fields which fit the boundary conditions. Since now n is complex and the allowed values of k are real, this gives rise to complex values for the frequency ν.

Suppose that for a particular mode we have found that k_a is an allowed value. Then in vacuum the resonator fields can execute undamped free oscillations of frequency $ck_a/2\pi$. But when the resonator is filled with leaky dielectric the frequency becomes,

$$\nu_a=\nu_a'+i\nu_a''=k_ac/2\pi n$$
$$=(k_ac/2\pi)(\epsilon'-i\epsilon'')^{-\frac{1}{2}}. \quad (5^23)$$

[3] Manning and Bell, Rev. Mod. Phys. 12, 215 (1940); W. Kauzman, Rev. Mod. Phys. 14, 12 (1942).

The physical meaning of the imaginary part ν_a'' is that the time factor now is

$$\exp(2\pi i\nu_a' t)\cdot\exp(-2\pi\nu_a'' t),$$

and the free oscillations are damped by the losses in the dielectric.

The situation thus closely resembles that in Section 9[1] where damping due to finite conductivity of the walls was considered. We can define a Q' factor which measures the dielectric damping in analogy with the definition of Q in ($9^1 1$). For a resonator with walls of perfect conductivity the damping factor will be $\exp(-\omega t/2Q')$ and

$$Q'=\nu_a'/2\nu_a''=(1/2)\cot\delta/2, \qquad (5^24)$$

where δ is the phase angle of the dielectric constant.

In a resonator where there are additional losses due to the finite conductivity of the walls the factor ($9^1 1$) with Q defined as in ($9^1 4$) will also affect the decay of the free oscillations and therefore the complete damping Q_c will be given by

$$1/Q_c=1/Q+1/Q'. \qquad (5^25)$$

From the form of this result it is evident that if the power factor of a dielectric is 1 percent (which is another way of saying that $\tan\delta=0.01$), then the Q value of a resonator filled with this material cannot exceed 100. Moreover in such a case the dielectric losses will be large compared to those in the walls, under ordinary circumstances.

We consider now the effect of leaky dielectric on a transmission line. A glance over the equations of Section 1[2] shows that they are satisfied for a dielectric on writing $\sqrt{\epsilon}\mathbf{E}$ in place of $\mathbf{E}$ and by writing

$$k=n\omega/c$$

and using the complex index of refraction in place of its previous real value, $n=1$. The complex index of refraction gives rise to a complex k which can be written

$$k=k'-ik''=(\omega/c)(\epsilon'-i\epsilon'')^{\frac{1}{2}}, \qquad (5^26)$$

which gives rise to damped propagation along the line. The current, for example, is now given by

$$I=I_0e^{-k''z}\cos(\omega t-k'z), \qquad (5^27)$$

and, therefore, the power level in the line dies off like $e^{-2k''z}$. Therefore the loss length L' for a line with imperfect dielectric is

$$L'=1/2k''=(\lambda/4\pi\sqrt{\epsilon})\csc(\delta/2), \qquad (5^28)$$

where L' is expressed in cm per e^{-1} power loss if λ is in cm. It should be noted that λ is the vacuum wave-length and $\lambda/\sqrt{\epsilon}$ is the wave-length in the medium.

Since this loss is additional to the loss arising from finite conductivity of the walls, the total effective L_c when both losses are present is given by

$$1/L_c=1/L+1/L', \qquad (5^29)$$

where L represents loss in the walls as in (4^24) and L' arises from dielectric loss.

6[2]. Reflection at Supports

Thin buttons of dielectric may be used to hold the center conductor in place in coaxial cable. Such buttons necessarily introduce wave reflections at each surface. However, by choosing the button spacing properly one may reduce the reflection to zero. Also, with a proper understanding of the effects of such buttons one may design micro-wave filters which are analogous to the recurrent-section lumped-constant wave filters in use at lower frequencies.

