Relativistic Field Theories of Elementary Particles*

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CONTENTS

PART I. TRANSFORMATION PROPERTIES OF THE FIELD EQUATIONS AND CONSERVATION LAWS

1. Units and notation

SINCE the requirements of relativity theory and quantum theory constitute the hypotheses of every theory it is natural to use $\hbar$, the Planck constant, divided by 2π, and c, the velocity of light in vacuum, as units. This means that all quantities are to be reduced to the dimensions of a power of *length* by multiplication with the requisite powers of $\hbar$ and c. For example, the quantity E in the following will mean the energy divided by $\hbar c$ a magnitude with a dimension cm^{-1}; $\mathbf{g}$ will mean the momentum in terms of the unit $\hbar$ and will likewise be of the dimension cm^{-1}; P, the angular momentum in the unit $\hbar$, is dimensionless; ϵ, the electric charge in the unit $(\hbar c)^{\frac{1}{2}}$ is dimensionless; the electromagnetic potential φ_i and the electromagnetic field strengths $f_{ik}=-f_{ki}$ have, in the unit $(\hbar c)^{\frac{1}{2}}$ the dimensions cm^{-1} and cm^{-2}, respectively. The reciprocal length associated with the rest mass m of a particle will be designated in the following by $\kappa=mc/\hbar$.

In accord with the above conventions we shall use the length $x_0=ct$ of the light path as the time coordinate; however, the imaginary time coordinate $x_4=ix_0=ict$ will also be employed. Thus tensor indices designated by small italic letters i, k, $\cdots$ which run from 1 to 4 involve the imaginary time coordinate. It is expedient to use in this connection a special rule for the transformation to complex conjugate quantities. For quantities with an index zero a star shall mean the conjugate complex in the ordinary sense (e.g., $s_0{}^*$ is the conjugate complex of the charge density s_0; s_i is the current vector). In general we shall mean by $U_{ik}{}^*\cdots$ the conjugate complex of $U_{ik}\cdots$ multiplied by $(-1)^n$ where n is the number of 4's among the $i, k\cdots$ (e.g., $s_4=is_0$, $s_4{}^*=is_0{}^*$).

Dirac's spinors u_ρ where $\rho=1, \cdots, 4$, are always given a Greek index which runs from 1 to 4; $u_\rho{}^*$ means the conjugate complex of u_ρ in the ordinary sense.

We shall denote wave functions, insofar as they are vectors or tensors, by the capital letter U with components U_i, U_{ik}, $\cdots$; the symmetry character of the tensors is sometimes separately specified. The electromagnetic and gravitational fields occupy special positions in that they are classical and the rest masses of the particles associated with them are zero; we shall therefore use for them the usual symbols $\varphi_i, f_{ik}=-f_{ki}$ and $g_{ik}=g_{ki}$, respectively.

The energy-momentum tensor is defined so that $-T_{44}$ and $-iT_{k4}$ where $k=1, 2, 3$, are, re-

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spectively, the energy density W and the momentum density G measured in terms of the natural units.

2. The variation principle and the energy-momentum tensor: gauge transformation and current vector

(a) *No external fields.*—First we shall consider all quantities as ordinary c numbers and proceed from a Lagrange function L which depends on any functions $q^{(r)}$ of x_i ($i=1, \cdots, 4$) and their first derivatives,

$$q_k^{(r)}=\partial q^{(r)}/\partial x_k, \tag{1}$$

but which does not depend explicitly on the coordinates x_i. Nothing special need be assumed here about the effect on $q^{(r)}$ of (proper) Lorentz transformations; the invariance of the real function L with respect to these transformations is sufficient. As is well known, the variation principle

$$\delta\int L(q_k^{(r)}, q^{(r)})d^4x=0, \tag{2}$$

in which the variation is assumed to be zero at the limits, determines the field equations

$$\sum_k (\partial/\partial x_k)[(\partial L/\partial q_k^{(r)})]-(\partial L/\partial q^{(r)})=0. \tag{3}$$

An energy-momentum tensor can be formed from the Lagrange function

$$T_{ik}=\sum_r (\partial L/\partial q_k^{(r)})q_i^{(r)}-L\delta_{ik}, \tag{4}$$

which, because of (3), satisfies the continuity equation

$$\sum_k \partial T_{ik}/\partial x_k=0. \tag{5}$$

We shall call the T_{ik} defined by (4) the *canonical* energy momentum tensor. It is in general not symmetric, nor is the energy density in general positive definite. In this connection it is necessary to bear in mind that for given values of the energy-momentum integrals the localization in space of the energy and momentum is uniquely determined only in the gravitation theory, where the production of the gravitational field gives the energy-momentum tensor a direct physical content.

