Cosmic-Ray Theory

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TABLE OF CONTENTS

Introduction

THE interaction of cosmic rays with matter gives rise to a great variety of secondary effects. Some involve nuclear transformations, as is clearly shown by the occurrence of "stars" on photographic plates exposed to the cosmic radiation. The production of mesotrons, too, is probably a nuclear process.

Nuclear processes, however, are comparatively rare and do not seem to play any essential role in the cosmic-ray effects which are most commonly observed, such as absorption and scattering of mesotrons, or shower production by electrons and photons. As far as we know, these phenomena can be interpreted as electromagnetic effects; i.e., as effects caused by the interaction between cosmic-ray particles and the electric fields of nuclei and electrons. A quantitative theory of the cosmic-ray effects caused by electromagnetic interaction has been developed under the assumption that the ordinary laws of quantum electrodynamics can be extrapolated to cosmic-ray energies. The results have proved extremely useful in disentangling some very complex cosmic-ray phenomena. It is hoped that they will be of further help in separating electromagnetic from non-electromagnetic cosmic-ray effects, as well as in determining the limits of validity of quantum electrodynamics.

The quanta of the primary cosmic radiation have exceedingly high energies. However, by the interaction of cosmic rays with matter, the pri-

TABLE I. *System of units.*

QUANTITY	DEFINITION	SYMBOL
Length	Centimeter	cm
Velocity	Velocity of light	c
Electric Charge	Charge of the electron	e
Potential Diff.	Volt	v

TABLE II. *Natural constants.*

Number of atoms per gram atom (Avogadro's number)	$N=6.02395\times10^{23}$
Classical radius of the electron	$r_0=2.8176\times10^{-13}$ cm
Fine structure constant	$\alpha=1/137.036$
Mass of the electron	$\mu_e=5.109\times10^5$ ev/c^2
Mass of the mesotron (tentatively)	$\mu=10^8$ ev/c^2
Mass of the proton	$\mu_H=9.315\times10^8$ ev/c^2

The constants in Table II have been computed with the following values for the fundamental constants:

electric charge of the electron (B40)	$e=4.802\times10^{-10}$ e.s.u.
specific charge of the electron (B40)	$e/mc=1.7591$ e.m.u./g
fine structure constant (B40)	$\alpha=e^2/\hbar c=1/137.036$
velocity of light (B29)	$c=2.99796\times10^{10}$ cm/sec.
Faraday (B29)	$F=9648.9$ e.m.u./g-equiv.

and from the formulae:

$$N=Fc/e,\quad r_0=e^2/mc^2,\quad \mu_e=(c/10^8)mc^2/e,\quad \mu_H=(c/10^8)c/F.$$

mary energy is subdivided into smaller and smaller quanta, until eventually it is completely transformed into excitation and ionization energy of atoms. It is convenient to set an arbitrary energy limit η_0 below which a ray ceases to belong to what we call cosmic radiation and its energy is considered as *dissipated*. We shall take $\eta_0=5\times10^6$ ev, which is approximately ten times the rest energy of the electron. Thus, most of the rays in the radioactive energy region are excluded from the "cosmic radiation." This limitation will enable us to use some simplifications which are allowed when the rest energy of the electron is small compared with the kinetic energies under consideration.

The main secondary effects produced by the passage of charged particles through matter are (a) *excitation of atoms*, (b) *ejection of electrons from atoms* and (c) *emission of photons*. The first two phenomena can be considered as the result of a direct interaction between the primary particle and atomic electrons; we shall refer to them as *non-radiative collision processes*, or, shortly, as *collision processes*. The emission of photons is caused by the acceleration of the primary particle in the Coulomb field of the nucleus; this process will be denoted as a *radiative collision process*, or, shortly, as a *radiation process*.

High energy photons traversing matter are known to undergo the following secondary processes. (a) *Photoelectric effect*, i.e., absorption by an atom accompanied by emission from the atom of a high energy electron. The photoelectric effect becomes less and less frequent with increasing photon energy, and can be disregarded entirely for energies larger than η_0. (b) *Compton effect*, i.e., scattering by free electrons. (c) *Pair production*, i.e., materialization of the photon energy into a positive and a negative electron.

The theoretical formulae concerning radiative and non-radiative collision processes, Compton effect, and pair production are summarized in the first part of the present paper together with a discussion of elastic scattering. The writers wish to thank Professor H. A. Bethe for helpful discussions of the items contained in this part. Part II describes the complex secondary effects (showers) arising from the repetition of the elementary processes considered in Part I. An attempt has been made to coordinate and supplement the somewhat fragmentary results on shower theory scattered in the literature.

We shall use throughout the paper the system of units suggested by one of us (R40a), which is based upon the fundamental units listed in Table I.

In this system, energies are measured in ev, momenta in ev/c, masses in ev/c^2. The *mass* of a particle is expressed by the same number which expresses its *rest energy* and its *characteristic momentum* (defined as the product mass times velocity of light). The symbol μ will be used to indicate any of these three quantities. We shall further use the symbols β, E and p to indicate the velocity, the kinetic energy, and the momentum of a particle, respectively. The energy of a photon will be denoted, in general, by W. Between β, E and p there exist the following well-known relations:

$$E=\mu/(1-\beta^2)^{\frac{1}{2}}-\mu, \tag{1.1}$$

$$E=(p^2+\mu^2)^{\frac{1}{2}}-\mu, \tag{1.2}$$

$$p=\mu\beta/(1-\beta^2)^{\frac{1}{2}}. \tag{1.3}$$

It may be appropriate to list here the values of the natural constants which will enter in our calculations (see Table II).

Part I.
Fundamental Processes

A. Collision Processes

§1. *Application of the Conservation Laws*

In this section and in the next we shall only consider collision processes which result in the ejection of electrons with energies large compared with the binding energies. In such processes the electrons can be considered as free, and the conservation laws of energy and momentum can be applied to determine the energy of the secondary electron as a function of the angle of emission. The result is expressed by Eq.(1.4), where μ is the mass of the primary particle, p its momentum before the collision, μ_e the mass of the electron, E' its energy after the collision, θ the angle between the initial trajectory of the primary particle and that of the secondary electron. The electron is supposed to be at rest before the collision.

$$E'=2\mu_e\frac{p^2\cos^2\theta}{[\mu_e+(p^2+\mu^2)^{\frac{1}{2}}]^2-p^2\cos^2\theta}. \quad (1.4)$$

E' increases with decreasing θ; thus, the *maximum transferable energy* E'_m is obtained putting in (1.4) $\theta=0$ (head-on collision):

$$E'_m=2\mu_e\frac{p^2}{\mu_e^2+\mu^2+2\mu_e(p^2+\mu^2)^{\frac{1}{2}}}. \quad (1.5)$$

When $\mu_e\ll\mu$ (mesotrons, protons), μ_e^2 in the denominator of (1.5) can be neglected. In the limiting cases of small and large energies, Eq. (1.5) can be further simplified as follows:

$p\ll\mu^2/\mu_e$

$$E'_m=2\mu_e\left(\frac{p}{\mu}\right)^2=2\mu_e\frac{\beta^2}{1-\beta^2}, \quad (1.5a)$$

$p\gg\mu^2/\mu_e$

$$E'_m=p\approx E. \quad (1.5b)$$

For comparatively small momenta [Eq. (1.5a)] the maximum transferable energy depends only on the velocity β of the primary particle. For very large momenta [Eq. (1.5b)], the maximum transferable energy approaches the primary energy itself. When $\mu=\mu_e$, E'_m is obviously equal to E for all energies. It may be noted that μ^2/μ_e is of the order of 2×10^{10} for mesotrons and of 2×10^{12} for protons. Since most of the observed cosmic-ray particles have momenta of the order of 10^9 ev/c, the condition for the validity of Eq. (1.5a) is fulfilled in many cases of practical importance.

§2. *Differential Collision Probability*

Let $\chi(E, E')dE'dx$ be the probability for a particle of mass μ, charge ±1 and energy E, traversing a thickness dx, to transfer an amount of energy between E' and $E'+dE'$ to a *free* electron. The function χ will be called the *differential collision probability*. It is convenient to measure dx in g/cm^2 and to introduce the constant

$$C=\pi N(Z/A)r_0^2=0.150(Z/A), \quad (1.6)$$

where Z and A are the atomic number and the atomic weight of the material. C represents the total "area" covered by the electrons contained in one g/cm^2, considered as spheres with radius r_0.

The interaction between charged particles and electrons is mainly determined by the electrostatic attraction or repulsion. Only for very large energies of the primary particle and very close collisions, other forces, connected with the spin, need to be taken into consideration. Classical treatment of the problem gives, *for small values of* E', the following expression for χ, known as the Rutherford formula:

$$\chi(E, E')dE'=(2C\mu_e/\beta^2)dE'/(E')^2. \quad (1.7)$$

According to this equation, the collision probability χ does not depend explicitly on the energy or on the mass of the primary particle, but only on its velocity β. The quantum-mechanical treatment leads again to Eq. (1.7) at the limit for small values of E'. The general expression for χ, however, is different from that which could be derived from classical mechanics, and depends in an essential way on the spin of the primary particle and on whether or not it is distinguishable from the secondary electron.

(a) ***Electrons.***—The collision probability for electrons with electrons has been calculated by Möller (M32) on the basis of the Dirac theory. When the energy E of the primary electron is large compared with μ_e, χ is given by the following expression:

$$\chi(E, E')dE' = 2C\mu_e dE'\left[\frac{E}{E'(E-E')} - \frac{1}{E}\right]^2. \quad (1.7a)$$

(For electrons, β is practically 1 for any E larger than η_0.)

Since it is not possible to distinguish between the primary and the secondary electron after the collision, Eq. (1.7a) must be interpreted as giving the probability of a collision which leaves one electron in the energy state E' and the other in the energy state $E-E'$. Thus, one takes into account all possible cases by letting E' vary from 0 to $E/2$ (not to E!). Eq. (1.7a) is symmetrical in E' and $E-E'$ and reduces to (1.7) for $E' \ll E$.

(b) ***Positrons.***—The collision probability for positrons with electrons has been calculated by Bhabha (B38). For $E \gg \mu_e$ it is

$$\chi(E, E')dE' = 2C\mu_e \frac{dE'}{(E')^2}\left[1 - 2\frac{E'}{E} + 3\left(\frac{E'}{E}\right)^2 - 2\left(\frac{E'}{E}\right)^3 + \left(\frac{E'}{E}\right)^4\right]. \quad (1.7b)$$

The difference between (1.7a) and (1.7b) arises from the fact that the exchange effect is different in the case of electrons from that in the case of positrons.

(c) ***Mesotrons.***—The spin of mesotrons is not yet known. Thus we write the expressions for the collision probabilities corresponding to the values of the spin 0, $\frac{1}{2}$ and 1. Quantities of the order of μ_e/μ are neglected in comparison with unity.

Spin 0 (see Bhabha, B38)

$$\chi(E, E')dE' = \frac{2C\mu_e}{\beta^2}\frac{dE'}{(E')^2}\left(1 - \beta^2\frac{E'}{E'_m}\right). \quad (1.7c)$$

Spin $\frac{1}{2}$ (see Bhabha, B38; Massey and Corben, M39)

$$\chi(E, E')dE' = \frac{2C\mu_e}{\beta^2}\frac{dE'}{(E')^2}\left[1 - \beta^2\frac{E'}{E'_m} + \frac{1}{2}\left(\frac{E'}{E+\mu}\right)^2\right]. \quad (1.7d)$$

Spin 1 (see Massey and Corben, M39; Oppenheimer, Snyder, and Serber, O40)

$$\chi(E, E')dE' = \frac{2C\mu_e}{\beta^2}\frac{dE'}{(E')^2}\left[\left(1 - \beta^2\frac{E'}{E'_m}\right)\left(1 + \frac{1}{3}\frac{E'}{E_c}\right) + \frac{1}{3}\left(\frac{E'}{E+\mu}\right)^2\left(1 + \frac{1}{2}\frac{E'}{E_c}\right)\right], \quad (1.7e)$$

where $E_c = \mu^2/\mu_e \approx 2\times 10^{10}$ ev.

As long as E' is small compared with both E and E_c, Eqs. (1.7d) and (1.7e) reduce to (1.7c), which means that the collision probability is independent of the spin. Eq. (1.7c), in turn, reduces to (1.7) when $E' \ll E'_m$. The difference between the collision probabilities becomes appreciable when E' is comparable with E_c or with E, and this can only happen when E itself is larger than E_c [see Eq. (1.5)]. For these large values of E', the collision probability is an increasing function of the spin. However, the difference between spin $\frac{1}{2}$ and spin 1 is much larger than the difference between spin 0 and spin $\frac{1}{2}$. Let us consider, in particular, the case $E' \ll E'_m$. The collision probabilities for spin 0 and spin $\frac{1}{2}$ are then given by the Rutherford formula (1.7), while the collision probability for spin 1 becomes

$$\chi(E, E')dE' = \frac{2C\mu_e}{\beta^2}\frac{dE'}{(E')^2}\left(1 + \frac{1}{3}\frac{E'}{E_c}\right). \quad (1.7f)$$

This expression contains an additional term which decreases with increasing energy as $1/E'$, whereas the Rutherford term decreases as $(1/E')^2$. For energies larger than $3E_c$ the additional term, which represents the interaction due to the spin, becomes larger than the Rutherford term, which represents the Coulomb interaction.

It may be noted that the influence of the spin on the collision probability of mesotrons manifests itself only for very close collisions. The theoretical predictions depend essentially on the hypothesis that the electromagnetic field of the

mesotron can be described in the ordinary way even at distances smaller than 10^{-13} cm from the "center" of the mesotron itself. So far, this hypothesis lacks any experimental support, although some theoretical justification for it can be found in Oppenheimer's arguments, which show that even for very close collisions the interaction between mesotron and electron remains small compared with the kinetic energy of the mesotron (see O40 and O41). At any rate, the validity of the formulae expressing the probabilities of large energy transfers from mesotrons to electrons cannot yet be considered as established.

(d) ***Protons.***—In the case of protons the interaction due to the spin can be neglected as long as the energy is small compared with $\mu_H^2/\mu_e \approx 2\times 10^{12}$ ev. Since this is practically always the case, we can use for protons the collision probability (1.7c).

§3. *Collision Loss*

Let $K_\eta(E)$ be the average energy loss per g/cm² caused by collisions in which secondary electrons of energy *larger* than η are produced. If η is large compared with the binding energy of the electrons, K_η can be obtained from the expression for the collision probability given in §2:

$$K_\eta(E)=\int_\eta^{E'_m} E'\chi(E,E')dE'. \tag{1.8}$$

For example, if one considers mesotrons or protons of moderate energy and assumes $\eta \ll E'_m$, Eqs. (1.7c) and (1.8) yield

$$K_\eta(E)=\frac{2C\mu_e}{\beta^2}\left[\log\frac{E'_m}{\eta}-\beta^2\right]. \tag{1.8a}$$

The calculation of the collision loss caused by small energy transfers requires a separate treatment because in these processes the electron cannot be considered as free. In other words, instead of calculating the transition probability for the system formed by the primary particle and a free electron, one has to calculate the transition probability for the system formed by the primary particle and an atom. The atom is considered initially in the ground level; its final state is an excited level which can belong either to the discrete or to the continuous spectrum (excitation or ionization). A theory of collision losses has been developed by Bethe on this basis, using Born's approximation (B30, B32; see also L37), and the result is represented by the following equation:

$$k_\eta(E)=\frac{2C\mu_e}{\beta^2}\left[\log\frac{2\mu_e\beta^2\eta}{(1-\beta^2)I^2(Z)}-\beta^2\right], \tag{1.9}$$

where k_η is the energy loss per g/cm² caused by collisions in which the energy transferred is *smaller* than η, and $I(Z)$ is the average ionization potential of an atom of atomic number Z. The function $I(Z)$ cannot be determined theoretically with any great accuracy, and the most reliable values of $I(Z)$ are probably those deduced from empirical data (see Livingston and Bethe, L37). Since, however, $I(Z)$ enters only in the logarithm, an approximate expression will be sufficient for our purpose and we shall use the following formula given by Bloch (B33):

$$I(Z)=I_H Z, \tag{1.10}$$

where $I_H=13.5$ ev is the ionization potential of hydrogen. Equation (1.9) is valid under the following conditions: (a) velocity β of the primary particle large compared with the velocities of atomic electrons, (b) η large compared with the binding energy and (c) η small compared with the maximum transferable energy E'_m.

For a given velocity β, k_η is independent of the mass of the primary particle. Since $p/\mu=\beta/(1-\beta^2)^{\frac{1}{2}}$, we may also say that k_η is a function of p/μ only. This function is graphically represented in Fig. 1 for $\eta=10^4$ ev and $Z=7.3$ (air). The initial decrease of k_η with increasing p/μ is caused by the factor $1/\beta^2$. It corresponds to the fact that, for a given impact parameter, the interaction between the passing particle and the atom becomes weaker as the time spent by the particle in the neighborhood of the atom becomes shorter. When β approaches its limiting value 1, the factor $1/\beta^2$ becomes practically constant and k_η increases with increasing momentum because of the factor $\beta^2/(1-\beta^2)$ in the logarithm. The reason for this increase has to be sought in the Lorentz contraction of the Coulomb

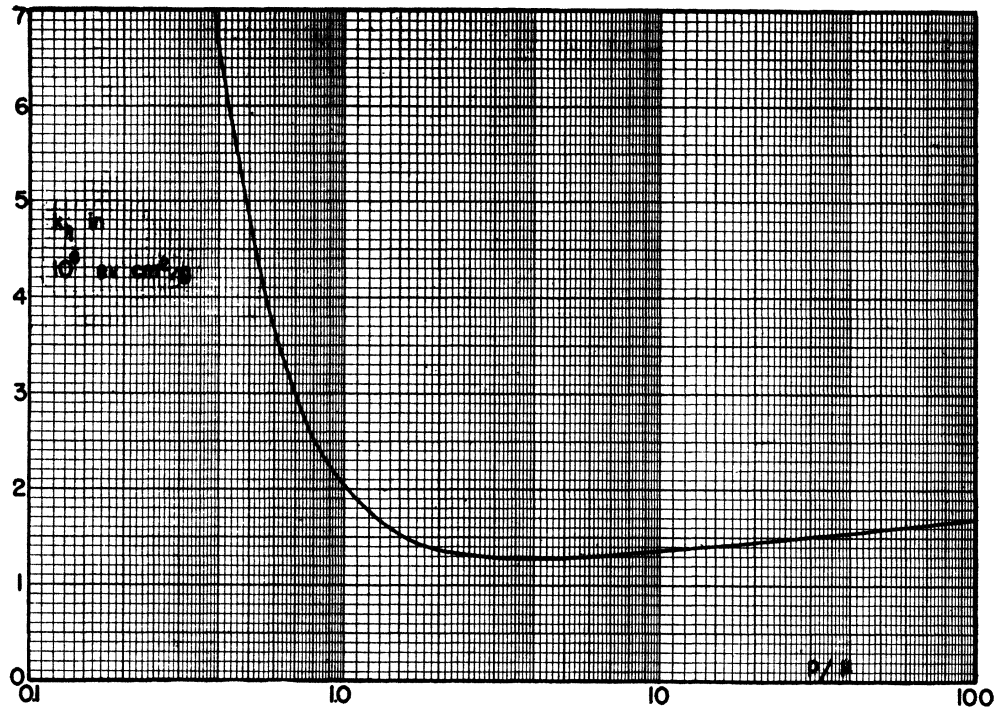

FIG. 1. Collision loss k_η for $\eta = 10^4$ ev, in air [from Eq. (1.9)]. Abscissa p/μ, ordinate 10^6 ev per g/cm². Valid for all particles with unit charge, provided $E'_m \gg 10^4$ ev.

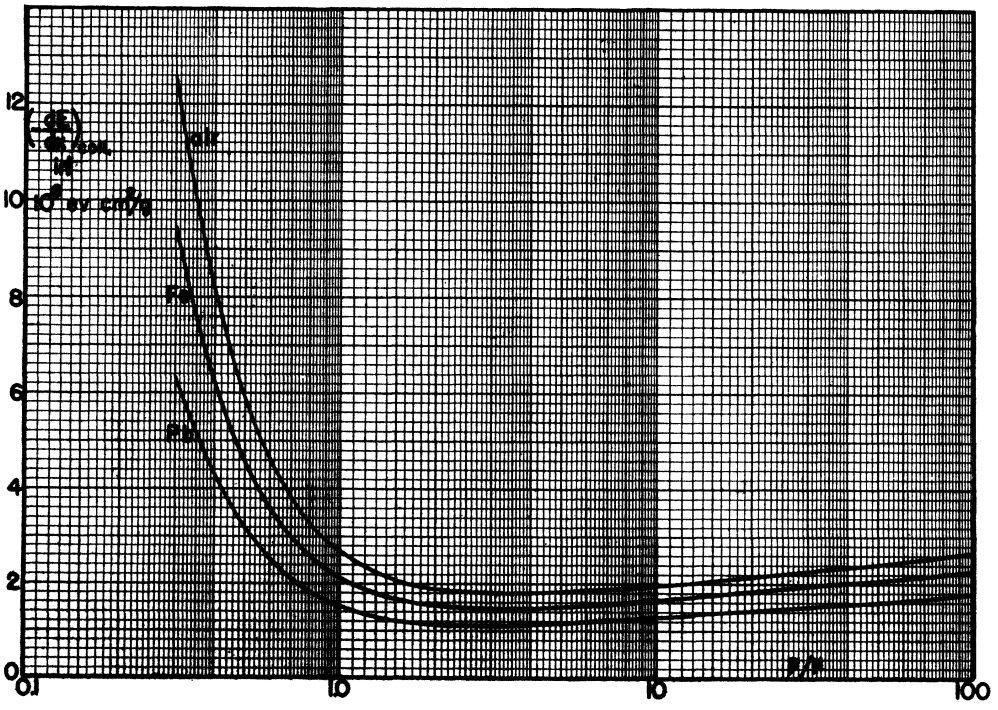

FIG. 2. Total collision loss $(-dE/dx)_{\text{coll}}$ of mesotrons in air, iron and lead [from Eqs. (1.11), (1.11a)]. Abscissa p/μ, ordinate 10^6 ev per g/cm². Calculated for $\mu = 10^8$ ev/c^2, but practically valid for values of μ from about 0.5×10^8 ev/c^2 to ∞.

field of the passing particle, which causes the electric effect of this particle to be felt at larger distances from its geometrical path.

Upon substitution of the numerical values of the constants, and considering only the extreme relativistic case, Eq. (1.9) becomes

$$k_\eta = 1.53\times 10^5 \frac{Z}{A}\left[21.4+\log\frac{\eta}{10^6} +2\log\frac{p}{\mu}-2\log Z\right]. \quad (1.9a)$$

The *total* energy loss $(-dE/dx)_{\text{coll}}$ is obtained by adding k_η and K_η. Thus, for mesotrons or protons with energy smaller than μ^2/μ_e, Eqs. (1.8a) and (1.9) give

$$\left(-\frac{dE}{dx}\right)_{\text{coll}} = \frac{2C\mu_e}{\beta^2}\left[\log\frac{2\mu_e\beta^2E'_m}{(1-\beta^2)I^2(Z)}-2\beta^2\right]. \quad (1.11)$$

The total energy loss for mesotrons in air, iron, and lead is plotted as a function of p/μ in Fig. 2.

If $p\ll\mu^2/\mu_e$, E'_m can be approximated by Eq. (1.5a), and (1.11) becomes

$$\left(-\frac{dE}{dx}\right)_{\text{coll}} = \frac{2C\mu_e}{\beta^2}\times\left[\log\frac{4\mu_e^2\beta^4}{(1-\beta^2)^2I^2(Z)}-2\beta^2\right]. \quad (1.11a)$$

Within the limits of validity of (1.11a), $(-dE/dx)_{\text{coll}}$, as well as k_η, is a function only of β, or of p/μ. In the extreme relativistic case, (1.11a) can be written as follows:

$$\left(-\frac{dE}{dx}\right)_{\text{coll}} = 1.53\times 10^5 \times\frac{Z}{A}\left[20.5+4\log\frac{p}{\mu}-2\log Z\right]. \quad (1.11b)$$

The total energy loss of electrons and positrons can easily be calculated using Eq. (1.9a) and Eqs. (1.7a) and (1.7b), respectively. The results are almost identical for both types of particles, and can be expressed in close approximation by the formula

$$\left(-\frac{dE}{dx}\right)_{\text{coll}} = 1.53\times 10^5 \times\frac{Z}{A}\left[20.2+3\log\frac{p}{\mu}-2\log Z\right]. \quad (1.12)$$

The *momentum* loss is easily obtained from the *energy* loss. Indeed, since $dp/dE=1/\beta$, it is simply

$$-\frac{dp}{dx} = -\frac{1}{\beta}\frac{dE}{dx}. \quad (1.13)$$

§4. *Density Effect*

So far, in investigating the interaction of charged particles with atoms, we have considered the latter as isolated. This is permitted when the particle travels in a gas. When the particle travels in a condensed material the atoms can still be considered as isolated in the case of a close collision, but no longer so when the impact parameter is larger than the atomic distances. For such distant collisions one has to take into account the screening of the electric field of the passing particle by the atoms of the medium. The screening reduces the interaction and decreases, therefore, the energy loss. Since distant collisions become more and more important as the velocity increases, the correction to be applied to the expression for the energy loss is an increasing function of the velocity. The influence of the density on the collision loss was first suggested by Swann (S38b) and quantitatively investigated by Fermi (F39, F40). According to Fermi, the quantity to be subtracted from the energy loss calculated by considering the atoms as isolated is given by the following formulae:

$$\text{for } \beta<\epsilon^{-\frac{1}{2}},\quad \Delta(\beta)=\frac{2C\mu_e}{\beta^2}\log\epsilon,$$

$$\text{for } \beta>\epsilon^{-\frac{1}{2}},\quad \Delta(\beta)=\frac{2C\mu_e}{\beta^2}\left[\log\frac{\epsilon-1}{1-\beta^2}+\frac{1-\epsilon\beta^2}{\epsilon-1}\right],$$

where ϵ is the dielectric constant of the medium, relative to vacuum.

A more refined analysis by Halpern and Hall (H40; see also W40) confirmed the existence of

the density dependence of the collision loss, but showed that the effect is considerably smaller than predicted by Fermi. No numerical calculations based upon the theory of Halpern and Hall have yet been published. But the authors, in a private communication to one of us, have indicated that in the case of mesotrons of 8×10^8 ev energy, the correction amounts to $(2C\mu_e/\beta^2)0.4$ for iron and $(2C\mu_e/\beta^2)0.55$ for lead. It would seem, therefore, that the density effect is completely negligible, at least as long as the kinetic energy is not very large compared with the rest energy.

In what follows, the correction for the density effect will be disregarded.

§5. *Range of Mesotrons and Protons*

Mesotrons and protons of moderate energy traversing matter lose energy almost exclusively by collision processes. Since the average energy transfer in each collision is small, a very large number of collisions is required to decrease the energy of the primary particle by any appreciable fraction. Consequently, the fluctuations in the energy loss are small and in a given material all particles of the same energy travel practically the same distance R before being stopped. This distance is called *the range*, and, as a function of momentum, satisfies the following differential equation:

$$\frac{dR}{dp}=-\frac{1}{(dp/dx)}, \tag{1.14}$$

which can be written as

$$\frac{dR}{d(p/\mu)}=\frac{\mu\beta^3}{2C\mu_eB}=\frac{\mu}{2C\mu_eB}\frac{(p/\mu)^3}{[1+(p/\mu)^2]^{\frac{3}{2}}}. \tag{1.14a}$$

In the case of energies small compared with μ^2/μ_e, B is only a function of p/μ (see §3):

$$B=\log\frac{4\mu_e^2(p/\mu)^4}{I^2(Z)}-2\frac{(p/\mu)^2}{1+(p/\mu)^2}. \tag{1.15}$$

Equation (1.14a) shows that, for a given value of p/μ, the range is directly proportional to the mass of the particle. B is a slowly varying function of p/μ and can be considered as a constant over a small momentum interval. Equation (1.14a) then yields (see H38)

$$R\left(\frac{p_2}{\mu}\right)-R\left(\frac{p_1}{\mu}\right)=\frac{\mu}{2C\mu_eB}\times\left\{\frac{(p_2/\mu)^2+2}{[(p_2/\mu)^2+1]^{\frac{1}{2}}}-\frac{(p_1/\mu)^2+2}{[(p_1/\mu)^2+1]^{\frac{1}{2}}}\right\}. \tag{1.16}$$

The range of particles with a given momentum p can be obtained by application of Eq. (1.16) to successive momentum intervals from 0 to p. In this way the graphs in Fig. 3 have been calculated. A small error is involved in these calculations because the expression (1.11a) for dE/dx is not valid when the velocity of the primary particle is reduced to a value comparable with the velocity of the atomic electrons. This error, however, is mostly negligible because the residual range of such a slow particle is only a small fraction of one g/cm² while the observed cosmic-ray particles have in general ranges of at least several g/cm². Only for the discussion of mesotron or proton tracks ending in the gas of a cloud chamber may a more accurate evaluation of R be desirable. The energy-range relation for protons of low energy has been calculated by Livingston and Bethe (L37) using the correct expression for dp/dx. R/μ, as deduced from the calculations of Livingston and Bethe, is plotted against p/μ in Fig. 4. According to our previous discussion, this graph can be used for mesotrons as well as for protons.

§6. *Primary Specific Ionization*

By primary specific ionization we mean the number of collisions per g/cm² resulting in the ionization of atoms. The primary specific ionization in hydrogen has been determined theoretically by Bethe and is given by the following equation:

$$j_P=\frac{2C\mu_e}{\beta^2}\frac{a}{I_0}\left[\log\frac{2\mu_e\beta^2}{(1-\beta^2)I_0}+b-\beta^2\right], \tag{1.17}$$

where $I_0=I_H=13.5$ ev is the ionization potential, and the constants a and b have the values $a=0.285$, $b=3.04$ (B33a). The primary specific ionization in gases other than hydrogen should still be represented by an equation of the type of Eq. (1.17), in which I_0 is the ionization potential of the *outer shell*, and with different

values of the constants a and b. No theoretical determination of a and b as a function of the atomic number is available. However, b should not change very much and, since it is small compared with the logarithm, in first approximation it can be considered as independent of Z. The only arbitrary constant is then a, which can be determined empirically by measurement of the primary specific ionization at a given velocity.

Equation (1.17) shows that the primary specific ionization is independent of the mass of the particle, for a given value of the velocity. Comparison with Eq. (1.9) indicates that k_η and j_P have a very similar functional dependence on β.

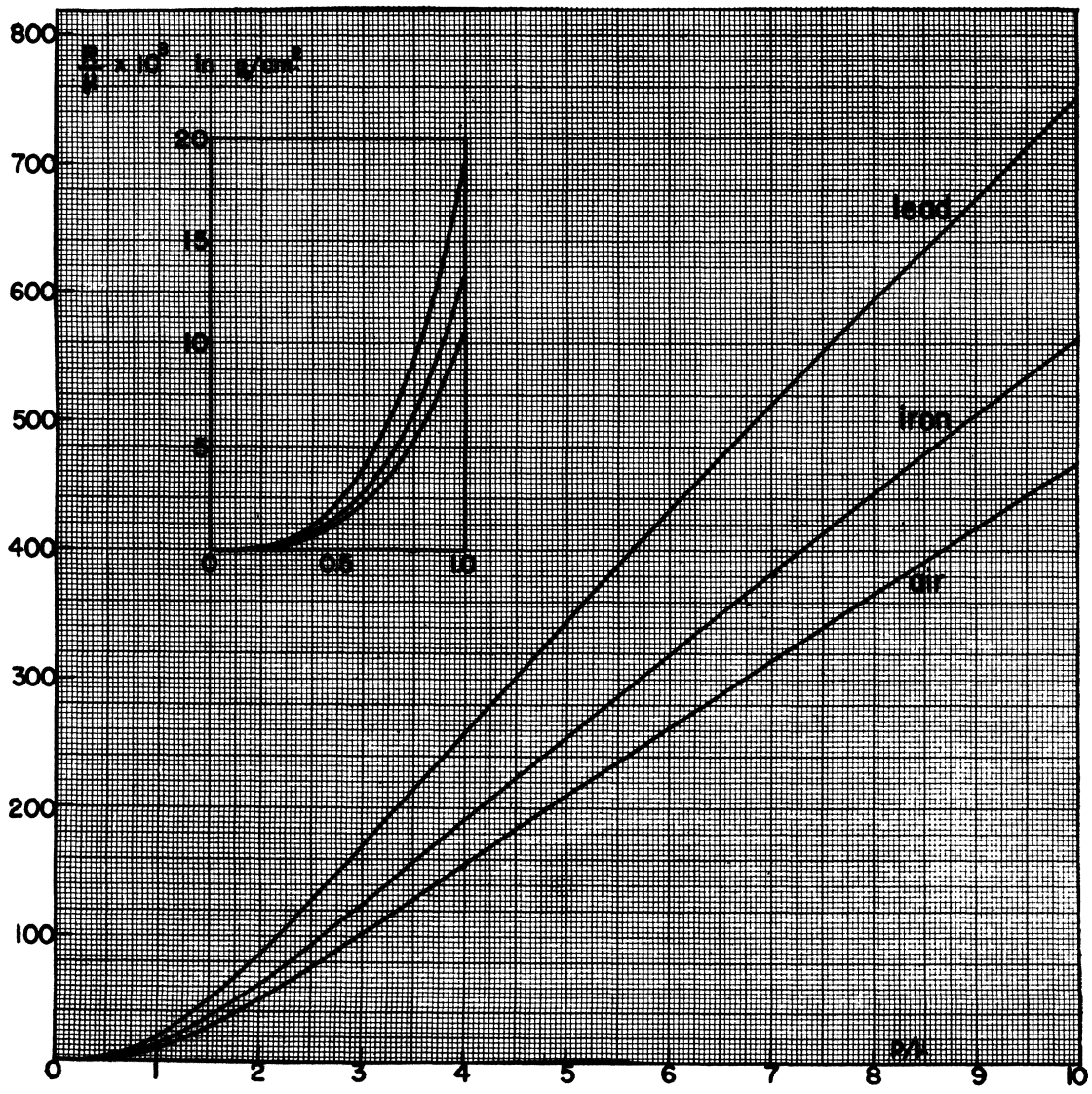

FIG. 3. Range of mesotrons or protons in air, iron and lead [from Eq. (1.16)]. Abscissa p/μ, ordinate $R\times 10^8/\mu$ in g/cm². For mesotrons with $\mu = 10^8$ ev/c^2, therefore, the ordinate represents the range in g/cm². The graphs are valid for particles with unit charge and arbitrary mass, provided other losses are negligible compared with the collision loss.