Let $n=\sqrt{\epsilon}$ be the refractive index of the dielectric material. At any place z along the cable there will be an advancing wave (propagated from left to right, toward $+z$) and a returning wave. Let A be equal to nV_a where V_a is the voltage amplitude of the advancing wave, and let B equal nV_b be a corresponding measure of the amplitude of the returning wave. Then the electrical condition at a given point is described by the two-component quantity $\begin{pmatrix}A\\B\end{pmatrix}$. This will be handled as a one-column two-row matrix in the calculations which follow.

We assume, as always, a time factor $e^{+i\omega t}$. The dependence of A on position is given by a factor e^{-ikz} where $k=n\omega/c$. Similarly, the dependence of B on position is given by a factor e^{+ikz}. Therefore the amplitudes $\begin{pmatrix}A\\B\end{pmatrix}$ at any point can be expressed in terms of those a distance z to the right, in the same medium, denoted by $\begin{pmatrix} A_1 \\ B_1 \end{pmatrix}$ by means of the matrix equation

$$\begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} e^{ikz} & 0 \\ 0 & e^{-ikz} \end{pmatrix} \begin{pmatrix} A_1 \\ B_1 \end{pmatrix}. \qquad (6^2 1)$$

For the reader who is not familiar with matrix algebra it may be remarked that the matrix equation

$$\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} c & d \\ e & f \end{pmatrix} \begin{pmatrix} g \\ h \end{pmatrix}$$

is simply a concise way of writing the two linear equations

$$a = cg + dh, \quad b = eg + fh.$$

In particular $(6^2 1)$ is a particular notation for the pair of equations

$$A = e^{ikz} A_1, \quad B = e^{-ikz} B_1.$$

The occurrence of the zeros in $(6^2 1)$ expresses the fact that A depends only on A_1 and not on B_1 which is the mathematical expression of the fact that there is no reflection produced along a uniform cable.

Now consider what occurs to $\begin{pmatrix} A \\ B \end{pmatrix}$ in going from left to right across an interface where the refractive index changes from 1 to n. The conditions to be fulfilled are that the radial electric vector must be continuous, and the circular magnetic field must be continuous. Letting $\begin{pmatrix} A_1 \\ B_1 \end{pmatrix}$ and $\begin{pmatrix} A_n \\ B_n \end{pmatrix}$ be the amplitudes on the two sides of the interface, we find these conditions are expressed by

$$A_1 + B_1 = n^{-1}(A_n + B_n), \quad A_1 - B_1 = A_n - B_n.$$

Solving these for A_1 and B_1 in terms of A_n and B_n we find the result can be written in matrix notation as

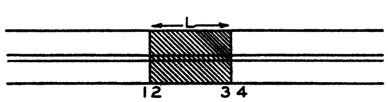


FIG. 4^2. Single-button support in coaxial cable.

$$\begin{pmatrix} A_1 \\ B_1 \end{pmatrix} = (1/2) \begin{pmatrix} n^{-1}+1 & n^{-1}-1 \\ n^{-1}-1 & n^{-1}+1 \end{pmatrix} \begin{pmatrix} A_n \\ B_n \end{pmatrix}. \qquad (6^2 2)$$

To bring out the physical significance of this result suppose the dielectric fills the cable to the right of the interface and that the cable is completely empty to the left of the interface. Suppose the cable is properly terminated so that $B_n = 0$. Then we have

$$A_1 = (1/2)(n^{-1}+1)A_n, \quad B_1 = (1/2)(n^{-1}-1)B_n.$$

The energy flow in the incident wave is proportional to A_1^2 or $(1/4)(n^{-1}+1)^2 A_n^2$, and in the reflected wave to $B_1^2 = (1/4)(n^{-1}-1)^2 A_n^2$. We assume for simplicity that n is real. The fraction of the incident energy that is reflected is

$$R = \frac{(n^{-1}-1)^2}{(n^{-1}+1)^2} = \frac{(n-1)^2}{(n+1)^2}. \qquad (6^2 3)$$

The equivalence of the two forms shows that the reflecting power at a single interface is the same whether the refractive index goes from 1 to n or from n to 1.