In the absence of interaction, the condition that the total energy be positive is a necessary one. We shall see, however, that in many cases this condition is capable of being fulfilled only in the q number theory. For this theory we shall assume that the order of the factors in the expressions for the physical quantities is at our disposal in the various special cases.

In addition to the canonical energy-momentum tensor T_{ik} there is the angular momentum tensor $M_{ij,k}=-M_{ji,k}$ which by means of

$$P_{ij}=-P_{ji}=\int M_{ij,4}d^4x \tag{6}$$

for $i, j=1, 2, 3$, defines the total angular momentum, and which likewise satisfies a continuity equation

$$\sum_k \partial M_{ij,k}/\partial x_k=0. \tag{7}$$

This tensor can be obtained most simply, with the use of the invariance of the Lagrange function with respect to Lorentz transformations (including the three-dimensional rotations), in the following way. By means of the infinitesimal Lorentz transformation

$$\delta x_i=\sum_j \delta w_{ij}x_j \quad \text{in which} \quad \delta w_{ij}=-\delta w_{ji}, \tag{8}$$

the quantities $q^{(r)}$ are transformed at a fixed space-time point according to

$$\delta q^{(r)}=\sum_{i<j}\sum_s S_{ij}^{(r,s)}q^{s}\delta w_{ij}. \tag{9}$$

This can be written

$$\delta q=\sum_{i<j} S_{ij,\mathrm{op}}q\delta w_{ij}. \tag{9a}$$

It is to be noted that the variation

$$\delta q=q'(x+\delta x)-q(x)$$

is different from

$$\begin{aligned}\delta^*(q)=q'(x)-q(x)&=\delta q-\sum_i q_i\delta x_i\\ &=\sum_{i<j}\delta w_{ij}(x_iq_j-x_jq_i+S_{ij,\mathrm{op}}q).\end{aligned} \tag{10}$$

It may also be seen that

$$\delta^* q_k=(\partial/\partial x_k)(\delta^* q). \tag{10a}$$

This relation does not hold for δq_k.

It is now easy to put the variation of $\int L dx$ into the form

$$\delta\int L dx=\int \delta^* L dx+\sum_k [\partial(L\delta x^{(k)})/\partial x_k]dx, \quad (11)$$

in which again we have set

$$\delta L=L'(x')-L(x); \quad \delta^* L=L'(x)-L(x).$$

If $\delta^* L$ can be obtained by an infinitesimal Lorentz transformation the variation of the integral must vanish because of the Lorentz invariance of L. By considering (3) and (10a) we find[1]

$$\frac{\partial}{\partial x_k}\sum_q\left[\frac{\partial L}{\partial q_k}\delta^* q+L\delta x_k\right]=0.$$

Using Eqs. (11) and (4), we can get from the equality

$$\sum_q (\partial L/\partial q_k)\delta^* q+L\delta x_k=\sum_{i<j} M_{ij,k}\delta w_{ij},$$

an expression for the angular momentum tensor:

$$M_{ij,k}=x_i T_{jk}-x_j T_{ik}+\sum_q (\partial L/\partial q_k)S_{ij,\mathrm{op}}q. \quad (12)$$

This completes the proof of the continuity Eq. (7) which, by use of Eq. (5), can be put in the form

$$-T_{ij}+T_{ij}+\sum_k \frac{\partial}{\partial x_k}\sum_q [(\partial L/\partial q_k)S_{ij,\mathrm{op}}q]=0. \quad (7a)$$

If we define, therefore, a tensor skew-symmetric in j and k,

$$f_{i,jk}=-f_{i,kj} \quad (13a)$$

by the relation

$$-f_{i,jk}+f_{j,ik}=\sum_q (\partial L/\partial q_k)S_{ij,\mathrm{op}}q, \quad (13b)$$

then

$$\theta_{ij}=T_{ij}+\sum_k (\partial f_{i,jk}/\partial x_k) \quad (14)$$

is, by (7a) and (13b), symmetric in i and j,

$$\theta_{ij}=\theta_{ji} \quad (14a)$$

and satisfies a continuity equation

$$\sum \partial\theta_{ik}/\partial x_k=0. \quad (14b)$$

[1] See W. Pauli, *Mathematical Encyclopedia*, Article on relativity, pp. 616, 627, especially Eqs. (170) and (181).