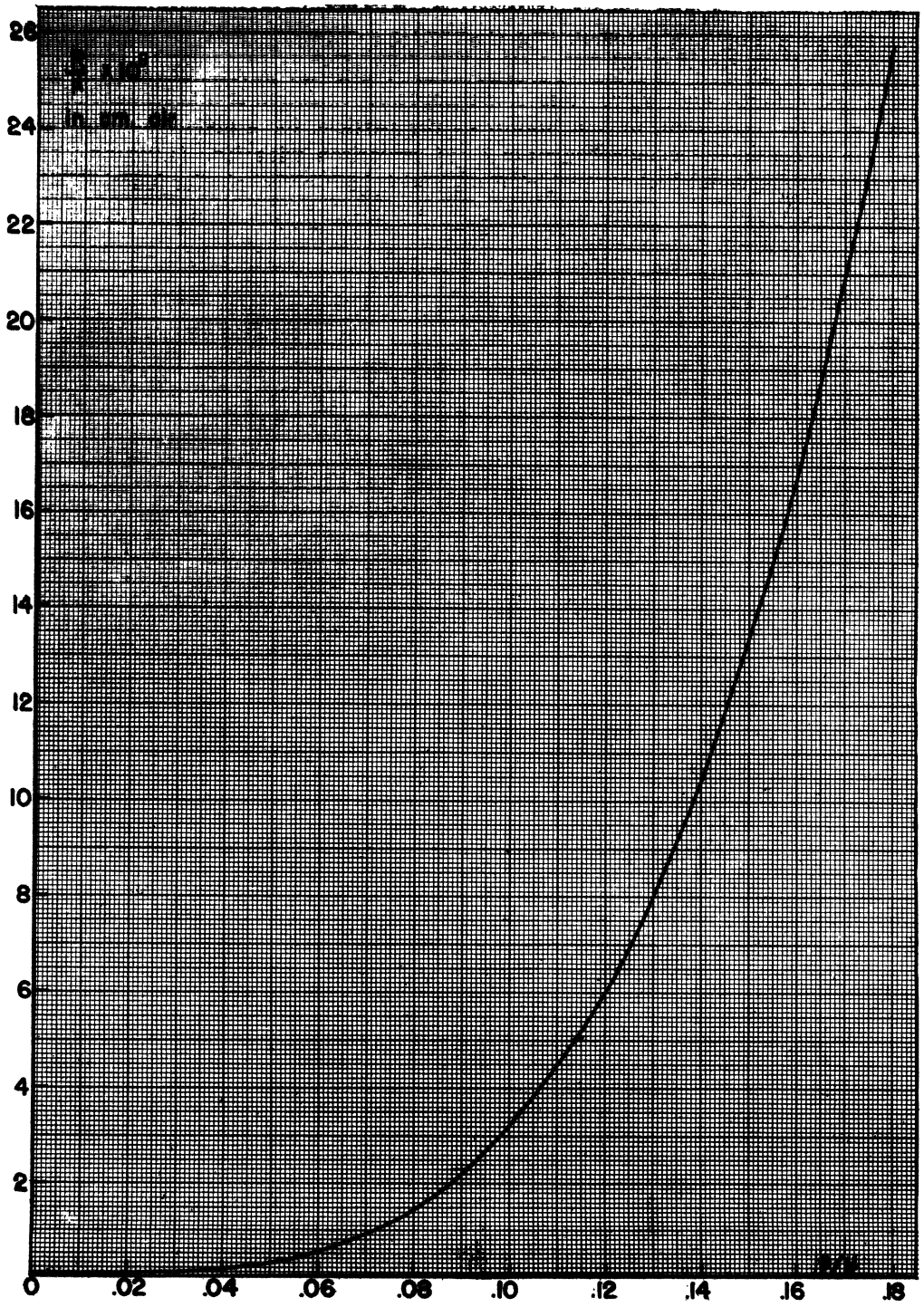

FIG. 4. Range of low energy mesotrons or protons in air (from the calculations of Livingston and Bethe, L37). Abscissa p/μ, ordinate $R\times 10^8/\mu$ in cm of air at N. T. P. For mesotrons with $\mu=10^8$ ev/c^2, therefore, the ordinate gives the range directly in cm of air. The graph is valid for particles with unit charge and arbitrary mass, provided other losses are negligible compared with the collision loss.

B. Compton Effect

§7. *Application of the Conservation Laws*

The Compton effect can be described as the collision between a photon and a free electron. Let W be the energy of the primary photon and suppose the electron to be initially at rest. As a result of the collision, the photon is scattered at an angle θ with a reduced energy W' and the electron acquires the energy $E=W-W'$.

The conservation laws of energy and momentum yield the following relation between W, W' and θ (Compton formula):

$$W'=W-E=\frac{W\mu_e}{\mu_e+W(1-\cos\theta)}. \quad (1.18)$$

W' decreases and E increases with increasing θ. It may be noted that, when $W\gg\mu_e$, W' is of the order of μ_e and E is of the order of W for all collisions except those for which $\cos\theta$ is very close to 1, and that the minimum value of W' is $\mu_e/2$.

§8. *Differential Scattering Probability*

Let $\kappa(W, W')dW'dx$ be the probability for a photon of energy W traversing a thickness dx g/cm² to undergo Compton collision in which the scattered photon has an energy between W' and $W'+dW'$. The function κ has been calculated by Klein and Nishina (K29) and is given by the following equation:

$$\kappa(W, W')dW'=\frac{C\mu_e}{W}\frac{dW'}{W'}\left[1+\left(\frac{W'}{W}\right)^2-\frac{W'}{W}\sin^2\theta\right], \quad (1.19)$$

where C is defined by Eq. (1.6) and θ by Eq. (1.18). When $W\gg\mu_e$, $W'/W\sin^2\theta$ is negligible compared with 1 because W'/W is much smaller than 1 except when θ is nearly 0. Thus (1.19) can be simplified into

$$\kappa(W, W')dW'=\frac{C\mu_e}{W}\frac{dW'}{W'}\left[1+\left(\frac{W'}{W}\right)^2\right]. \quad (1.19a)$$

It appears from Eq. (1.19a) that the scattering probability decreases rapidly with increasing W'; i.e., with decreasing E.

§9. *Total Scattering Probability*

Let Σ_c be the total probability for a photon of energy W to undergo Compton scattering in a thickness of dx g/cm². Σ_c is obtained by integration of $\kappa dW'$ from $\mu_e/2$ to W. When $W\gg\mu_e$, we get, using Eq. (1.19a):

$$\Sigma_c(W)=\frac{C\mu_e}{W}\left[\log\frac{2W}{\mu_e}+\frac{1}{2}\right]. \quad (1.20)$$

The expression (1.20) for the total scattering probability has been calculated assuming Eq. (1.19a) to be valid for all values of W'. Actually Eq. (1.19a) is only valid for those collisions in which the energy of the recoil electron is large compared with the binding energy, because otherwise the electron cannot be considered as free. The error, however, is negligible because of the small number of recoil electrons of low energy produced by Compton effect. This contrasts with the case of collision processes, in which most of the secondary electrons have small energies, so that it is necessary in the latter case to take into consideration the binding forces. A plot of the total scattering probability as a function of energy is given in Figs. 13 and 13a (air and lead).

C. Radiation Processes

§10. *General Remarks*

The emission of photons by charged particles is closely connected with their deflection in the electric field of the nucleus. According to the classical electromagnetic theory, a charged particle emits electromagnetic waves whenever it undergoes an acceleration, and the intensity of the emitted radiation is directly proportional to the square of the acceleration. According to the quantum theory, a collision of a charged particle with a nucleus may or may not be accompanied by the emission of radiation, the latter case (elastic scattering) being the more probable. However, some qualitative results on the radiation probability can easily be derived by classical considerations. Thus, classically, for a given distance of approach the acceleration is directly proportional to the charge of the nucleus; it follows that the radiation loss will be

a rapidly increasing function of the atomic number. On the other hand, the acceleration is inversely proportional to the mass μ of the particle; thus the radiation loss of electrons will be much larger than that of heavier particles like mesotrons or protons. We may also say that the intensity of the field in which a particle radiates appreciably increases as the mass of the particle is increased and, therefore, that the radiation losses of heavy particles arise only from much closer impacts than those responsible for the radiation losses of electrons. Also, the impact parameter of radiative collisions increases as the energy of the particle increases, because of the Lorentz contraction of the electric field of the moving particle.

The distance from the nucleus at which radiation phenomena occur plays an essential role in the development of the theory. If this distance is large compared with the nuclear radius and small compared with the atomic radius, the field acting on the particle during the radiation process can be considered as the Coulomb field of a point charge Z concentrated in the nucleus. If the distance is of the order of the atomic radius or larger, the screening of the nuclear field by the outer electrons must be taken into account. If, finally, the distance is of the order of the nuclear radius, the nuclear field cannot be considered any longer as that of a point charge.

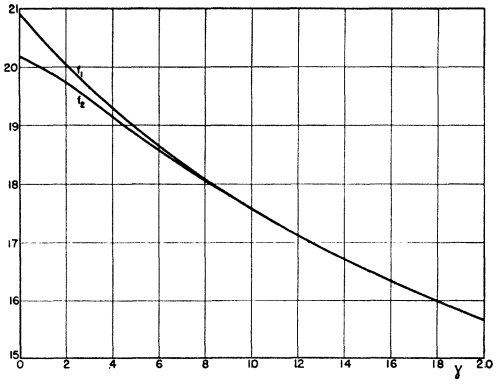

FIG. 5. Functions $f_1(\gamma)$ and $f_2(\gamma)$ in Eqs. (1.24) and (1.43) (from Bethe and Heitler, B34).

In the radiation processes the energy W of the emitted photon is not determined by its angle of emission because the nucleus takes part of the momentum. However, it can be proved that the average angle of emission of photons of energy W by particles of mass μ is of the order of μ/W.

§11. *Differential Radiation Probability of Electrons*

We shall first consider the radiation processes of positive or negative electrons. Let $U=E+\mu_e$ be the *total* energy of the electron, inclusive of the rest energy. Let $\Phi(U, v)dvdx$ be the probability for an electron of energy U traversing a thickness of dx g/cm^2 to emit a photon with fractional energy between v and $v+dv$, the fractional energy v being defined as the ratio W/U of the energy of the secondary photon to the total energy of the primary electron. Thus the maximum value of v is $1-(\mu_e/U)$. Since we only consider energies large compared with μ_e, in most cases the total energy U can be identified with the kinetic energy E.

The radiation losses of electrons take place at distances from the nucleus which are always large compared with the nuclear radius. Thus, the nuclear field can be described as that of a point charge. The screening effect of the outer electrons has been calculated by Bethe and Heitler (B34) on the basis of the Fermi-Thomas model of the atom. It turns out that the influence of the screening on a particular radiation process is determined by the quantity

$$\gamma=100\frac{\mu_e}{U}\frac{v}{1-v}Z^{-\frac{1}{3}} \tag{1.21}$$

and increases with decreasing γ. For $\gamma\gg1$ the screening can be practically neglected. In the case $\gamma\approx0$ we shall call the screening "complete." For a given value of v, γ decreases with increasing U. Thus, if the primary energy is large enough, the screening can be considered complete for practically all energies of the emitted photon.

If U is assumed to be large compared with μ_e, the expressions for $\Phi(U, v)$ for various ranges of γ are found to be as follows:

no screening, $\gamma \gg 1$

$$\Phi(U, v)dv = 4\alpha \frac{N}{A} Z^2 r_0^2 \frac{dv}{v} [1 + (1-v)^2 - \tfrac{2}{3}(1-v)] \left[\log \left(\frac{2U}{\mu_e} \frac{1-v}{v} \right) - \frac{1}{2} \right]; \tag{1.22}$$

complete screening, $\gamma \approx 0$

$$\Phi(U, v)dv = 4\alpha \frac{N}{A} Z^2 r_0^2 \frac{dv}{v} \{ [1 + (1-v)^2 - \tfrac{2}{3}(1-v)] \log (183 Z^{-\frac{1}{3}}) + \tfrac{1}{9}(1-v) \}; \tag{1.23}$$

intermediate cases

$\gamma < 2$

$$\Phi(U, v)dv = 4\alpha \frac{N}{A} Z^2 r_0^2 \frac{dv}{v} \left\{ [1 + (1-v)^2] \left[\frac{f_1(\gamma)}{4} - \frac{1}{3} \log Z \right] - \tfrac{2}{3}(1-v) \left[\frac{f_2(\gamma)}{4} - \frac{1}{3} \log Z \right] \right\}, \tag{1.24}$$

$2 < \gamma < 15$

$$\Phi(U, v)dv = 4\alpha \frac{N}{A} Z^2 r_0^2 \frac{dv}{v} [1 + (1-v)^2 - \tfrac{2}{3}(1-v)] \left[\log \left(\frac{2U}{\mu_e} \frac{1-v}{v} \right) - \frac{1}{2} - c(\gamma) \right]. \tag{1.25}$$

The functions $f_1(\gamma)$, $f_2(\gamma)$, and $c(\gamma)$ are given by Fig. 5 and Table III. Equation (1.23) shows that when the screening can be considered as complete the probability for a given fractional energy loss does not depend on the primary energy U.

§12. *Average Radiation Loss of Electrons*

The average energy loss per g/cm² due to radiation processes is given by the equation

$$-(dE/dx)_{\text{rad}} = U \int_0^{1-\mu_e/U} v\Phi(U, v)dv. \tag{1.26}$$

At the limit for small and large energy, respectively, Eqs. (1.22) and (1.23) can be used and the energy loss becomes:

$$\mu_e \ll U \ll 137 \mu_e Z^{-\frac{1}{3}}, \quad -(dE/dx)_{\text{rad}} = 4\alpha \frac{N}{A} Z^2 r_0^2 \cdot U \left(\log \frac{2U}{\mu_e} - \frac{1}{3} \right), \tag{1.27}$$

$$U \gg 137 \mu_e Z^{-\frac{1}{3}}, \quad -(dE/dx)_{\text{rad}} = 4\alpha \frac{N}{A} Z^2 r_0^2 \cdot U \left[\log (183 Z^{-\frac{1}{3}}) + \frac{1}{18} \right]. \tag{1.28}$$

For the intermediate cases the integral in (1.26) must be evaluated numerically. It is seen that the average radiation loss increases with increasing energy and is proportional to the energy for large energies.

§13. *Radiation Length. Simplified Formulae*

We shall define as *radiation length* the thickness X_0, where

$$\frac{1}{X_0} = 4\alpha \frac{N}{A} Z^2 r_0^2 \log 183 Z^{-\frac{1}{3}}, \tag{1.29}$$

and shall denote by t a thickness measured in radiation lengths. We shall then introduce the differential radiation probability per radiation length

$$\varphi(U, v) = X_0 \Phi(U, v) \tag{1.30}$$

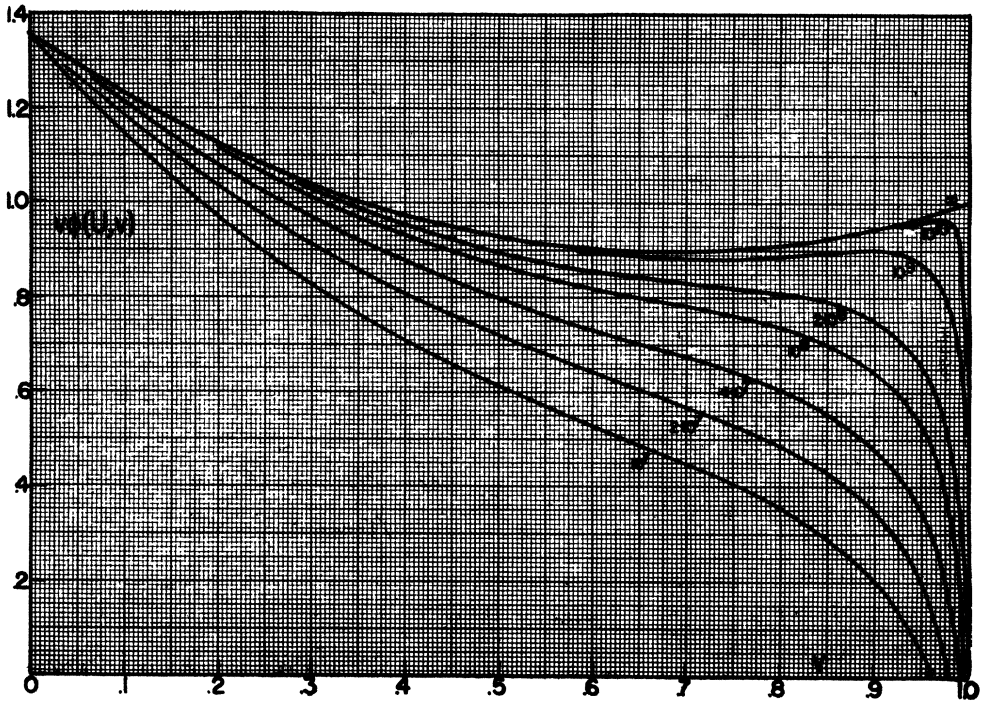

FIG. 6. Differential radiation probability per radiation length of air for electrons of various energies. Abscissa, $v = W/U$ ordinate, $v\varphi(U, v)$. The numbers attached to the curves indicate the energy U of the primary electron.

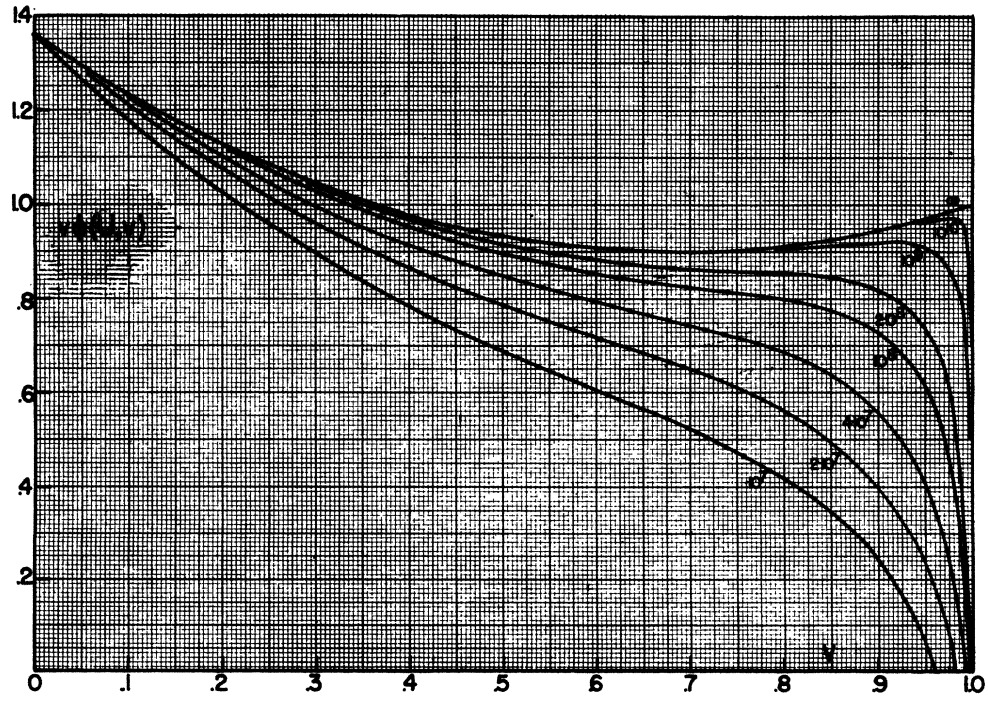

FIG. 7. Differential radiation probability per radiation length of lead for electrons of various energies. Abscissa, $v = W/U$; ordinate, $v\varphi(U, v)$. The numbers attached to the curves indicate the energy U of the primary electron.

TABLE III. *Numerical values of the function $c(\gamma)$ in Eqs. (1.25) and (1.44) (from Bethe and Heitler, B34).*

γ	2	2.5	3	4	5	6	8	10	15
$c(\gamma)$	0.21	0.16	0.13	0.09	0.065	0.05	0.03	0.02	0.01

and the average fractional energy loss per radiation length

$$-\frac{1}{E}\left(\frac{dE}{dt}\right)_{\text{rad}} = -X_0 \cdot \frac{1}{E}\left(\frac{dE}{dx}\right)_{\text{rad}} \approx \int_0^1 v\varphi(U, v)dv. \tag{1.31}$$

The function $v\varphi(U, v)$ is plotted against v for various U and two substances (air and lead) in Figs. 6 and 7. The average *fractional* energy loss $-(1/E)(dE/dt)_{\text{rad}}$ is given as a function of energy in Fig. 9. It appears that the description of radiation phenomena is only slightly dependent on atomic number when thicknesses are measured in radiation lengths. Moreover, the dependence on atomic number is reduced with increasing energy and disappears almost entirely for large energies. In fact, in the case of complete screening the differential radiation probability per radiation length is

$$\varphi_0(v)dv = [1+(1-v)^2-(1-v)(\tfrac{2}{3}-2b)](dv/v) \tag{1.23a}$$

and the average fractional energy loss

$$-(1/E)(dE/dt) = 1+b, \tag{1.28a}$$

where $b = 1/[18 \log (183Z^{-\frac{1}{3}})]$. b is very small compared with 1 and its value ranges from 0.012 to 0.015 when Z changes from 7.3 (air) to 82 (lead). Thus, no appreciable error is made by taking $b = 0.0135$ for all elements.

Equations (1.23a) and (1.28a) are also valid for substances other than pure elements, provided we take

$$1/X_0 = p_1/X_1 + p_2/X_2 + \cdots, \tag{1.29a}$$

where p_1, p_2, $\cdots$, are the fractional weights of the various components and X_1, X_2, $\cdots$, the corresponding radiation lengths. The values of X_0 for various substances are listed in Table IV (see p. 271).

In many instances it is convenient to substitute, in place of the correct expressions for the radiation probability, approximate expressions which have a simpler mathematical form. Three of these simplified expressions are given in the following equations:

$$\varphi_1(v)dv = dv/v, \tag{1.32}$$

$$\varphi_2(v)dv = -\frac{1}{\log 2}\frac{dv}{\log (1-v)}, \tag{1.33}$$

$$\varphi_3(v)dv = \frac{4}{3}(1-v)\frac{dv}{v}. \tag{1.34}$$

Inspection of Fig. 8 indicates to what degree the simplified formulae approach the correct expressions for various energies.

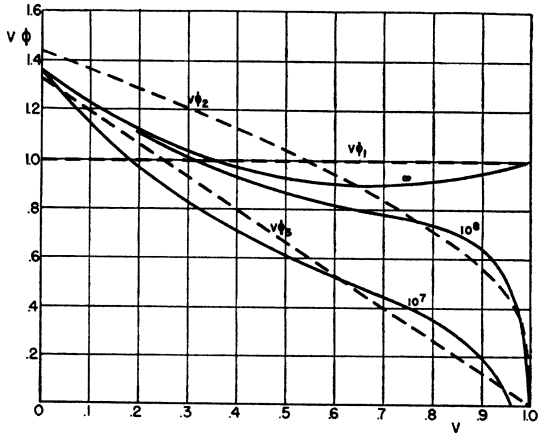

FIG. 8. Comparison between the approximate expressions φ_1, φ_2, φ_3 for the differential radiation probability (dashed lines) and the correct expression for the same quantity at various energies in air (solid lines).

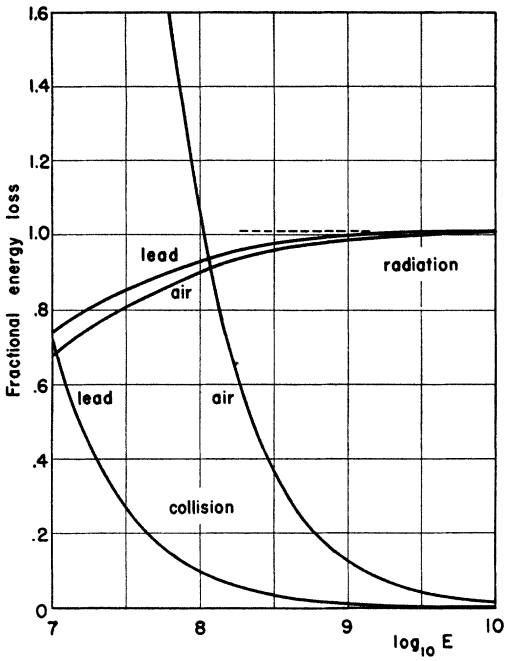

FIG. 9. Fractional energy loss by collision $-(1/E)\times(dE/dt)_{\text{coll}}$ and fractional energy loss by radiation $-(1/E)(dE/dt)_{\text{rad}}$ for electrons per radiation length of air and of lead.

§14. *Comparison between Radiation Loss and Collision Loss. Fluctuations in the Radiation Loss*

As already pointed out, the average energy loss by radiation increases rapidly with increasing energy, while the average energy loss by collision is practically a constant. Thus, at large energies radiation losses are much more important than collision losses, while at small energies the reverse is true. In Fig. 9 the curves giving the fractional energy loss by collision in one radiation length of air and lead are drawn for comparison with the corresponding radiation losses. It is apparent that the energy at which the radiation loss overtakes the collision loss decreases with increasing atomic number.

Another characteristic difference between radiation losses and collision losses is caused by the fact that the energy loss by radiation occurs in fewer and larger steps than the energy loss by collision. Thus, while electrons of a given energy traversing a given thickness all lose practically the same energy by collision, there is a considerable straggling in their energy loss by radiation. We may ask what the probability is for an electron of initial energy U_0 to have an energy between U and $U+dU$ after traversing a thickness t. A solution of this problem has been given by Bethe and Heitler (B34), using the simplified formula (1.33) for the radiation probability, and is expressed by

$$w(U)dU=\frac{dU}{U_0}\frac{(\log U_0/U)^{(t/\log 2)-1}}{\Gamma(t/\log 2)}. \tag{1.35}$$

Equation (1.35) is valid when collision losses can be neglected.

§15. *Radiation Processes of Mesotrons*

The emission of photons by mesotrons takes place at much smaller distances from the nucleus than the emission of photons by electrons. Therefore, in the theory of radiation processes of mesotrons the screening of the nuclear field by the outer electrons can be neglected to a greater extent than in the corresponding theory for electrons. Instead, it is necessary to take into account the fact that the nuclear field at distances smaller than the nuclear radius cannot be considered as the Coulomb field of a point charge.

In radiation phenomena the spin of the mesotron plays an even more essential role than in collision processes (see §2). This is so because the average impact parameter for radiation is much smaller than for collision, and because the spin-dependent forces have a small range. According to Christy and Kusaka (C41) the differential radiation probabilities per g/cm² for mesotrons of spin 0, ½, and 1, respectively, are as follows:

Spin 0

$$\Phi(U,v)dv=\alpha\frac{N}{A}Z^2r_0^2\left(\frac{\mu_e}{\mu}\right)^2\frac{16}{3}\frac{1-v}{v}dv\left[\log\left(\frac{12}{5}\frac{1-v}{v}\frac{U}{\mu Z^{\frac{1}{3}}}\right)-\frac{1}{2}\right]. \tag{1.36}$$

Spin $\frac{1}{2}$

$$\Phi(U,v)dv=\alpha\frac{N}{A}Z^2r_0^2\left(\frac{\mu_e}{\mu}\right)^2\frac{16}{3}\left(\frac{3v}{4}+\frac{1-v}{v}\right)dv\left[\log\left(\frac{12}{5}\frac{1-v}{v}\frac{U}{\mu Z^{\frac{1}{3}}}\right)-\frac{1}{2}\right]. \tag{1.37}$$

Spin 1

$$\Phi(U,v)dv=\alpha\frac{N}{A}Z^2r_0^2\left(\frac{\mu_e}{\mu}\right)^2dv\left\{\left[\frac{16}{3}\frac{1-v}{v}+\frac{13}{12}v-\frac{5}{24}\frac{v^3}{1-v}\right]\log\left(\frac{2\pi}{5}\frac{1-v}{v}\frac{U}{\mu Z^{\frac{1}{3}}}\right)-\frac{v}{1-v}\left(\frac{10-10v+3v^2}{8}\right)\right.$$

$$\left.-\frac{52}{9}\frac{1-v}{v}+\frac{v}{1-v}\left(\frac{34-34v+7v^2}{24}\right)\log^2\left(\frac{2\pi}{5}\frac{1-v}{v}\frac{U}{\mu Z^{\frac{1}{3}}}\right)+\frac{\pi U}{5\mu Z^{\frac{1}{3}}}\left(\frac{2-2v+7v^2}{12}\right)\right\}, \tag{1.38}$$

where U is the total energy of the primary mesotron and $v=W/U$ is the fractional energy of the emitted photon. The above formulae have been calculated on the assumption that $U\gg\mu$, that the screening of the outer electrons may be neglected, and that the electric potential of the nuclear field may be considered to be that of a point charge for distances larger than the nuclear radius d and constant for distances smaller than d. d was taken as equal to $(5/6)(r_0\mu_e/\alpha\mu)Z^{\frac{1}{3}}=0.57r_0Z^{\frac{1}{3}}$. The expression for the radiation probability for mesotrons of spin $\frac{1}{2}$ is very similar to that for electrons (see Eq. (1.22)). The factor $(\mu_e/\mu)^2$ is due to the difference in mass, and the factor $(6/5)Z^{-\frac{1}{3}}$ in the logarithm is connected with the cut-off of the nuclear field at d. The neglect of screening sets an upper limit $U\approx5\times10^{11}$ ev to the validity of the expressions for the radiation probabilities for spin 0 and spin $\frac{1}{2}$ (Eqs. 1.36 and 1.37). The radiation probability for spin 1 (Eq. 1.38) is less affected by the screening, but includes terms describing processes which cannot legitimately be computed by the existing theories when U is larger than about 2×10^{10} ev. Thus (1.38) is only valid for primary energies smaller than this limit. A minimum estimate of the radiation probability for $U>2\times10^{10}$ ev can be obtained, according to Christy and Kusaka, neglecting the doubtful phenomena altogether. By doing so, Eq. (1.38) becomes

$$\Phi(U,v)dv=\alpha\frac{N}{A}Z^2r_0^2\left(\frac{\mu_e}{\mu}\right)^2dv\left\{\left[\frac{16}{3}\frac{1-v}{v}+\frac{13}{12}v-\frac{5}{24}\frac{v^3}{1-v}\right]\log\left(\frac{2\pi}{5}\frac{1-v}{v}\frac{U}{\mu Z^{\frac{1}{3}}}\right)\right.$$

$$-\frac{v}{1-v}\left(\frac{10-10v+3v^2}{8}\right)-\frac{52}{9}\frac{1-v}{v}+\frac{v}{1-v}\left(\frac{34-34v+7v^2}{24}\right)\left[\log^2\left(\frac{2\pi}{5}\frac{1-v}{v}\frac{U}{\mu Z^{\frac{1}{3}}}\right)\right.$$

$$\left.\left.-\log^2\left(\frac{\pi}{5A}\frac{U}{\mu Z^{\frac{1}{3}}}\right)\right]+\left[A+A\log\left(\frac{\pi}{5A}\frac{U}{\mu Z^{\frac{1}{3}}}\right)\right]\left(\frac{2-2v+7v^2}{12}\right)\right\}, \tag{1.38a}$$

where A is a constant of the order of $1/\alpha=137$.

Comparison between Eqs. (1.36), (1.37) and (1.38) or (1.38a) indicates that the probability of large radiation losses is much greater for mesotrons of spin 1 than for mesotrons of spin $\frac{1}{2}$, and somewhat greater for mesotrons of spin $\frac{1}{2}$ than for mesotrons of spin 0. Hence the probability of large radiation losses by mesotrons depends on the spin in the same way as the probability of large collision losses. Numerical evaluation of the formulae shows that, for any value of the spin, large energy transfers are more likely to occur by radiation than by collision. The *total* energy loss, however, is mainly determined by collision processes up to much larger energies than in the case of electrons.

It must be emphasized that the above conclusions on the radiation losses of mesotrons do not have the same degree of certainty as do the corresponding results on the radiation losses of electrons, because the theory involves the properties of the electromagnetic field of mesotrons at distances smaller than 10^{-13} cm from the mesotron (see §2).

D. Pair Production

§16. *General Remarks*

A high energy photon, traversing the intense electric field in the neighborhood of a nucleus, has a certain probability of transforming itself into a positive and a negative electron. Conservation of energy yields the following relation between the energy W of the primary photon and the *total* energies U and U' of the electron pair:

$$U+U'=W, \quad \text{or} \quad u+u'=1 \tag{1.39}$$

if we introduce the fractional energies $u=U/W$ and $u'=U'/W$. The fractional energies u and u' vary from μ_e/W to $(1-\mu_e/W)$. Since the energies under consideration are large compared with μ_e, in most cases the total energies U, U' can be identified with the kinetic energies E, E'.

The process of pair production can be looked upon as a photoelectric effect, whereby an electron is raised from a state of negative energy to a state of positive energy leaving a "hole" in the infinite distribution of negative-energy electrons. The theory of pair production is closely related to the theory of radiation processes. Indeed, in the case of a radiation process an electron makes a transition between two states of positive energy and a photon is emitted. In the case of pair production a photon is absorbed and causes an electron to make a transition from a state of negative energy to a state of positive energy.

The energy distribution between the two electrons of a pair is not determined by the direction of their motion with respect to that of the primary photon, because the nucleus takes part of the momentum. However, it can be proved that the average angle of emission of an electron of energy U is of the order of μ_e/U.

§17. *Differential Probability for Pair Production*

Let $\Psi(W, u)dudx$ be the probability for a photon of energy W traversing a thickness of dx g/cm^2 to produce a pair, in which the positron has a fractional energy between u and $u+du$. As in radiation phenomena, it is important to consider the distance from the nucleus at which the pair production takes place, because of the screening of the nuclear field by the outer electrons. The influence of the screening is determined by the quantity

$$\gamma=100\frac{\mu_e}{W}\frac{1}{u(1-u)}Z^{-\frac{1}{3}} \tag{1.40}$$

and decreases with increasing γ. We shall neglect the screening when $\gamma\gg1$ and we shall call the screening "complete" when $\gamma\approx0$. For a given value of u, γ decreases with increasing W. Thus, for large energies of the primary photon the screening can be considered as complete for all processes of pair production.

Under the assumption that $W\gg\mu_e$, the expressions for $\Psi(W, u)$ for various ranges of γ have been given by Bethe and Heitler as follows:

no screening, $\gamma\gg1$

$$\Psi(W, u)du=4\alpha\frac{N}{A}Z^2r_0^2du\left[u^2+(1-u)^2+\tfrac{2}{3}u(1-u)\right]\left[\log\frac{2W}{\mu_e}u(1-u)-\frac{1}{2}\right]; \tag{1.41}$$

complete screening, $\gamma\approx0$

$$\Psi(W, u)du=4\alpha\frac{N}{A}Z^2r_0^2du\left\{\left[u^2+(1-u)^2+\tfrac{2}{3}u(1-u)\right]\log\left(183Z^{-\frac{1}{3}}\right)-\tfrac{1}{9}u(1-u)\right\}; \tag{1.42}$$

intermediate cases

$\gamma<2$

$$\Psi(W, u)du=4\alpha\frac{N}{A}Z^2r_0^2du\left\{[u^2+(1-u)^2]\left[\frac{f_1(\gamma)}{4}-\frac{1}{3}\log Z\right]+\tfrac{2}{3}u(1-u)\left[\frac{f_2(\gamma)}{4}-\frac{1}{3}\log Z\right]\right., \quad (1.43)$$

$2<\gamma<15$

$$\Psi(W, u)du=4\alpha\frac{N}{A}Z^2r_0^2du[u^2+(1-u)^2+\tfrac{2}{3}u(1-u)]\left[\log\frac{2W}{\mu_e}u(1-u)-\tfrac{1}{2}-c(\gamma)\right]. \quad (1.44)$$

The functions $f_1(\gamma)$, $f_2(\gamma)$ and $c(\gamma)$ are the same as those which enter in the expressions (1.24) and (1.25) for the radiation probabilities, and are given by Fig. 5 and Table III. The functions Ψ are symmetrical with respect to u and $(1-u)$; i.e., with respect to the energy of the electron and that of the positron. In the case of complete screening [Eq. (1.42)], ψ is a function of the fractional energy u only.

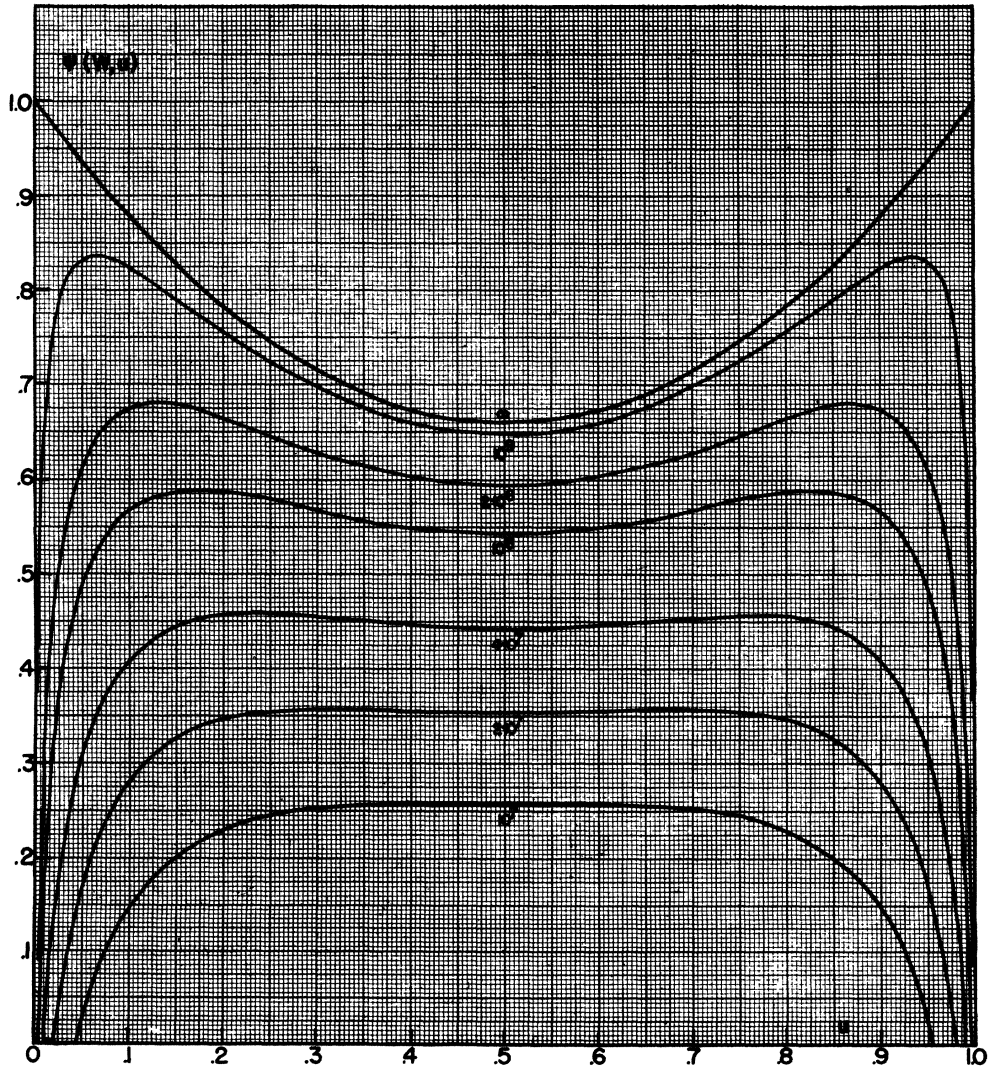

FIG. 10. Differential probability of pair production per radiation length of air, for photons of various energies. Abscissa, $u=U/W$; ordinate, $\psi(W, u)$. The numbers attached to the curves indicate the energy W of the primary photon.