As a numerical example, if the dielectric is polystyrene for which $\epsilon = 2.7$, we have

$$n = 1.65, \quad R = 5.8 \text{ percent}.$$

With such a large amount of reflection at a single interface it is obviously important to take steps to produce destructive interference between waves reflected from the different interfaces in a cable.

At an interface where the index changes from n to 1 we find, analogous to $(6^2 2)$,

$$\begin{pmatrix} A_n \\ B_n \end{pmatrix} = (1/2) \begin{pmatrix} n+1 & n-1 \\ n-1 & n+1 \end{pmatrix} \begin{pmatrix} A_1 \\ B_1 \end{pmatrix}. \qquad (6^2 4)$$

This matrix is the reciprocal of the one occurring in $(6^2 2)$ as it should be. The rule for multiplying matrices will be needed in verifying this statement and in the following calculations. It is this: If

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} e & f \\ g & h \end{pmatrix} \begin{pmatrix} i & j \\ k & l \end{pmatrix},$$

then

$$a = ei + fk, \quad b = ej + fl, \quad c = gi + hk, \quad d = gj + hl.$$

Consider now the over-all effect of a single button of thickness L. (See Fig. 4^2.) By means of (6^24) we can express $\begin{pmatrix} A_3 \\ B_3 \end{pmatrix}$ in terms of $\begin{pmatrix} A_4 \\ B_4 \end{pmatrix}$. Then (6^21) gives $\begin{pmatrix} A_2 \\ B_2 \end{pmatrix}$ in terms of $\begin{pmatrix} A_3 \\ B_3 \end{pmatrix}$ and finally $\begin{pmatrix} A_1 \\ B_1 \end{pmatrix}$ in terms of $\begin{pmatrix} A_2 \\ B_2 \end{pmatrix}$ is given by (6^22). Hence the over-all expression for $\begin{pmatrix} A_1 \\ B_1 \end{pmatrix}$ in terms of $\begin{pmatrix} A_4 \\ B_4 \end{pmatrix}$ is given by

$$\begin{pmatrix} A_1 \\ B_1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} n^{-1}+1 & n^{-1}-1 \\ n^{-1}-1 & n^{-1}+1 \end{pmatrix}\begin{pmatrix} e^{ia} & 0 \\ 0 & e^{-ia} \end{pmatrix} \times \frac{1}{2}\begin{pmatrix} n+1 & n-1 \\ n-1 & n+1 \end{pmatrix}\begin{pmatrix} A_4 \\ B_4 \end{pmatrix}$$

where $a=\omega nL/c$. Multiplying together the three matrices (remembering that the order of the factors is important) we find a single matrix representing the effect of a single button,

$$\frac{1}{4n}\begin{pmatrix} [(n+1)^2e^{ia}-(n-1)^2e^{-ia}] & -2i(n^2-1)\sin a \\ 2i(n^2-1)\sin a & [(n+1)^2e^{-ia}-(n-1)^2e^{ia}] \end{pmatrix}. \qquad (6^25)$$

This will be called the one-button matrix. For most purposes it is more convenient to write (6^25) in the form

$$\begin{pmatrix} A_1 \\ B_1 \end{pmatrix} = \begin{pmatrix} P_1 & Q_1^* \\ Q_1 & P_1^* \end{pmatrix}\begin{pmatrix} A_4 \\ B_4 \end{pmatrix},$$

with

$$P_1=\cos a+i\frac{n^2+1}{2n}\sin a, \qquad (6^26)$$

$$Q_1=i\frac{n^2-1}{2n}\sin a.$$

From this we find that the reflecting power of a single button is

$$R_1=\frac{(n^2-1)^2\sin^2 a}{(n^2+1)^2\sin^2 a+4n^2\cos^2 a}. \qquad (6^27)$$

For polystyrene, $n=1.65$, this is

$$R_1=\frac{2.89\sin^2 a}{13.7\sin^2 a+10.8\cos^2 a}.$$

From these results we see that the reflecting power vanishes if $a=m\pi$, that is for L such as to give an integral number of half wave-lengths in the material. Maximum reflection occurs for an odd integral number of quarter wave-lengths in the material. For polystyrene the maximum is 21 percent.