From (13a, b) it follows uniquely that

$$f_{i,jk}=\sum_q \tfrac{1}{2}[-S_{ij,\mathrm{op}}q(\partial L/\partial q_k)-S_{ki,\mathrm{op}}q(\partial L/\partial q_j)+S_{jk,\mathrm{op}}q(\partial L/\partial q_i)]. \quad (13c)$$

It is to be emphasized that the tensor θ_{ij} is symmetric only by virtue of the field Eqs. (3). Moreover, we have, by (12) and (13b),

$$x_i\theta_{jk}-x_j\theta_{ik}=M_{ij,k}+\sum_l (\partial/\partial x_l)(x_i f_{j,kl}-x_j f_{i,kl}). \quad (14c)$$

The equality of the total energy, momentum integrals over space-like volumes calculated from T_{i4} and θ_{i4} follows from (14):

$$\int\theta_{i4}dV=\int T_{i4}dV. \quad (15a)$$

From (14c) we get, similarly, for the angular momentum tensor

$$P_{ij}=\int M_{ij,4}dV=\int(x_i\theta_{j4}-x_j\theta_{i4})dV. \quad (15b)$$

The general definition (14) of the symmetric energy-momentum tensor has been given by Belinfante[2] and Rosenfeld.[3] Since the localization of the energy plays a role principally in the theory of gravitation it is an important fact that the energy momentum tensor defined in the gravitation theory[4] goes over into the one used above in the particular case of special relativity. It is to be noted, however, that the energy density obtained from the θ_{ik}, *viz.*, $-\theta_{44}$, is positive definite only in special cases.

In order to prepare for the introduction of external electromagnetic fields, it is useful to divide the field quantities q into the complex quantities $U(x)$, their complex conjugates $U^*(x)$ which are to be considered as independent of $U(x)$, and the real quantities $V(x)$. Every sum over q then decomposes into sums over U, U^* and V, so that, for example, the energy mo-

[2] F. J. Belinfante, Physica **6**, 887 (1939). For the relation to gravitation theory see Physica **7**, 305 (1940).

[3] L. Rosenfeld, *Memoires de l'Academie Roy. Belgique* **6**, 30 (1940).

[4] D. Hilbert, *Gsch. Nach. Math. Phys.* (1915), p. 395. For generalization to spinors see H. Weyl, Zeits. f. Physik **56**, 330 (1929).

mentum tensor assumes the form

$$T_{ik}=\sum_r (U_i^{*(r)}(\partial L/\partial U_k^{*(r)})+(\partial L/\partial U_k^{(r)})U_i^{(r)} +(\partial L/\partial V_k^{(r)})V_i^{(r)})-L\delta_{ik}. \quad (16)$$

We introduce as possible transformations on the $U^{(r)}$, $U^{*(r)}$, even in the absence of external fields, a change in phase which is constant in space and time:

$$U^{(r)}\to U^{(r)}e^{i\alpha}, \quad U^{*(r)}\to U^{*(r)}e^{-i\alpha}. \quad (17)$$

We postulate that the Lagrange function L shall be invariant with respect to such phase changes with an arbitrary constant value of α. Differentiation of L with respect to the phase then gives the relation

$$\sum_r [U^{*(r)}(\partial L/\partial U^{*(r)})+\sum_k U_k^{*(r)}(\partial L/\partial U_k^{*(r)})] =\sum_r [U^{(r)}(\partial L/\partial U^{(r)}) +\sum_k U_k^{(r)}(\partial L/\partial U_k^{(r)})]. \quad (18)$$

This makes it possible to define a vector s_k

$$s_k=\epsilon i \sum [(\partial L/\partial U_k^{(r)})U^{(r)} - U^{*(r)}(\partial L/\partial U_k^{*(r)})], \quad (19)$$

where ϵ is a constant. As can easily be seen, s_k satisfies the continuity equation

$$\sum_k (\partial s_k/\partial x_k)=0. \quad (20)$$

We interpret s_k as the electromagnetic current. It can, of course, be defined uniquely only if the external electromagnetic fields are given. The real fields, which permit no phase transformation of the form (17), describe particles which in general cannot be sources of electromagnetic fields and which therefore have neither electrostatic nor magnetostatic properties. The electromagnetic field itself, which is associated in the particle picture with photons, is described however by the real fields.