§18. *Total Probability for Pair Production*

Let $\Sigma(W)dx$ be the total probability for a photon with energy W to produce a pair in a thickness of dx g/cm^2; i.e.,

$$\Sigma(W)=\int_{\mu_e/W}^{1-\mu_e/W}\Psi(W,u)du. \tag{1.45}$$

At the limit for small and large energies, respectively, Eqs. (1.41) and (1.42) yield

$\mu_e \ll W \ll 137\mu_e Z^{-\frac{1}{3}}$

$$\Sigma(W)=4\alpha\frac{N}{A}Z^2r_0^2\left(\frac{7}{9}\log\frac{2W}{\mu_e}-\frac{109}{54}\right), \tag{1.46}$$

$W \gg 137\mu_e Z^{-\frac{1}{3}}$

$$\Sigma(W)=\Sigma_0=4\alpha\frac{N}{A}Z^2r_0^2\left[\frac{7}{9}\log(183Z^{-\frac{1}{3}})-\frac{1}{54}\right]. \tag{1.47}$$

For the intermediate cases the integral in (1.45) must be evaluated numerically. Equation (1.47) shows that the total probability for pair production at large energies is a constant in a given material.

§19. *Probabilities per Radiation Length. Simplified Expressions*

The probabilities for pair production can conveniently be expressed in terms of the radiation length defined in §13. Let

$$\psi(W,u)=X_0\Psi(W,u) \tag{1.48}$$

be the differential probability for pair production per radiation length and

$$\sigma(W)=X_0\Sigma(W) \tag{1.49}$$

be the total probability for pair production per radiation length. The function ψ is plotted against u for various W in Figs. 10 and 11 (air and lead). The function σ is plotted against W in Figs. 13 and 13a (air and lead). The analytical expressions for ψ and σ in the case of complete screening are

$$\begin{aligned}\psi(W,u)du&=\psi_0(u)du\\&=[u^2+(1-u)^2+(\tfrac{2}{3}-2b)u(1-u)]du,\end{aligned} \tag{1.42a}$$

$$\sigma(W)=\sigma_0=\frac{7}{9}-\frac{b}{3} \tag{1.47a}$$

where b is the same as in Eqs. (1.23a) and (1.28a).

It appears that the equations describing pair production, like those describing radiation processes, depend only slightly on the atomic number, when thicknesses are measured in radiation lengths, and are entirely independent of Z at the limit for large energies. The differential probability $\psi(W,u)$ for a given W does not change very much with u. Therefore, in first approximation, one can use the simplified expression

$$\psi_1(W,u)du=\sigma(W)du \tag{1.50}$$

or, at high energies,

$$\psi_1(W,u)du=\sigma_0 du\approx\frac{7}{9}du. \tag{1.50a}$$

The approximation (1.50) is good at low energies. At intermediate energies a better approximation is given by the following equation:

$$\psi_2(W,u)du=\frac{\sigma(W)}{\sigma_0}\psi_0(u)du. \tag{1.50b}$$

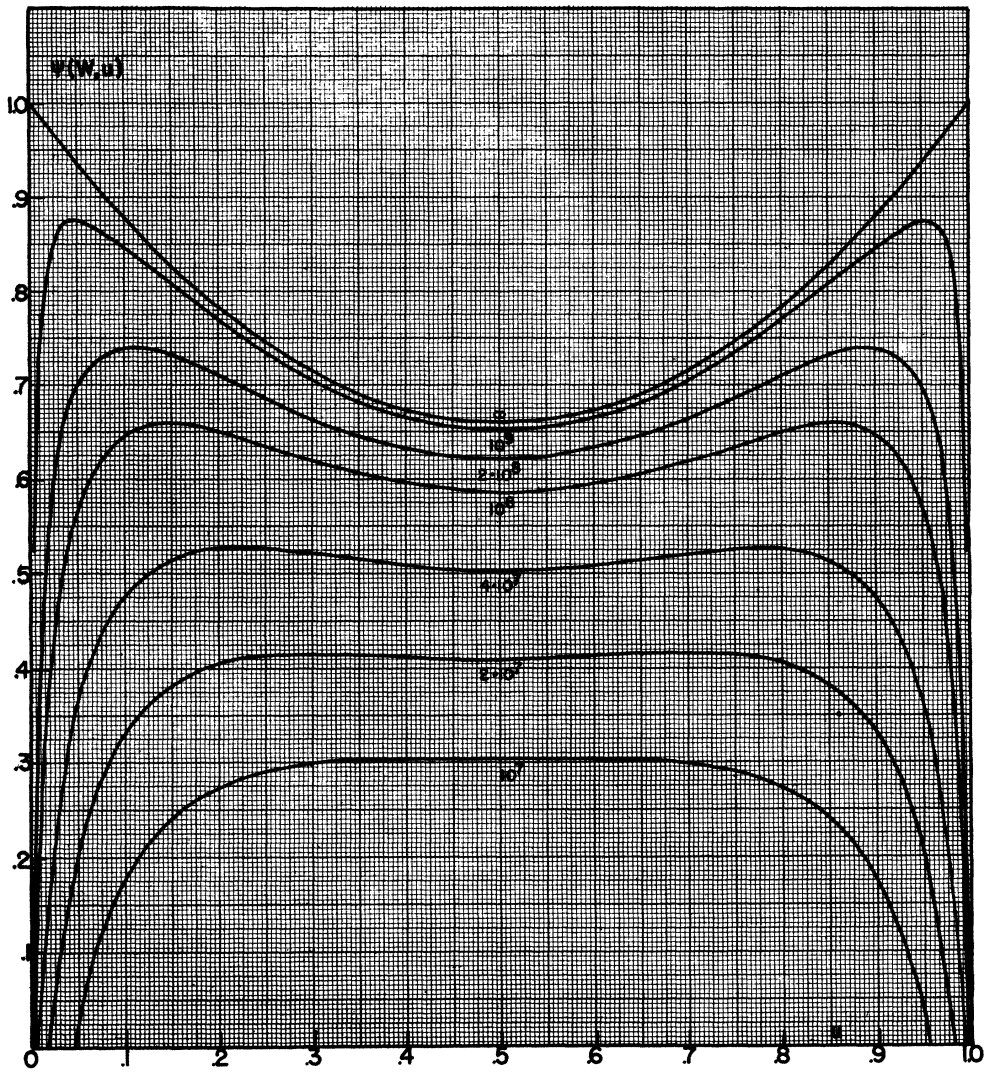

FIG. 11. Differential probability of pair production per radiation length of lead, for photons of various energies. Abscissa, $u = U/W$; ordinate, $\psi(W, u)$. The numbers attached to the curves indicate the energy W of the primary photon.

A comparison between the approximate expressions ψ_1, ψ_2 and the correct expression ψ is found in Fig. 12.

§20. *Comparison between Pair Production and Compton Effect*

The total probability for Compton scattering (see §9) decreases rapidly with increasing photon energy, while the total probability for pair production is a slowly increasing function of the energy. Thus, at large energies most of the photons are absorbed by pair production, while at small energies most of the photons are absorbed by Compton effect. The absorption of photons by pair production and Compton effect in lead and air are compared in Figs. 13 and 13a. It is seen that the energy at which the pair production becomes dominant decreases with increasing atomic number.

E. Scattering

§21. *Differential Probability for Elastic Scattering*

It has already been pointed out that when a charged particle passes near a nucleus it undergoes a deflection which, in most cases, is not accompanied by loss of energy. This phenomenon, called *elastic scattering*, is caused by the same electric interaction between the passing particle and the Coulomb field of the nucleus, which also determines the radiation processes (see §10).

In investigating the elastic scattering, the nuclei will be considered as fixed point charges of magnitude Z. We shall denote by $\xi(\Theta)d\omega dx$ the probability that a particle of charge ± 1, momentum p and velocity β, traversing a thickness of dx g/cm^2, undergoes a nuclear collision which deflects its trajectory into the solid angle $d\omega$, at an angle Θ to its original motion. In the calculation of ξ, the spin of the incident particle must be taken into account. The expressions for ξ corresponding to the values 0, $\frac{1}{2}$, and 1 of spin are, respectively:

Spin 0 (see Williams, W39)

$$\xi(\Theta)d\omega = N\frac{Z^2}{A}\frac{r_0^2}{4}\frac{\mu_e^2}{p^2\beta^2}\frac{d\omega}{\sin^4\frac{1}{2}\Theta}. \tag{1.51}$$

Spin $\frac{1}{2}$ (see Mott, M29)

$$\xi(\Theta)d\omega = N\frac{Z^2}{A}\frac{r_0^2}{4}\frac{\mu_e^2}{p^2\beta^2}(1-\beta^2\sin^2\tfrac{1}{2}\Theta)\frac{d\omega}{\sin^4\frac{1}{2}\Theta}. \tag{1.51a}$$

Spin 1 (see Massey and Corben, M39)

$$\xi(\Theta)d\omega = N\frac{Z^2}{A}\frac{r_0^2}{4}\frac{\mu_e^2}{p^2\beta^2}\left(1+\frac{1}{6}\frac{p^2\beta^2}{\mu^2}\sin^2\Theta\right)\frac{d\omega}{\sin^4\frac{1}{2}\Theta}. \tag{1.51b}$$

It is seen that the probability of large deflections is different for different values of the spin. For small deflections, however, the terms depending on spin are negligible and one can use for particles of any spin the following expression:

$$\xi(\Theta)d\omega = 4N\frac{Z^2}{A}r_0^2\frac{\mu_e^2}{p^2\beta^2}\frac{d\omega}{\Theta^4}. \tag{1.51c}$$

The finite size of the nucleus and the screening of the nuclear field by the outer electrons limit the validity of the above equations for large and for small values of Θ, respectively. Following Williams (W39), we may take into account the finite size of the nucleus by assuming that its electric charge, instead of being concentrated in a point, is distributed in a sphere of radius d. It can be shown that this assumption does not affect materially the calculated value of the scattering probability ξ for $\Theta < \lambda\!\!\!^{-}/d$, while it causes ξ to go rapidly to zero for $\Theta > \lambda\!\!\!^{-}/d$, where $\lambda\!\!\!^{-}$ is the De Broglie wave-length of the incident particle divided by 2π. Similarly, if we put $a = (137)^2 r_0 Z^{-\frac{1}{3}}$ (a may be denoted as the atomic radius), it can be proved that the screening of the nuclear field by the outer electrons does not affect the scattering probability ξ for $\Theta > \lambda\!\!\!^{-}/a$, while it causes ξ practically to vanish as soon as Θ becomes smaller than $\lambda\!\!\!^{-}/a$. Taking for d the value $d = 0.57 r_0 Z^{\frac{1}{3}}$, as in §15, we can assume that ξ is given by Eqs. (1.51) to (1.51c) for $\Theta_{\min} < \Theta < \Theta_{\max}$, where

$$\Theta_{\min} = \lambda\!\!\!^{-} Z^{\frac{1}{3}}/(137)^2 r_0, \quad \Theta_{\max} = \lambda\!\!\!^{-}/0.57 r_0 Z^{\frac{1}{3}} \tag{1.52}$$

and is practically zero for $\Theta < \Theta_{\min}$ or $\Theta > \Theta_{\max}$.

It is, of course, possible that forces other than electromagnetic may play a role in the scattering of cosmic-ray particles. This possibility, however, will not be taken into consideration.

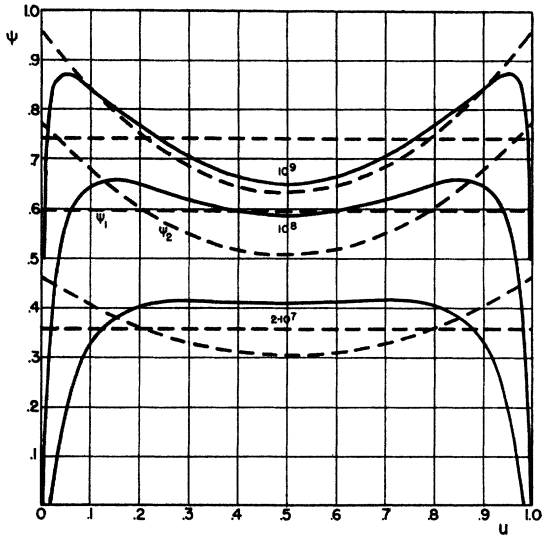

FIG. 12. Comparison between the approximate expressions ψ_1, ψ_2 for the differential probability of pair production (dashed lines) and the correct expression for the same quantity (solid lines). The graphs have been calculated for lead and for W equal to 2×10^7, 10^8, 10^9 ev.

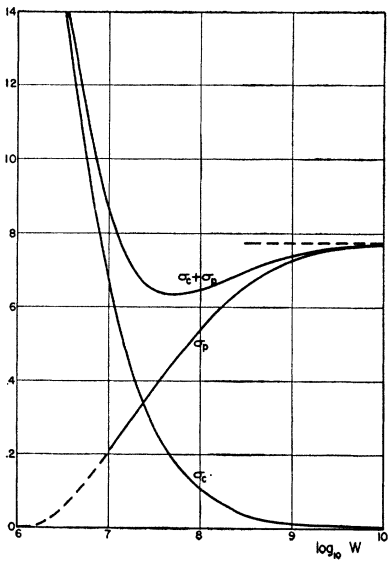

FIG. 13. Total probability per radiation length of air for pair production (σ_p) and for Compton scattering (σ_c) as a function of the photon energy W. The sum of the two probabilities ($\sigma_p+\sigma_c$), giving the total absorption coefficient per radiation length, is also indicated. For $W<10^7$ ev, σ_p cannot be calculated with the formulae given in the text, which are only valid when $W\gg\mu_e$, and a more accurate equation must be used (see B34).

§22. *Multiple Scattering. Calculation of the Mean Square Angle of Scattering*

The deflection that a cosmic-ray particle undergoes in traversing a plate of finite thickness may be caused either by a single collision, or by many subsequent collisions. It can be proved that large deflections are more likely to occur in single collisions, while small deflections are generally caused by many collisions.

The result of single collisions is referred to as *single scattering*, the result of a small number of collisions as *plural scattering*, and the result of a large number of collisions as *multiple scattering*. The theory of single scattering, which is of importance for large deflections, is contained in the formulae given in the foregoing section. The theory of plural scattering is very complicated and will not be investigated here. The theory of multiple scattering, which is of importance for small deflections, can be treated in a simple way by statistical methods. We shall develop it using Eq. (1.51c) as the expression for the elementary scattering probability. This is allowed because only small deflections play an important role in multiple scattering. Since Eq. (1.51c) holds for all particles with unit charge, the results will be valid for electrons, mesotrons and protons indifferently.

As a first step, we want to calculate the mean square angle of scattering in an infinitesimal layer dx.

$$\langle\Theta^2\rangle_{\mathrm{Av}(dx)}=dx\int_{\Theta_{\min}}^{\Theta_{\max}}\Theta^2\xi(\Theta)\cdot2\pi\Theta d\Theta. \tag{1.53}$$

It follows immediately from (1.51c) and (1.52) that

$$\begin{aligned}\langle\Theta^2\rangle_{\mathrm{Av}(dx)}&=dx\cdot8\pi N\frac{Z^2}{A}r_0^2\frac{\mu_e^2}{p^2\beta^2}\log\frac{\Theta_{\max}}{\Theta_{\min}}\\&=dx\cdot16\pi N\frac{Z^2}{A}r_0^2\frac{\mu_e^2}{p^2\beta^2}\log(181Z^{-\frac{1}{3}}).\end{aligned} \tag{1.53a}$$

It will be noted that $\langle\Theta^2\rangle_{Av(dx)}$ depends on atomic number in much the same way as the radiation loss of electrons [see Eq. (1.28)]. The difference between the numerical factors (183 and 181) in the logarithm cannot be considered as real, and has a negligible effect on the result. Hence, the description of scattering phenomena will be simplified by measuring thicknesses in radiation lengths [see (1.29)]. Indeed, the mean square angle of scattering in a thickness of dt radiation lengths becomes independent of atomic number, and is given by

$$\langle\Theta^2\rangle_{Av(dt)}=4\pi\cdot 137\mu_e^2 dt/p^2\beta^2. \tag{1.53b}$$

If we introduce the constant E_s with the dimension of an energy:

$$E_s=\mu_e(4\pi 137)^{\frac{1}{2}}=21\times 10^6 \text{ ev}, \tag{1.54}$$

Eq. (1.53b) becomes

$$\langle\Theta^2\rangle_{Av(dt)}=(E_s^2/p^2\beta^2)dt. \tag{1.53c}$$

According to a general rule on the superposition of small and independent deviations, the mean square value of Θ in a finite thickness t can be obtained by integration of (1.53c) from 0 to t.

$$\langle\Theta^2\rangle_{Av(t)}=E_s^2\int_0^t\frac{dt'}{p^2\beta^2}. \tag{1.55}$$

If the energy loss is negligible, p and β can be considered as constant, and (1.55) reduces to

$$\langle\Theta^2\rangle_{Av(t)}=(E_s^2/p^2\beta^2)t. \tag{1.55a}$$

In general, Eq. (1.55) can be written as follows:

$$\langle\Theta^2\rangle_{Av(t)}=E_s^2\int_{p_2}^{p_1}\frac{1}{-(dp/dt)}\frac{dp}{p^2\beta^2}, \tag{1.55b}$$

where $(-dp/dt)$ is the momentum loss per radiation length and p_1 and p_2 are the momenta at the thicknesses 0 and t, respectively. In the case of mesotrons, according to Eqs. (1.11a) and (1.13),

$$-\frac{dp}{dt}=X_0\frac{2C\mu_e B(p/\mu)}{\beta^3}, \tag{1.56}$$

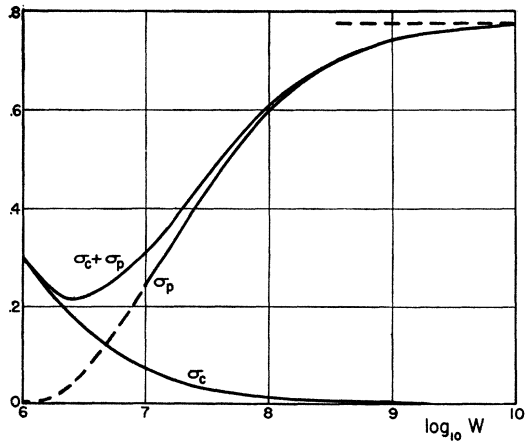

FIG. 13a. σ_p, σ_c and $\sigma_p+\sigma_c$ for lead.

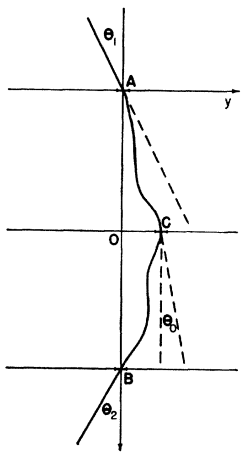

FIG. 14. Illustrating the calculation of the sidewise distribution at the center of the trajectory of cosmic-ray particles passing through two fixed points.

where B is a slowly varying function of p/μ given by Eq. (1.15). If one considers B as a constant, Eq. (1.55b) yields

$$\langle\Theta^2\rangle_{\text{Av}(t)}=\frac{1}{X_0}\frac{E_s^2}{2C\mu_e B}\int_{p_2}^{p_1}\frac{\beta}{p^2}dp$$

$$=\frac{1}{X_0}\frac{E_s^2}{2C\mu_e B}\cdot\frac{1}{\mu}\int_{p_2/\mu}^{p_1/\mu}\frac{d(p/\mu)}{p/\mu[(p/\mu)^2+1]^{\frac{1}{2}}}$$

$$=\frac{1}{X_0}\frac{E_s^2}{2C\mu_e B}\cdot\frac{1}{\mu}\log\left[\frac{1+[1+(p_2/\mu)^2]^{\frac{1}{2}}}{p_2/\mu}\times\frac{p_1/\mu}{1+[1+(p_1/\mu)^2]^{\frac{1}{2}}}\right].$$

With the same approximation (constant B), p_1 and p_2 are related to the thickness t by the equation (see 1.16)

$$t=\frac{1}{X_0}\frac{\mu}{2C\mu_e B}\left[\frac{(p_1/\mu)^2+2}{[(p_1/\mu)^2+1]^{\frac{1}{2}}}-\frac{(p_2/\mu)^2+2}{[(p_2/\mu)^2+1]^{\frac{1}{2}}}\right].$$

It follows that

$$\langle\Theta^2\rangle_{\text{Av}(t)}=\frac{E_s^2}{\mu^2}t\frac{\log\left[\frac{1+y_2}{1+y_1}\frac{p_1}{p_2}\right]}{y_1+1/y_1-y_2-1/y_2},\tag{1.57}$$

where $y=[(p/\mu)^2+1]^{\frac{1}{2}}=(E+\mu)/\mu$ is the *total* energy divided by the rest energy. In the practical cases, the thickness t as well as the final momentum p_2 is known, and the initial momentum p_1 can be obtained from the graph in Fig. 3.

Instead of considering the total deflection Θ, it is often more convenient to consider the projection θ of the deflection on a plane containing the initial trajectory. It can easily be shown that the mean square value of θ is one-half the mean square value of Θ. Therefore,

$$\langle\theta^2\rangle_{\text{Av}(t)}=\tfrac{1}{2}E_s^2\int_0^t dt'/p^2\beta^2\tag{1.58}$$

or, if the energy loss is negligible,

$$\langle\theta^2\rangle_{\text{Av}(t)}=\tfrac{1}{2}E_s^2 t/p^2\beta^2.\tag{1.58a}$$

§23. *The Distribution Function**

Let us consider a parallel and infinitely narrow beam of cosmic-ray particles incident on a plate of some scattering substance. The particles are all supposed to have the same energy, and their energy loss in the scattering substance is neglected. We ask for the space and angular distributions of the beam after traversal of a thickness t of the scattering substance.

Let us take a system of Cartesian coordinates with the origin at the point of incidence and one of the axes in the direction of the motion of the incident particles. This axis will be denoted as the t axis, while the other two will be the y and z axes, respectively. Let us consider the projection of the motion of the particles on the (t, y) plane and let $F(t, y, \theta)dyd\theta$ be the number of particles at the thickness t having a lateral displacement (y, dy) and traveling at an angle $(\theta, d\theta)$ with the t axis. For reasons of symmetry, the same function F describes also the space and angular distribution in the

* The developments in this article follow closely a lecture given by Professor Fermi at the University of Chicago in the summer of 1940 and include some unpublished results. The writers wish to express their sincere appreciation to Professor Fermi for allowing them to make use of these results.

(t, z) plane. Since deflections in the two orthogonal directions y and z are independent of each other, the number of particles having a lateral displacement with components (y, dy) and (z, dz) and an angular deflection with components $(\theta_y, d\theta_y)$ and $(\theta_z, d\theta_z)$ at the thickness t is given by $F(t, y, \theta_y) \cdot F(t, z, \theta_z) dy dz d\theta_y d\theta_z$. We want to calculate the distribution function F under the usual assumption that the angle θ is small.

Let $\rho_{\Delta t}(\theta)d\theta$ be the probability that a particle traversing the thickness Δt will be deflected through an angle $(\theta, d\theta)$. The deflection θ is not necessarily caused by a single collision. Hence, the function $\rho(\theta)$ is not immediately related to the function $\xi(\Theta)$ defined previously. By its definition, $\rho(\theta)$ satisfies the following equations:

$$\begin{aligned}
&\rho_{\Delta t}(\theta) = \rho_{\Delta t}(-\theta), \\
&\int_{-\infty}^{\infty} \rho_{\Delta t}(\theta) d\theta = 1, \\
&\int_{-\infty}^{\infty} \theta \rho_{\Delta t}(\theta) d\theta = 0, \qquad (1.59) \\
&\int_{-\infty}^{\infty} \theta^2 \rho_{\Delta t}(\theta) d\theta = \langle \theta^2 \rangle_{\mathrm{Av}(\Delta t)} = \frac{1}{2} \frac{E_s^2}{p^2\beta^2} \Delta t.
\end{aligned}$$

Since ρ has a very sharp maximum at $\theta = 0$ and goes rapidly to zero on both sides of the maximum, the integrals can be extended from $-\infty$ to $+\infty$.

We ask now for the change that the function F undergoes in the layer from t to $t+\Delta t$. The function F changes because both the space distribution and the angular distribution of particles are modified by the traversal of the layer Δt. The space distribution is modified because particles traveling at an angle θ undergo a lateral displacement $\theta \Delta t$ in the layer Δt. The scattering in this layer represents only a second order effect and can be disregarded as far as the change in the space distribution is concerned. It follows that the particles having a lateral displacement y at the thickness $t+\Delta t$ are those which had a lateral displacement $y - \theta \Delta t$ at the thickness t. Hence, neglecting the change in the angular distribution,

$$F(t+\Delta t, y, \theta) = F(t, y - \theta \Delta t, \theta) = F(t, y, \theta) - \theta \Delta t (\partial F / \partial y).$$

In order to calculate the effect of the change in the angular distribution, let us consider two angular intervals $(\theta, d\theta)$ and $(\theta', d\theta')$. There are $F(t, y, \theta') dy d\theta'$ particles in $(\theta', d\theta')$ at the thickness t with a lateral displacement (y, dy), and a fraction $\rho_{\Delta t}(\theta - \theta') d\theta$ of these particles is scattered into the angular interval $(\theta, d\theta)$ while traversing the layer Δt. Hence, if we neglect the change in the space distribution,

$$F(t+\Delta t, y, \theta) = \int_{-\infty}^{\infty} F(t, y, \theta') \rho_{\Delta t}(\theta - \theta') d\theta'.$$

Since $\rho_{\Delta t}(\theta - \theta')$ is different from zero only for very small values of the argument, F can be developed in Taylor series of $(\theta - \theta')$. Dropping the terms beyond the second order and taking into account Eqs. (1.59), we obtain

$$F(t+\Delta t, y, \theta) = F(t, y, \theta) + \frac{1}{4} \frac{E_s^2}{p^2\beta^2} \frac{\partial^2 F}{\partial \theta^2} \Delta t.$$

Hence the total change of the distribution function F in the layer Δt is

$$-\theta \frac{\partial F}{\partial y} \Delta t + \frac{1}{4} \frac{E_s^2}{p^2\beta^2} \frac{\partial^2 F}{\partial \theta^2} \Delta t.$$

If we set

$$w = 2p\beta/E_s \tag{1.60}$$

we obtain for F the differential equation

$$\frac{\partial F}{\partial t} = -\theta\frac{\partial F}{\partial y} + \frac{1}{w^2}\frac{\partial^2 F}{\partial \theta^2}. \tag{1.61}$$

It will be noted that, for large values of p, w represents twice the energy of the incident particle in units of the characteristic energy E_s, while for small values of p, w represents four times the energy in the same units.

We look for a solution of Eq. (1.61) which corresponds to a single incident particle. The function F will then represent the *probability* for a certain lateral displacement and a certain angular deflection at the thickness t. It can easily be proved that such a solution (given by Fermi; see footnote, p. 265) is

$$F(t, y, \theta) = \frac{\sqrt{3}}{2\pi}\frac{w^2}{t^2}\exp\left[-w^2\left(\frac{\theta^2}{t} - \frac{3y\theta}{t^2} + \frac{3y^2}{t^3}\right)\right]. \tag{1.62}$$

Indeed, it is seen upon substitution that (1.62) satisfies (1.61). That the boundary conditions are also fulfilled will be made apparent by what follows.

By integrating the distribution function F over y, one obtains a function $G(t, \theta)$ which represents the angular distribution irrespective of lateral displacement.

$$G(t, \theta) = \int_{-\infty}^{\infty} F(t, y, \theta)dy = \frac{1}{2\sqrt{\pi}}\frac{w}{t^{\frac{1}{2}}}\exp\left[-\tfrac{1}{4}(w^2\theta^2/t)\right]. \tag{1.63}$$

Similarly, by integrating the function F over θ one obtains a function $H(t, y)$ which represents the distribution in space, irrespective of angle.

$$H(t, y) = \int_{-\infty}^{\infty} F(t, y, \theta)d\theta = \frac{\sqrt{3}}{2\sqrt{\pi}}\frac{w}{t^{\frac{3}{2}}}\exp\left[-\tfrac{3}{4}(w^2y^2/t^3)\right]. \tag{1.64}$$

It follows from (1.63) and (1.64) that, for all values of t:

$$\int_{-\infty}^{\infty} G(t, \theta)d\theta = \int_{-\infty}^{\infty} H(t, y)dy = 1. \tag{1.65}$$

Moreover, at the limit for $t=0$, G is zero for all values of θ except $\theta=0$, and H is zero for all values of y except $y=0$; i.e.,

$$G(0, \theta) = \delta(\theta), \quad H(0, y) = \delta(y), \tag{1.66}$$

where δ is Dirac's improper function.

This proves that the solution (1.62) actually corresponds to a single particle incident at $t=0$, $y=0$ in the direction of the t axis.

Equation (1.63) shows that, at every thickness, the angular distribution irrespective of position is Gaussian. The mean square angle of scattering is given by

$$\langle\theta^2\rangle_{Av(t)} = 2t/w^2 = \tfrac{1}{2}E_s^2 t/p^2\beta^2 \tag{1.67}$$

in agreement with (1.58a).

Similarly, Eq. (1.64) shows that at every thickness the distribution in space, irrespective of angle, is Gaussian. The mean square displacement is

$$\langle y^2\rangle_{Av(t)}=\frac{2t^3}{3w^2};\quad \text{i.e.,}\quad \langle y^2\rangle_{Av(t)}=\frac{1}{6}\frac{E_s^2}{p^2\beta^2}t^3=\frac{1}{3}t^2\langle\theta^2\rangle_{Av(t)}. \tag{1.68}$$

It may be noted that, if we only consider those particles which have a certain displacement at a given thickness, their angular distribution is *not* Gaussian. The same remark applies to the space distribution of particles which have a certain angular deflection at a given thickness.

The distribution function (1.62) can be used to solve various problems arising in the discussion of cosmic-ray experiments. Suppose, for instance, that a beam of cosmic-ray particles is known to pass through two points A and B (more precisely, through two small areas at A and B) and one wants to determine the sidewise distribution of the trajectories with respect to the straight line connecting A and B at half the distance between A and B. Let us take A as the origin of a system of coordinates and AB as the t axis (see Fig. 14). As before, we shall consider the projection of the trajectories on a (t, y) plane through AB and suppose that the angles of the trajectories with the t axis are small. We draw a straight line perpendicular to the segment AB through its center O and consider a point C of this line at a distance y_0 from O. Let $t_0=(AO)=(OB)$ and let us consider three elementary segments, dy_1 at $A(t=0)$, dy_0 at $C(t=t_0)$ and dy_2 at $B(t=2t_0)$. Suppose cosmic-ray particles are incident at A uniformly in the different directions; we ask for the probability $f(y_0)dy_0$ of a sidewise displacement (y_0, dy_0) at $t=t_0$.

Let $Kd\theta_1dy_1$ be the number of particles incident upon dy_1 in the angular interval $(\theta_1, d\theta_1)$. The distance of C from their original trajectory is $y_0-t_0\theta_1$; hence, the number of these particles going through dy_0 at an angle $(\theta_0, d\theta_0)$ is

$$Kd\theta_1dy_1F(t_0, y_0-t_0\theta_1, \theta_0-\theta_1)d\theta_0dy_0$$

and the total number of particles coming from any direction and going through dy_0 at the angle $(\theta_0, d\theta_0)$ is*

$$Kdy_1dy_0d\theta_0\int_{-\infty}^{\infty}F(t_0, y_0-t_0\theta_1, \theta_0-\theta_1)d\theta_1=Kdy_1dy_0\frac{\sqrt{3}}{2\sqrt{\pi}}\frac{w}{t_0^{3/2}}\exp\left[-\tfrac{3}{4}(w^2/t_0^3)(y_0-t_0\theta_0)^2\right]d\theta_0.$$

A particle passing through dy_0 at an angle θ_0 has a probability $H(t_0, -y_0-t_0\theta_0)dy_2$ of going through dy_2 at B. Hence the number of particles going through dy_1, dy_0 and dy_2 is

$$Kdy_1dy_0dy_2\left(\frac{3}{8\pi}\right)^{1/2}\frac{w}{t_0^{5/2}}\exp\left[-\frac{3}{2}\frac{w^2y_0^2}{t_0^3}\right],$$

while the total number of particles going through dy_1 and dy_2 is

$$\frac{1}{2t_0}Kdy_1dy_2.$$

The probability $f(y_0)dy_0$ is the ratio between the two above numbers and is therefore given by

$$f(y_0)dy_0=\left(\frac{3}{2\pi}\right)^{1/2}\frac{w}{t_0^{3/2}}\exp\left[-\frac{3}{2}\frac{w^2y_0^2}{t_0^3}\right]dy_0.$$

(Fermi; see footnote, p. 265.)

* This result becomes an immediate consequence of Eq. (1.64) if one imagines the particle as moving in the opposite direction.

Part II.

Multiplicative Showers

§24. *General Remarks*

It has been shown in the first section that charged particles traversing matter lose energy by *collision* and by *radiation*. Most of the energy lost by collision is spent in exciting the atoms or ejecting from the atoms electrons of small energy, and must be regarded as dissipated, according to our definition (see Introduction). The energy lost by radiation, on the contrary, is fairly uniformly distributed among secondary photons of all energies from zero up to the energy of the primary particle itself. For electrons of small energy and for mesotrons of practically all energies the collision losses are more important than radiation losses. Hence the interaction of mesotrons or of low energy electrons with matter results mainly in an energy dissipation. Electrons of large energy, however, lose most of their energy by radiation. Hence by the interaction of high energy electrons with matter only a small fraction of the energy is dissipated, while a large portion is spent in the production of photons of high energy. The secondary photons, in turn, undergo materialization or Compton collision. In either process electrons are produced of energy comparable with that of the photon. These new electrons radiate more photons, which again materialize into electron pairs or produce Compton electrons. At every new step the number of particles increases and their average energy decreases. As the process goes on, an increasing number of particles falls below the limiting energy η_0, until eventually the energy of the primary electron is completely dissipated.

The phenomenon outlined is called a *multiplicative shower*, or a *cascade shower*. It is clear that a shower can be initiated by a high energy photon as well as by a high energy electron. Mesotrons, too, can give rise to a secondary shower by producing a high energy electron or photon.

The theory of cascade showers was first developed independently by Carlson and Oppenheimer (C37) and by Bhabha and Heitler (B37). Further contributions were published by Landau and Rumer (L38), Snyder (S38), Serber (S38a), Nordheim and Hebb (N39), and others (see F37, I38, E38, H38, A38, T39, A40, N40, S40). The mathematical aspect of the theory has recently been discussed in detail by Scott (S41).

In our discussion we will first consider the problem as unidimensional; i.e., we will assume that all shower particles (electrons and photons) move in the same direction as the primary particle which has produced the shower. The spread of a shower caused by scattering will be discussed later (§44). One is justified in treating the problem of the longitudinal development of a shower separately from the problem of its lateral spread because the change in path length due to scattering is, in general, negligible.

§25. *Definitions, Notations, and Approximations*

Since the phenomena mainly responsible for the generation of showers are radiation processes and pair production, it is convenient, in the following discussion, to measure thicknesses in radiation lengths (see §13). We shall further introduce the following definitions and notations.