From a mechanical point of view $L=\frac{1}{8}''$ is a good thickness for ordinary coaxial cable. With polystyrene and a 15-cm vacuum wave-length, this makes $a=12°$ and so R_1 is about 1.1 percent. It is worth noting that these same beads would give an extremely low reflecting power if the cable is used at considerably longer wave-lengths. That is why the problem of reflection from the beads is not such an important one in ultra-high frequency work as it is in the micro-wave region.

7^2. Chokes and By-Pass Condensers

Suppose we wish to continue a transmission line as a circuit for low frequency currents while having the high frequency power not go beyond a certain point. A suitable element for this is called a choke. In Fig. 5^2 suppose that a cup-shaped member is attached to the inner conductor of a coaxial cable as shown. Suppose the load impedance as regarded from the closed end of the cup is Z_L. If the length of the cup is L, the impedance presented at the open end of the cup is, from (2^25),

$$Z'=Z_2\frac{Z_L\cos kL+iZ_2\sin kL}{iZ_L\sin kL+Z_2\cos kL}, \qquad (7^21)$$

where Z_2 is the characteristic impedance of the

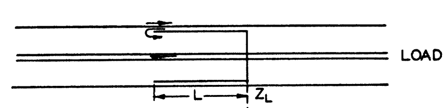


FIG. 5^2. High frequency choke with voltage node at far end of cup.

element of concentric line formed by the outer conductor and the outside wall of the cup.

Likewise the input impedance presented at the open end of the cup is

$$Z''=iZ_1 \tan kL,$$

where Z_1 is the characteristic impedance of the line formed by the inner conductor and the inside

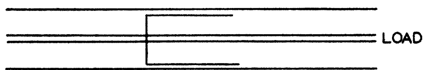


FIG. 6[2]. High frequency choke with voltage node at near end of cup.

wall of the cup. Regarded from the cross section at the open end of the cup, these two impedances are in series, for the currents flow as marked in the sketch and the total voltage drop is the sum of that over the two elements. Hence the total input impedance is

$$Z=Z'+Z''. \tag{7·2}$$

If the length of the cup is a quarter wave-length then

$$Z=iZ_1\infty+Z_2{}^2/Z_L=\infty. \tag{7·3}$$

Thus the impedance at the open end of the cup is infinite. Hence there will be total reflection at the cup of a radiofrequency wave in such a way that there is a voltage loop and a current node at the mouth of the cup, exactly as if the line terminated there in an open circuit.

Next let us consider the same structure with the cup turned the other way. (See Fig. 6[2].) In this case the cup impedance is $iZ_1 \tan kL$ and this is in series with the load impedance Z_L. Therefore the resultant of the two is $(Z_L+iZ_1 \tan kL)$. If the length of the cup is a quarter wave-length, this is infinite. Viewed from the bottom end of the cup this infinite impedance becomes a zero impedance. Therefore, a wave coming from the left is totally reflected with a voltage node at the outside of the bottom of the cup, just as if the cup constituted a complete short circuit of the end of the line.

In other circumstances one may need to have a break in the line for low frequency currents while not interfering with the flow of the high frequency power. This can be done as in Fig. 7[2]. If the overlapping portion of the separated outer conductors is equal to a quarter wave-length then the infinite impedance at the open end between the two outer conductors transforms to zero impedance between the two outer conductors at the left end of the overlap. Hence the current flow in the large outer conductor is carried on in the inner one without a voltage drop, so the wave goes on, although low frequency currents are blocked by the lack of contact between the two outer conductors.