We have, for simplicity, assumed that all the complex fields contained in L belong to particles of the same charge. If we wanted particles of different charges to be connected through these fields it would be necessary to require the invariance of L with respect to transformations like (17) for the various $U^{(r)}$ with different values α_r of the phase α for the different fields associated with the different charges, which have to be proportional to the charges.

(*b*) *The presence of external electromagnetic fields.*—We expressly assume that all field equations are contained in, or follow from, the relations (3) which are a consequence of the variation principle, so that it is unnecessary to add any supplementary conditions. With this assumption it is possible to introduce an external electromagnetic field by replacing the operation $\partial/\partial x_k$ when applied to the $U^{(r)}$ in the Lagrange function and the wave equations, by the operator

$$D_k=(\partial/\partial x_k)-i\epsilon\varphi_k \quad (21)$$

and when applied to the $U^{*(r)}$ by the operator which is the complex conjugate of the D_k

$$D_k^*=(\partial/\partial x_k)+i\epsilon\varphi_k. \quad (21')$$

The operator $\partial/\partial x_k$ is left unchanged when applied to real fields $V^{(r)}$; in the following therefore we shall not refer to these fields. The φ_k is the electromagnetic potential (with dimensions as given in §1) and ϵ is the charge of the particles measured in the unit $(\hbar c)^{\frac{1}{2}}$. The field strengths are given by

$$f_{ik}=(\partial\varphi_k/\partial x_i)-(\partial\varphi_i/\partial x_k); \quad (22)$$

the existence of the field exhibits itself in the non-commutation of the operators D_k:

$$D_iD_k-D_kD_i=-i\epsilon f_{ik}; \quad D_i^*D_k^*-D_k^*D_i^*=i\epsilon f_{ik}. \quad (22a)$$

The new Lagrange function is thus obtained by changing the meaning of the $U_k^{(r)}$ and $U_k^{*(r)}$ in the unchanged function

$$L(U_k^{(r)},\ U^{(r)},\ U_k^{*(r)},\ U^{*(r)}).$$

In the new function we have

$$U_k^{(r)}=D_kU^{(r)}; \qquad U_k^{*(r)}=D_k^*U^{*(r)}. \quad (1')$$

The equations which derive from the variation principle (2) for fixed φ_k take the form

$$D_k^*(\partial L/\partial U_k^{(r)})-(\partial L/\partial U^{(r)})=0; \quad D_k(\partial L/\partial U_k^{*(r)})-(\partial L/\partial U^{*(r)})=0. \quad (3')$$

In the derivative with respect to $U^{(r)}$ we always keep the $U_k^{(r)}$ (not the $\partial U^{(r)}/\partial x_k$) constant. However, if, in addition to these equations, supplementary conditions form a part of the theory, the new supplementary conditions obtained by the substitutions (21) and (21′) may not be compatible with the other equations without additional terms.

The theory obtained in this manner is invariant with respect to the gauge transformation

$$U^{(r)}\rightarrow U^{(r)}e^{i\alpha};\quad U^{*(r)}\rightarrow U^{*(r)}e^{-i\alpha};\quad (23a)$$

$$\varphi_k\rightarrow\varphi_k-\frac{1}{\epsilon}i\frac{\partial\alpha}{\partial x_k},\quad (23b)$$

where now α may be an arbitrary function of position. This is always correct if the Lagrange function in the absence of external fields is invariant with respect to the transformation (17) with constant phase. For, from (1′) it follows that

$$U_k^{(r)}\rightarrow U_k^{(r)}e^{i\alpha};\quad U_k^{*(r)}\rightarrow U_k^{*(r)}e^{-i\alpha}\quad (23c)$$

also holds for the transformation (23a, b). Furthermore it follows from the gauge invariance that

$$\begin{aligned}(\partial L/\partial U_k^{(r)})&\rightarrow(\partial L/\partial U_k^{(r)})e^{-i\alpha};\\ (\partial L/\partial U^{(r)})&\rightarrow(\partial L/\partial U^{(r)})e^{-i\alpha};\end{aligned}\quad (24)$$

$$\begin{aligned}(\partial L/\partial U_k^{*(r)})&\rightarrow(\partial L/\partial U_k^{*(r)})e^{i\alpha};\\ (\partial L/\partial U^{*(r)})&\rightarrow(\partial L/\partial U^{*(r)})e^{i\alpha}.\end{aligned}\quad (24^+)$$

It is because of this that the operators D^* in the first and D in the second Eq. (3′) are consistent.