(*a*) *Differential electron spectrum*, $\pi(E, t)dE$: average number of electrons (positive and negative) at the thickness t with energy between E and $E+dE$. If we want to specify that the shower has been produced by an electron of energy E_0 or by a photon of energy W_0, we shall write $\pi(E_0, E, t)dE$ or $\pi(W_0, E, t)dE$, respectively.

(*b*) *Differential photon spectrum*, $\gamma(W, t)dW$: average number of photons at the thickness t with energy between W and $W+dW$($\gamma(E_0, W, t)dW$ or $\gamma(W_0, W, t)dW$ for a shower produced by an electron of energy E_0 or by a photon of energy W_0).

(*c*) *Integral electron spectrum*, $\Pi(E, t)$: average number of electrons with energy larger than E at the thickness t($\Pi(E_0, E, t)$ or $\Pi(W_0, E, t)$ for a shower produced by an electron of energy E_0 or by a photon of energy W_0).

$$\Pi(E, t) = \int_E^\infty \pi(E', t)dE'. \qquad (2.1)$$

(*d*) *Integral photon spectrum*, $\Gamma(W, t)$: average number of photons with energy larger than W at the thickness $t(\Gamma(E_0, W, t)$ or $\Gamma(W_0, W, t)$ for a shower produced by an electron of energy E_0 or by a photon of energy W_0).

$$\Gamma(W, t)=\int_W^\infty \gamma(W', t)dW'. \tag{2.2}$$

(*e*) *Differential electron track length*, $z_\pi(E)dE$: total distance traveled by all shower electrons while their energy lies between E and $E+dE$.

$$z_\pi(E)=\int_0^\infty \pi(E, t)dt. \tag{2.3}$$

Similarly,

$$z_\gamma(W)=\int_0^\infty \gamma(W, t)dt \tag{2.4}$$

gives the *differential photon track length*, $z_\gamma(W)dW$.

(*f*) *Integral electron track length*, $z_\Pi(E)$: total distance traveled by all shower electrons while their energy is larger than E.

$$z_\Pi(E)=\int_E^\infty z_\pi(E')dE'=\int_0^\infty \Pi(E, t)dt. \tag{2.5}$$

Similarly,

$$z_\Gamma(W)=\int_W^\infty z_\gamma(W')dW'=\int_0^\infty \Gamma(W, t)dt \tag{2.6}$$

defines the *integral photon track length*, $Z_\Gamma(W)$.

The functions z_π, z_γ, z_Π, and z_Γ are easier to calculate than the corresponding functions π, γ, Π, Γ, and knowledge of them is sufficient to solve several problems; for instance, to determine the number of electrons and photons in equilibrium with a mesotron beam (see §40).

(*g*) *Specific ionization*, $j(t)dt$: total number of ion pairs produced by all shower particles in the layer between t and $t+dt$.

(*h*) *Total ionization*, J: total number of ion pairs produced by all shower particles until the absorption of the shower is completed.

$$J=\int_0^\infty j(t)dt. \tag{2.7}$$

(*i*) *Center of gravity:* The position $\bar{t}_\pi(E)$ of the center of gravity of shower electrons of energy E is defined by

$$\bar{t}_\pi(E)=\frac{\int_0^\infty t\pi(E, t)dt}{\int_0^\infty \pi(E, t)dt}=\frac{1}{z_\pi(E)}\int_0^\infty t\pi(E, t)dt. \tag{2.8}$$

Similar expressions give the positions of the centers of gravity of photons of energy $W(\bar{t}_\gamma(W))$, of electrons with energy larger than $E(\bar{t}_\Pi(E))$, of photons with energy larger than $W(\bar{t}_\Gamma(W))$ and of the ionization $(\bar{t}_i)$.

(*j*) *Longitudinal spread of a shower*, τ: The longitudinal spread $\tau_\pi(E)$ of shower electrons of energy E is given by

$$[\tau_\pi(E)]^2=\frac{\int_0^\infty [t-\bar{t}_\pi(E)]^2\pi(E, t)dt}{\int_0^\infty \pi(E, t)dt}$$

$$=\frac{1}{z_\pi(E)}\int_0^\infty t^2\pi(E, t)dt-[\bar{t}_\pi(E)]^2. \tag{2.9}$$

Similar expressions define the quantities $\tau_\gamma(W)$, $\tau_\Pi(E)$, $\tau_\Gamma(W)$ and τ_j, which are, respectively, the longitudinal spread of photons of energy W, that of electrons of energy larger than E, that of photons of energy larger than W, and that of the ionization.

The quantities defined above describe the *average* behavior of showers. The actual behavior of an individual shower may differ considerably from the average. Thus, the problem arises to determine the *probability* for a certain behavior (for instance, the probability that N electrons with energy larger than E are found at the thickness t, etc.). This problem is referred to as the *fluctuation problem* and has not yet been solved satisfactorily. It will be discussed briefly in §43.

Even the discussion of the average behavior of showers cannot be carried out mathematically without certain simplifications. We have pointed out repeatedly that for large energies the dominant processes are radiation losses of electrons and pair production by photons. When the energy is decreased, collision losses become important and for still smaller energies the

Compton effect has to be taken into account. It is convenient to introduce a quantity ϵ called *critical energy*, which is defined as the energy dissipated by collision in one radiation length, by electrons of energy ϵ. Remembering that for cosmic-ray electrons energy and momentum are practically identical, Eq. (1.9a) with $\eta=5\times10^6$ yields the following equation for ϵ

$$\epsilon=1.53\times10^5\frac{Z}{A}X_0 \times\left(23.0+2\log\frac{\epsilon}{\mu_e}-2\log Z\right). \quad (2.10)$$

The values of ϵ for various substances are listed in Table IV. In first approximation, ϵ turns out to be inversely proportional to the atomic number, since X_0 goes approximately as A/Z^2.

As long as we confine our attention to energies large compared with the critical energy, the theory of showers can be developed considering only radiation phenomena and pair production. Furthermore, if the energies under consideration are also large compared with $137\mu_eZ^{-\frac{1}{3}}$, radiation phenomena and pair production can be described by the asymptotic formulae for complete screening. In what follows we shall call "*approximation A*" the approximation in which collision processes and Compton effect are neglected, and the asymptotic formulae are used to describe radiation processes and pair production.

For energies in the neighborhood of the critical energy, the Compton effect can still be disregarded, but the collision processes must be taken into account. These, however, do not contribute appreciably to the production of high energy secondary electrons, hence it will be sufficient to consider only their influence on the energy loss of electrons. Since the collision loss does not change rapidly with energy, it can be taken as constant and equal to ϵ ev per radiation length. We shall call "*approximation B*" the approximation in which the Compton effect is neglected, the collision loss is described as a constant energy dissipation and the asymptotic formulae for radiation processes and pair production are used.

For energies small compared with the critical energy, both the Compton effect and collision processes contribute considerably to the absorption, as well as to the production of shower particles. In air, for instance (see Fig. 13), a photon of 2.4×10^7 ev has the same probability for Compton effect and for materialization. The ratio between the probabilities of an electron of energy E' being produced by collision, and by materialization of a photon of energy W, is approximately given [according to Eqs. (1.7) and (1.50)] by

$$\frac{\text{collision}}{\text{materialization}}=\frac{2CX_0}{\sigma(W)}\cdot\frac{\mu_eW}{(E')^2}.$$

In air, where the average energy of shower particles is of the order of 10^8 ev, and considering that electrons and photons are present in comparable number, electrons in the neighborhood of the limiting energy $\eta_0=5\times10^6$ ev are much more likely to be produced by collision processes of electrons than by materialization of photons.

It will be shown that the shower problem under approximation A can be solved completely. When the energies under consideration are not too close to the initial energy, an analytical procedure can be used, on the basis of principles laid down by Carlson and Oppenheimer, Landau and Rumer (§§26–30). For energies close to the initial energy one can apply a method of successive approximations, developed by Bhabha and Heitler (§31).

The shower problem under approximation B

TABLE IV. *Atomic number (Z), atomic weight (A), radiation length (X_0), and critical energy (ϵ) for various substances.*

SUBSTANCE	Z	A	X_0 (G/CM²)	ϵ (10^6 EV)
Hydrogen	1	1	138	815
Carbon	6	12	52	120
Nitrogen	7	14	45	103
Oxygen	8	16	39.7	90
Aluminum	13	27	26.3	52
Argon	18	39.9	20.8	37
Iron	26	55.84	14.4	25
Copper	29	63.57	13.3	22.4
Lead	82	207.2	5.9	7.0
Air	N 76.9%* O 21.8% A 1.3%		43	98
Water	H 11.1%* O 88.9%		43	111

* By weight.

TABLE V. *List of references for the calculation of the various quantities describing showers.*

QUANTITY	REFERENCE	REMARKS
$\pi(E_0, E, t)$	Eq. (2.55)	Appr. A; $\epsilon \ll E \ll E_0$
	Eq. (2.98)	Appr. B; $\sim 2\epsilon < E \ll E_0$
$\pi(W_0, E, t)$	Eq. (2.58)	Appr. A; $\epsilon \ll E \ll W_0$
	§36	Appr. B; $\sim 2\epsilon < E \ll W_0$
$\gamma(E_0, W, t)$	Eq. (2.56), Fig. 16	Appr. A; $\epsilon \ll W \ll E_0$
	Table XI	Appr. A; W of the order of E_0
	Eq. (2.99)	Appr. B; $\sim 2\epsilon < W \ll E_0$
$\gamma(W_0, W, t)$	Eq. (2.59)	Appr. A; $\epsilon \ll W \ll W_0$
	Table XII	Appr. A; W of the order of W_0
	§36	Appr. B; $\sim 2\epsilon < W \ll W_0$
$\Pi(E_0, E, t)$	Eq. (2.57), Fig. 15	Appr. A; $\epsilon \ll E \ll E_0$
	Table IX	Appr. A; E of the order of E_0
	Eq. (2.100)	Appr. B; $\sim 2\epsilon < E \ll E_0$
	Eq. (2.104), Fig. 19	Appr. B; $E = 0$
$\Pi(W_0, E, t)$	Eq. (2.60)	Appr. A; $\epsilon \ll E \ll W_0$
	Table X	Appr. A; E of the order of W_0
	§36	Appr. B; $\sim 2\epsilon < E \ll W_0$
	Eq. (2.105)	Appr. B; $E = 0$
$J(E_0)$, $J(W_0)$	Eq. (2.108)	
$j(E_0, t)$, $j(W_0, t)$	Eq. (2.110)	Appr. B
$z_\pi(E_0, E)$, $z_\gamma(E_0, W)$	Eq. (2.44)	Appr. A; $\epsilon \ll \frac{E}{W} \ll E_0$
	Eq. (2.96)	Appr. B; $\sim 2\epsilon < \frac{E}{W} \ll E_0$
$z_\Pi(E_0, E)$	Eq. (2.44)	Appr. A; $\epsilon \ll E \ll E_0$
	Eq. (2.96)	Appr. B; $\sim 2\epsilon < E \ll E_0$
	Eq. (2.106)	Appr. B; $E = 0$
$z_\pi(W_0, E)$, $z_\gamma(W_0, W)$	Eq. (2.45)	Appr. A; $\epsilon \ll \frac{E}{W} \ll W_0$
	Eq. (2.97)	Appr. B; $\sim 2\epsilon < \frac{E}{W} \ll W_0$
$z_\Pi(W_0, E)$	Eq. (2.45)	Appr. A; $\epsilon \ll E \ll W_0$
	Eq. (2.97)	Appr. B; $\sim 2\epsilon < E \ll W_0$
	Eq. (2.107)	Appr. B; $E = 0$
$i_\pi(E_0, E)$, $i_\gamma(E_0, W)$, $i_\pi(W_0, E)$, $i_\gamma(W_0, W)$	Eq. (2.46)	Appr. A; $\epsilon \ll \frac{E}{W} \ll \frac{E_0}{W_0}$
	§36	Appr. B; $\sim 2\epsilon < \frac{E}{W} \ll \frac{E_0}{W_0}$
$i_\Pi(E_0, E)$, $i_\Pi(W_0, E)$	Eq. (2.46)	Appr. A; $\epsilon \ll E \ll \frac{E_0}{W_0}$
	§36	Appr. B; $\sim 2\epsilon < E \ll \frac{E_0}{W_0}$
	Eqs. (2.106), (2.107)	Appr. B; $E = 0$
$\tau_\pi(E_0, E)$, $\tau_\gamma(E_0, W)$, $\tau_\pi(W_0, E)$, $\tau_\gamma(W_0, W)$	Eq. (2.47)	Appr. A; $\epsilon \ll \frac{E}{W} \ll \frac{E_0}{W_0}$
	§36	Appr. B; $\sim 2\epsilon < \frac{E}{W} \ll \frac{E_0}{W_0}$
$\tau_\Pi(E_0, E)$, $\tau_\Pi(W_0, E)$	Eq. (2.47)	Appr. A; $\epsilon \ll E \ll \frac{E_0}{W_0}$
	§36	Appr. B: $\sim 2\epsilon < E \ll \frac{E_0}{W_0}$
	Eq. (2.106), (2.107)	Appr. B; $E = 0$

has been partially solved in the case of showers produced by a primary particle with energy large compared with the critical energy. Following a method first developed by Snyder, it is possible to calculate the specific ionization as a function of depth (§§32, 34, 35, and 37) and the energy spectrum of shower particles down to energies of about 2ϵ (§§32, 33, 34, and 35).

Some calculations have been performed on the low energy end of the spectrum where neither approximation *A* nor approximation *B* can be used. In these calculations, however, the primary energy was still regarded as large compared with ϵ (§38). No theoretical data are available on showers produced by primary electrons or photons with energy of the order of the critical energy, although it has been shown by Dresden, Scott, and Uhlenbeck that the problem can be solved, at least in principle, by a method of successive approximations similar to that of Bhabha and Heitler (D41; see also Scott, S41).

For convenience of the reader, we list in Table V the fundamental quantities describing the shower, along with references to the formulae, tables, and graphs by which they can be evaluated. The integral photon spectrum Γ and the other functions related to Γ are of no great practical use and will not be calculated explicitly.

A. Shower Theory under Approximation *A*

§26. *The Diffusion Equations*

We will here take into consideration only radiation phenomena and pair production, using the asymptotic formulae for complete screening to describe the probabilities of these effects. According to our previous discussion, such a theory will give accurate results for shower particles of energy large compared with both ϵ and $137\mu_e Z^{-\frac{1}{3}}$, the most stringent limitation being the first one in the case of light elements and the second in the case of heavy elements.

The rest energy of the electron will be considered as negligible compared with the kinetic energy, hence no distinction will be made between the kinetic energy and the total energy of the electron.

The equations satisfied by the differential spectra $\pi(E, t)$ and $\gamma(W, t)$ can be obtained as follows. In a given thickness dt the number of electrons with energy between E and $E+dE$ undergoes a change because of the following effects:

(a) Photons with energy W larger than E produce a certain number of electrons of both signs in the energy range (E, dE). This number is

$$dEdt\cdot 2\int_E^\infty \gamma(W,t)\psi_0\left(\frac{E}{W}\right)\frac{dW}{W}=dEdt\cdot 2\int_0^1 \gamma\left(\frac{E}{u},t\right)\psi_0(u)\frac{du}{u},$$

where ψ_0 is given by Eq. (1.42a) and $u=E/W$.

(b) Some electrons with energy E' larger than E enter the interval (E, dE) by radiating part of their energy. Their number is

$$dEdt\int_E^\infty \pi(E',t)\varphi_0\left(\frac{E'-E}{E'}\right)\frac{dE'}{E'}=dEdt\int_0^1 \pi\left(\frac{E}{1-v},t\right)\varphi_0(v)\frac{dv}{1-v},$$

where φ_0 is given by Eq. (1.23a) and $v=(E'-E)/E'$.

(c) Some electrons initially in the interval (E, dE) leave this interval by radiation loss. Their number is

$$\pi(E,t)dEdt\int_0^E \varphi_0\left(\frac{W}{E}\right)\frac{dW}{E}=\pi(E,t)dEdt\int_0^1 \varphi_0(v)dv,$$

where $v=W/E$. Both integrals (b) and (c) happen to be divergent, because $\varphi_0(v)$ behaves as $1/v$ ("infra-red catastrophe"). Their difference, however, remains finite.

In the thickness dt the number of photons with energy between W and $W+dW$ undergoes a change because of the following effects:

(a) Electrons with energy E larger than W radiate a certain number of photons in the energy interval (W, dW). This number is

$$dWdt\int_W^\infty \pi(E,t)\varphi_0\left(\frac{W}{E}\right)\frac{dE}{E}=dWdt\int_0^1 \pi\left(\frac{W}{v},t\right)\varphi_0(v)\frac{dv}{v},$$

where $v=W/E$.

(b) Some photons initially in the interval (W, dW) are absorbed by pair production. According to Eq. (1.47a) their number is

$$dWdt\times\gamma(W,t)\sigma_0.$$

Therefore

$$\frac{\partial\pi(E,t)}{\partial t}=2\int_0^1\gamma\left(\frac{E}{u},t\right)\psi_0(u)\frac{du}{u}-\int_0^1\left[\pi(E,t)-\frac{1}{1-v}\pi\left(\frac{E}{1-v},t\right)\right]\varphi_0(v)dv, \tag{2.11}$$

$$\frac{\partial\gamma(W,t)}{\partial t}=\int_0^1\pi\left(\frac{W}{v},t\right)\varphi_0(v)\frac{dv}{v}-\sigma_0\gamma(W,t). \tag{2.12}$$

The functions φ_0 and ψ_0 do not depend on the atomic number, hence the solutions of the Eqs. (2.11) and (2.12) are the same for all substances, provided, of course, we measure the thickness in radiation lengths. The functions φ_0 and ψ_0 depend only on the ratio between the primary energy and that of the emitted particle. Hence any solution of Eqs. (2.11), (2.12) remains valid if all energies are multiplied by a constant factor.

§27. *Elementary Solutions*

We want to show that the diffusion equations (2.11) and (2.12) have solutions in which the variables energy and thickness are separated, of the type

$$\pi(E,t)=F_\pi(E)f(t),\qquad \gamma(W,t)=F_\gamma(W)f(t). \tag{2.13}$$

Upon substitution in the diffusion equations, one obtains

$$F_\pi(E)\frac{df(t)}{dt}=f(t)\left\{2\int_0^1 F_\gamma\left(\frac{E}{u}\right)\psi_0(u)\frac{du}{u}-\int_0^1\left[F_\pi(E)-\frac{1}{1-v}F_\pi\left(\frac{E}{1-v}\right)\right]\varphi_0(v)dv\right\},$$

$$F_\gamma(W)\frac{df(t)}{dt}=f(t)\left\{\int_0^1 F_\pi\left(\frac{W}{v}\right)\varphi_0(v)\frac{dv}{v}-\sigma_0F_\gamma(W)\right\}.$$

The quantities in curled brackets are functions of E or W only. Therefore the ratio $\dot f(t)/f(t)$ does not depend on t, but is equal to some constant, λ.

$$df(t)/dt=\lambda f(t).$$

Hence

$$f(t)=\text{const. } e^{\lambda t}$$

and

$$\begin{aligned}\lambda F_\pi(E)&=2\int_0^1 F_\gamma\left(\frac{E}{u}\right)\psi_0(u)\frac{du}{u}-\int_0^1\left[F_\pi(E)-\frac{1}{1-v}F_\pi\left(\frac{E}{1-v}\right)\right]\varphi_0(v)dv,\\ \lambda F_\gamma(W)&=\int_0^1 F_\pi\left(\frac{W}{v}\right)\varphi_0(v)\frac{dv}{v}-\sigma_0F_\gamma(W).\end{aligned} \tag{2.14}$$

Equations (2.14) can be solved, and consequently the diffusion equations can be satisfied by functions of the form (2.13). Before discussing the solution in the present particular case, we want to point out that this conclusion is also valid when other terms, describing collision processes and Compton effect, are included in the diffusion equations, and when more complicated expressions are used for the probabilities of radiation phenomena and pair production. In other words, it is always possible to find solutions of the diffusion equations in which the number of shower particles varies exponentially with depth, while the proportion of electrons and photons and the shape of their energy spectra remain unchanged as the depth changes. These solutions will be referred to as *elementary solutions* of the diffusion equations.

Equations (2.14) are satisfied by power functions of the energy,

$$F_\pi(E)=aE^{-(s+1)}, \quad F_\gamma(W)=bW^{-(s+1)}, \tag{2.15}$$

where s is a positive number.

Inserting (2.15) in (2.14), we get

$$\lambda a=-A(s)a+B(s)b, \qquad \lambda b=C(s)a-\sigma_0 b, \tag{2.16}$$

where

$$\begin{aligned} A(s)&=\int_0^1[1-(1-v)^s]\varphi_0(v)dv, \quad C(s)=\int_0^1 v^s\varphi_0(v)dv,\\ B(s)&=2\int_0^1 u^s\psi_0(u)du, \qquad\qquad \sigma_0=\int_0^1\psi_0(u)du. \end{aligned} \tag{2.17}$$

In order that (2.16) may be solved for a and b, λ must satisfy the quadratic equation

$$[\lambda+A(s)](\lambda+\sigma_0)-B(s)C(s)=0. \tag{2.18}$$

Hence for every value of the exponent s there are two possible values of λ:

$$\begin{aligned} \lambda_1(s)&=-\frac{A(s)+\sigma_0}{2}+\tfrac{1}{2}\{[A(s)-\sigma_0]^2+4B(s)C(s)\}^{\frac{1}{2}},\\ \lambda_2(s)&=-\frac{A(s)+\sigma_0}{2}-\tfrac{1}{2}\{[A(s)-\sigma_0]^2+4B(s)C(s)\}^{\frac{1}{2}}. \end{aligned} \tag{2.19}$$

The ratio between the coefficients a and b is

$$\frac{a_1}{b_1}=\frac{B(s)}{A(s)+\lambda_1(s)}=\frac{\sigma_0+\lambda_1(s)}{C(s)}, \quad \text{or} \quad \frac{a_2}{b_2}=\frac{B(s)}{A(s)+\lambda_2(s)}=\frac{\sigma_0+\lambda_2(s)}{C(s)} \tag{2.20}$$

according to the two possible choices for λ. In conclusion, we have the two following sets of elementary solutions of the diffusion equations (2.11) and (2.12):

$$\begin{aligned} \pi(E,t)&=a_1E^{-(s+1)}\exp[\lambda_1(s)t], \quad \gamma(W,t)=\frac{a_1C(s)}{\sigma_0+\lambda_1(s)}W^{-(s+1)}\exp[\lambda_1(s)t],\\ \pi(E,t)&=a_2E^{-(s+1)}\exp[\lambda_2(s)t], \quad \gamma(W,t)=\frac{a_2C(s)}{\sigma_0+\lambda_2(s)}W^{-(s+1)}\exp[\lambda_2(s)t], \end{aligned} \tag{2.21}$$

where a_1 and a_2 are arbitrary constants, and λ_1 and λ_2 are given by Eqs. (2.19).

Explicit expressions for $A(s)$, $B(s)$, and $C(s)$ are easily obtained by substituting in (2.17) the

expressions for $\varphi_0(v)$ and $\psi_0(u)$ given by (1.23a) and (1.42a). The result is

$$A(s)=\left(\frac{4}{3}+2b\right)\left(\frac{d}{ds}\log s!+\gamma-1+\frac{1}{s+1}\right)+\frac{1}{2}-\frac{1}{(s+1)(s+2)}$$
$$=1.36\frac{d}{ds}\log(s+1)!-\frac{1}{(s+1)(s+2)}-0.0750,$$
$$B(s)=2\left[\frac{1}{s+1}-\left(\frac{4}{3}+2b\right)\left(\frac{1}{s+2}-\frac{1}{s+3}\right)\right]=2\left[\frac{1}{s+1}-\frac{1.36}{(s+2)(s+3)}\right], \tag{2.17a}$$
$$C(s)=\left(\frac{4}{3}+2b\right)\left(\frac{1}{s}-\frac{1}{s+1}\right)+\frac{1}{s+2}=\frac{1}{s+2}+\frac{1.36}{s(s+1)},$$
$$\sigma_0=\frac{7}{9}-\frac{b}{3}=0.773.$$

Approximate expressions of simpler mathematical form can be obtained by using (1.32) and (1.49a) instead of (1.23a) and (1.42a). They are

$$A(s)=(d/ds)\log s!+0.5772,\quad B(s)=2\sigma_0/(s+1),\quad C(s)=1/s,\quad \sigma_0=7/9. \tag{2.17b}$$

In Eqs. (2.17a) and (2.17b),

$$\frac{d}{ds}\log s!=\int_0^1 dx-\gamma$$

indicates the logarithmic derivative of the gamma-function and $\gamma=0.5772$ is Euler's constant, while $b=0.0135$ is defined in §13. The values of $A(s)$, $B(s)$, $C(s)$, $\lambda_1(s)$, $\lambda_2(s)$, for various values of s are listed in Table VII.

The elementary solutions (2.19) have a direct physical meaning. Suppose a beam of electrons and photons is incident at $t=0$, and let $\pi(E,0)dE$ be the number per second of incident electrons with energy between E and $E+dE$, while $\gamma(W,0)$ is the number per second of incident photons with energy between W and $W+dW$. Suppose that, for energies larger than a certain limit η, both $\pi(E,0)$ and $\gamma(W,0)$ are represented by a power law with the same exponent:

$$\pi(E,0)=K_\pi E^{-(s+1)},\quad \gamma(W,0)=K_\gamma W^{-(s+1)}.$$

The differential spectra of electrons and photons for E and W larger than η and for any thickness t are then given by linear combinations of the two elementary solutions corresponding to the exponent $s+1$, namely,

$$\pi(E,t)=E^{-(s+1)}[a_1\exp[\lambda_1(s)t]+a_2\exp[\lambda_2(s)t]],$$
$$\gamma(W,t)=W^{-(s+1)}\left[\frac{a_1C(s)}{\sigma_0+\lambda_1(s)}\exp[\lambda_1(s)t]+\frac{a_2C(s)}{\sigma_0+\lambda_2(s)}\exp[\lambda_2(s)t]\right], \tag{2.22}$$

where the constants a_1 and a_2 are chosen so as to satisfy the boundary conditions

$$a_1+a_2=K_\pi,\quad a_1\frac{C(s)}{\sigma_0+\lambda_1(s)}+a_2\frac{C(s)}{\sigma_0+\lambda_2(s)}=K_\gamma.$$

Thus, for energies larger than η, the energy distribution of electrons and photons is independent of thickness traversed. The variation of number with depth is different for electrons and photons and depends on the initial proportion of the two types of shower particles. For large thicknesses, however, the terms containing $\exp[\lambda_2(s)t]$ become negligible compared with the terms containing

$\exp[\lambda_1(s)t]$, because λ_2 is always negative and larger in absolute value than λ_1. Hence for $t \gg 1$, Eqs. (2.22) reduce to

$$\pi(E, t) = a_1 E^{-(s+1)} \exp[\lambda_1(s)t], \quad \gamma(W, t) = \frac{a_1 C(s)}{\sigma_0 + \lambda_1(s)} W^{-(s+1)} \exp[\lambda_1(s)t], \tag{2.22a}$$

which indicate that, at sufficiently large thicknesses, the numbers of electrons and photons are in a constant ratio and both numbers vary exponentially with depth, though the rate of change with depth depends on s and hence on the initial energy distribution. According to Table VII, λ_1 is positive for $s<1$, zero for $s=1$, and negative for $s>1$, approaching $-\sigma_0 = -0.7733$ as s approaches infinity. In the first case the number of shower particles increases with depth, in the second case it remains constant, and in the last case it decreases. The case in which the number of shower particles is independent of depth is particularly important, and the corresponding spectra of electrons and photons will be referred to as *normal spectra*. Under approximation A the normal spectra are given by

$$\pi_0(E) = a_1/E^2, \quad \gamma_0(W) = \frac{a_1 C(1)}{\sigma_0} \cdot \frac{1}{W^2} = \frac{1.31 a_1}{W^2}. \tag{2.23}$$

§28. *Mellin and Laplace integrals of* π, γ, II

We encounter a more difficult mathematical problem when we try to solve the diffusion equations (2.11), (2.12) under arbitrary boundary conditions, in particular under the boundary conditions corresponding to a single incident electron or photon. As in many problems of a similar type, a powerful method of attack is offered by functional transformations. In the following discussion we shall make frequent use of the transformations known as the *Mellin* and the *Laplace* transformations. For convenience of the reader, we summarize in Appendix I the fundamental properties of these transformations.

Let us consider the Mellin integrals of π and γ with respect to the energy; i.e., the quantities

$$\mathfrak{M}_\pi(s, t) = \int_0^\infty E^s \pi(E, t) dE, \quad \mathfrak{M}_\gamma(s, t) = \int_0^\infty W^s \gamma(W, t) dW, \tag{2.24}$$

where s is a complex parameter. $\mathfrak{M}_\pi$ and $\mathfrak{M}_\gamma$ are defined for all those values of s which make the integrals converge. Since π and γ are identically zero for energies larger than the primary energy, we only need to consider the convergence of the integrals at the lower limit. It follows that the field of convergence of $\mathfrak{M}_\pi$ and $\mathfrak{M}_\gamma$ is the half plane defined by $R(s) > s_0$, where $R(s)$ indicates the real part of s and s_0 is a real constant. It will be shown later than $s_0 = 0$.

Let us consider also the Laplace integrals of π and γ with respect to t; i.e., the quantities

$$\mathfrak{L}_\pi(E, \lambda) = \int_0^\infty e^{-\lambda t} \pi(E, t) dt, \quad \mathfrak{L}_\gamma(W, \lambda) = \int_0^\infty e^{-\lambda t} \gamma(W, t) dt. \tag{2.25}$$

$\mathfrak{L}_\pi$ and $\mathfrak{L}_\gamma$ are defined for all those values of the complex parameter λ which make the integrals converge. The field of convergence is defined by $R(\lambda) > \lambda_0$, where λ_0 is a real constant. It will be shown later that $\lambda_0 = -\sigma_0$.

We shall finally consider the Mellin integrals with respect to energy of the Laplace integrals with respect to thickness; i.e., the quantities

$$\mathfrak{N}_\pi(s, \lambda) = \int_0^\infty E^s dE \int_0^\infty e^{-\lambda t} \pi(E, t) dt, \quad \mathfrak{N}_\gamma(s, \lambda) = \int_0^\infty W^s dW \int_0^\infty e^{-\lambda t} \gamma(W, t) dt, \tag{2.26}$$

which are functions of the two complex parameters s and λ.

The reason for introducing the transforms $\mathfrak{M}$, $\mathfrak{L}$, $\mathfrak{N}$ is that they are more easily determined than the original functions π and γ. Some properties of showers can be derived directly from the knowledge of the above integrals, while the functions π and γ themselves can be obtained from the transforms, by using the inversion formulae given in Appendix I.

The boundary conditions of the problem are determined by the functions $\pi(E, 0)$ and $\gamma(W, 0)$, which describe the radiation incident at $t=0$. We shall consider in particular the boundary conditions representing:

(a) a single primary electron of energy E_0

$$\pi(E, 0)=\delta(E-E_0); \qquad \gamma(W, 0)=0; \tag{2.27}$$

(b) a single primary photon of energy W_0

$$\pi(E, 0)=0; \qquad \gamma(W, 0)=\delta(W-W_0), \tag{2.28}$$

where δ is Dirac's improper function.

The Mellin integrals, as pointed out by Landau and Rumer (L38), satisfy a simple system of differential equations, which can be obtained by multiplying both sides of (2.11) and (2.12) by E^s and W^s, respectively, and integrating with respect to energy from 0 to ∞. The integrals on the right-hand sides can be transformed as follows:

$$\int_0^\infty E^s dE \cdot 2\int_0^1 \gamma(E/u, t)\psi_0(u)(du/u)$$

$$=\int_0^\infty W^s\gamma(W, t)dW\cdot 2\int_0^1 u^s\psi_0(u)du, \quad (W=E/u)$$

$$=\mathfrak{M}_\gamma(s, t)\cdot B(s),$$

$$\int_0^\infty E^s dE\int_0^1\left[\pi(E, t)-\frac{1}{1-v}\pi\left(\frac{E}{1-v}, t\right)\right]\varphi_0(v)dv$$

$$=\int_0^\infty \pi(E, t)E^s dE\int_0^1 \varphi_0(v)dv-\int_0^\infty (E')^s\pi(E', t)dE'\int_0^1 (1-v)^s\varphi_0(v)dv, \quad [E'=E/(1-v)]$$

$$=\int_0^\infty E^s\pi(E, t)dE\int_0^1 [1-(1-v)^s]\varphi_0(v)dv,$$

$$=\mathfrak{M}_\pi(s, t)\cdot A(s),$$

$$\int_0^\infty W^s dW\int_0^1 \pi(W/v, t)\,\varphi_0(v)\,(dv/v)=\int_0^\infty E^s\pi(E, t)dE\int_0^1 v^s\varphi_0(v)dv, \quad (E=W/v)$$

$$=\mathfrak{M}_\pi(s, t)\cdot C(s).$$

One finally obtains

$$\frac{\partial\mathfrak{M}_\pi(s, t)}{\partial t}=-A(s)\mathfrak{M}_\pi(s, t)+B(s)\mathfrak{M}_\gamma(s, t), \quad \frac{\partial\mathfrak{M}_\gamma(s, t)}{\partial t}=C(s)\mathfrak{M}_\pi(s, t)-\sigma_0\mathfrak{M}_\gamma(s, t). \tag{2.29}$$

If one now multiplies both sides of (2.29) by $e^{-\lambda t}$ and integrates with respect to t from 0 to ∞, one gets the following system of algebraic equations for $\mathfrak{N}_\pi$ and $\mathfrak{N}_\gamma$:

$$\begin{aligned}\lambda\mathfrak{N}_\pi(s, \lambda)-\mathfrak{M}_\pi(s, 0)&=-A(s)\mathfrak{N}_\pi(s, \lambda)+B(s)\mathfrak{N}_\gamma(s, \lambda),\\ \lambda\mathfrak{N}_\gamma(s, \lambda)-\mathfrak{M}_\gamma(s, 0)&=C(s)\mathfrak{N}_\pi(s, \lambda)-\sigma_0\mathfrak{N}_\gamma(s, \lambda),\end{aligned} \tag{2.30}$$

where $\mathfrak{M}_\pi(s, 0)$ and $\mathfrak{M}_\gamma(s, 0)$ are the Mellin integrals of $\pi(E, 0)$ and $\gamma(W, 0)$. In particular,

(a) for an incident electron of energy E_0

$$\mathfrak{M}_\pi(s, 0) = E_0{}^s, \quad \mathfrak{M}_\gamma(s, 0) = 0; \tag{2.31}$$

(b) for an incident photon of energy W_0

$$\mathfrak{M}_\pi(s, 0) = 0, \quad \mathfrak{M}_\gamma(s, 0) = W_0{}^s. \tag{2.32}$$

The solutions of Eqs. (2.30) are:

(a) for incident electron of energy E_0

$$\begin{aligned}
\mathfrak{N}_\pi(E_0, s, \lambda) &= \frac{(\sigma_0+\lambda)E_0{}^s}{[A(s)+\lambda](\sigma_0+\lambda)-B(s)C(s)} = \frac{(\sigma_0+\lambda)E_0{}^s}{[\lambda-\lambda_1(s)][\lambda-\lambda_2(s)]}, \\
\mathfrak{N}_\gamma(E_0, s, \lambda) &= \frac{C(s)E_0{}^s}{[A(s)+\lambda](\sigma_0+\lambda)-B(s)C(s)} = \frac{C(s)E_0{}^s}{[\lambda-\lambda_1(s)][\lambda-\lambda_2(s)]};
\end{aligned} \tag{2.33}$$

(b) for incident photon of energy W_0

$$\begin{aligned}
\mathfrak{N}_\pi(W_0, s, \lambda) &= \frac{B(s)W_0{}^s}{[A(s)+\lambda](\sigma_0+\lambda)-B(s)C(s)} = \frac{B(s)W_0{}^s}{[\lambda-\lambda_1(s)][\lambda-\lambda_2(s)]}, \\
\mathfrak{N}_\gamma(W_0, s, \lambda) &= \frac{[A(s)+\lambda]W_0{}^s}{[A(s)+\lambda](\sigma_0+\lambda)-B(s)C(s)} = \frac{[A(s)+\lambda]W_0{}^s}{[\lambda-\lambda_1(s)][\lambda-\lambda_2(s)]},
\end{aligned} \tag{2.34}$$

where $\lambda_1(s)$ and $\lambda_2(s)$ are the functions defined by Eqs. (2.19).