Another way of doing the same thing is to put circular flanges on the ends of the two portions of outer conductor of the same size, as in Fig. 8[2]. In this case the correct radius of the flanged ends has to be calculated as follows. Suppose $r=a$ is the radius of the outer conductor; then we must have $E_z=0$ at $r=a$, so there will be no potential drop across the gap between the flanges, just as if the outer conductor were continuous. We have, quite generally,

$$E_z=AJ_0(kr)+BN_0(kr),$$

where J_0 and N_0 are the two Bessel functions of zero order. The requirement $E_z=0$ at $r=a$ gives the equation

$$AJ_0(ka)+BN_0(ka)=0,$$

which determines the ratio of B to A. At the outer radius of the flange the radial current must

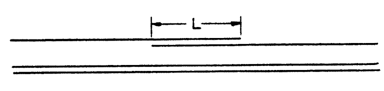


FIG. 7[2]. Low frequency choke, suitable for rotating parts.

sink to zero, giving $H_\varphi=0$ which requires that $\partial E_z/\partial r=0$. This gives

$$AJ_0'(kb)+BN_0'(kb)=0,$$

which is the equation to determine b, the outer flange radius.

As a specific example, suppose $a=0.5$ cm and we are dealing with $\lambda=3$ cm, giving $k=2.08$ and

$ka=1.04$. We have $J_0(ka)=0.7473$ and $N_0(ka)=0.1188$ and, therefore, if we write

$$E_z=0.1188J_0(kr)-0.7473N_0(kr),$$

we have a suitable expression which vanishes at $r=a$. We have now to find the value of kb such that $\partial E_z/\partial r=0$. This is best done by making a graph of E_z against r from standard tables of Bessel functions. In this way we find the function has a maximum at $kb=2.4$ or $b=1.15$ cm, as the proper outer radius of the flanges.

8². Transmission Line Resonators

Any finite section of transmission line may be used either by itself or in connection with lumped inductance and capacity to make resonant circuits. First let us consider a length z of transmission line which is closed at one end and open at the other. The impedance at the open end is, by (2²7),

$$Z=iZ_0 \tan kz.$$

The free oscillations must be such that there is zero current flowing even though there is a finite voltage amplitude. Hence, the natural resonant frequencies will be such as to make the impedance at the open end be infinite. Therefore, the resonant values of k are

$$kz=(n+\tfrac{1}{2})\pi$$

which can be written

$$z=(n/2+\tfrac{1}{4})\lambda, \qquad (8^21)$$

where n is an integer. The resonance of lowest frequency is such that the length is a quarter wave-length.

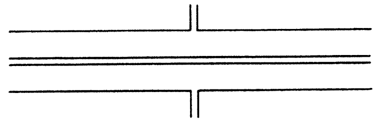


FIG. 8². Low frequency choke, for fixed cables.

If the resonator is closed at both ends then the frequency has to be such as to give zero impedance at either end. This means one must have $kz=n\pi$, and the length must be an integral number of half wave-lengths. This is in agreement with the field theory treatment given in Section 5¹.

Suppose now that there is a condenser of capacity C (farad) across the otherwise open end of a line that is closed at the other end as in the sketch. The shorted line is in series with the condenser so the input impedance at the terminals 1,2 is the sum of the separate impedances.

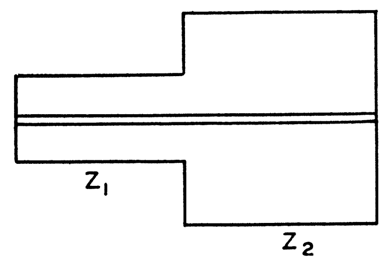


FIG. 9². Resonator made from two shorted sections of line.

It is, therefore,

$$Z=iZ_0 \tan kz+1/i\omega C, \qquad (8^22)$$

and the resonances are the frequencies for which $Z=0$, since in the actual resonator in which the terminals 1,2 are joined together current must flow there without a potential drop. This condition gives the equation,

$$kz \tan kz=z/cCZ_0. \qquad (8^23)$$

If C is small the roots of (8²3) are close to those given by (8²1). If L and C and Z_0 are given the possible values of k can be conveniently found by graphing $x \tan x$ against x from which the allowed values of kz are readily found. From such a graph it can be immediately seen that an increase of the capacity has the general effect of reducing all of the resonant frequencies. This approximate treatment based on transmission line theory should be compared with the field theory discussion given in Section 7¹.