We should like in particular to note the difference between fields like $U^{(r)}$, $U^{*(r)}$ which under the gauge group suffers a transformation of the type (23a) which we shall call the gauge transformation of the first type, and fields, such as the electromagnetic field, the potentials of which undergo gauge transformations of the second type (23b). This distinction is manifested through the fact that only expressions which are bilinear in U and U^* are associated with physically measurable quantities even when the associated field is quantized according to the Bose statistics. On the other hand the real fields V and the electromagnetic field strengths (when quantized according to the Bose statistics) are measurable quantities. From this it follows that, in principle, only gauge invariant quantities can be obtained by direct measurement. (The importance of the fact that the rest mass of the photons must be exactly zero for transformations of the second type is discussed in Part II, §§2(c) and 2(e).)

The following formal remark is of use for calculations in connection with the current vector and energy-tensor. Let f^* be an arbitrary function of the $U^{(r)}$, $U^{*(r)}$, $U_k^{(r)}$, $U_k^{*(r)}$ which is multiplied by $e^{-i\alpha}$ in a gauge transformation of the first type, and g another function of these quantities which is multiplied by $e^{i\alpha}$. Then

$$(\partial/\partial x_k)(f^*g)=(D_k^*f^*)g+f^*(D_kg),$$

since the terms which involve φ_k cancel. Similarly we have for the derivative of the gauge invariant Lagrange function L with respect to x_k:

$$\begin{aligned}(\partial L/\partial x_k)=\textstyle\sum_r\big[&(\partial L/\partial U^{(r)})D_kU^{(r)}\\ &+\textstyle\sum_i(\partial L/\partial U_i^{(r)})D_kD_iU^{(r)}\\ &+(\partial L/\partial U^{*(r)})D_k^*U^{*(r)}\\ &+\textstyle\sum_i(\partial L/\partial U_i^{*(r)})D_k^*D_i^*U^{*(r)}\big].\end{aligned}$$

The terms which include φ cancel out as a result of relation (18) which remains valid when the meaning of $U_k^{(r)}$, $U_k^{*(r)}$ is changed according to (1′).

We define now the current vector s_k and the energy tensor T_{ik} as before by expressions (19) and (4). As a result of (3′) the continuity Eq. (20) for the current continues to hold. From the energy-momentum tensor we obtain by making use of (3′) and the expression for $\partial L/\partial x_k$:

$$\begin{aligned}\partial T_{ik}/\partial x_k=\textstyle\sum_r\{&[(D_kD_i-D_iD_k)U^{(r)}](\partial L/\partial U_k^{(r)})\\ &+[(D_k^*D_i^*-D_i^*D_k^*)U^{*(r)}](\partial L/\partial U^{*(r)})\}.\end{aligned}$$

From (20) and (19) it follows, therefore, that

$$\partial T_{ik}/\partial x_k=f_{ik}s_k.\quad (25)$$

It is necessary that this equation hold for the energy tensor of the original U field where there is an external electromagnetic field since it expresses the existence of the Lorentz force. It finally justifies our looking upon s_k as the electromagnetic four-vector current.

We have not yet considered the generation of the electromagnetic field by means of the U field. The above formulation suggests that this generation can be obtained by the Maxwell equation $\partial f_{ik}/\partial x_k = s_i$ since then the continuity equation

$$(\partial/\partial x_k)(T_{ik}+S_{ik})=0$$

is satisfied where T_{ik} is the energy tensor of the U field and

$$S_{ik}=f_{ir}f_{kr}-\tfrac{1}{4}f_{rs}f_{rs}\delta_{ik}$$

is that of the electromagnetic field. However, the application of the particle picture, or the second quantization of the U field to this formulation of the rule for the generation of the electromagnetic field gives rise to the known difficulties of the infinite self energy; these difficulties have not yet been overcome.

We include here a discussion of the possibility of introducing additional terms in the Lagrange function which depend explicitly on the field strengths f_{ik} and which are consistent with the postulate of gauge invariance. The original definition of the current still applies and its continuity equation holds, but in place of (25) we have[5]

$$\partial T_{ik}/\partial x_k = f_{ik}s_k-\tfrac{1}{2}(\partial L/\partial f_{rs})(\partial f_{rs}/\partial x_i).$$

This makes additional terms in T_{ik} and s_k necessary, since for the new quantities T_{ik}' and s_k' the relation

$$\partial T_{ik}'/\partial x_k = f_{ik}s_k' \tag{25'}$$

must be valid.