Either by directly solving Eqs. (2.29) with the boundary conditions (2.31) or (2.32), or by applying the inverse Laplace transformation to $\mathfrak{N}_\pi$ and $\mathfrak{N}_\gamma$ (see Appendix I), one easily obtains:

(a) for a primary electron of energy E_0,

$$\begin{aligned}
\mathfrak{M}_\pi(E_0, s, t) &= \frac{E_0{}^s}{\lambda_1(s)-\lambda_2(s)}\{[\sigma_0+\lambda_1(s)]\exp[\lambda_1(s)t]-[\sigma_0+\lambda_2(s)]\exp[\lambda_2(s)t]\}, \\
\mathfrak{M}_\gamma(E_0, s, t) &= \frac{C(s)E_0{}^s}{\lambda_1(s)-\lambda_2(s)}\{\exp[\lambda_1(s)t]-\exp[\lambda_2(s)t]\};
\end{aligned} \tag{2.35}$$

(b) for a primary photon of energy W_0,

$$\begin{aligned}
\mathfrak{M}_\pi(W_0, s, t) &= -\frac{W_0{}^s}{C(s)}\frac{[\sigma_0+\lambda_1(s)][\sigma_0+\lambda_2(s)]}{\lambda_1(s)-\lambda_2(s)}\{\exp[\lambda_1(s)t]-\exp[\lambda_2(s)t]\}, \\
\mathfrak{M}_\gamma(W_0, s, t) &= -\frac{W_0{}^s}{\lambda_1(s)-\lambda_2(s)}\{[\sigma_0+\lambda_2(s)]\exp[\lambda_1(s)t]-[\sigma_0+\lambda_1(s)]\exp[\lambda_2(s)t]\}.
\end{aligned} \tag{2.36}$$

For any given value of t, the functions on the right-hand sides of Eqs. (2.35) and (2.36) are regular for $s>0$ and tend to infinity at the limit for $s=0$, where $\lambda_1(s)$ becomes positively infinite. This shows that the field of convergence of $\mathfrak{M}_\pi$ and $\mathfrak{M}_\gamma$ is the positive half plane. The expressions (2.35) and (2.36) represent $\mathfrak{M}_\pi$ and $\mathfrak{M}_\gamma$ only in this half plane, and give the analytical continuation of $\mathfrak{M}_\pi$ and $\mathfrak{M}_\gamma$ in the negative half plane.

The Laplace integrals $\mathfrak{L}_\pi$ and $\mathfrak{L}_\gamma$ are obtained by applying the inverse Mellin transformation to $\mathfrak{N}_\pi$ and $\mathfrak{N}_\gamma$ (see Appendix I) as follows:

(a) primary electron of energy E_0

$$\mathfrak{L}_\pi(E_0, E, \lambda)=\frac{1}{2\pi i}\cdot\frac{1}{E}\int_{\delta-i\infty}^{\delta+i\infty}\left(\frac{E_0}{E}\right)^s\frac{(\sigma_0+\lambda)ds}{[\lambda-\lambda_1(s)][\lambda-\lambda_2(s)]},$$
$$\mathfrak{L}_\gamma(E_0, W, \lambda)=\frac{1}{2\pi i}\cdot\frac{1}{W}\int_{\delta-i\infty}^{\delta+i\infty}\left(\frac{E_0}{W}\right)^s\frac{C(s)ds}{[\lambda-\lambda_1(s)][\lambda-\lambda_2(s)]}; \tag{2.37}$$

(b) primary photon of energy W_0

$$\mathfrak{L}_\pi(W_0, E, \lambda)=\frac{1}{2\pi i}\cdot\frac{1}{E}\int_{\delta-i\infty}^{\delta+i\infty}\left(\frac{W_0}{E}\right)^s\frac{B(s)ds}{[\lambda-\lambda_1(s)][\lambda-\lambda_2(s)]},$$
$$\mathfrak{L}_\gamma(W_0, W, \lambda)=\frac{1}{2\pi i}\cdot\frac{1}{W}\int_{\delta-i\infty}^{+i\infty}\left(\frac{W_0}{W}\right)^s\frac{[A(s)+\lambda]ds}{[\lambda-\lambda_1(s)][\lambda-\lambda_2(s)]}. \tag{2.38}$$

The integration paths in the above integrals run parallel to the imaginary axis, to the right of all the poles. For $\lambda>-\sigma_0$, there is only one pole on the positive half plane of s; this pole lies on the real axis and is defined by the equation

$$\lambda_1(s)=\lambda. \tag{2.39}$$

It can easily be shown that there is an infinite number of poles on the negative half plane because of the behavior of the logarithmic derivative of the factorial function, which enters in $A(s)$.

Let us consider the expression for $\mathfrak{L}_\pi$ in Eq. (2.37). For $E<E_0$, the integrand tends to zero at the limit for $s=-\infty$. Hence, deforming the contour to the left, the integral can be expressed as the sum of the residues at all the poles. Each pole contributes a term proportional to $(E_0/E)^s$. For the pole on the positive real axis, s is real and positive, whereas for those in the negative half plane, $R(s)<0$. The latter poles contribute terms which are negligible with respect to the first one if $E\ll E_0$. A similar conclusion applies to the other Laplace integrals. Hence, for energies small compared with the primary energy, $\mathfrak{L}_\pi$ and $\mathfrak{L}_\gamma$ are given by:

(a) for a primary electron of energy E_0,

$$\mathfrak{L}_\pi(E_0, E, \lambda)=-\frac{1}{E}\left(\frac{E_0}{E}\right)^s\frac{\sigma_0+\lambda_1(s)}{[\lambda_1(s)-\lambda_2(s)]\lambda'_1(s)},$$
$$\mathfrak{L}_\gamma(E_0, W, \lambda)=-\frac{1}{W}\left(\frac{E_0}{W}\right)^s\frac{C(s)}{[\lambda_1(s)-\lambda_2(s)]\lambda'_1(s)}; \tag{2.40}$$

(b) for a primary photon of energy W_0,

$$\mathfrak{L}_\pi(W_0, E, \lambda)=-\frac{1}{E}\left(\frac{W_0}{E}\right)^s\frac{B(s)}{[\lambda_1(s)-\lambda_2(s)]\lambda'_1(s)},$$
$$\mathfrak{L}_\gamma(W_0, W, \lambda)=-\frac{1}{W}\left(\frac{W_0}{W}\right)^s\frac{A(s)+\lambda_1(s)}{[\lambda_1(s)-\lambda_2(s)]\lambda'_1(s)}. \tag{2.41}$$

In the above equations, s is defined by the condition (2.39), hence λ coincides with $\lambda_1(s)$. $\lambda'_1(s)$ indicates the derivative of λ_1 with respect to s and is given as a function of s in Table VII. For $\lambda=-\sigma_0$, $s=+\infty$ and the expressions (2.40) and (2.41) go to infinity. It follows that the half plane of convergence of the Laplace integrals is to the right of the point $\lambda=-\sigma_0$.

We write also the expressions for the integrals of Π, which are easily obtained using Eq. (A. 8) of Appendix I*:

* The expressions for the Mellin integrals have been given by Landau and Rumer (L38). The expressions for the Laplace integrals have not previously been published.

(a) primary electron of energy E_0

$$\mathfrak{M}_{\Pi}(E_0, s-1, t)=-\frac{1}{s}\frac{E_0{}^s}{\lambda_1(s)-\lambda_2(s)}\{[\sigma_0+\lambda_1(s)]\exp[\lambda_1(s)t]-[\sigma_0+\lambda_2(s)]\exp[\lambda_2(s)t]\},$$
$$\mathfrak{L}_{\Pi}(E_0, E, \lambda)=-\frac{1}{s}\left(\frac{E_0}{E}\right)^s\frac{\sigma_0+\lambda_1(s)}{[\lambda_1(s)-\lambda_2(s)]\lambda'_1(s)}; \tag{2.42}$$

(b) primary photon of energy W_0

$$\mathfrak{M}_{\Pi}(W_0, s-1, t)=-\frac{W_0{}^s}{sC(s)}\frac{[\sigma_0+\lambda_1(s)][\sigma_0+\lambda_2(s)]}{\lambda_1(s)-\lambda_2(s)}\exp[\lambda_1(s)t]-\exp[\lambda_2(s)t],$$
$$\mathfrak{L}_{\Pi}(W_0, E, \lambda)=-\frac{1}{s}\left(\frac{W_0}{E}\right)^s\frac{B(s)}{[\lambda_1(s)-\lambda_2(s)]\lambda'_1(s)}. \tag{2.43}$$

§29. *Track Length, Center of Gravity, and Longitudinal Spread*

The expressions for the track length z, the position of the center of gravity $\bar{t}$, and the longitudinal spread τ corresponding to π, γ, or Π are immediately derived from the expressions for the Laplace integrals of the same functions. Indeed, it is easily seen that

$$z=(\mathfrak{L})_{\lambda=0},\quad \bar{t}=\left(-\frac{1}{\mathfrak{L}}\frac{\partial\mathfrak{L}}{\partial\lambda}\right)_{\lambda=0},$$
$$\tau^2=\left[\frac{1}{\mathfrak{L}}\frac{\partial^2\mathfrak{L}}{\partial\lambda^2}-\left(\frac{1}{\mathfrak{L}}\frac{\partial\mathfrak{L}}{\partial\lambda}\right)^2\right]_{\lambda=0}=\left[\frac{\partial}{\partial\lambda}\left(\frac{1}{\mathfrak{L}}\frac{\partial\mathfrak{L}}{\partial\lambda}\right)\right]_{\lambda=0}$$

Using the results of the foregoing section and the numerical values listed in Table VII, one gets:*

(a) primary electron of energy E_0

$$z_{\pi}(E_0, E)=\frac{\sigma_0}{\lambda_2(1)\lambda'_1(1)}\frac{E_0}{E^2}=0.437\frac{E_0}{E^2},$$
$$z_{\gamma}(E_0, W)=\frac{C(1)}{\lambda_2(1)\lambda'_1(1)}\frac{E_0}{W^2}=0.572\frac{E_0}{W^2}, \tag{2.44}$$
$$z_{\Pi}(E_0, E)=\frac{\sigma_0}{\lambda_2(1)\lambda'_1(1)}\frac{E_0}{E}=0.437\frac{E_0}{E};$$

(b) primary photon of energy W_0

$$z_{\pi}(W_0, E)=\frac{B(1)}{\lambda_2(1)\lambda'_1(1)}\frac{W_0}{E^2}=0.437\frac{W_0}{E^2},$$
$$z_{\gamma}(W_0, W)=\frac{A(1)}{\lambda_2(1)\lambda'_1(1)}\frac{W_0}{W^2}=0.572\frac{W_0}{W^2}, \tag{2.45}$$
$$z_{\Pi}(W_0, E)=\frac{B(1)}{\lambda_2(1)\lambda'_1(1)}\frac{W_0}{E}=0.437\frac{W_0}{E}.$$

* The expressions for the track lengths have been given by Nordheim and Hebbs (N39). The other expressions have not previously been published.

The t's have expressions of the form

$$t = -\frac{1}{\lambda'_1(1)} y + h = 1.01y + h, \tag{2.46}$$

where y is the logarithm of the ratio between the initial energy (E_0 or W_0) and the energy under consideration (E or W) and h has, for the various functions, the values listed in Table VI. The τ's are given by expressions of the form

$$\tau^2 = -\frac{\lambda''_1(1)}{[\lambda'_1(1)]^3} y + k = 1.61y + k, \tag{2.47}$$

TABLE VI. *Values of h in Eq. (2.46) and of k in Eq. (2.47).*

Quantity	$i_\pi(E_0, E)$	$i_\gamma(E_0, W)$	$i_\Pi(E_0, E)$	$i_\pi(W_0, E)$	$i_\gamma(W_0, W)$	$i_\Pi(W_0, E)$
h	1.0	1.2	0.03	1.8	2.0	0.8
k	−0.1	1.0	−0.7	1.1	2.1	0.5

where y has the same meaning as in (2.46), and the values of k are given also in Table VI. It may be noted that the dependence of the track lengths on energy is the same as for the normal spectra defined in §27. The above expressions, of course, break down when the energies approach that of the primary particle.

§30. *Differential and Integral Spectra*

The expressions for the differential and integral spectra of electrons and photons can be obtained from Eqs. (2.35), (2.36), (2.42) and (2.43), by using the inversion formula of the Mellin transformation. For instance,

$$\pi(E, t)dE = \frac{1}{2\pi i}\frac{dE}{E}\int_{\delta-i\infty}^{\delta+i\infty} E^{-s}\mathfrak{M}_\pi(s, t)ds,$$

where the integration path is any line parallel to the imaginary axis in the positive half plane. Similar expressions hold for γ and Π. Introducing the logarithm of the energy, one gets:

(a) for a primary electron of energy E_0,

$$\pi(E_0, E, t)dE = -\frac{dy}{2\pi i}\int_{\delta-i\infty}^{\delta+i\infty} dsH_1(s)\exp[ys+\lambda_1(s)t] - \frac{dy}{2\pi i}\int_{\delta-i\infty}^{\delta+i\infty} dsH_2(s)\exp[ys+\lambda_2(s)t], \tag{2.48}$$

where

$$y = \log\left(\frac{E_0}{E}\right), \quad H_1(s) = \frac{\sigma_0+\lambda_1(s)}{\lambda_1(s)-\lambda_2(s)}, \quad H_2(s) = -\frac{\sigma_0+\lambda_2(s)}{\lambda_1(s)-\lambda_2(s)},$$

$$\gamma(E_0, W, t)dW = -\frac{dy}{2\pi i}\int_{\delta-i\infty}^{\delta+i\infty} dsL(s)\{\exp[ys+\lambda_1(s)t-\tfrac{1}{2}\log s] - \exp[ys+\lambda_2(s)t-\tfrac{1}{2}\log s]\}, \tag{2.49}$$

where

$$y = \log\left(\frac{E_0}{W}\right) \quad \text{and} \quad L(s) = \frac{\sqrt{s}\,C(s)}{\lambda_1(s)-\lambda_2(s)}.$$

$$\Pi(E_0, E, t)=\frac{1}{2\pi i}\int_{\delta-i\infty}^{\delta+i\infty} dsH_1(s)\exp\left[ys+\lambda_1(s)t-\log s\right] + \frac{1}{2\pi i}\int_{\delta-i\infty}^{\delta+i\infty} dsH_2(s)\exp\left[ys+\lambda_2(s)t-\log s\right], \quad (2.50)$$

where y, $H_1(s)$ and $H_2(s)$ are as in Eq. (2.48);

(b) for a primary photon of energy W_0,

$$\pi(W_0, E, t)dE=-\frac{dy}{2\pi i}\int_{\delta-i\infty}^{\delta+i\infty} dsM(s)\{\exp\left[ys+\lambda_1(s)t+\tfrac{1}{2}\log s\right]-\exp\left[ys+\lambda_2(s)t+\tfrac{1}{2}\log s\right]\}, \quad (2.51)$$

where

$$y=\log\left(\frac{W_0}{E}\right) \quad \text{and} \quad M(s)=-\frac{[\sigma_0+\lambda_1(s)][\sigma_0+\lambda_2(s)]}{\sqrt{s}\,C(s)[\lambda_1(s)-\lambda_2(s)]}.$$

$$\gamma(W_0, W, t)dW=-\frac{dy}{2\pi i}\int_{\delta-i\infty}^{\delta+i\infty} dsH_2(s)\exp\left[ys+\lambda_1(s)t\right]-\frac{dy}{2\pi i}\int_{\delta-i\infty}^{\delta+i\infty} dsH_1(s)\exp\left[ys+\lambda_2(s)t\right], \quad (2.52)$$

where $y=\log(W_0/W)$ and $H_1(s)$, $H_2(s)$ are as in Eq. (2.48):

$$\Pi(W_0, E, t)=\frac{1}{2\pi i}\int_{\delta-i\infty}^{\delta+i\infty} dsM(s)\{\exp(ys+\lambda_1(s)t-\tfrac{1}{2}\log s)-\exp(ys+\lambda_2(s)t-\tfrac{1}{2}\log s)\}, \quad (2.53)$$

where y, $M(s)$ are as in Eq. (2.51).

In the preceding equations each integrand has been written as the product of a function which changes slowly with $s(H_1(s)$, $H_2(s)$, $L(s)$ or $M(s))$ and an exponential term of the general form $\exp(ys+\lambda(s)t-n\log s)$. The exponential term has a saddle point at the point $\bar{s}$ defined by the equation

$$y+\lambda'(\bar{s})t-n/\bar{s}=0. \quad (2.54)$$

The point $\bar{s}$ lies on the positive real axis, and the exponent has a sharp minimum there as one goes along the real axis. But since it is true for any analytic function $f(x+iy)$ that

$$(\partial^2 f/\partial x^2)+(\partial^2 f/\partial y^2)=0$$

the exponential term must have an equally sharp maximum at the point $\bar{s}$, as one progresses at right angles to the real axis, which is the direction of the path of integration. If we make the integration path go through the point $\bar{s}$, the most important contribution to the integral will come from a portion of the integration path in the immediate neighborhood of $\bar{s}$. Hence we can treat the slowly varying functions of s as constants, evaluating them at the saddle point. We are left with integrals of the type

$$\frac{1}{2\pi i}\int_{\bar{s}-i\infty}^{\bar{s}+i\infty}\exp\left[ys+\lambda(s)t-n\log s\right]ds,$$

which can be calculated by developing the exponent in a Taylor series of powers of $(s-\bar{s})$ and neglecting terms of order greater than two. Using (2.54) we get

$$ys+\lambda(s)t-n\log s=y\bar{s}+\lambda(\bar{s})t-n\log\bar{s}+[\lambda''(\bar{s})t+n/\bar{s}^2][(s-\bar{s})^2/2].$$

Since the path of integration is a straight line parallel to the imaginary axis, we can set $(s-\bar{s})=ix$,

and the integral becomes

$$\frac{1}{2\pi}\exp\left[y\bar{s}+\lambda(\bar{s})t-n\log\bar{s}\right]\int_{-\infty}^{\infty}\exp\left\{-(x^2/2)[\lambda''(\bar{s})t+n/\bar{s}^2]\right\}dx$$

$$=\frac{1}{(2\pi)^{\frac{1}{2}}}\frac{1}{[\lambda''(\bar{s})t+n/\bar{s}^2]^{\frac{1}{2}}}\exp\left[y\bar{s}+\lambda(\bar{s})t-n\log\bar{s}\right]=\frac{1}{(2\pi)^{\frac{1}{2}}}\frac{1}{\bar{s}^n}\frac{(e^v)^{\bar{s}}}{[\lambda''(\bar{s})t+n/\bar{s}^2]^{\frac{1}{2}}}\exp\left[\lambda(\bar{s})t\right].$$

Thus each of the functions π, γ, and Π is expressed as the sum of two quantities, proportional to $\exp[\lambda_1(\bar{s})t]$ and $\exp[\lambda_2(\bar{s})t]$, respectively. Since λ_2 is always negative and larger in absolute value than λ_1, the second term can be neglected when t is not too small. We finally obtain, dropping the bar on the s,*

(a) for a primary electron of energy E_0

$$\pi(E_0,E,t)dE=\frac{1}{(2\pi)^{\frac{1}{2}}}\frac{H_1(s)}{[\lambda''_1(s)t]^{\frac{1}{2}}}\left(\frac{E_0}{E}\right)^s\frac{dE}{E}\exp\left[\lambda_1(s)t\right],$$
$$t=-\frac{1}{\lambda'_1(s)}\log\left(\frac{E_0}{E}\right), \tag{2.55}$$

$$\gamma(E_0,W,t)dW=\frac{1}{(2\pi)^{\frac{1}{2}}}\frac{1}{\sqrt{s}}\frac{L(s)}{[\lambda''_1(s)t+(1/2s^2)]^{\frac{1}{2}}}\left(\frac{E_0}{W}\right)^s\frac{dW}{W}\exp\left[\lambda_1(s)t\right],$$
$$t=-\frac{1}{\lambda'_1(s)}\left[\log\left(\frac{E_0}{W}\right)-\frac{1}{2s}\right], \tag{2.56}$$

$$\Pi(E_0,E,t)=\frac{1}{(2\pi)^{\frac{1}{2}}}\frac{1}{s}\frac{H_1(s)}{[\lambda''_1(s)t+(1/s^2)]^{\frac{1}{2}}}\left(\frac{E_0}{E}\right)^s\exp\left[\lambda_1(s)t\right],$$
$$t=-\frac{1}{\lambda'_1(s)}\left[\log\left(\frac{E_0}{E}\right)-\frac{1}{s}\right]; \tag{2.57}$$

(b) for a primary photon of energy W_0

$$\pi(W_0,E,t)dE=\frac{1}{(2\pi)^{\frac{1}{2}}}\sqrt{s}\frac{M(s)}{[\lambda''_1(s)t-(1/2s^2)]^{\frac{1}{2}}}\left(\frac{W_0}{E}\right)^s\frac{dE}{E}\exp\left[\lambda_1(s)t\right],$$
$$t=-\frac{1}{\lambda'_1(s)}\left[\log\left(\frac{W_0}{E}\right)+\frac{1}{2s}\right], \tag{2.58}$$

$$\gamma(W_0,W,t)dW=\frac{1}{(2\pi)^{\frac{1}{2}}}\frac{H_2(s)}{[\lambda''_1(s)t]^{\frac{1}{2}}}\left(\frac{W_0}{W}\right)^s\frac{dW}{W}\exp\left[\lambda_1(s)t\right],$$
$$t=-\frac{1}{\lambda'_1(s)}\log\left(\frac{W_0}{W}\right), \tag{2.59}$$

$$\Pi(W_0,E,t)=\frac{1}{(2\pi)^{\frac{1}{2}}}\frac{1}{\sqrt{s}}\frac{M(s)}{[\lambda''_1(s)t+(1/2s^2)]^{\frac{1}{2}}}\left(\frac{W_0}{E}\right)^s\exp\left[\lambda_1(s)t\right],$$
$$t=-\frac{1}{\lambda'_1(s)}\left[\log\left(\frac{W_0}{E}\right)-\frac{1}{2s}\right]. \tag{2.60}$$

* See Iwanenko and Sokolow (I38), Snyder (S38), Schönberg (S40).

TABLE VII.

s	$A(s)$	$B(s)$	$C(s)$	$\lambda_2(s)$	$\lambda_1(s)$	$\lambda'_1(s)$	$\lambda''_1(s)$
0.0	0.0000	1.546	∞	$-\infty$	$+\infty$	$-\infty$	$+\infty$
0.1	0.1520	1.400	12.842	−4.715	+3.789	−25.005	—
0.2	0.2863	1.280	6.123	−3.330	2.270	− 9.488	+75
0.3	0.4067	1.180	3.923	−2.749	1.569	− 5.415	+26
0.4	0.5152	1.095	2.846	−2.415	1.127	− 3.654	12.5
0.5	0.6146	1.022	2.214	−2.201	0.813	− 2.693	7.6
0.6	0.706	0.959	1.802	−2.055	0.576	− 2.093	4.95
0.7	0.791	0.905	1.513	−1.953	0.389	− 1.685	3.50
0.8	0.870	0.855	1.3014	−1.878	0.235	− 1.389	2.55
0.9	0.943	0.812	1.1400	−1.824	0.108	− 1.1660	1.97
1.0	1.0135	0.7733	1.0135	−1.7868	0.000	− 0.9908	1.5634
1.1	1.078	0.7383	0.9112	−1.760	−0.092	− 0.8501	1.275
1.2	1.142	0.7065	0.8276	−1.744	−0.171	− 0.7333	1.060
1.3	1.200	0.6778	0.7580	−1.734	−0.239	− 0.6362	0.893
1.4	1.257	0.6514	0.6988	−1.732	−0.298	− 0.5531	0.764
1.5	1.311	0.6272	0.6484	−1.734	−0.350	− 0.4825	0.655
1.6	1.363	0.6049	0.6047	−1.741	−0.395	− 0.4214	0.565
1.7	1.412	0.5842	0.5666	−1.751	−0.435	− 0.3691	0.487
1.8	1.460	0.5650	0.5329	−1.762	−0.470	− 0.3238	0.423
1.9	1.506	0.5473	0.5032	−1.780	−0.500	− 0.2841	0.370
2.0	1.550	0.5306	0.4767	−1.797	−0.526	− 0.2498	0.320
2.1	1.592	0.5148	0.4528	−1.816	−0.550	− 0.2202	0.277
2.2	1.634	0.5004	0.4313	−1.837	−0.570	− 0.1943	0.241
2.3	1.674	0.4866	0.4117	−1.859	−0.589	− 0.1719	0.210
2.4	1.713	0.4736	0.3940	−1.882	−0.605	− 0.1523	0.182
2.5	1.750	0.4614	0.3776	−1.904	−0.619	− 0.1354	0.159
2.6	1.787	0.4499	0.3627	−1.928	−0.632	− 0.1205	0.138
2.7	1.821	0.4389	0.3489	−1.951	−0.643	− 0.1077	0.120
2.8	1.857	0.4285	0.3362	−1.977	−0.654	− 0.0964	0.107
2.9	1.892	0.4186	0.3243	−2.003	−0.663	− 0.0863	0.093
3.0	1.923	0.4093	0.3134	−2.026	−0.671	− 0.0777	0.080
4.0	2.211	0.3352	0.2347	−2.264	−0.720	− 0.0307	
5.0	2.448	0.2847	0.1882	−2.480	−0.742	− 0.0146	
6.0	2.648	0.2479	0.1574	−2.669	−0.752	− 0.0080	
7.0	2.822	0.2198	0.1354	−2.837	−0.759	− 0.0048	
8.0	2.977	0.1975	0.1189	−2.988	−0.763	− 0.0031	
9.0	3.115	0.1794	0.1060	−3.123	−0.765	− 0.0021	
10.0	3.239	0.1644	0.0957	−3.246	−0.766	− 0.0015	

Equations (2.55) to (2.60) represent the solution of our problem. Numerical calculations are carried out as follows. Given the primary energy E_0 or W_0 and the energy E or W at which we want to determine any of the functions π, γ, or Π, we first calculate the values of t for a set of values of the parameter s, and then enter corresponding values of s and t in the expression for the function required. The quantities $\lambda_1(s)$, $\lambda'_1(s)$, $\lambda''_1(s)$, $H_1(s)$, $H_2(s)$, $L(s)$ and $M(s)$ which appear in Eqs. (2.55) to (2.60) are given as functions of s in Tables VII and VIII.

The expressions for the differential and integral spectra depend only on the thickness t and on the ratio of the initial energy to the energy of the observed shower particles, which result agrees with that predicted in §26. For a given value of this ratio, the intensity of the differential as well as of the integral spectrum increases at first with increasing t, goes through a maximum and then decreases again. The optimum thickness T coincides approximately with the value of t which makes the function $\exp(ys+\lambda_1 t-n\log s)$ a maximum, because the other terms change slowly with t. Thus T is defined by the equation

$$[y+\lambda'_1(s)T-(n/s)](\partial s/\partial t)_{t=T}+\lambda_1(s)=0,$$

which gives, remembering that Eq. (2.54) is satisfied,

$$\lambda_1(s)=0.$$

Hence, according to Table VII,

$$s=1, \qquad T=-(y-n)/\lambda'_1(1)=1.01(y-n). \tag{2.61}$$

The maximum values of the functions π, γ and Π are easily obtained by putting $s=1$, $\lambda_1=0$, $t=T$ in Eqs. (2.55) to (2.60). The maximum values and the optimum thicknesses are explicitly given in Eqs. (2.62) to (2.67):

(a) Incident electron of energy E_0

$$T_\pi(E_0, E)=1.01 \log (E_0/E), \tag{2.62}$$

$$\pi(E_0, E, T_\pi)dE=\frac{0.137}{[\log (E_0/E)]^{\frac{1}{2}}}\frac{E_0}{E^2}dE,$$

$$T_\gamma(E_0, W)=1.01[\log (E_0/W)-\tfrac{1}{2}], \tag{2.63}$$

$$\gamma(E_0, W, T_\gamma)dW=\frac{0.180}{[\log (E_0/W)-0.18]^{\frac{1}{2}}}\frac{E_0}{W^2}dW,$$

$$T_\Pi(E_0, E)=1.01[\log (E_0/E)-1], \tag{2.64}$$

$$\Pi(E_0, E, T_\Pi)=\frac{0.137}{[\log (E_0/E)-0.37)}\frac{E_0}{E}.$$

(b) Incident photon of energy W_0

$$T_\pi(W_0, E)=1.01[\log (W_0/E)+\tfrac{1}{2}], \tag{2.65}$$

$$\pi(W_0, E, T_\pi)dE=\frac{0.137}{[\log (W_0/E)+0.18]^{\frac{1}{2}}}\frac{W_0}{E^2}dE,$$

$$T_\gamma(W_0, W)=1.01 \log (W_0/W), \tag{2.66}$$

$$\gamma(W_0, W, T_\gamma)dW=\frac{0.180}{[\log (W_0/W)]^{\frac{1}{2}}}\frac{W_0}{W^2}dW,$$

$$T_\Pi(W_0, E)=1.01[\log (W_0/E)-\tfrac{1}{2}], \tag{2.67}$$

$$\Pi(W_0, E, T_\Pi)=\frac{0.137}{[\log (W_0/E)-0.183]^{\frac{1}{2}}}\frac{W_0}{E}.$$

Note that the optimum thickness T in either case is about one radiation length greater for π than for Π. This is to be expected, since the maximum for energies larger than E occurs at smaller depths than the maximum for energy E. It is seen also that the maximum number of electrons with energy greater than E is proportional to the ratio between the initial energy and E. The expressions for the optimum thickness T may be compared with the corresponding expressions for the position of the center of gravity $\bar{t}$, given in Eqs. (2.46). It is seen that $\bar{t}$ exceeds T by a quantity independent of energy, which indicates that the increase of number of particles with thickness before the maximum is more rapid than the decrease beyond the maximum.

In the neighborhood of a given energy and of a given thickness the differential energy spectra of electrons and photons are approximately represented by a power law with exponent

TABLE VIII.

s	$H_1(s)$	$H_2(s)$	$L(s)$	$M(s)$
0.0	0.500	0.500	0.469	0.533
0.1	0.537	0.463	0.478	0.521
0.2	0.543	0.457	0.489	0.507
0.3	0.542	0.458	0.498	0.499
0.4	0.536	0.464	0.508	0.489
0.5	0.526	0.474	0.520	0.480
0.6	0.513	0.487	0.531	0.471
0.7	0.496	0.504	0.541	0.463
0.8	0.477	0.523	0.551	0.453
0.9	0.456	0.544	0.560	0.443
1.0	0.4328	0.5672	0.5672	0.4328
1.1	0.408	0.592	0.573	0.422
1.2	0.383	0.617	0.576	0.410
1.3	0.357	0.643	0.578	0.397
1.4	0.331	0.669	0.577	0.384
1.5	0.306	0.694	0.574	0.370
1.6	0.281	0.719	0.568	0.355
1.7	0.257	0.743	0.561	0.340
1.8	0.235	0.765	0.554	0.325
1.9	0.213	0.787	0.542	0.310
2.0	0.194	0.806	0.530	0.295
2.1	0.176	0.824	0.518	0.280
2.2	0.160	0.840	0.505	0.266
2.3	0.145	0.855	0.492	0.252
2.4	0.132	0.868	0.478	0.240
2.5	0.120	0.880	0.465	0.227
2.6	0.109	0.891	0.451	0.215
2.7	0.099	0.901	0.438	0.204
2.8	0.090	0.910	0.425	0.193
2.9	0.082	0.918	0.412	0.183
3.0	0.075	0.925	0.401	0.173
4.0	0.034	0.966	0.304	0.108
5.0	0.018	0.982	0.242	0.073

$-(s+1)$ and the variation of the number of particles with depth follows approximately an exponential law with exponent $\lambda_1 t$. The relation between the "absorption coefficient," $-\lambda_1$, describing the variation with depth, and the exponent $-(s+1)$ describing the energy distribution is the same as in the case of the elementary solutions discussed in §27. In particular, the energy distribution of electrons and photons at the maximum is approximately represented by the *normal spectrum* (2.23), which corresponds to $\lambda_1 = 0$.

The approximations made in calculating the differential and integral spectra limit the validity of the expressions obtained to values of t not smaller than about one radiation length and energies not too close to the initial energy. This is evident from the very fact that for $t=0$ some

TABLE IX. *Integral electron spectrum* $\Pi(E_0, E)$ *for electron initiated showers, calculated by the method of successive collisions under approximation A.* $l=t/\log 2$ *(from Arley, A38).*

l	$\log E_0/E=2$	4	6	8	10
0.2	1.00	1.05	1.08	1.11	1.14
0.4	1.03	1.17	1.30	1.43	1.55
0.6	1.06	1.35	1.64	1.93	2.26
0.8	1.09	1.58	2.09	2.64	3.23
1.0	1.12	1.84	2.65	3.54	4.56
2.0	1.09	3.35	6.78	11.7	18.3
3.0	0.93	4.56	12.64	26.8	50.2
5.0	0.50	5.17	26.21	80.5	206.

TABLE X. *Integral electron spectrum* $\Pi(W_0, E)$ *for photon initiated showers, calculated by the method of successive collisions under approximation A.* $l=t/\log 2$ *(from Arley and Eriksen, A40).*

l	$\log W_0/E=2$	4	6	8	10
0.2	0.200	0.234	0.242	0.246	0.248
0.4	0.350	0.432	0.462	0.482	0.502
0.6	0.484	0.636	0.716	0.782	0.848
0.8	0.598	0.858	1.01	1.16	1.32
1.0	0.694	1.07	1.36	1.65	1.96
2.0	0.926	2.26	4.08	6.44	9.38
3.0	0.940	3.40	8.44	16.6	30.0
5.0	0.494	4.70	19.6	56.4	144.

TABLE XI. *Differential photon spectrum for electron initiated showers, calculated by the method of successive collisions under approximation A.* $l=t/\log 2$*; the figures listed in the table represent* $W\gamma(E_0, W)$ *(from Arley and Eriksen, A40).*

l	$\log E_0/W=2$	4	6	8	10
0.2	0.129	0.130	0.133	0.136	0.138
0.4	0.244	0.251	0.265	0.275	0.293
0.6	0.352	0.389	0.428	0.468	0.504
0.8	0.453	0.543	0.622	0.684	0.795
1.0	0.550	0.724	0.867	1.02	1.22
2.0	0.933	1.91	3.47	5.19	7.41
3.0	1.02	3.16	7.94	14.8	25.1
5.0	0.850	5.89	18.6	50.1	113.

TABLE XII. *Differential photon spectrum for photon initiated showers, calculated by the method of successive collisions under approximation A.* $l=t/\log 2$*; the figures listed in the table represent* $W\gamma(W_0, W)$ *(from Arley and Eriksen, A40).*

l	$\log W_0/W=2$	4	6	8	10
0.2	0.0136	0.016	0.016	0.016	0.016
0.4	0.0483	0.0556	0.0588	0.0596	0.0607
0.6	0.0979	0.120	0.130	0.137	0.140
0.8	0.158	0.193	0.225	0.240	0.262
1.0	0.224	0.299	0.350	0.389	0.450
2.0	0.535	0.923	1.40	2.00	3.06
3.0	0.768	2.09	4.23	6.92	11.6
5.0	0.776	4.32	13.4	31.6	83.1

of the functions reduce to the δ-function, which cannot be represented by an analytical expression.

An application of the formulae developed in the present article is found in the graphs given in Figs. 15 and 16, representing $\Pi(E_0, E, t)$ and $W\gamma(E_0, W, t)$ as functions of t for various values of E_0/E and E_0/W, respectively.

§31. *The Method of Successive Collisions*

Bhabha and Heitler have developed a method of approach to the problem of cosmic-ray showers which differs from the analytical method described in the preceding articles and can be characterized as the method of successive collisions.

The procedure is as follows. Given an electron of energy E_0 incident at $t=0$, one calculates the probability $f_0(E_0, E, t)$ of this electron having an energy larger than E at the depth t. Then one computes the number of photons with energy larger than E emitted by the electron at various points of its path, and the number $f_1(E_0, E, t)$ of electrons produced by these photons, called electrons of the *first generation*, which reach the thickness t with energy larger than E. In a similar way the electrons of the second and succeeding generations are evaluated. The total number of electrons with energy greater than E at the thickness t is finally expressed as the sum of the numbers of electrons of the various generations:

$$\Pi(E_0, E, t)=f_0(E_0, E, t)+f_1(E_0, E, t)+f_2(E_0, E, t)+\cdots. \quad (2.68)$$

The series converges fairly well only when E is not much smaller than E_0 and t not larger than a few radiation lengths. This is, on the other hand, the case in which the expressions derived analytically break down. Thus the two methods are complementary.