A resonator can be made as in Fig. 9² by joining together a length z_1 of shorted line of characteristic impedance Z_1 and a shorted length z_2 of characteristic impedance Z_2. The input impedance presented by this combination to terminals mounted on the disconnected outer conductors is then

$$i(Z_1 \tan kz_1 + Z_2 \tan kz_2)$$

which must vanish at the resonant frequencies. These can easily be located graphically by plotting $Z_1 \tan kz_1$ and $-Z_2 \tan kz_2$ against k and noting the values of k at which the two curves intersect.

Exercise: Calculate the lowest resonant frequency of the resonator shown in Fig. 10² (figure of revolution about the horizontal center line), where the dimensions are, in inches, $a=2$, $b=3$, $c=1$, $d=\frac{1}{2}$, and $e=\frac{1}{4}$. *Answer*: 468 megacycle/sec.

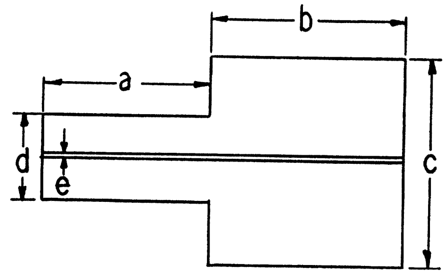


FIG. 10². Resonator with dimensions for exercise.

9². Tapered Lines

By a tapered line is meant one in which the proportions change along the length of the line —for example, a coaxial cable with a variable ratio of inner to outer diameter.[4]

It will be supposed that the dimensions in a section are all small compared with the wavelength. The theory may be developed by using (1²14) and (1²15) as a starting point. Since in practice tapered lines will be used only in short transition sections we shall neglect losses, that is, assume $R=0$ and $G=0$.

The generalization now being considered is that L and C are here functions of z. The basic line equations are:

$$\begin{aligned}(\partial V/\partial z) &= -L(\partial I/\partial t),\\ (\partial I/\partial z) &= -C(\partial V/\partial t),\end{aligned} \tag{9²1}$$

where the units are: V, volt; z, cm; L, henry/cm; I, ampere; t, sec.; and C, farad/cm. Assuming harmonic time dependence through the factor $e^{i\omega t}$ we find that V and I satisfy the following differential equations:

$$\begin{aligned}V'' - \frac{d \log L}{dz} V' + \omega^2 LCV &= 0,\\ I'' - \frac{d \log C}{dz} I' + \omega^2 LCI &= 0.\end{aligned} \tag{9²2}$$

[4] Eckart, Zeits. fur Hochfrequenztechnik **55**, 173 (1940) gives a very general treatment of the theory. Other important references are: Ballantine, J. Frank. Inst. **203**, 561 (1927); Wheeler and Murnaghan, Phil. Mag. **6**, 146 (1928); Starr, Proc. I. R. E. **20**, 1052 (1932); Burrows, Bell Sys. Tech. J. **17**, 555 (1938); Wheeler, Proc. I. R. E. **27**, 65 (1939).

If $\epsilon=\mu=1$ we have $LC=1/c^2$ and the characteristic impedance of the line Z in ohms is related to L and C by the expressions, $Z=cL=1/cC$. Therefore the two logarithmic derivatives appearing in (9²2) can be expressed in terms of the logarithmic derivative of Z. With $k=\omega/c$ we have

$$\begin{aligned}V'' - \frac{d \log Z}{dz} V' + k^2 V &= 0,\\ I'' + \frac{d \log Z}{dz} I' + k^2 I &= 0.\end{aligned} \tag{9²3}$$

It is only necessary to discuss one of these, since, if the solution is known for $V(z)$, that for $I(z)$ may be obtained from the first of (9²1) in the form

$$I = (i/\omega L)(\partial V/\partial z) = (i/kZ)(\partial V/\partial z). \tag{9²4}$$

We have now to discuss the properties of the first of (9²3) which determines the variation along the line of the potential difference between conductors in the line. For a line of uniform properties $d \log Z/dz = 0$ and the equation reduces to one which is satisfied by e^{+ikz} or e^{-ikz} giving the usual propagation of undistorted harmonic waves at the velocity of light. The term in the derivative of V can be transformed away by writing

$$V = \sqrt{Z}\, U, \tag{9²5}$$

in which case the differential equation for U is

$$U'' + [k^2 + (Z''/2Z) - (3Z'^2/4Z^2)]U = 0. \tag{9²6}$$

There are two special cases in which the equation for V can be solved in terms of known functions.