The s_k can be found most easily from the equation which arises from the variation of φ_k,

$$\delta\int Ld^4x=-\int s_k'\delta\varphi_k d^4x;$$

namely,

$$s_k'=s_k-(\partial/\partial x_l)(\partial L/\partial f_{kl}). \tag{26}$$

The new term satisfies the continuity equation so that

$$\partial s_k'/\partial x_k=0 \tag{20'}$$

also holds. Equation (25′) is now fulfilled if we put

$$T_{ik}'=T_{ik}-f_{ir}\partial L/\partial f_{rk}. \tag{27}$$

This can be established by the use of the Maxwell equations

$$\partial f_{ks}/\partial x_i+\partial f_{ik}/\partial x_s+\partial f_{si}/\partial x_k=0,$$

which arise from (22).

The use of such additional terms for the description of particles which have a magnetic moment will be discussed in Part II, §§2(d) and 3(a).

Part II. Special Fields

1. The wave fields of particles without spin

(a) *The wave equation, current vector, and energy-momentum tensor.*—The simplest example of a relativistically invariant wave equation is the scalar equation

$$\Box U-\kappa^2 U=0, \tag{1}$$

in which $\Box$ is the operator

$$\Box=\Delta-(\partial^2/\partial x_0^2)=\sum_{i=1}^{4}(\partial^2/\partial x_i^2) \tag{2}$$

and $\kappa=mc/\hbar$ where m is the rest-mass and $\hbar$ is the quantum of action divided by 2π. We do not require that U be real.

The wave Eq. (1) can be obtained from the variation principle

$$\delta\int Ld^4x=0$$

(d^4x = four-dimensional volume element) if we use for the Lagrange function

$$L=(\partial U^*/\partial x_i)(\partial U/\partial x_i)+\kappa^2U^*U. \tag{3}$$

From this we get for the current vector by Eq. (19) (I)

$$s_k=\epsilon i((\partial U^*/\partial x_k)U-(\partial U/\partial x_k)U^*),^6 \tag{4}$$

in which ϵ is the charge of the particles measured in the natural unit $(\hbar c)^{\frac{1}{2}}$. This current vector satisfies the continuity equation

$$\partial s_k/\partial x_k=0. \tag{5}$$

The energy-momentum tensor T_{ik} is defined in this case by

$$T_{ik}=\frac{\partial U^*}{\partial x_i}\frac{\partial U}{\partial x_k}+\frac{\partial U^*}{\partial x_k}\frac{\partial U}{\partial x_i}-L\delta_{ik}. \tag{6}$$

[5] The factor $\frac{1}{2}$ in the additional term arises from the fact that the summations over r and s are independent. With this rule we have for the variation of the field strengths

$$\delta L=\tfrac{1}{2}(\partial L/\partial f_{rs})\delta f_{rs}.$$

[6] This sequence of factors proves convenient in the q number theory since it avoids a zero-point charge.

This likewise satisfies the continuity equation

$$\partial T_{ik}/\partial x_k = 0. \tag{7}$$

The T_{ik} is symmetric; furthermore the energy density $-T_{44}$ is positive definite—an important property:

$$\begin{aligned} W = -T_{44} = &-\frac{\partial U^*}{\partial x_4}\frac{\partial U}{\partial x_4} \\ &+\text{grad } U^* \cdot \text{grad } U + \kappa^2 U^* U \\ = &\frac{\partial U^*}{\partial x_0}\frac{\partial U}{\partial x_0} \\ &+\text{grad } U^* \cdot \text{grad } U + \kappa^2 U^* U. \end{aligned} \tag{8}$$

It is often useful to transform the wave equation of the second order into a system of wave equations of the first order as follows:

$$U_k = \partial U/\partial x_k, \quad \partial U_k/\partial x_k = \kappa^2 U. \tag*{$\langle 1\rangle$}$$

This form of the equations has a greater similarity to the form of those in the vector theory which will be discussed later. Furthermore they can be derived from a variational principle by using the Lagrange function

$$L = (\partial U^*/\partial x_i) U_i + U_i^* (\partial U/\partial x_i) - U_i^* U_i + \kappa^2 U^* U, \tag*{$\langle 3\rangle$}$$

the U_k, U and their complex conjugates are to be independently varied.