In the original calculations of Bhabha and Heitler, as well as in the calculations of Arley, who extended Bhabha and Heitler's results, the following approximations were made. The probability f_0 of an electron having an energy larger than a certain fraction of its original energy after traversing a thickness t was calculated from Eq. (1.35). The numbers of photons and electrons produced in a given thickness were evaluated from the simplified expressions (1.32) and (1.50a) for the probability of radiation processes and the probability of pair production. Some numerical results are given in Tables IX–XII.

B. Shower Theory under Approximation *B*

§32. *The Diffusion Equations*

In the present section we assume that all shower electrons lose by collision an amount ϵ of energy per radiation length. ϵ is regarded as independent of energy, and the production of high speed electrons by collision is neglected. Radiation processes and pair production are described by the asymptotic formulae for complete screening, unless otherwise stated. Compton effect is still disregarded. From the above approximations we may expect accurate results down to energies of the order of the critical energy, at least for substances of low atomic number. For substances of high atomic number, the validity of the theory is limited at small energies by the breakdown of the asymptotic formulae for radiation phenomena and pair production.

The inclusion of the collision loss merely adds a term to Eq. (2.11). Every electron, traversing the thickness dt, loses by collision an amount ϵ of energy. Because of this energy loss, a number $\pi(E+dE)\epsilon dt$ of electrons enters the energy interval (E, dE) from the upper boundary, and a number $\pi(E)\epsilon dt$ of electrons leaves the same energy interval from the lower boundary. Hence the net variation in the number of electrons between E and $E+dE$ caused by the collision loss is

$$[\pi(E+dE)-\pi(E)]\epsilon dt=\epsilon(\partial\pi/\partial E)dEdt$$

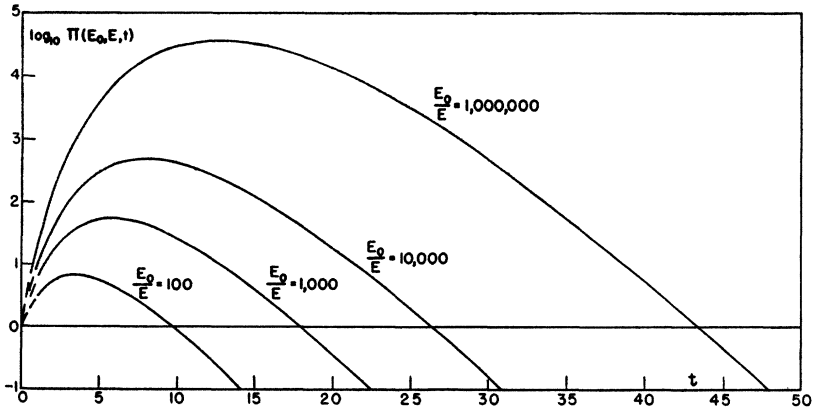

FIG. 15. Integral electron spectra $\Pi(E_0, E, t)$ for electron initiated showers, calculated under approximation A, according to Eq. (2.57).

and Eq. (2.11) becomes

$$\frac{\partial\pi(E,t)}{\partial t}=2\int_0^1\gamma\left(\frac{E}{u},t\right)\psi_0(u)\frac{du}{u}-\int_0^1\left[\pi(E,t)-\frac{1}{1-v}\pi\left(\frac{E}{1-v},t\right)\right]\varphi_0(v)dv+\epsilon\frac{\partial\pi(E,t)}{\partial E}. \tag{2.69}$$

The collision loss does not affect $\partial\gamma/\partial t$. Hence the second diffusion equation (2.12) remains unchanged:

$$\frac{\partial\gamma(W,t)}{\partial t}=\int_0^1\pi\left(\frac{W}{v},t\right)\varphi_0(v)\frac{dv}{v}-\sigma_0\gamma(W,t). \tag{2.12}$$

§33. *Elementary Solutions*

We want first to investigate the elementary solutions of Eqs. (2.69), (2.12); i.e., the solutions of the type (see §27)

$$\pi(E,t)=F_\pi(E)e^{\lambda t},\quad \gamma(W,t)=F_\gamma(W)e^{\lambda t}. \tag{2.70}$$

Upon substitution we obtain

$$\begin{aligned}\lambda F_\pi(E)&=2\int_0^1 F_\gamma(E/u)\psi_0(u)(du/u)-\int_0^1\left[F_\pi(E)-\frac{1}{1-v}F_\pi\left(\frac{E}{1-v}\right)\right]\varphi_0(v)dv+\epsilon\frac{dF_\pi(E)}{dE},\\ (\sigma_0+\lambda)F_\gamma(W)&=\int_0^1 F_\pi(W/v)\varphi_0(v)(dv/v).\end{aligned} \tag{2.71}$$

For energies large compared with ϵ, the collision term can be neglected and Eqs. (2.71) go over into (2.14), which are solved by the power functions (2.15). This suggests the following expressions for $F_\pi(E)$ and $F_\gamma(W)$:

$$\begin{aligned}F_\pi(E)&=aE^{-(s+1)}p(s,E/\epsilon),\\ F_\gamma(W)&=bW^{-(s+1)}g(s,W/\epsilon),\end{aligned} \tag{2.72}$$

where s is a positive number and the functions p and g tend to 1 for energies large compared with ϵ. It follows that λ must satisfy (2.18) and has, for each value of s, two possible values, λ_1 and λ_2, as given by (2.19). The corresponding ratios between the coefficients a and b are given by (2.20). Substituting the expressions (2.72) in the diffusion equations (2.71) and taking (2.20) into account,

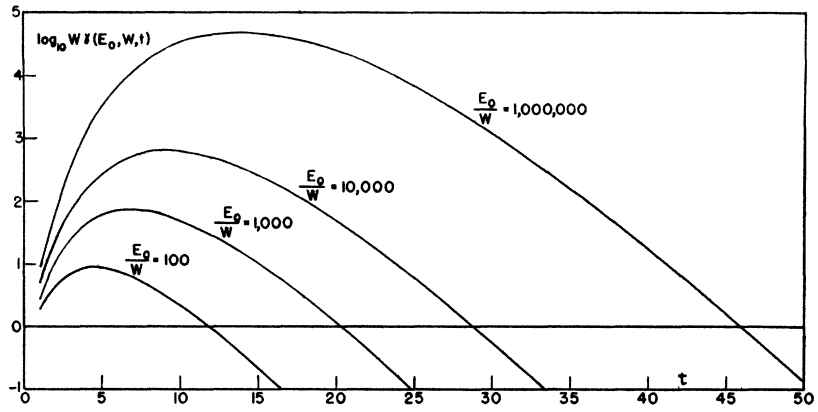

FIG. 16. Differential photon spectra for electron initiated showers, calculated under approximation A, according to Eq. (2.56). The ordinate gives $\log_{10} W\gamma(E_0, W, t)$.

one obtains

$$\lambda p(s, E/\epsilon) = 2\frac{C(s)}{\sigma_0+\lambda(s)}\int_0^1 g\left(s, \frac{E}{\epsilon u}\right)u^s\psi_0(u)du - \int_0^1\left[p\left(s, \frac{E}{\epsilon}\right)\right.$$
$$\left.-(1-v)^s p\left(s, \frac{E}{\epsilon(1-v)}\right)\right]\varphi_0(v)dv - (s+1)p\left(s, \frac{E}{\epsilon}\right)\frac{\epsilon}{E} + \frac{\partial p(s, E/\epsilon)}{\partial(E/\epsilon)}, \quad (2.73)$$

$$C(s)g(s, W/\epsilon) = \int_0^1 v^s p(s, W/\epsilon v)\varphi_0(v)dv.$$

Elimination of g yields

$$\lambda p(s, E/\epsilon) = \frac{2}{\sigma_0+\lambda(s)}\int_0^1 u^s\psi_0(u)du\int_0^1 v^s\varphi_0(v)p(s, E/\epsilon uv)dv - \int_0^1\{p(s, E/\epsilon)$$
$$-(1-v)^s p[s, E/\epsilon(1-v)]\}\varphi_0(v)dv - (s+1)p(s, E/\epsilon)\frac{\epsilon}{E} + \frac{\partial p(s, E/\epsilon)}{\partial(E/\epsilon)} \quad (2.74)$$

or, putting $E/\epsilon = x$,

$$\lambda p(s, x) = \frac{2}{\sigma_0+\lambda(s)}\int_0^1 u^s\psi_0(u)du\int_0^1 v^s\varphi_0(v)p(s, x/uv)dv$$
$$-\int_0^1\{p(s, x) - (1-v)^s p[s, x/(1-v)]\}\varphi_0(v)dv - (s+1)\frac{p(s, x)}{x} + \frac{\partial p(s, x)}{\partial x}. \quad (2.74a)$$

The solution of Eq. (2.74a) can be found by performing a Mellin transformation in x. This procedure leads to an expression of the following form:*

$$p(s, x) = \frac{1}{2\pi i}\int_{-\delta-i\infty}^{-\delta+i\infty}\frac{\Gamma(-r)\Gamma(s+r+1)}{\Gamma(s+1)}K(s, r)x^{-r}dr, \quad (2.75)$$

where $K(s, r)$ is a function of the complex variable r and of the parameter s, and satisfies the recurrence equation

$$\left[\lambda(s) + A(s+r) - \frac{B(s+r)C(s+r)}{\sigma_0+\lambda(s)}\right]K(s, r) = rK(s, r-1) \quad (2.76)$$

* See Snyder (S38); Serber (S38a).

with the boundary condition

$$K(s, 0)=1. \tag{2.77}$$

The integration path in Eq. (2.75) runs parallel to the imaginary axis, to the left of the pole at $r=0$ (where $\Gamma(-r)$ becomes infinite) and to the right of the pole at $r=-(s+1)$ (where $\Gamma(r+s+1)$ becomes infinite). The details of the derivation are given in Appendix II. For each value of s there are, of course, two solutions $p_1(s, x)$ and $p_2(s, x)$ and, therefore, two functions $K_1(s, r)$ and $K_2(s, r)$, corresponding to the two possible choices for λ.

The second equation (2.73) yields

$$g(s, x)=\frac{1}{2\pi i}\int_{-\delta-i\infty}^{-\delta+i\infty}\frac{\Gamma(-r)\Gamma(s+r+1)}{\Gamma(s+1)}\frac{C(s+r)}{C(s)}K(s, r)x^{-r}dr, \tag{2.78}$$

where $x=W/\epsilon$ and the integration path runs to the left of the pole at $r=0$ and to the right of the pole at $r=-s$.

Substituting (2.75) and (2.78) in (2.72) and remembering (2.20), we obtain the following expressions for $F_\pi(E)$ and $F_\gamma(W)$:

$$\begin{aligned}
F_\pi(E)&=a\epsilon^{-(s+1)}\frac{1}{2\pi i}\int_{-\delta-i\infty}^{-\delta+i\infty}\frac{\Gamma(-r)\Gamma(s+r+1)}{\Gamma(s+1)}K(s, r)x^{-(r+s+1)}dr,\\
F_\gamma(W)&=\frac{a}{\sigma_0+\lambda(s)}\epsilon^{-(s+1)}\frac{1}{2\pi i}\int_{-\delta-i\infty}^{-\delta+i\infty}\frac{\Gamma(-r)\Gamma(s+r+1)}{\Gamma(s+1)}C(s+r)K(s,\ r)x^{-(r+s+1)}dr,
\end{aligned} \tag{2.79}$$

where $\lambda(s)$ and $K(s, r)$ are either $\lambda_1(s)$, $K_1(s, r)$ or $\lambda_2(s)$, $K_2(s, r)$.

The expression for the integral electron spectrum $F_{II}(E)$ is obtained by integration of $F_\pi(E)$ with respect to the energy.

$$F_{II}(E)=\int_E^\infty F_\pi(E')dE'=\epsilon\int_x^\infty F_\pi(\epsilon x')dx'=\frac{a}{s}\epsilon^{-s}\frac{1}{2\pi i}\int_{-\delta-i\infty}^{-\delta+i\infty}\frac{\Gamma(-r)\Gamma(s+r)}{\Gamma(s)}K(s, r)x^{-(r+s)}dr, \tag{2.80}$$

where the integration path is to the left of the pole at $r=0$ and to the right of the pole at $r=-s$.

The complex integrals in Eqs. (2.79) and (2.80) are difficult to compute in the general case. However, when $x\ll 1$ the integrands go rapidly to zero with decreasing r, and the integrals can be evaluated by the method of residues, deforming the contour of integration to the left. The behavior in the neighborhood of $x=0$ is determined by the residue at the first pole to the left of the integration path, which is found at $r=-(s+1)$ for F_π and at $r=-s$ for F_γ and F_{II}. Considering only this residue, one finds

$$\begin{aligned}
F_\pi(E)&=\epsilon^{-(s+1)}[q_1(s)+q_2(s)\log(\epsilon/E)],\\
F_\gamma(W)&=\epsilon^{-s}\frac{1}{W}q_3(s),\\
F_{II}(E)&=\epsilon^{-s}q_4(s),
\end{aligned} \tag{2.81}$$

where

$$\begin{aligned}
q_2(s)&=\frac{a}{s}\frac{2.1}{\sigma_0+\lambda(s)}K(s,\ -s),\\
q_3(s)&=\frac{a}{s}\frac{1.36}{\sigma_0+\lambda(s)}K(s,\ -s),\\
q_4(s)&=\frac{a}{s}K(s,\ -s).
\end{aligned} \tag{2.82}$$

The quantity $q_1(s)$ in the equation for F_π can obviously be disregarded in comparison with the logarithmic term for sufficiently small E. The expression for $q_1(s)$ is rather complicated and will not be given explicitly.

The function $K(s, -s)$ can be calculated as explained in Appendix II. The results for $K=K_1$ are represented in Fig. 17.

It is seen that, when the energy tends to zero, the integral spectrum of electrons tends to a finite value, the differential spectrum of electrons diverges as $-\log E$, and the differential spectrum of photons diverges as $1/W$. That the value of $F_\Pi(0)$ is finite is an obvious consequence of the energy loss by collision. It may be recalled that the expression for $F_\Pi(E)$ calculated neglecting the collision loss diverges as $1/E^s$ at $E=0$.

One may question the physical significance of the above results, since the assumptions under which they have been obtained are not valid for energies small compared with ϵ. However, the conclusions regarding the behavior of F_π, F_γ and F_Π at the limit for zero energy are qualitatively correct. Moreover, it will be shown that the integral spectrum of electrons calculated under the present approximation yields an accurate evaluation of the specific ionization (see §38).

An alternative way to solve Eq. (2.74) is suggested by the following considerations (R41). Roughly speaking, the effect of the collision loss is to reduce the energy of each shower electron by a certain amount, proportional to ϵ. Therefore, we may try to find an approximate solution of Eqs. (2.69), (2.12) by substituting $E+m\epsilon$ for E in the solution of Eqs. (2.11), (2.12); i.e., by putting

$$F_\pi(E)=a(E+m\epsilon)^{-(s+1)}=a\left[E\left(1+m\frac{\epsilon}{E}\right)\right]^{-(s+1)}. \tag{2.83}$$

An obvious refinement of Eq. (2.83) is

$$\begin{gathered} F_\pi(E)=a\left[\mu\left(s,\frac{\epsilon}{E}\right)E\right]^{-(s+1)}, \\ \mu\left(s,\frac{\epsilon}{E}\right)=1+m_1(s)\frac{\epsilon}{E}+m_2(s)\left(\frac{\epsilon}{E}\right)^2+\cdots \end{gathered} \tag{2.84}$$

and we are thus led to the following expression for p:

$$p(s, E/\epsilon)=[\mu(s, \epsilon/E)]^{-(s+1)}. \tag{2.85}$$

The coefficients in the expansion for p are computed by making use of Eq. (2.74). The result for m_1 and m_2 is

$$m_1=1/F(s, 1), \quad m_2=m_1^2\frac{s+2}{2}-m_1\frac{s+2}{F(s, 2)}, \tag{2.86}$$

where

$$F(s, n)=\lambda(s)+A(s+n)-\frac{B(s+n)C(s+n)}{\sigma_0+\lambda(s)}.$$

There are, of course, two functions (μ_1 and μ_2) for each value of the exponent s, corresponding to the two values (λ_1 and λ_2) of λ. The two first coefficients $m_1(s)$ and $m_2(s)$ in the expansion for μ_1 are given as functions of s in Fig. 18. Evaluation of the terms of higher order seems to indicate that the series is only partially convergent. However, for energies not smaller than about twice the critical energy, one obtains a fairly good approximation by breaking off the series at the terms of second order in ϵ/E.

Expressions similar to (2.84) can be found for the functions $F_\gamma(W)$ and $F_{II}(E)$. We shall put

$$F_\gamma(W)=a\frac{C(s)}{\sigma_0+\lambda(s)}\frac{1}{W}\left[\nu\left(s,\frac{\epsilon}{W}\right)W\right]^{-s},$$
$$g(s,\,W/\epsilon)=[\nu(s,\,\epsilon/W)]^{-s}, \tag{2.87}$$
$$\nu\left(s,\frac{\epsilon}{W}\right)=1+n_1(s)\frac{\epsilon}{W}+n_2(s)\left(\frac{\epsilon}{W}\right)^2+\cdots,$$

and

$$F_{II}(E)=(a/s)[\rho(s,\,\epsilon/E)E]^{-s},$$
$$\rho\left(s,\frac{\epsilon}{E}\right)=1+r_1(s)\frac{\epsilon}{E}+r_2(s)\left(\frac{\epsilon}{E}\right)^2+\cdots. \tag{2.88}$$

The expression for F_γ is suggested by the behavior of the differential photon spectrum at small energies [see Eqs. (2.81)]. The coefficients in the series for ν and ρ can easily be calculated in terms of the coefficients in the series for μ, by using the second of Eqs. (2.73) and the relation

$$\int_E^\infty[\mu(s,\,\epsilon/E')E']^{-(s+1)}dE'=(1/s)[\rho(s,\,\epsilon/E)E]^{-s},$$

which follows immediately from the definition of ρ. It is found that

$$n_1=\frac{s+1}{s}\frac{C(s+1)}{C(s)}m_1,$$
$$n_2=\frac{s+1}{2}n_1^2+\frac{C(s+2)}{C(s)}\left[\frac{s+1}{s}m_2-\frac{(s+1)(s+2)}{2s}m_1^2\right], \tag{2.89}$$
$$r_1=m_1,$$
$$r_2=\frac{s+1}{s+2}m_2.$$

Again, there are two values of ν (ν_1 and ν_2) and two values of ρ (ρ_1 and ρ_2) for every value of s. The first two coefficients in the expansions for ν_1 and ρ_1 are given in Fig. 18.

§34. *Correction for the Deviation of ψ from ψ_0*

It has been pointed out repeatedly that the formulae for complete screening do not represent accurately the probabilities for pair production and radiation phenomena when the energy is not large compared with $137\mu_e Z^{-\frac{1}{3}}$. Inspection of Figs. 9, 13 and 13a shows that the discrepancy is more pronounced in the case of pair production than in the case of radiation processes. Putting the exact expressions for φ and ψ in the diffusion equations would complicate the mathematical problem considerably. However, in the particular case of the normal spectrum ($\lambda=0$) it is possible to correct, at least approximately, for the deviation of the actual probability for pair production from its asymptotic value. To this purpose, we write the Eqs. (2.71) for the case of the normal spectrum, substituting for σ_0 and $\psi_0(u)$ the more accurate expressions $\sigma(W)$ and $[\sigma(W)/\sigma_0]\psi_0(u)$ (see §19):

$$0=2\int_0^1\frac{\sigma(E/u)}{\sigma_0}F_\gamma\left(\frac{E}{u}\right)\psi_0(u)\frac{du}{u}-\int_0^1\left[F_\pi(E)-\frac{1}{1-v}F_\pi\left(\frac{E}{1-v}\right)\right]\varphi_0(v)dv+\epsilon\frac{dF_\pi(E)}{dE},$$
$$\sigma(W)F_\gamma(W)=\int_0^1 F_\pi(W/v)\varphi_0(v)(dv/v).$$

The above equations are formally identical to (2.71) for $\lambda=0$ if we consider $F_\pi(E)$ and $[\sigma(W)/\sigma_0]F_\gamma(W)$ as the unknown functions instead of $F_\pi(E)$ and $F_\gamma(W)$. Therefore, the solution is obtained from (2.72) by changing $F_\gamma(W)$ into $[\sigma(W)/\sigma_0]F_\gamma(W)$. One sees that the expression for the differential spectrum of electrons remains unchanged, while that for the differential spectrum of photons is multiplied by the ratio between the asymptotic value σ_0 of the absorption coefficient and its value for the energy W under consideration.

§35. *Solution for a Single Incident Electron or Photon*

It has been proved in §30 that, when the energy loss by collision is neglected, the shower produced by a single incident electron of energy E_0 is described by the functions

$$\begin{aligned}\pi(E_0, E, t)&=\frac{1}{2\pi i}\int_{\delta-i\infty}^{\delta+i\infty}ds\left[\frac{\sigma_0+\lambda_1(s)}{\lambda_1(s)-\lambda_2(s)}\exp[\lambda_1(s)t]-\frac{\sigma_0+\lambda_2(s)}{\lambda_1(s)-\lambda_2(s)}\exp[\lambda_2(s)t]\right]\frac{E_0{}^s}{E^{s+1}},\\ \gamma(E_0, W, t)&=\frac{1}{2\pi i}\int_{\delta-i\infty}^{\delta+i\infty}ds\frac{C(s)}{\lambda_1(s)-\lambda_2(s)}\exp[\lambda_1(s)t]-\exp[\lambda_2(s)t]\frac{E_0{}^s}{W^{s+1}},\end{aligned}\tag{2.90}$$

which give the differential spectra of electrons and photons at various depths. The functions π and γ can be considered as linear combinations of elementary solutions of the type (2.21), where, however, the parameter s is now a complex quantity. The coefficients are

$$a_1=\frac{ds}{2\pi i}\frac{\sigma_0+\lambda_1(s)}{\lambda_1(s)-\lambda_2(s)}E_0{}^s \quad\text{and}\quad a_2=\frac{-ds}{2\pi i}\frac{\sigma_0+\lambda_2(s)}{\lambda_1(s)-\lambda_2(s)}E_0{}^s.$$

As first indicated by Snyder (S38), for $E_0\gg\epsilon$ one can obtain an approximate solution of the shower problem including collision loss by substituting in Eqs. (2.90) the elementary solutions of the diffusion equations (2.69), (2.12), in place of the elementary solutions of the diffusion equations (2.11), (2.12). This substitution gives

$$\begin{aligned}\pi(E_0,E,t)=\frac{1}{2\pi i}\int_{\delta-i\infty}^{\delta+i\infty}ds\Bigg\{&\frac{\sigma_0+\lambda_1(s)}{\lambda_1(s)-\lambda_2(s)}p_1\left(s,\frac{E}{\epsilon}\right)\exp[\lambda_1(s)t]\\&-\frac{\sigma_0+\lambda_2(s)}{\lambda_1(s)-\lambda_2(s)}p_2\left(s,\frac{E}{\epsilon}\right)\exp[\lambda_2(s)t]\Bigg\}\frac{E_0{}^s}{E^{s+1}},\end{aligned}\tag{2.91}$$

$$\gamma(E_0,W,t)=\frac{1}{2\pi i}\int_{\delta-i\infty}^{\delta+i\infty}ds\frac{C(s)}{\lambda_1(s)-\lambda_2(s)}\left\{g_1\left(s,\frac{W}{\epsilon}\right)\exp[\lambda_1(s)t]-g_2\left(s,\frac{W}{\epsilon}\right)\exp[\lambda_2(s)t]\right\}\frac{E_0{}^s}{W^{s+1}}.$$

The functions π and γ defined by Eqs. (2.91) are linear combinations of the elementary solutions of the diffusion equations (2.69), (2.12); hence π and γ are themselves solutions of these equations. We want to investigate the behavior of these functions at $t=0$. Putting $t=0$ in Eqs. (2.91), we obtain

$$\begin{aligned}\pi(E_0, E, 0)&=\frac{1}{2\pi i}\int_{\delta-i\infty}^{\delta+i\infty}ds\left[\frac{\sigma_0+\lambda_1(s)}{\lambda_1(s)-\lambda_2(s)}p_1\left(s,\frac{E}{\epsilon}\right)-\frac{\sigma_0+\lambda_2(s)}{\lambda_1(s)-\lambda_2(s)}p_2\left(s,\frac{E}{\epsilon}\right)\right]\frac{E_0{}^s}{E^{s+1}},\\ \gamma(E_0, W, 0)&=\frac{1}{2\pi i}\int_{\delta-i\infty}^{\delta+i\infty}ds\frac{C(s)}{\lambda_1(s)-\lambda_2(s)}\left[g_1\left(s,\frac{W}{\epsilon}\right)-g_2\left(s,\frac{W}{\epsilon}\right)\right]\frac{E_0{}^s}{W^{s+1}}.\end{aligned}\tag{2.92}$$

The quantities p_1, p_2 and g_1, g_2 differ from 1 by terms of the order of ϵ/E and ϵ/W, respectively. If in (2.92) we set $p_1=p_2=1$ and $g_1=g_2=1$, $\pi(E_0, E, 0)$ and $\gamma(E_0, W, 0)$ reduce to $\delta(E-E_0)$ and zero, respectively; i.e., we return to the earlier treatment neglecting collision loss. For energies large compared with ϵ, therefore, $\pi(E_0, E, 0)$ coincides approximately with $\delta(E-E_0)$ and $\gamma(E_0, W, 0)$

is approximately zero. Moreover, when $E>E_0$ and $W>E_0$ the integrals in Eqs. (2.92) may be evaluated by deforming the contour to the right, because the integrands go to zero for $R(s)=+\infty$. Since the integrands have no poles in the positive half plane, $\pi(E_0, E, 0)$ and $\gamma(E_0, W, 0)$ are in any case identically zero for energies larger than E_0. On the other hand, both $\pi(E_0, E, 0)$ and $\gamma(E_0, W, 0)$ may differ considerably from zero for energies of the order of ϵ or smaller. We conclude that, when $E_0 \gg \epsilon$, the expressions for π and γ given by Eqs. (2.91) satisfy *approximately* the boundary conditions describing a single incident electron of energy E_0. More precisely, Eqs. (2.91) represent the shower produced by a primary electron of energy E_0 accompanied by a virtual distribution of electrons and photons, which does not extend beyond E_0 and has appreciable intensity only in the neighborhood of the critical energy, ϵ. Such a shower cannot differ appreciably from the shower produced by a single electron of energy E_0 because the contribution of primaries of small energy to the further development of the shower is negligible.

The expressions for the differential spectra of electrons and photons in the case of a single incident photon can be found by exactly the same procedure.

§36. *Explicit Expressions*

Explicit expressions for the various quantities describing a shower can be derived from the formulae developed in the preceding section.* We indicate shortly the procedure and give the final results.

The Laplace integrals of π and γ are obtained by multiplying the corresponding functions by $e^{-\lambda t}$ and integrating with respect to t from 0 to ∞. Thus, for instance,

$$\mathfrak{L}_\pi(E_0, E, \lambda)=\frac{1}{2\pi i}\int_0^\infty e^{-\lambda t}dt\int_{\delta-i\infty}^{\delta+i\infty}ds\left[\frac{\sigma_0+\lambda_1(s)}{\lambda_1(s)-\lambda_2(s)}p_1\left(s,\frac{E}{\epsilon}\right)e^{\lambda_1(s)t}-\frac{\sigma_0+\lambda_2(s)}{\lambda_1(s)-\lambda_2(s)}p_2\left(s,\frac{E}{\epsilon}\right)e^{\lambda_2(s)t}\right]\frac{E_0{}^s}{E^{s+1}}.$$

The integration path for the integral with respect to s can be chosen to the right of the point s defined by $\lambda_1(s)=\lambda$. Then $R(\lambda_1-\lambda)<0$ and the integration with respect to t can be performed first. This yields

$$\mathfrak{L}_\pi(E_0, E, \lambda)=\frac{1}{2\pi i}\int_{\delta-i\infty}^{\delta+i\infty}ds\left[\frac{\sigma_0+\lambda_1(s)}{\lambda_1(s)-\lambda_2(s)}\frac{p_1(s, E/\epsilon)}{\lambda-\lambda_1(s)}-\frac{\sigma_0+\lambda_2(s)}{\lambda_1(s)-\lambda_2(s)}\frac{p_2(s, E/\epsilon)}{\lambda-\lambda_2(s)}\right]\frac{E_0{}^s}{E^{s+1}}.$$

The complex integral can be evaluated in the same way as that entering in the expression for the Laplace integral calculated neglecting the collision loss. When E is sufficiently small compared with E_0, the integral is practically equal to the residue of the integrand at the point s defined by $\lambda_1(s)=\lambda$ and therefore

$$\mathfrak{L}_\pi(E_0, E, \lambda)=\mathfrak{L}_\pi{}^{(0)}(E_0, E, \lambda)p_1(s, E/\epsilon), \tag{2.93}$$

where $\mathfrak{L}_\pi{}^{(0)}(E_0, E, \lambda)$ is the Laplace integral of $\pi(E_0, E, t)$ calculated with neglect of the collision loss, and is given by (2.40).

For energies larger than about 2ϵ one can use the expression (2.85) for p_1, hence

$$\mathfrak{L}_\pi(E_0, E, \lambda)=-\frac{\sigma_0+\lambda_1(s)}{[\lambda_1(s)-\lambda_2(s)]\lambda'_1(s)}\frac{E_0{}^s}{[\mu_1(s, \epsilon/E)E]^{s+1}} \tag{2.93a}$$

and similarly [see (2.40), (2.42)],

$$\begin{aligned}\mathfrak{L}_\gamma(E_0, W, \lambda)&=-\frac{C(s)}{[\lambda_1(s)-\lambda_2(s)]\lambda'_1(s)}\frac{E_0{}^s}{[\nu_1(s, \epsilon/W)W]^s}\cdot\frac{1}{W},\\ \mathfrak{L}_\Pi(E_0, E, \lambda)&=-\frac{1}{s}\frac{\sigma_0+\lambda_1(s)}{[\lambda_1(s)-\lambda_2(s)]\lambda'_1(s)}\frac{E_0{}^s}{[\rho_1(s, \epsilon/E)E]^s}.\end{aligned} \tag{2.94}$$

* The expression for the integral electron spectrum at $E=0$ has been given by Snyder (S38) and Serber (S38a). The expressions for the other quantities have not previously been published.

Analogous expressions hold for a photon-initiated shower [see (2.41), (2.43)]:

$$\begin{aligned}
\mathfrak{L}_\pi(W_0, E, \lambda) &= -\frac{B(s)}{[\lambda_1(s)-\lambda_2(s)]\lambda'_1(s)}\,\frac{W_0^{\,s}}{[\mu_1(s, \epsilon/E)E]^{s+1}},\\
\mathfrak{L}_\gamma(W_0, W, \lambda) &= -\frac{A(s)+\lambda_1(s)}{[\lambda_1(s)-\lambda_2(s)]\lambda'_1(s)}\,\frac{W_0^{\,s}}{[\nu_1(s, \epsilon/W)W]^{s}}\cdot\frac{1}{W},\\
\mathfrak{L}_\Pi(W_0, E, \lambda) &= -\frac{1}{s}\,\frac{B(s)}{[\lambda_1(s)-\lambda_2(s)]\lambda'_1(s)}\,\frac{W_0^{\,s}}{[\rho_1(s, \epsilon/E)E]^{s}}.
\end{aligned} \tag{2.95}$$

The expressions for the track lengths are given by the Laplace integrals for $\lambda=0$:

(a) *Primary electron of energy* E_0

$$\begin{aligned}
z_\pi(E_0, E) &= 0.437E_0/[\mu_1(1, \epsilon/E)E]^2,\\
z_\gamma(E_0, W) &= 0.572E_0/[\nu_1(1, \epsilon/W)W^2],\\
z_\Pi(E_0, E) &= 0.437E_0/[\rho_1(1, \epsilon/E)E].
\end{aligned} \tag{2.96}$$

(b) *Primary photon of energy* W_0

$$\begin{aligned}
z_\pi(W_0, E) &= 0.437W_0/[\mu_1(1, \epsilon/E)E]^2,\\
z_\gamma(W_0, W) &= 0.572W_0/[\nu_1(1, \epsilon/W)W^2],\\
z_\Pi(W_0, E) &= 0.437W_0/[\rho_1(1, \epsilon/E)E].
\end{aligned} \tag{2.97}$$

The expressions for the position of the center of gravity and for the longitudinal spread can be obtained by the method developed in §29. One readily sees that, if the derivatives of μ, ν and ρ with respect to s are neglected, Eqs. (2.46) and (2.47) are still valid, provided one takes $y=\log(E_0/\mu E)$ [or $y=\log(W_0/\mu E)$] for the case of the differential electron spectrum, $y=\log(E_0/\nu W)$ [or $y=\log(W_0/\nu W)$] for the case of the differential photon spectrum, and $y=\log(E_0/\rho E)$ [or $y=\log(W_0/\rho E)$] for the case of the integral electron spectrum.

The complex integrals entering in the expressions for the differential spectra can be evaluated by the saddle point method. For energies not smaller than the critical energy the functions p and g can be considered as slowly varying functions of s. Using again for p and g the expressions (2.85), (2.87) and following the procedure developed in §30, one obtains

$$\pi(E_0, E, t)dE = \frac{1}{(2\pi)^{\frac{1}{2}}}\,\frac{H_1(s)}{[\lambda''_1(s)t]^{\frac{1}{2}}}\,\frac{E_0^{\,s}}{[\mu_1(s, \epsilon/E)E]^{s+1}}\exp[\lambda_1(s)t]dE, \tag{2.98}$$

$$t = -\frac{1}{\lambda'_1(s)}\log\left(\frac{E_0}{E}\right),$$

$$\gamma(E_0, W, t)dW = \frac{1}{(2\pi)^{\frac{1}{2}}}\,\frac{1}{\sqrt{s}}\,\frac{L(s)}{[\lambda''_1(s)t+(1/2s^2)]^{\frac{1}{2}}}\,\frac{E_0^{\,s}}{[\nu_1(s, \epsilon/W)W]^{s}}\,\frac{1}{W}\exp[\lambda_1(s)t]dW, \tag{2.99}$$

$$t = -\frac{1}{\lambda'_1(s)}\left[\log\left(\frac{E_0}{W}\right)-\frac{1}{2s}\right],$$

$$\Pi(E_0, E, t) = \frac{1}{(2\pi)^{\frac{1}{2}}}\,\frac{1}{s}\,\frac{H_1(s)}{[\lambda''_1(s)t+(1/s^2)]^{\frac{1}{2}}}\,\frac{E_0^{\,s}}{[\rho_1(s, \epsilon/E)E]^{s}}\exp[\lambda_1(s)t], \tag{2.100}$$

$$t = -\frac{1}{\lambda'_1(s)}\left[\log\left(\frac{E_0}{E}\right)-\frac{1}{s}\right].$$

Similarly, the expressions for $\pi(W_0, E, t)$, $\gamma(W_0, W, t)$ and $\Pi(W_0, E, t)$ are obtained by substituting $(\mu_1 E)^{-(s+1)}$ for $E^{-(s+1)}$ in (2.58), $(\nu_1 W)^{-s}W^{-1}$ for $W^{-(s+1)}$ in (2.59) and $(\rho_1 E)^{-s}$ for E^{-s} in (2.60).

It is seen that the number of shower particles is reduced by the collision loss. For instance, the number of electrons with energy above E is found to be equal to the number of electrons with energy above $\rho_1 E$ calculated without considering the collision loss.