Exponential Line

The simplest special case is that in which the line is tapered in such a way that the characteristic impedance varies exponentially along the line. Suppose

$$Z(z) = Z_0 \exp (2k_0 z), \tag{9²7}$$

then the differential equation for U becomes

$$U'' + (k^2 - k_0^2) U = 0. \tag{9²8}$$

and, therefore, the solutions for U depend on the sign of

$$k'^2 = k^2 - k_0^2.$$

If k'^2 is positive the solution for U is undamped and oscillatory and there is real wave propagation along the line, with the voltage amplitude building up exponentially as one goes in the direction in which the characteristic impedance increases. But if k'^2 is negative the solution for U is a real exponential function and the wave is attenuated in going along the line. Such an exponentially tapered line therefore behaves like a high-pass filter. It passes only those waves for which k is greater than k_0. Therefore the cut-off frequency is greater for more rapid rates of taper.

Suppose we have a wave traveling toward $+z$. The voltage is represented by

$$V = V_0 \exp (k_0 z) \cdot \exp [i(\omega t - k' z)],$$

and, therefore, by (9²4), the current is represented by

$$I = (V_0/Z_0) \frac{(k' + ik_0)}{k} \times \exp(-k_0 z) \cdot \exp [i(\omega t - k'z)].$$

The ratio of voltage to current at any place gives the impedance of a load which could terminate the line at that place without producing a reflected wave. This terminating impedance is

$$Z_i = \frac{k}{(k' + ik_0)} Z_0 \exp (2k_0 z). \tag{9²9}$$

This terminating impedance must therefore be somewhat reactive although its phase angle tends to zero if the frequency used is large compared with the cut-off frequency so k' is large compared to k_0.

Let us consider a particular example. Suppose it is desired to design a transition section of coaxial cable to pass from a characteristic impedance of 50 ohms to a characteristic impedance of 100 ohms in a meter of line length. If the inner conductor is the same throughout and is 125 mils in diameter, then the diameter of the outer conductor at the two ends must be 288 mils and 660 mils, respectively. Since the transition takes place in one meter we have $200k_0 = x \ln 2$ or $k_0 = 3.47 \cdot 10^{-3}$ cm^{-1}. Therefore the cut-off wave-length is $2\pi/k_0 = 1810$ cm.

If this transition section is used for radiation of 15-cm wave-length, or $k = 0.418$ cm^{-1} then it can be calculated that the phase-angle of the terminating impedance is less than one degree.

Line With Z Varying a Power of z

Another case which can be treated in terms of known functions is that in which $Z(z)$ is a simple power of z measured from some origin. Suppose

$$Z(z) = Z_1 z^n, \tag{9²10}$$

where Z_1 is the characteristic impedance at a point at unit distance from the place where Z would vanish if this law were valid everywhere. In practice, one will be dealing with finite sections of tapered line for which $z \neq 0$, say the portion extending from $z = +a$ to $z = +b$; hence, no difficulty arises from the vanishing or negative values of Z seemingly implied by (9²10).

For this case $d \log Z/dz = n/z$ and (9²3) becomes

$$V'' - (n/z) V' + k^2 V = 0, \tag{9²11}$$

an equation which can be solved in terms of Bessel functions. The solution is

$$V(z) = z^m Z_m(kz) \text{ with } m = (1-n)/2, \tag{9²12}$$

where $Z_m(kz)$ stands for the general Bessel function of order m. By making use of known properties of Bessel functions it is therefore possible to make a detailed study of tapered lines of this kind.