We mention finally the theory which is dual to $\langle 1\rangle$. In this the scalar U is replaced by a pseudo-scalar U_{klmn} which is antisymmetric in all indices and the vector U_k by a pseudo-vector U_{klm} which is also antisymmetric in all indices. The equations analogous to $\langle 1\rangle$ are then

$$\begin{aligned} U_{lmn} &= \frac{\partial U_{klmn}}{\partial x_k}; \\ \frac{\partial U_{lmn}}{\partial x_k} - \frac{\partial U_{mnk}}{\partial x_l} + \frac{\partial U_{nkl}}{\partial x_m} - \frac{\partial U_{klm}}{\partial x_n} &= \kappa^2 U_{klmn}. \end{aligned} \tag{9}$$

(*b*) *Eigenstates in momentum space. Charge conjugate solutions.*—It is known that the most general solution of (1) can be written as a sum of plane waves. If we introduce a large cube with edge length L so that the solutions are periodic in a cubic lattice of length L, the components of the wave numbers must be integral multiples of $2\pi/L$.

The wave Eq. (1) requires that the propagation vectors $(k_0, \mathbf{k})$* of the waves satisfy the well known relation

$$k_0^2 = k^2 + \kappa^2. \tag{10}$$

k_0 will always be defined as the positive root:

$$k_0 = +(k^2+\kappa^2)^{\frac{1}{2}}. \tag{10$'$}$$

We may now write

$$\begin{aligned} U^*(x, x_0) = (V)^{-\frac{1}{2}} \textstyle\sum_k (2k_0)^{-\frac{1}{2}} &\times \{U_+^*(k) \exp[i(-\mathbf{k}\cdot\mathbf{x} + k_0 x_0)] \\ &+ U_-(k) \exp[i(\mathbf{k}\cdot\mathbf{x} - k_0 x_0)]\}, \end{aligned} \tag{11}$$

$$\begin{aligned} U(x, x_0) = (V)^{-\frac{1}{2}} \textstyle\sum_k (2k_0)^{-\frac{1}{2}} &\times \{U_+(k) \exp[i(\mathbf{k}\cdot\mathbf{x} - k_0 x_0)] \\ &+ U_-^*(k) \exp[i(-\mathbf{k}\cdot\mathbf{x} + k_0 x_0)]\}. \end{aligned} \tag{11*}$$

The notation is chosen so that the amplitudes of the Fourier decomposition which are provided with a star are multiplied by $\exp(ik_0x_0)$ while those without the star are multiplied by $\exp(-ik_0x_0)$; the $(2k_0)^{-\frac{1}{2}}$ is always to be taken as positive.

From (5) and (7) we have for the total energy, total momentum and total charge

$$E = -\int T_{44} dV = \textstyle\sum_k k_0 [U_+^*(k) U_+(k) + U_-(k) U_-^*(k)], \tag{12}$$

$$\mathbf{G} = \textstyle\sum_k \mathbf{k} [U_+^*(k) U_+(k) + U_-(k) U_-^*(k)], \tag{13}$$

$$e = \frac{1}{i}\int s_4 dV = \epsilon \textstyle\sum_k [U_+^*(k) U_+(k) - U_-^*(k) U_-(k)]. \tag{14}$$

Equation (14) shows that the eigenvibrations of negative frequency in the U^* (and positive frequency in U) belong to states of negative charge. This is in agreement with the fact that the sign of the current vector changes when U and U^* are interchanged in (4) whereas the energy momentum tensor remains unchanged. We can say, therefore, that the solution U^* of

* In cases where no ambiguity exists the (vector) quantities **k** and **x** will be indicated by italic type.

the wave equation which is the conjugate complex of a given solution $U(x, x_0)$ is the "charge conjugate solution." This is also in accord with the treatment of the problem by means of the general principle of Part I when external electromagnetic fields are present. The principle requires that the wave Eq. (1) be replaced by

$$\sum_i D_i^2 U - \kappa^2 U = 0. \quad (15)$$

This equation remains correct if U and e are replaced by U^* and $-e$, since then D_k goes over to D_k^*.[7]

In §3(*c*) it is shown that in the case of half integer spins the relation between complex conjugate and charge conjugate solutions is somewhat more complicated.

The theory given here must be associated with particles without spin since for a given $\mathbf{k}$ and a given sign of k_0 there exists only one eigenstate.

(*c*) *Quantization.*—We do not wish to base the following discussion explicitly on the canonical formalism because an unnecessarily sharp distinction between time and space is introduced in this formalism, and this is convenient only in the absence of supplementary conditions involving the canonical variables at a given instant. We use here a generalization of the method of quantization first used by Jordan and Pauli in the case of the electromagnetic field.[8] Moreover we shall require that the relation

$$dF/dx_0 = i[H, F] \quad (16)$$

be valid for every physical quantity F which does not explicitly depend on the time. H is the Hamiltonian operator which expresses the total energy divided by $\hbar c$.