For energies small compared with ϵ, the expression (2.75) for p can be used. One obtains for $\pi(E_0, E, t)$

$$\pi(E_0, E, t) = -\frac{1}{4\pi^2}\int_{\delta-i\infty}^{\delta+i\infty} ds \frac{\sigma_0+\lambda_1(s)}{\lambda_1(s)-\lambda_2(s)} E_0^{\,s}\epsilon^{-(s+1)} \exp[\lambda_1(s)t] \int_{-\delta-i\infty}^{-\delta+i\infty} \frac{\Gamma(-r)\Gamma(s+r+1)}{\Gamma(s+1)} K_1(s, r)x^{-(s+r+1)}dr \tag{2.101}$$

$$+\frac{1}{4\pi^2}\int_{\delta-i\infty}^{\delta+i\infty} ds \frac{\sigma_0+\lambda_2(s)}{\lambda_1(s)-\lambda_2(s)} E_0^{\,s}\epsilon^{-(s+1)} \exp[\lambda_2(s)t] \int_{-\delta-i\infty}^{\delta+i\infty} \frac{\Gamma(-r)\Gamma(s+r+1)}{\Gamma(s+1)} K_2(s, r)x^{-(s+r+1)}dr$$

and for the integral electron spectrum

$$\Pi(E_0, E, t) = -\frac{1}{4\pi^2}\int_{\delta-i\infty}^{\delta+i\infty} \frac{ds}{s} \frac{\sigma_0+\lambda_1(s)}{\lambda_1(s)-\lambda_2(s)} \left(\frac{E_0}{\epsilon}\right)^s \exp[\lambda_1(s)t] \int_{-\delta-i\infty}^{-\delta+i\infty} \frac{\Gamma(-r)\Gamma(s+r)}{\Gamma(s)}$$

$$\cdot K_1(s, r)x^{-(s+r)}dr + \frac{1}{4\pi^2}\int_{\delta-i\infty}^{\delta+i\infty} \frac{ds}{s} \frac{\sigma_0+\lambda_2(s)}{\lambda_1(s)-\lambda_2(s)} \left(\frac{E_0}{\epsilon}\right)^s \exp[\lambda_2(s)t]$$

$$\cdot \int_{-\delta-i\infty}^{-\delta+i\infty} \frac{\Gamma(-r)\Gamma(s+r)}{\Gamma(s)} K_2(s, r)x^{-(s+r)}dr. \tag{2.102}$$

The integrals with respect to r in the expression for Π are practically equal to the residues of the integrands at $r=-s$ if $E \ll \epsilon$ (see §33). At the limit for $E=0$, we have accurately

$$\Pi(E_0, 0, t) = \frac{1}{2\pi i}\int_{\delta-i\infty}^{\delta+i\infty} \frac{ds}{s} \frac{\sigma_0+\lambda_1(s)}{\lambda_1(s)-\lambda_2(s)} \left(\frac{E_0}{\epsilon}\right)^s K_1(s, -s) \exp[\lambda_1(s)t]$$

$$-\frac{1}{2\pi i}\int_{\delta-i\infty}^{\delta+i\infty} \frac{ds}{s} \frac{\sigma_0+\lambda_2(s)}{\lambda_1(s)-\lambda_2(s)} \left(\frac{E_0}{\epsilon}\right)^s K_2(s, -s) \exp[\lambda_2(s)t]. \tag{2.103}$$

The second term in (2.103) can be neglected with respect to the first, and this can be evaluated by the saddle point method, in which we consider $K_1(s, -s)$ as a slowly varying function of s. The calculation is exactly similar to that leading to Eq. (2.57), and the result is

$$\Pi(E_0, 0, t) = \frac{1}{(2\pi)^{\frac{1}{2}}} \frac{1}{s} \frac{H_1(s)K_1(s, -s)}{[\lambda''_1(s)t+(1/s^2)]^{\frac{1}{2}}} \left(\frac{E_0}{\epsilon}\right)^s \exp[\lambda_1(s)t],$$

$$t = -\frac{1}{\lambda'_1(s)}\left[\log\left(\frac{E_0}{\epsilon}\right) - \frac{1}{s}\right]. \tag{2.104}$$

Similarly one obtains

$$\Pi(W_0, 0, t) = \frac{1}{(2\pi)^{\frac{1}{2}}} \frac{1}{\sqrt{s}} \frac{M(s)K_1(s, -s)}{[\lambda''_1(s)t+(1/2s^2)]^{\frac{1}{2}}} \left(\frac{W_0}{\epsilon}\right)^s \exp[\lambda_1(s)t],$$

$$t = -\frac{1}{\lambda'_1(s)}\left[\log\left(\frac{W_0}{E}\right) - \frac{1}{2s}\right]. \tag{2.105}$$

It may be noted that the expressions for the integral spectrum at $E=0$ calculated under approximation B closely resemble the expressions for the integral spectrum at $E=\epsilon$ calculated under approximation A. A graphical representation of $\Pi(E_0, 0, t)$ as a function of t for various values of E_0/ϵ is given in Fig. 19.

Multiplication of (2.103) by $e^{-\lambda t}$ and integration from $t=0$ to $t=\infty$ yields an expression for the Laplace integral of $\Pi(E_0, 0, t)$, which is very similar to (2.37) and can be evaluated in the same way. From the expression for the Laplace integral, the expressions for the track length, the position of the center of gravity and the longitudinal spread are readily obtained. We give the results without further discussion.

(a) Primary electron of energy E_0

$$\Psi_\Pi(E_0, 0, \lambda) = -\frac{1}{s}\frac{\sigma_0+\lambda_1(s)}{[\lambda_1(s)-\lambda_2(s)]\lambda'_1(s)}K_1(s, -s)\left(\frac{E_0}{\epsilon}\right)^s,$$

$$z_\Pi(E_0, 0) = 0.437K_1(1, -1)E_0/\epsilon = E_0/\epsilon, \tag{2.106}$$

$$i_\Pi(E_0, 0) = 1.01 \log (E_0/\epsilon) + 0.4,$$

$$\tau_\Pi^2(E_0, 0) = 1.61 \log (E_0/\epsilon) - 0.2.$$

(b) Primary photon of energy W_0

$$\Psi_\Pi(W_0, 0, \lambda) = -\frac{1}{s}\frac{B(s)}{[\lambda_1(s)-\lambda_2(s)]\lambda'_1(s)}K_1(s, -s)\left(\frac{W_0}{\epsilon}\right)^s,$$

$$z_\Pi(W_0, 0) = 0.437K_1(1, -1)W_0/\epsilon = W_0/\epsilon, \tag{2.107}$$

$$i_\Pi(W_0, 0) = 1.01 \log (W_0/\epsilon) + 1.2,$$

$$\tau_\Pi^2(W_0, 0) = 1.61 \log (W_0/\epsilon) + 0.9.$$

The equations for $z_\Pi(E_0, 0)$ and $z_\Pi(W_0, 0)$ express the obvious fact that the track length of all shower electrons equals the initial energy divided by the collision loss per unit length (see W39a, R40). This result is obtained from the present calculations in which terms proportional to negative powers of E_0/ϵ or W_0/ϵ are neglected, and confirms that the method is accurate when E_0 or W_0 is large compared with ϵ (see S38).

C. Total and Specific Ionization. The Low Energy End of the Spectrum

§37. *Total Ionization*

It is a well-known experimental fact that the total number of ion pairs produced when an alpha- or a beta-particle is completely absorbed by a gas is proportional to the energy of the particle, and, for a given energy, is the same for both types of particles. Hence, the total number of ion pairs produced by an alpha- or beta-particle of energy E_0 is given by

$$J = E_0/V_0, \tag{2.108}$$

where V_0 represents the average energy spent per ion pair produced and depends only on the nature of the gas.

The reason for V_0 being independent of the energy and of the nature of the ionizing particle is made clear by the following considerations. When the primary particle is absorbed by the gas, its energy is spent in exciting the atoms and producing secondary rays partly by collision, partly by radiation phenomena. The secondary rays will excite more atoms and produce tertiary electrons and photons, and so on. It is clear that an electron will continue to lose energy by inelastic collision as long as its energy is larger

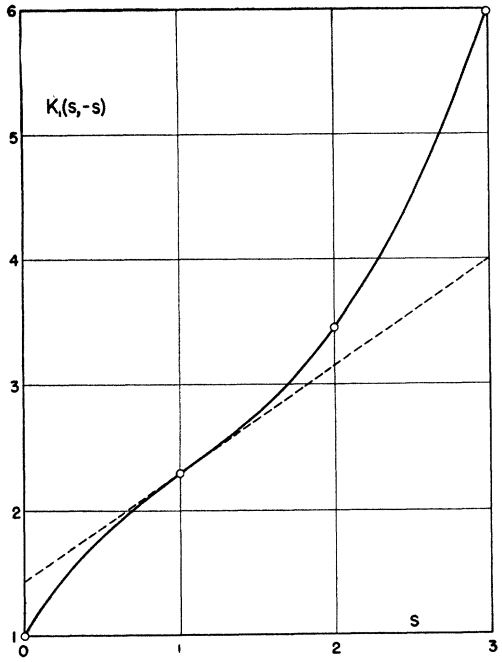

FIG. 17. Function $K_1(s, -s)$ defined by Eqs. (2.76), (2.77). The curve has been drawn using the values of K_1 listed in Table XV (Appendix II). The slope at $s=1$ (as given by Eq. (A. 16)) is indicated by the dashed line.

than the lowest excitation potential of the atoms, and that any photon will readily be absorbed by photoelectric effect as long as its energy is larger than the minimum ionization potential. On the other hand, if an atom is brought to a highly excited state by inelastic collision of an electron or by absorption of a quantum, it promptly loses the excitation energy by emitting a photon or an Auger electron. It is seen that the degradation of the initial energy goes on until nothing else is left but a certain number of atoms in the lowest ionized level, and a certain number of electrons and photons of a few ev energy. The fraction of the initial energy which is used in producing ionization depends essentially on the relative probability for excitation and ionization of the atoms. It is not appreciably affected by the nature of the primary particle nor by its energy because, in any case, most of the ionization and excitation processes are produced by secondary electrons of small energy. Hence the approximate proportionality of the number of ion pairs to the primary energy expressed by Eq. (2.108).

The above considerations make us confident that the proportionality, which is experimentally established for energies up to a few million ev, will still hold for particles of much larger energy, such as those which form the cosmic radiation. Thus we shall use Eq. (2.108) to calculate the total ionization produced either by an electron or by a photon of energy E_0. The values of the constant V_0 for various gases, as deduced from experiments with alpha- and beta-particles, are summarized in Table XIII. It is seen that V_0 is not a smooth function of the atomic number. It would be very difficult to calculate V_0 theoretically, although its general trend can easily be understood. It may be noted that V_0 is particularly small in those gases in which excitation is unlikely compared with ionization. On the contrary, the value of the ionization potential does not influence V_0 very strongly, as might be expected at first.

§38. *Specific Ionization*

Consider the shower produced by a single primary electron or photon of energy E_0 and consider all shower particles with energy greater than η which are present at the thickness $t(\eta < E_0)$. Let $\rho_\eta(t)\delta t$ be the total amount of energy given up by all these particles to secondaries of energy smaller than η in the layer $(t, \delta t)$. The energy of any particle falling below the limit η during the traversal of the layer δt is included in

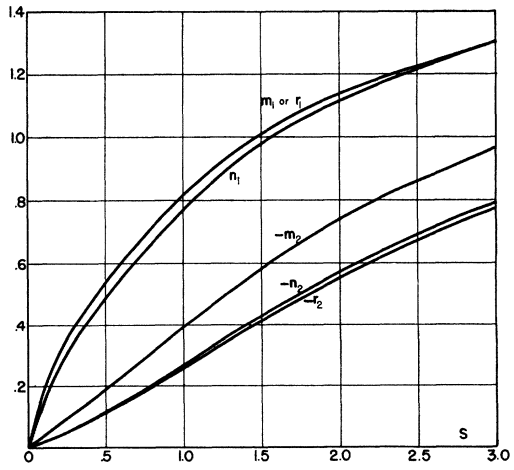

FIG. 18. First and second coefficients in the series for μ_1, ν_1, ρ_1 (Eqs. 2.84, 2.87, 2.88) as functions of s.

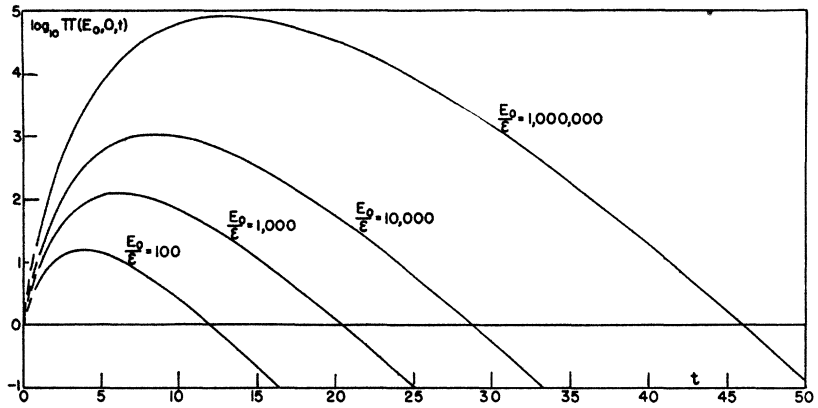

FIG. 19. "Total" number of electrons $\Pi(E_0, 0, t)$ for electron initiated showers, calculated under approximation *B*, according to Eq. (2.104). Multiplication by ϵ/V_0 yields the specific ionization $j(E_0, t)$ [see Eq. (2.110)].

$\rho_\eta(t)\delta t$. If η coincides with the limiting energy η_0, ρ_η becomes what we have agreed to call the *energy dissipation*. The integrated energy loss $\int_0^\infty \rho_\eta(t)dt$ is obviously independent of η and equal to the initial energy. However, the energy loss in a given layer depends on η. If η, for instance, is decreased, a larger number of rays comes into consideration, while the energy given to secondaries of energy smaller than η by each ray is decreased. The two effects work in opposite directions, but do not necessarily compensate each other.

The energy $\rho_\eta(t)\delta t$ lost by the shower particles in a given layer δt manifests itself ultimately in the production of a number $\rho_\eta(t)\delta t/V_0$ of ion pairs. In general, not all of these ions will be produced in the layer δt itself, because part of the energy $\rho_\eta\delta t$ is given to secondary rays which travel some distance before being absorbed. On the other hand, secondary rays generated in preceding layers may penetrate the layer δt and produce some ionization in it. It is clear that the difference between $\rho_\eta(t)\delta t/V_0$ and the actual number of ion pairs formed in δt tends to zero for very small η. More specifically, the difference is negligible when η is so small that the number and energy distribution of shower particles can be considered as constant over a distance equal to the range of particles of energy η. In conclusion, the general expression for the specific ionization is

$$j(t) = \frac{1}{V_0} \lim_{\eta \to 0} \rho_\eta(t). \qquad (2.109)$$

The specific ionization $j(t)$ can be calculated explicitly if we introduce again the simplifying assumptions made in the preceding section B; more specifically, if we describe radiation phenomena and pair production by the asymptotic formulae for complete screening, neglect the Compton effect and consider the energy lost by collision as given up in infinitesimal amounts at the rate of ϵ ev per radiation length.

The quantity $\rho_\eta(t)$ may be expressed by

$$\rho_\eta(t)\delta t = \delta t[\epsilon\Pi(\eta, t) + \epsilon\eta\pi(\eta, t) + (\text{terms depending on radiation and pair production})],$$

where the first term represents the energy loss of particles with energy greater than η, and the second term represents the energy of the particles which fall below the limit η while traversing δt, because of collision losses. As η tends to zero, the first term tends to $\epsilon\Pi(0, t)\delta t$, while the other terms vanish. Hence

$$j(t) = \Pi(0, t)\epsilon/V_0. \qquad (2.110)$$

$\Pi(0, t)$ represents the total number of electrons at t, irrespective of energy, and is given by Eqs. (2.104) and (2.105) for the cases of a primary electron of energy E_0 and of a primary photon of energy W_0, respectively.

The center of gravity $\bar{t}_j$ and the longitudinal spread of the ionization τ_j are given by the same formulae which give the center of gravity and the longitudinal spread of the total number of electrons $\Pi(0, t)$ [see Eqs. (2.106), (2.107)].

Despite the drastic simplifications used, the final expression for the specific ionization is probably accurate. In fact, the behavior of

TABLE XIII. *Average energy spent per ion pair produced in various gases [from Rutherford, Chadwick and Ellis, Radiations from Radioactive Substances (1930), p. 81].*

GAS	Z	V_0 (EV)
Hydrogen	1	33.0
Helium	2	27.8
Nitrogen	7	35.0
Oxygen	8	32.3
Neon	10	27.4
Argon	18	25.4
Krypton	36	22.8
Xenon	54	20.8
Air	—	35.0

shower particles of large energy is correctly accounted for, including the production of low energy electrons and photons by high energy particles. The behavior of electrons and photons of small energy is not correctly described; i.e., some error is made in evaluating the distribution in space of the ionization caused by the absorption of low energy electrons and photons. This error, however, cannot affect the final result seriously because the range of low energy electrons and photons is small compared with the range of the shower itself, hence the exact value of the range is of no great importance, provided the total number of ions formed is correctly computed.

§39. *The Low Energy End of the Spectrum*

The determination of the energy distribution of electrons and photons with energy small compared with the critical energy constitutes a very difficult problem. In the low energy region, as already pointed out, the asymptotic expressions for the probabilities of pair production and radiation cannot be used, and the Compton effect as well as the production of secondary electrons by collision must be taken into consideration. The consequence is that the equations become so involved as to discourage any attempt at analytical solution. Therefore, no general expressions for the low energy end of the spectrum can be given and one must treat every single problem separately, by methods of numerical approximations.

The determination of the track lengths z_π and z_γ is particularly important in connection with the problem of the energy distribution of electrons and photons in equilibrium with a harder radiation (see §40), and also because the functions z_π and z_γ describe the properties of a shower in the neighborhood of the maximum. In the case of showers produced by primary electrons or photons with energy large compared with ϵ, Eqs. (2.96), (2.97) can be used to calculate z_π and z_γ down to an energy η two or three times as large as the critical energy ϵ. For smaller values of the energy, the functions z_π and z_γ can be calculated by the following procedure. One first calculates the number of electrons and photons of energy E' smaller than η created by electrons and photons of energy larger than η. This can be done easily: one multiplies the production probabilities per unit length by the differential track lengths of electrons and photons with energy larger than η, and then integrates over the energy. The asymptotic formulae for the probabilities of pair production and radiation can be used. The number of electrons which drift through the upper boundary η because of the energy dissipation by collision is given by $\epsilon z_\pi(\eta)$. The differential track length of the low energy electrons directly produced by high energy particles (they may be denoted as electrons of the first generation) can then be obtained by multiplying the number of electrons produced in the various energy intervals (E', dE') by the distance they travel while their energy is reduced from $E+dE$ to $E(E<E')$. Similarly, the differential track length of the photons of the first generation is obtained by dividing the number of photons produced in the various energy intervals (W, dW) by the total absorption coefficient of photons of energy W. In a similar way, one can calculate the track lengths of the particles of the second and of the successive generations, where by particles of the nth generation we understand the particles produced by particles of the $(n-1)$th generation. In these calculations, of course, one must use either the correct expressions for the probabilities of the various phenomena involved, or simplified expressions which give a good approximation at low energies.

Calculations along this line have been carried out by Bethe* and confirm the qualitative conclusions reached under approximation B [see

* We are greatly indebted to Professor Bethe for making available to us his results, which are not yet published.

Eq. (2.81)]; i.e., that at the limit for small energies the differential spectrum of electrons diverges logarithmically, while the differential spectrum of photons behaves as $1/W$. Bethe's numerical results on the energy distribution of low energy photons in air are given in Fig. 20.

D. Electrons and Photons in Equilibrium with a Harder Radiation

§40. *General Method*

Let us suppose that electrons or photons are produced in matter by some kind of primary rays, like mesotrons, which are more penetrating than the electrons or photons themselves. The secondary electrons or photons will multiply into showers and an equilibrium condition between the primary rays and the shower particles will be reached as soon as the primary rays have traversed a sufficient thickness.

In order to have the most general case, we shall assume that electrons as well as photons are produced, with arbitrary energy distributions. Let $n_\pi(E, t_0)dEdt_0$ be the average number of electrons with energy (E, dE) generated in the layer (t_0, dt_0) and $n_\gamma(W, t_0)dWdt_0$ the average number of photons with energy (W, dW) generated in the same layer. Let us consider the shower produced by *all* electrons and photons generated in the layer (t_0, dt_0). The differential spectra of electrons and photons, which describe this shower at a distance t' from the layer (t_0, dt_0), are obviously proportional to dt_0 and will be denoted by $dt_0\pi(E, t_0, t')$ and $dt_0\gamma(W, t_0, t')$. These two functions satisfy the diffusion equations of the multiplication theory [Eqs. (2.11), (2.12) if we follow approximation A, or Eqs. (2.69), (2.12) if we follow approximation B] and the boundary conditions

$$\pi(E, t_0, 0) = n_\pi(E, t_0),$$
$$\gamma(W, t_0, 0) = n_\gamma(W, t_0). \tag{2.111}$$

Let $\pi_s(E, t)$ and $\gamma_s(W, t)$ be the differential spectra of electrons and photons observed at the thickness t. These functions are given by

$$\pi_s(E, t) = \int_0^\infty \pi(E, t-t', t')dt',$$
$$\gamma_s(W, t) = \int_0^\infty \gamma(W, t-t', t')dt'. \tag{2.112}$$

The upper limits of integration can be taken as $+\infty$ if one assumes that, at the thickness t, the amount of matter traversed by the primary rays is larger than the maximum range of showers.

In order to proceed any further, one has to make some definite assumption about the variation of the rate of production of electrons and photons with depth. In many cases one can assume that the intensity of the primary radiation, and consequently the rate of production of secondary rays, does not change appreciably over a distance equal to the maximum range of showers. In this case one can put in Eqs. (2.112) $\pi(E, t-t', t') = \pi(E, t, t')$ and $\gamma(W, t-t', t') = \gamma(W, t, t')$ and one obtains

$$\pi_s(E, t) = \int_0^\infty \pi(E, t, t')dt' = z_\pi(E, t),$$
$$\gamma_s(W, t) = \int_0^\infty \gamma(W, t, t')dt' = z_\gamma(W, t), \tag{2.113}$$

where $z_\pi(E, t)dt$ and $z_\gamma(W, t)dt$ represent the differential track lengths of electrons and photons for the shower originated by the incident radiation (2.111) in (t, dt).

When the variation of the primary intensity with depth cannot be neglected, one can often approach the actual conditions closely by assuming that the rate of production of electrons

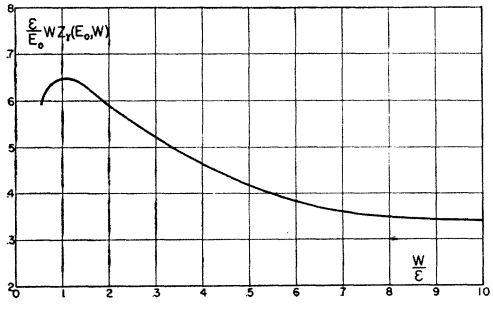

FIG. 20. Low energy end of the photon spectrum, according to Bethe. Multiplying the ordinates by $(E_0/\epsilon) \times (dW/W)$ gives the differential track length $z_\gamma(E_0, W)dW$.

and photons decreases exponentially with depth; i.e.,

$$n_\pi(E, t_0+t') = n_\pi(E, t_0)\exp(-\mu t'),$$
$$n_\gamma(W, t_0+t') = n_\gamma(W, t_0)\exp(-\mu t'). \tag{2.114}$$

It follows that

$$\pi(E, t-t', t') = \pi(E, t, t')\exp(\mu t'),$$
$$\gamma(W, t-t', t') = \gamma(W, t, t')\exp(\mu t'), \tag{2.115}$$

and

$$\pi_s(E, t) = \int_0^\infty e^{\mu t'}\pi(E, t, t')dt' = \mathfrak{L}_\pi(E, -\mu),$$
$$\gamma_s(W, t) = \int_0^\infty e^{\mu t'}\gamma(W, t, t')dt' = \mathfrak{L}_\gamma(W, -\mu), \tag{2.116}$$

where $\mathfrak{L}_\pi(E, -\mu)$ and $\mathfrak{L}_\gamma(W, -\mu)$ are the Laplace integrals for $\lambda = -\mu$ of the functions π and γ at $t_0 = t$. Hence the problem of determining π_s and γ_s is reduced to the problem of determining the Laplace integrals of the functions π and γ which satisfy the diffusion equations of the multiplication theory and the boundary conditions (2.111). For $\lambda = 0$, the Laplace integrals coincide with the track lengths, which give the solution of our problem in the case that the variation of the primary intensity with depth can be neglected. It may be recalled that the Laplace integrals converge only when $\lambda > -\sigma_0$. Physically this means that an equilibrium condition between the primary rays and the secondary shower particles exists only when the "absorption coefficient" μ of the primary rays is smaller than the absorption coefficient of the shower photons.

§41. *Spectra at Large Energies*

If we limit ourselves to shower particles of large energies, approximation A can be used and the Laplace integrals in Eq. (2.116) can be calculated following step by step the procedure developed in §28.

The Mellin integrals with respect to energy of the Laplace integrals with respect to t are defined again by Eqs. (2.30) where, however, $\mathfrak{M}_\pi(s, 0)$ and $\mathfrak{M}_\gamma(s, 0)$ are given by

$$\mathfrak{M}_\pi(s, 0) = \int_0^\infty E^s n_\pi(E, t)dE,$$
$$\mathfrak{M}_\gamma(s, 0) = \int_0^\infty W^s n_\gamma(W, t)dW, \tag{2.117}$$

rather than by (2.31) or (2.32).

The solution of Eqs. (2.30) is

$$\mathfrak{N}_\pi(s, \lambda) = \frac{(\sigma_0+\lambda)\mathfrak{M}_\pi(s, 0)+B(s)\mathfrak{M}_\gamma(s, 0)}{[\lambda-\lambda_1(s)][\lambda-\lambda_2(s)]},$$
$$\mathfrak{N}_\gamma(s, \lambda) = \frac{C(s)\mathfrak{M}_\pi(s, 0)+[A(s)+\lambda]\mathfrak{M}_\gamma(s, 0)}{[\lambda-\lambda_1(s)][\lambda-\lambda_2(s)]}, \tag{2.118}$$

and the inversion formula of the Mellin transformation yields

$$\mathfrak{L}_\pi(E, \lambda) = \frac{1}{2\pi i}\int_{\delta-i\infty}^{\delta+i\infty}\frac{(\sigma_0+\lambda)\mathfrak{M}_\pi(s, 0)+B(s)\mathfrak{M}_\gamma(s, 0)}{[\lambda-\lambda_1(s)][\lambda-\lambda_2(s)]}\frac{1}{E^{s+1}}ds,$$
$$\mathfrak{L}_\gamma(W, \lambda) = \frac{1}{2\pi i}\int_{\delta-i\infty}^{\delta+i\infty}\frac{C(s)\mathfrak{M}_\pi(s, 0)+[A(s)+\lambda]\mathfrak{M}_\gamma(s, 0)}{[\lambda-\lambda_1(s)][\lambda-\lambda_2(s)]}\frac{1}{W^{s+1}}ds. \tag{2.119}$$

In calculating $\mathfrak{L}_\pi(E, \lambda)$ for a certain value E of the energy, we can take n_π and n_γ as equal to zero for energies smaller than E, because shower particles of energy E cannot be produced by primary

electrons or photons of energy smaller than E. Hence $\mathfrak{M}_\pi(s, 0)$ and $\mathfrak{M}_\gamma(s, 0)$ become

$$\mathfrak{M}_\pi(s, 0) = \int_E^\infty E_0{}^s n_\pi(E_0, t) dE_0 = \mathfrak{E}_E{}^{(s)},$$
$$\mathfrak{M}_\gamma(s, 0) = \int_E^\infty W_0{}^s n_\gamma(W_0, t) dW_0 = \mathfrak{W}_E{}^{(s)}, \tag{2.120}$$

where $\mathfrak{E}_E{}^{(s)}$ and $\mathfrak{W}_E{}^{(s)}$ represent, respectively, the sums of the sth powers of the energies given to electrons and photons of energy larger than E by the hard component, in one radiation length. The complex integral in the expression for $\mathfrak{L}_\pi$ can be evaluated by taking the residues at all the poles to the left of the integration path. Each pole contributes a term of the form

$$f_1(s)\frac{\mathfrak{E}_E{}^{(s)}}{E^{s+1}} + f_2(s)\frac{\mathfrak{W}_E{}^{(s)}}{E^{s+1}}.$$

There is only one pole on the positive real axis, at the point defined by $\lambda_1(s) = \lambda$, and the residues at the poles in the negative half plane can be neglected with respect to this one, if the average energy of the secondary electrons and photons with energy larger than E is sufficiently large compared with E. A similar conclusion applies to the expression for $\mathfrak{L}_\gamma$. One then obtains

$$\mathfrak{L}_\pi(E, \lambda) = -\frac{\sigma_0 + \lambda_1(s)}{[\lambda_1(s) - \lambda_2(s)]\lambda'_1(s)} \frac{\mathfrak{E}_E{}^{(s)}}{E^{s+1}} - \frac{B(s)}{[\lambda_1(s) - \lambda_2(s)]\lambda'_1(s)} \frac{\mathfrak{W}_E{}^{(s)}}{E^{s+1}},$$
$$\mathfrak{L}_\gamma(W, \lambda) = -\frac{C(s)}{[\lambda_1(s) - \lambda_2(s)]\lambda'_1(s)} \frac{\mathfrak{E}_W{}^{(s)}}{W^{s+1}} - \frac{A(s) + \lambda_1(s)}{[\lambda_1(s) - \lambda_2(s)]\lambda'_1(s)} \frac{\mathfrak{W}_W{}^{(s)}}{W^{s+1}}, \tag{2.121}$$

where s is defined by $\lambda_1(s) = \lambda$.

The corresponding expression for the Laplace integral of the integral spectrum is

$$\mathfrak{L}_\Pi(E, \lambda) = -\frac{1}{s}\frac{\sigma_0 + \lambda_1(s)}{[\lambda_1(s) - \lambda_2(s)]\lambda'_1(s)} \frac{\mathfrak{E}_E{}^{(s)}}{E^s} - \frac{1}{s}\frac{B(s)}{[\lambda_1(s) - \lambda_2(s)]\lambda'_1(s)} \frac{\mathfrak{W}_E{}^{(s)}}{E^s}. \tag{2.122}$$

For $\lambda = 0$, $s = 1$ and Eqs. (2.121), (2.122) give

$$z_\pi(E) = \frac{0.437}{E^2}(\mathfrak{E}_E{}^{(1)} + \mathfrak{W}_E{}^{(1)}),$$
$$z_\gamma(W) = \frac{0.572}{W^2}(\mathfrak{E}_W{}^{(1)} + \mathfrak{W}_W{}^{(1)}), \tag{2.123}$$
$$z_\Pi(E) = \frac{0.437}{E}(\mathfrak{E}_E{}^{(1)} + \mathfrak{W}_E{}^{(1)}),$$

where $(\mathfrak{E}_E{}^{(1)} + \mathfrak{W}_E{}^{(1)})$ and $(\mathfrak{E}_W{}^{(1)} + \mathfrak{W}_W{}^{(1)})$ are the total energies given to secondaries of energy larger than E and W, respectively, by the hard component, per radiation length. These expressions are very similar to those for the track lengths in showers produced by a single incident electron or photon [see Eqs. (2.44) and (2.45)].

§42. *Specific Ionization. Spectra at Low Energies*

Let us denote by *total specific ionization*, j_T, the total number of ion pairs produced by the primary particle and by all its secondary rays per radiation length. If the variation of the primary intensity over a distance equal to the maximum range of showers can be neglected, j_T is obviously equal to the total energy loss of the primary particle divided by V_0:

$$j_T=\frac{1}{V_0}\left(-\frac{dE}{dt}\right)_{\text{total}} \tag{2.124}$$

The total specific ionization must not be confused with the primary specific ionization j_P defined in §6, which represents the number of ion pairs produced by the primary particle directly.

The problem of calculating the total specific ionization in the case that the variation of the primary intensity with depth cannot be neglected has not yet been solved satisfactorily. The main difficulty arises from the fact that, in general, the average energy of all electrons and photons produced by the primary rays directly is not large compared with the critical energy and, therefore, the procedure developed in §§35, 36, 38 cannot be used. The same difficulty is encountered when one tries to calculate the energy spectra of secondary electrons and photons in the neighborhood of the critical energy ϵ.

An approximate method for determining the track lengths, which can be used also when the initial energy is not large compared with ϵ, has been developed by Tamm and Belenky (T39). The approximation consists in replacing the actual diffusion equations with slightly different equations, which can be more easily handled; a device which had been used previously by Carlson and Oppenheimer (C37). We refer the reader to the original paper for further details.

E. Fluctuations. Lateral Spread of Showers

§43. *Fluctuations**

It has already been mentioned that the problem of determining the probability for a certain deviation from the average behavior of a shower has not yet received a satisfactory solution. We shall, therefore, limit ourselves to a few remarks.

Let $\bar{N}(E_0, t)$ be the total number of particles irrespective of energy which is found, on the average, at the thickness t when a primary particle of energy E_0 is incident at $t=0$. If the shower particles were independent of each other, the probability $P(N)$ of N rather than $\bar{N}$ particles being present at t would be represented by the Poisson law

$$P(N)=e^{\bar{N}}(\bar{N})^N/N! \tag{2.125}$$

which gives for the mean square deviation from the average

$$\langle(N-\bar{N})^2\rangle_{\text{Av}}=\langle N^2\rangle_{\text{Av}}-(\bar{N})^2=\bar{N}. \tag{2.126}$$

The particles of a shower, however, are not independent, since they arise from the same primary ray. Hence the use of Eqs. (2.125) and (2.126) is not justified, although there are some reasons to believe that the Poisson distribution may be roughly valid at large thicknesses (see, for instance, E38).

Furry (F37) made an attempt to determine the function $P(N)$ using a simplified model for the shower phenomenon. In the Furry model the shower particles are considered to be all of the same nature and it is assumed that each particle traversing a thickness dt of matter has a definite probability, proportional to dt, of splitting into two equal particles. The collision loss is neglected. These assumptions lead to a value of $\bar{N}$ which increases exponentially with thickness, and to the following expressions for $P(N)$ and $[\langle(N-\bar{N})^2\rangle_{\text{Av}}]^{\frac{1}{2}}$:

$$P(N)=\frac{1}{\bar{N}}\left[1-\frac{1}{\bar{N}}\right]^{N-1}, \tag{2.127}$$

$$\langle(N-\bar{N})^2\rangle_{\text{Av}}=\bar{N}(\bar{N}-1). \tag{2.128}$$

It is seen that according to the Furry model the fluctuations are much larger than those calculated from the Poisson formula. Indeed, the mean square deviation is approximately proportional to $(\bar{N})^2$ rather than to $\bar{N}$. The most serious source of error in Furry's calculations arises from neglecting the collision loss. The

* See for discussions of this problem Furry (F37), Arley (A38), Euler (E38), Nordsieck, Lamb, and Uhlenbeck (N40).

effect of the collision loss becomes more and more noticeable with increasing thickness, as shown by the fact that the average number of shower particles $\bar{N}$ should pass through a maximum and then decrease again instead of increasing continuously as the Furry model would indicate. Therefore, we may expect Eqs. (2.127) and (2.128) to be approximately valid only for small thicknesses.

Recently Nordsieck, Lamb, and Uhlenbeck (N40) attacked the fluctuation problem from a more general standpoint, using again the Furry model but taking into account, at least roughly, the energy loss by collision. They did not reach any closed expression for $P(N)$ but were able to calculate some values for the mean square deviation. The results are given in Table XIV, where $z=\log(E_0/0.4\epsilon)$ and $\sigma=[\langle N^2\rangle_{Av}-(\bar{N})^2]/\bar{N}$.

The quantity σ is 1 according to Poisson's formula, and $\bar{N}-1$ according to Furry's formula. It is seen that the actual value lies between these two extremes, except for $t=4z$. On the other hand, the authors state that their results for $t=4z$ are the least accurate. Hence the conclusion seems justified that, in general, the fluctuations are intermediate between those predicted by the Poisson and Furry formulae.

TABLE XIV. *Mean square deviation from the average number of particles in a shower (from Nordsieck and others, N40).* $z=\log(E_0/0.4\epsilon)$; $\sigma=[\langle N^2\rangle_{Av}-(\bar{N})^2]/\bar{N}$.

	$z=4.75$			$z=7.0$	
t	$\bar{N}$	σ	t	$\bar{N}$	σ
$2z$	20.4	9.6	$2z$	159	42
$3z$	9.7	6.5	$3z$	60	29
$4z$	2.5	4.4	$4z$	9	16.5

§44. *Lateral Spread of Showers*

We have considered so far only the longitudinal development of showers. The shower particles, however, do not travel exactly in the same direction as the primary particle which has produced the shower, because of the finite angle of emission of electrons and photons and because of the multiple scattering of electrons. Following Eq. (1.53c), and remembering that for high energy electrons energy and momentum are practically identical, we find that the root mean square angle of scattering of electrons of energy E in one radiation length is given by E_s/E, where $E_s=21\times10^6$ ev. This angle may be compared with the average angle of emission, which is, both for electrons and photons, of the order of μ_e/E (see §§10 and 16). It is seen that the average angle of emission is much smaller than the average angle of scattering in one radiation length. Since the average distance between two emission processes is of the order of magnitude of a radiation length, it follows that only scattering needs to be taken into account in calculating the spread of showers.

Let us consider a shower propagating in a homogeneous medium. Since the scattering is inversely proportional to energy, the shower particles of high energy remain concentrated in a narrow cone. The electrons of low energy, which are produced all along the central core, are scattered away from the axis of the shower. The smaller the energy, the larger is the angle of divergence. On the other hand, low energy shower particles are readily absorbed and do not contribute to the further development of the shower. Therefore, the lateral extension of the shower does not continue to increase as the shower develops in matter, but very soon reaches a limit determined by the range of low energy shower particles, and remains thence constant.

Calculations on the spread of showers have been carried out by Euler and Wergeland (E40) and by Bethe (B41). A consistent theory has been developed recently by Nordheim (N41) along the following lines. Consider an electron of energy E_0 incident at $t=0$ and let $\pi(E_0, E, t)dE$ be the average number of shower electrons with energy between E and $E+dE$ at the thickness t. These electrons will travel at various angles with respect to the trajectory of the primary electron and will be found at various distances from the axis of the shower. We want to calculate the mean square angle of deflection $\langle\Theta^2\rangle_{Av}$ and the mean square distance from the axis $\langle X^2\rangle_{Av}$. Let us consider the shower at an intermediate level t' and let $\pi(E_0, E', t')dE'$ be the number of electrons with energy (E', dE') at t'. In traversing the thickness dt', these electrons undergo a certain amount of scattering and the mean square angle of deflection is $(E_s/E')^2dt'$. The same

electrons produce a number

$$\pi(E_0, E', t')dE' \cdot \pi(E', E, t-t')dE$$

of electrons of energy (E, dE) at the thickness t. The angle and the position of all of the descendent electrons is affected by the scattering which the parent electrons have undergone in the layer (t', dt'). Such scattering contributes a term equal to $(E_s/E')^2 dt$ to the mean square angle of deviation of each descendent electron at t, and a term equal to $(E_s/E')^2(t-t')^2 dt'$ to its mean square distance from the core. Since the descendent electrons under consideration are a fraction

$$\frac{\pi(E_0, E', t')dE' \cdot \pi(E', E, t-t')}{\pi(E_0, E, t)}$$

of the total number of electrons of energy (E, dE) at t, it follows that $\langle\Theta^2\rangle_{Av}$ and $\langle X^2\rangle_{Av}$ can be calculated by the following integrals:

$$\begin{aligned}\langle\Theta^2\rangle_{Av} &= \frac{1}{\pi(E_0, E, t)} \int_E^{E_0} dE' \int_0^t dt' \left(\frac{E_s}{E'}\right)^2 \\ &\quad \times \pi(E_0, E', t')\pi(E', E, t-t'), \\ \langle X^2\rangle_{Av} &= \frac{1}{\pi(E_0, E, t)} \int_E^{E_0} dE' \int_0^t dt'(t-t')^2 \\ &\quad \times \left(\frac{E_s}{E'}\right)^2 \pi(E_0, E', t')\pi(E', E, t-t').\end{aligned} \tag{2.129}$$

Thus, the problem of determining $\langle\Theta^2\rangle_{Av}$ and $\langle X^2\rangle_{Av}$ is solved, at least in principle, if the function $\pi(E_0, E, t)$ describing the longitudinal development of the shower is known. The function $\pi(E_0, E, t)$ can be calculated accurately if we limit ourselves to energies sufficiently large to justify the use of approximation A (see Part II, Section A). In this case, evaluation of the integrals (2.129) yields

$$\begin{aligned}\langle\Theta^2\rangle_{Av} &= 0.55(E_s/E)^2 \text{ (radians)}^2, \\ \langle X^2\rangle_{Av} &= 0.60(E_s/E)^2 \text{ (radiation lengths)}^2.\end{aligned} \tag{2.130}$$

The problem of determining $\langle\Theta^2\rangle_{Av}$ and $\langle X^2\rangle_{Av}$ for low energy particles is more difficult because the function $\pi(E_0, E, t)$ cannot be given in a closed form. Numerical calculations have been carried out by Nordheim for the case of air, which is particularly important because of the so-called Auger showers. The root mean square distance from the axis of the shower, for all electrons irrespective of energy, turns out to be of the order of 100 meters in air at N. T. P. This figure is in agreement with the results obtained by Bethe (B41), while the calculations of Euler and Wergeland (E40) had given a much lower value for the spread.

As for the distribution functions, it is assumed that both the angular distribution and the space distribution of shower particles of a given energy can roughly be represented by a Gaussian law. The distribution, of course, is not even approximately Gaussian when particles of all energies are taken into consideration.

Appendix I

(a) *The Laplace Integral*

Given a function $f(x)$ of a real variable x, the *Laplace integral* or *Laplace transform* of $f(x)$ is defined by the equation

$$\mathfrak{L}_f(\lambda) = \int_0^\infty e^{-\lambda x} f(x)dx \tag{A. 1}$$

where λ is a *complex* parameter. We assume the integral to be convergent at the lower limit. If the integral converges at the upper limit for a certain value λ_a of λ, it also converges for all values of λ for which $R(\lambda) > R(\lambda_a)$, where R indicates the real part. On the other hand, if the integral diverges for a certain value λ_b of λ, it also diverges for all values of λ for which $R(\lambda) < R(\lambda_b)$. Hence, in general, the function $\mathfrak{L}_f(\lambda)$ is defined in the half plane to the right of a straight line parallel to the imaginary axis.

The following formulae can easily be proved:

$$\mathfrak{L}_{f'}(\lambda) = \lambda\mathfrak{L}_f(\lambda) - f(0), \tag{A. 2}$$

where $f' = df/d\lambda$;

$$\mathfrak{L}_F(\lambda) = -\frac{1}{\lambda}[\mathfrak{L}_f(\lambda) - F(0)], \tag{A. 3}$$

where $F(x) = \int_x^\infty f(x')dx'$;

$$\left[\frac{d^n}{d\lambda^n}\mathfrak{L}_f(\lambda)\right]_{\lambda=0} = (-1)^n \int_0^\infty x^n f(x)dx. \tag{A. 4}$$

An important property of the Laplace transformation is that, under not very restrictive conditions, the correspondence between $f(x)$ and $\mathfrak{L}_f(\lambda)$ established by Eq. (A. 1) is unique; i.e., there is only one function $f(x)$ which has $\mathfrak{L}_f(\lambda)$ as its Laplace integral. If $\mathfrak{L}_f(\lambda)$ is known, $f(x)$ can be determined by the following inversion formula:

$$f(x) = \frac{1}{2\pi i} \int_c e^{\lambda x} \mathfrak{L}_f(\lambda)d\lambda, \tag{A. 5}$$

where the integration path c is a straight line running parallel to the imaginary axis, in the half plane of convergence of $\mathfrak{L}_f(\lambda)$.

(b) The Mellin Integral

Given a function $f(y)$ of a real variable y, the *Mellin integral*, or *Mellin transform* of $f(y)$ is defined by the equation

$$\mathfrak{M}_f(s)=\int_0^\infty y^s f(y)dy, \tag{A. 6}$$

where s is a *complex* parameter. If the integral diverges at the lower limit for a certain value s_a of s, it also diverges for all values of s for which $R(s)<R(s_a)$. If the integral diverges at the upper limit for a certain value s_b of s, it also diverges for all values of s for which $R(s)>R(s_b)$. Hence, if the Mellin integral converges anywhere, its field of convergence is a strip bounded by two straight lines parallel to the imaginary axis.

The following formulae can easily be proved:

$$\mathfrak{M}_{f'}(s)=-s\mathfrak{M}_f(s-1), \tag{A. 7}$$

where $f'=df/dx$;

$$\mathfrak{M}_F(s)=\frac{1}{s+1}\mathfrak{M}_f(s+1), \tag{A. 8}$$

where $F(y)=\int_y^\infty f(y')dy'$.

The Mellin transformation, like the Laplace transformation, is in general unique and can be inverted by the following formula:

$$f(y)=\frac{1}{2\pi i}\int_C y^{-(s+1)}\mathfrak{M}_f(s)ds, \tag{A. 9}$$

where the integration path C runs parallel to the imaginary axis within the strip of convergence.

Appendix II

Let

$$\mathfrak{M}_p(s, r)=\int_0^\infty x^r p(s, x)dx$$

be the Mellin integral of the function p defined by Eq. (2.74a). Since $\lim_{x\to\infty} p=1$, the integral is convergent at the upper limit only for $R(r)<-1$. Hence, the field of convergence of $\mathfrak{M}_p(s, r)$ will be a strip extending from $r=-1$ to the first singularity to the left of this point.

By multiplying both sides of Eq. (2.74a) by x^r and integrating with respect to x from 0 to ∞ one obtains for $\mathfrak{M}_p(s, r)$ the following recurrence equation:

$$\left[\lambda(s)+A(s+r+1)-\frac{B(s+r+1)C(s+r+1)}{\sigma_0+\lambda(s)}\right]\times\mathfrak{M}_p(s, r)=-(s+r+1)\mathfrak{M}_p(s, r-1). \tag{A. 10}$$

The function $\mathfrak{M}_p(s, r)$ has a singularity at $r=-1$, as already pointed out and as indicated by the fact that the coefficient of $\mathfrak{M}_p(s, r)$ in (A. 10) vanishes for $r=-1$. It follows immediately from Eq. (A. 10) that other singularities occur at $r=0$ and at $r=n$ where n is any positive integer. Again, at $r=-(s+1)$ the coefficient of $\mathfrak{M}_p(s, r)$ is infinite and the coefficient of $\mathfrak{M}_p(s, r-1)$ is zero. This suggests that $\mathfrak{M}_p(s, r)$ has singularities at $r=-(s+2)$ and at $r=-(s+n)$ where n is any positive integer larger than 1. It is convenient to eliminate part of the singularities by writing $\mathfrak{M}_p(s, r)$ as the product of a function having simple poles at $r=-1, 0, 1, \cdots, r=-(s+2), r=-(s+3), \cdots$, and a new unknown function. We set therefore

$$\mathfrak{M}_p(s, r-1)=\frac{\Gamma(-r)\Gamma(s+r+1)}{\Gamma(s+1)}K(s, r), \tag{A. 11}$$

where $\Gamma(y)$ is the gamma-function and has simple poles at $y=0, -1, -2, \cdots$.

Substituting (A. 11) into (A. 10) and remembering that $y\Gamma(y)=\Gamma(y+1)$, we obtain

$$\left[\lambda(s)+A(s+r+1)-\frac{B(s+r+1)C(s+r+1)}{\sigma_0+\lambda(s)}\right]\times K(s, r+1)=(r+1)K(s, r)$$

or

$$\left[\lambda(s)+A(s+r)-\frac{B(s+r)C(s+r)}{\sigma_0+\lambda(s)}\right]\times K(s, r)=rK(s, r-1). \tag{A. 12}$$*

Suppose that Eq. (A. 12) has been solved. Introducing the solution into (A. 11) and applying the inverse Mellin transformation, one gets

$$p(s, x)=\frac{1}{2\pi i}\int_{-\delta-i\infty}^{-\delta+i\infty}x^{-r}\mathfrak{M}_p(s, r-1)dr=\frac{1}{2\pi i}\int_{-\delta-i\infty}^{-\delta+i\infty}x^{-r}\frac{\Gamma(-r)\Gamma(s+r+1)}{\Gamma(s+1)}K(s, r)dr, \tag{A. 13}$$

where the integration path runs parallel to the imaginary axis, to the left of the pole at $r=0$ and to the right of the pole at $r=-(s+1)$. At the limit for $x=\infty$, the integrand in Eq. (A. 13) vanishes for all values of r with a positive real part. Hence, the integral is equal to the residue at $r=0$, which is given by $2\pi iK(s, 0)$. Since

$$\lim_{x\to\infty} p(s, x)=1,$$

it follows that

$$K(s, 0)=1.$$

For $r=0$ the coefficients of both $K(s, r)$ and $K(s, r-1)$ in (A. 12) vanish and the ratio between $K(s, -1)$ and $K(s, 0)$ is given by

$$\frac{K(s, -1)}{K(s, 0)}=\lim_{r\to 0}\left\{\frac{1}{r}\left[\lambda(s)+A(s+r)-\frac{B(s+r)C(s+r)}{\sigma_0+\lambda(s)}\right]\right\}=\frac{dA(s)}{ds}-\frac{1}{\sigma_0+\lambda(s)}\frac{d}{ds}[B(s)C(s)],$$

or, since $K(s, 0)=1$,

$$K(s, -1)=\frac{dA(s)}{ds}-\frac{1}{\sigma_0+\lambda(s)}\frac{d}{ds}[B(s)C(s)]. \tag{A. 14}$$

By repeated application of Eq. (A. 12) one can easily obtain the values of $K(s, r)$ for any real integral value of r. Thus, for instance,

$$K(s, -2)=-K(s, -1)\left[\lambda(s)+A(s-1)-\frac{B(s-1)C(s-1)}{\sigma_0+\lambda(s)}\right],$$

$$K(s, -3)=-\tfrac{1}{2}K(s, -2)\left[\lambda(s)+A(s-2)-\frac{B(s-2)C(s-2)}{\sigma_0+\lambda(s)}\right],$$

$$\frac{1}{K(s, 1)}=\lambda(s)+A(s+1)-\frac{B(s+1)C(s+1)}{\sigma_0+\lambda(s)}$$

$$\frac{1}{K(s, 2)}=\frac{1}{2K(s, 1)}\left[\lambda(s)+A(s+2)-\frac{B(s+2)C(s+2)}{\sigma_0+\lambda(s)}\right].$$

Recurrence equations can also be found for the derivatives $\partial K(s, r)/\partial s$ and $\partial K(s, r)/\partial r$, by differentiating (A. 12) with respect to s and with respect to r, respectively. Since $K(s, 0)=1$, we have the condition $\partial K(s, 0)/\partial s=0$. Starting

* See Snyder (S38); Serber (S38a).

from this value and using the recurrence equation, one can determine $\partial K(s, r)/\partial s$ for an arbitrary value of s and any integer value of r. On the other hand, Eq. (A. 12) gives directly the logarithmic derivative of $K(s, r)$ with respect to r for a sufficiently large value of r. Indeed, for r_1 sufficiently large,

$$\frac{1}{K(s, r_1)}\left(\frac{\partial K(s, r)}{\partial r}\right)_{r=r_1} \approx -\frac{1}{2}\left[\log \frac{F(s, r_1)}{r_1}+\log \frac{F(s, r_1+1)}{r_1+1}\right], \quad \text{(A. 15)}$$

where

$$F(s, r)=\lambda(s)+A(s+r)-\frac{B(s+r)C(s+r)}{\sigma_0+\lambda(s)}.$$

Once $\partial K(s, r)/\partial r$ is known for a particular integer value of r, it can be calculated for any other integer value using the recurrence equation.

In the practical applications, one is mainly interested in the values of the function K_1 for $r=-s$; i.e., in the function of one variable $K_1(s, -s)$. The derivative of this function with respect to s is given by

$$\frac{dK(s, -s)}{ds}=\left(\frac{\partial K(s, r)}{\partial s}\right)_{r=-s}-\left(\frac{\partial K(s, r)}{\partial r}\right)_{r=-s}.$$

TABLE XV.

s	$K_1(s, -s)$
0	1.00
1	2.29
2	3.45
3	5.98
4	10.61
5	18.15
6	29.19

The values of $K_1(s, -s)$ for $s=0$, 1, 2, 3, 4, 5, and 6 are given in Table XV. The derivative of $K_1(s, -s)$ has been calculated explicitly for $s=1$ and is given by

$$\left(\frac{dK_1(s, -s)}{ds}\right)_{s=1}=0.85. \quad \text{(A. 16)}$$

Using the values of $K_1(s, -s)$ listed in Table XV and the value of the derivative at $s=1$ given by (A. 16), it is possible to draw a fairly accurate interpolation curve, on which the values of $K_1(s, -s)$ for non-integer values of s can be read. Such a curve is represented in Fig. 17.

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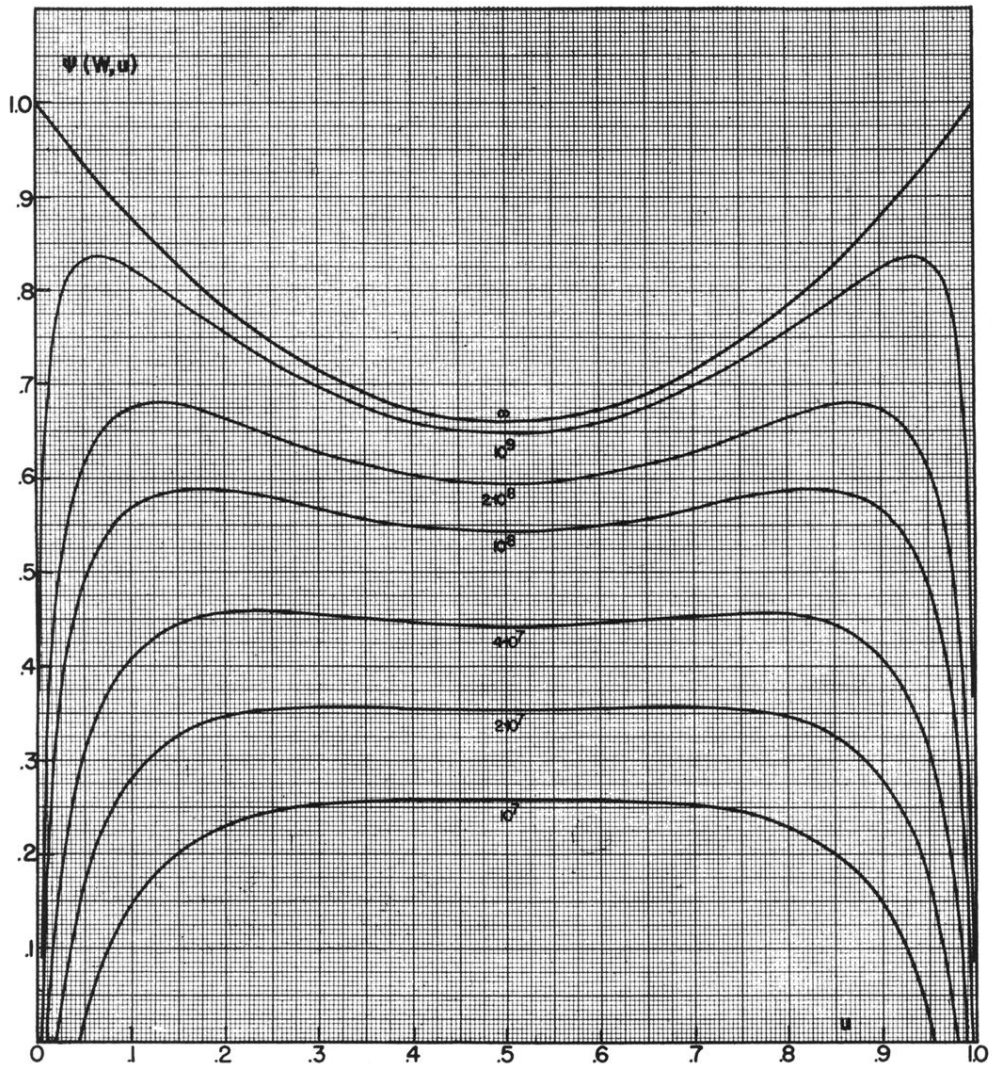

FIG. 10. Differential probability of pair production per radiation length of air, for photons of various energies. Abscissa, $u = U/W$; ordinate, $\psi(W, u)$. The numbers attached to the curves indicate the energy W of the primary photon.

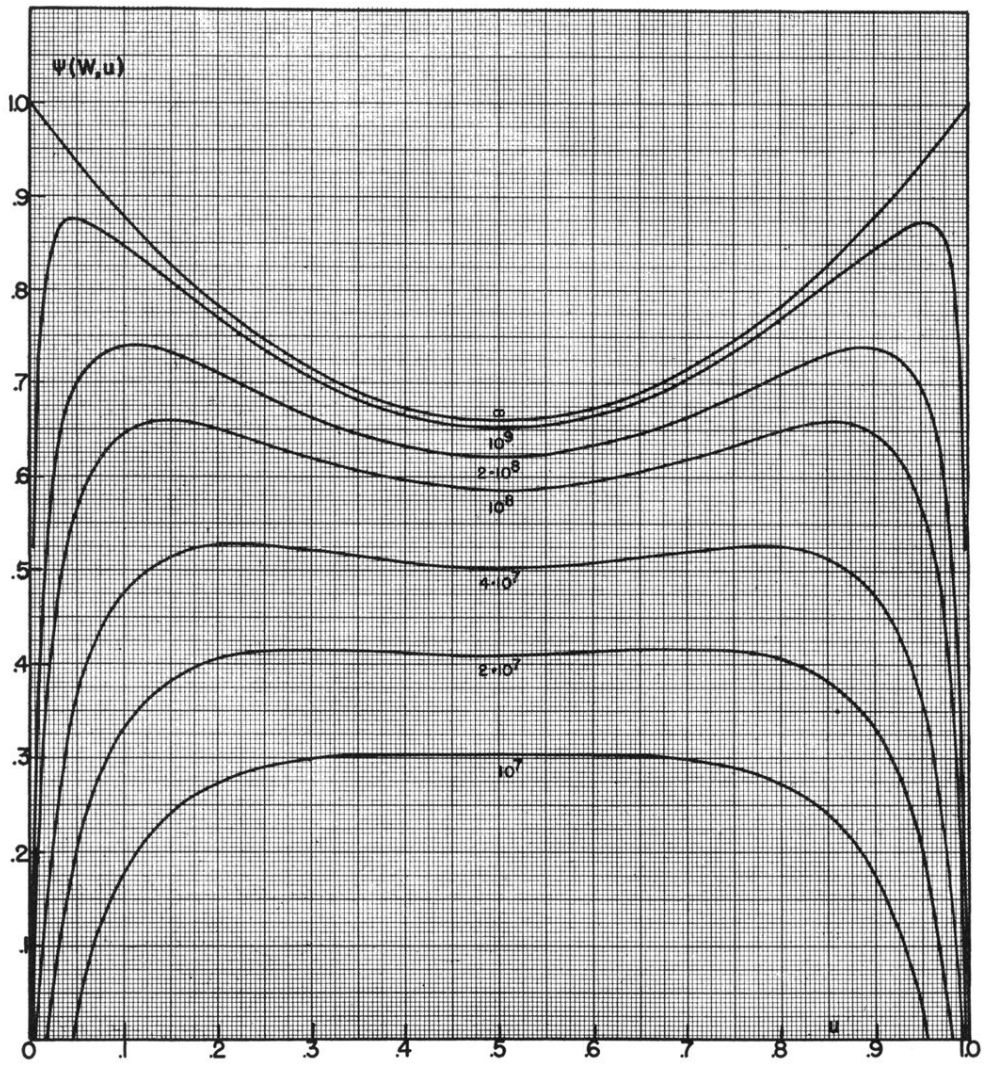

FIG. 11. Differential probability of pair production per radiation length of lead, for photons of various energies. Abscissa, $u = U/W$; ordinate, $\psi(W, u)$. The numbers attached to the curves indicate the energy W of the primary photon.

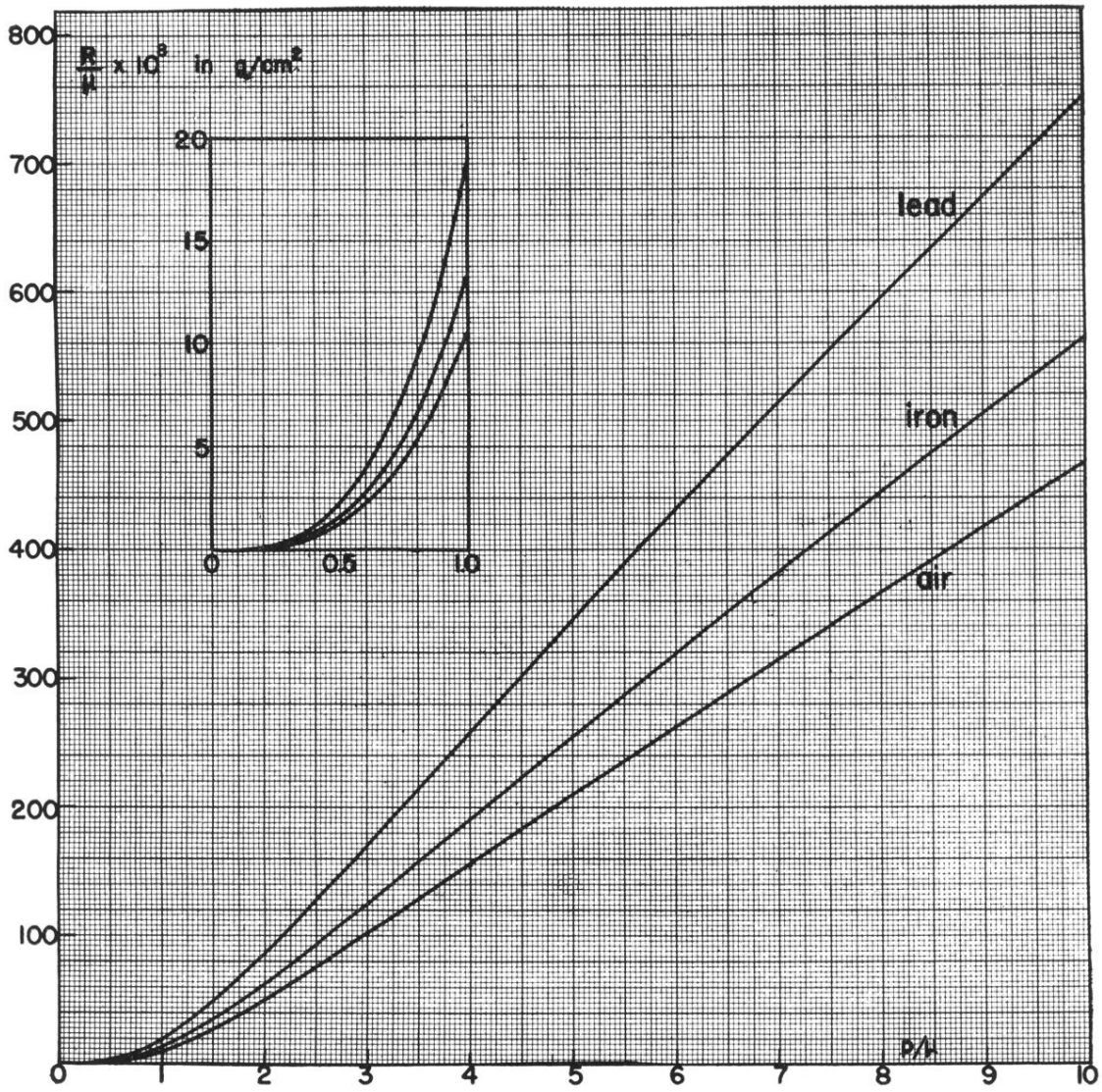

FIG. 3. Range of mesotrons or protons in air, iron and lead [from Eq. (1.16)]. Abscissa p/μ, ordinate $R\times 10^8/\mu$ in g/cm². For mesotrons with $\mu = 10^8$ ev/c^2, therefore, the ordinate represents the range in g/cm². The graphs are valid for particles with unit charge and arbitrary mass, provided other losses are negligible compared with the collision loss.

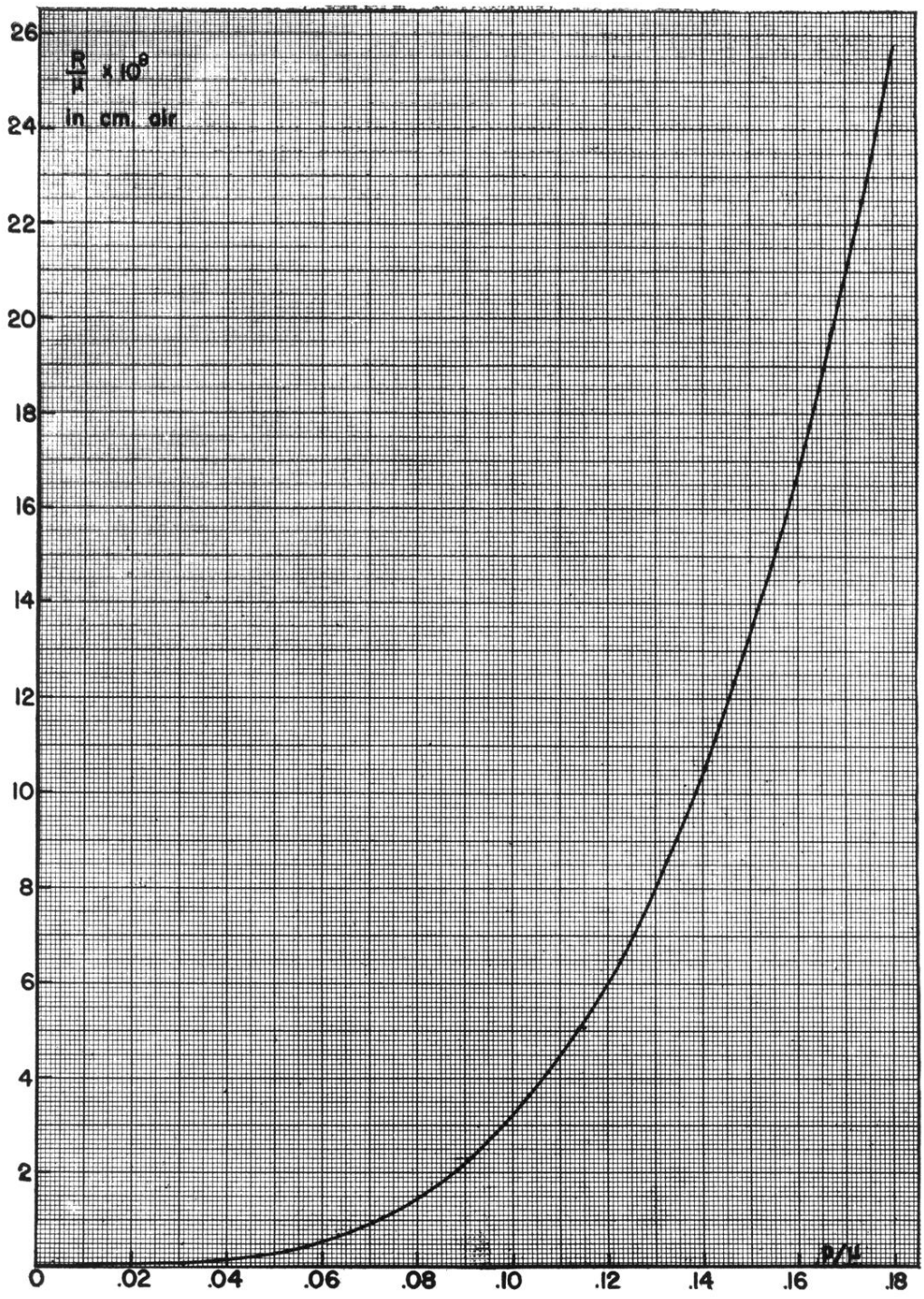

FIG. 4. Range of low energy mesotrons or protons in air (from the calculations of Livingston and Bethe, L37). Abscissa p/μ, ordinate $R\times 10^8/\mu$ in cm of air at N. T. P. For mesotrons with $\mu = 10^8$ ev/c^2, therefore, the ordinate gives the range directly in cm of air. The graph is valid for particles with unit charge and arbitrary mass, provided other losses are negligible compared with the collision loss.

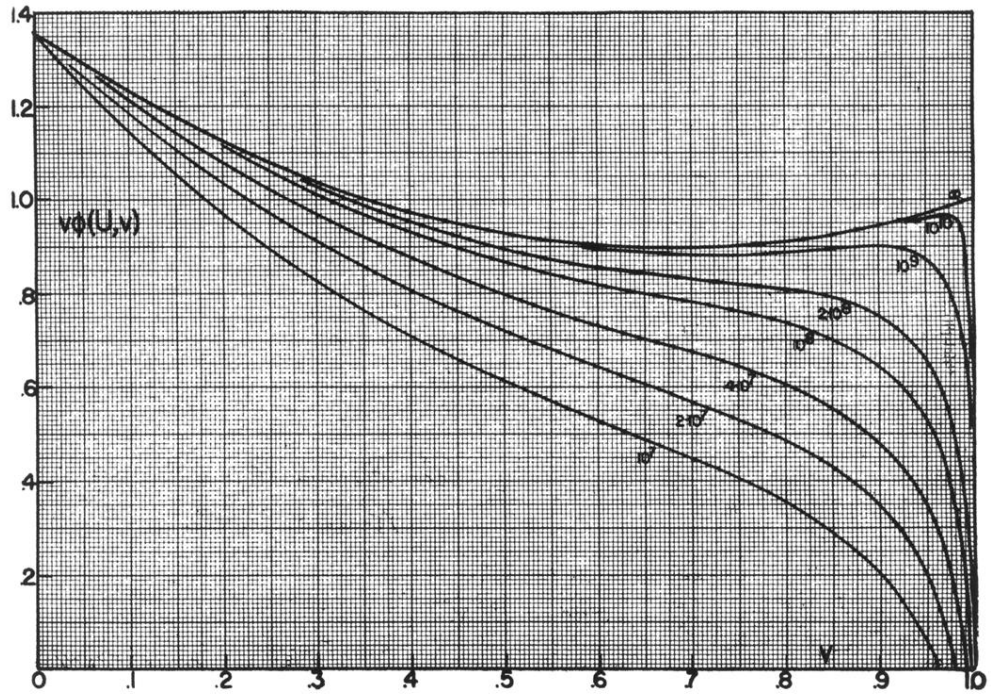

FIG. 6. Differential radiation probability per radiation length of air for electrons of various energies. Abscissa, $v = W/U$ ordinate, $v\varphi(U, v)$. The numbers attached to the curves indicate the energy U of the primary electron.

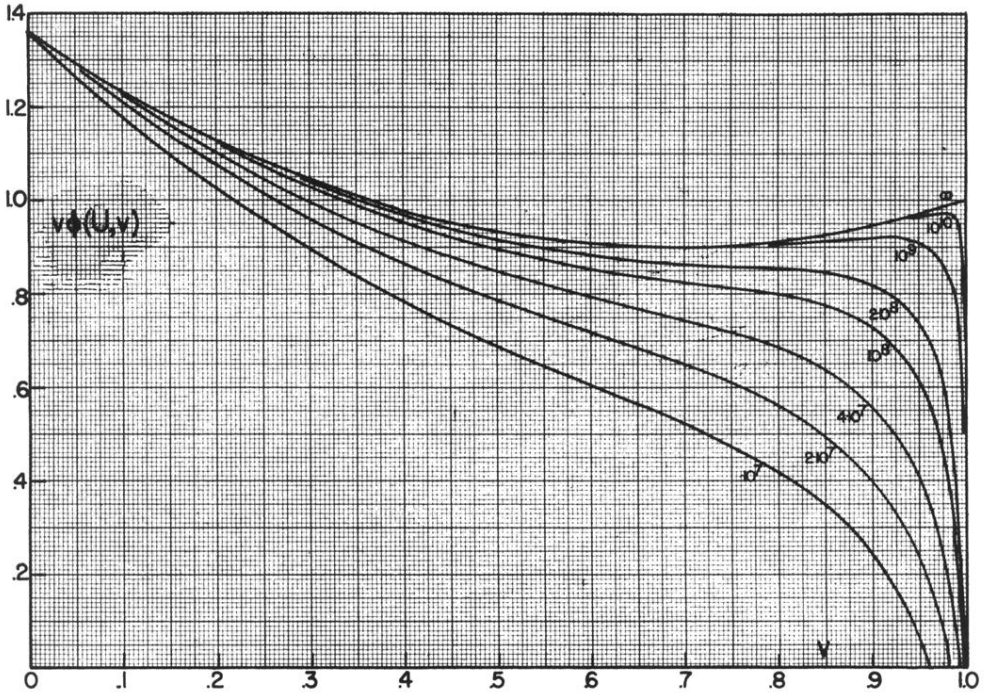

FIG. 7. Differential radiation probability per radiation length of lead for electrons of various energies. Abscissa, $v = W/U$; ordinate, $v\varphi(U, v)$. The numbers attached to the curves indicate the energy U of the primary electron.