The $U_+(k)$, $U_+^*(k)$, $U_-(k)$, $U_-^*(k)$ defined by (11) and (11*) contain the time explicitly; this is not the case for the quantities.

$$u_\pm(k) = U_\pm(k) \exp(-ik_0x_0); \quad u_\pm^*(k) = U_\pm^*(k) \exp(+ik_0x_0), \quad (17)$$

since these can be expressed in terms of U, U^*, $\partial U/\partial x_0$, $\partial U^*/\partial x_0$ without the explicit introduction of x_0.[9] Since the $U_\pm(k)$ are constant in the force free case, we get from (16) the relations

$$i[H, u_\pm(k)] = -ik_0 u_\pm(k); \quad i[H, u_\pm^*(k)] = ik_0 u_\pm^*(k)$$
and
$$[H, U_\pm(k)] = -k_0 U_\pm(k); \quad [H, U_\pm^*(k)] = k_0 U_\pm^*(k). \quad (18)$$

We first discuss the quantization according to the Bose statistics. The values of all bracket symbols of the type

$$[U(k), U(k')], \quad [U^*(k), U^*(k')], \quad [U(k), U^*(k')],$$

in which the U and U^* can be given $+$ or $-$ indices in an arbitrary way, follow uniquely from (12) and (18) if one further assumption is included, namely, that these bracket symbols are themselves c numbers. In fact, from (12) and (18) it follows that only the last of the above bracket symbols is different from zero and this one only if $\mathbf{k}=\mathbf{k}'$ and the $+$ or $-$ indices for the two quantities are the same. We find from (18) that

$$[U_+(k), U_+^*(k)] = [U_-(k), U_-^*(k)] = 1. \quad (19)$$

From the relations we find by familiar processes that the eigenvalues of

$$N_+(k) = U_+^*(k) U_+(k); \quad N_-(k) = U_-^*(k) U_-(k) \quad (20)$$

are the positive integers (including zero). It is this that makes the familiar transition to the particle picture possible. From (12), (13) and (14) it follows (as can be shown by familiar methods) that $N_+(k)$ and $N_-(k)$ belong to the charge $+\epsilon$ and the charge $-\epsilon$, respectively, and that both belong to the momentum $+\mathbf{k}$. We see from (12) that for every value of $\mathbf{k}$ there is a zero-point energy of the vacuum of one quantum, k_0; thus the zero-point energy per eigenstate present

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[7] For the theory of pair production based on this theory see W. Pauli and V. Weisskopf, Helv. Phys. Acta **7**, 809 (1934).

[8] The logical development of this method, including the interaction between particles, is given by the formalism of Dirac with more than one time-variable. See P. A. M. Dirac, *Quantum Mechanics* (Oxford, 1935), second edition.

[9] The quantities $U_\pm(k)$, $U_\pm^*(k)$ are also very important when there is an interaction of the U field with other fields. If, in this case, the Hamiltonian function is given by $H_0+\Omega$ where H_0 is the Hamiltonian operator for the force free case, and Ω is the interaction energy, Eq. (16) requires simply that

$$dU_\pm(k)/dx_0 = i[\Omega, U_\pm(k)]; \quad dU_\pm^*(k)/dx_0 = i[\Omega, U_\pm^*(k)].$$

in the vacuum is a half quantum, $\frac{1}{2}k_0$, as is the case in the electromagnetic field.

When we go over to the bracket symbols of the field functions $U(x, x_0)$ themselves we find on using (11) and (19) that

$$i[U(x, x_0), U(x', x_0')] = i[U^*(x, x_0), U^*(x', x_0')] = 0 \quad (21)$$

and that

$$i[U(x, x_0), U^*(x', x_0')] = i[U^*(x, x_0), U(x', x_0')] = D(x-x', x_0-x_0'). \quad (21)$$

The D function in (21) is defined by the equation

$$D(x, x_0) = \frac{1}{(2\pi)^3}\int d^3k e^{i\mathbf{k}\cdot\mathbf{x}} \frac{\sin k_0 x_0}{k_0} \quad (22)$$

and k_0 is defined by

$$k_0 = (k^2+\kappa^2)^{\frac{1}{2}}.$$

The form of the D function is uniquely determined by the requirement that it satisfy the wave equation

$$\Box D - \kappa^2 D = 0 \quad (22')$$

and the relations

$$D(x, 0) = 0 \quad (\partial D/\partial x_0)_{x_0=0} = \delta(\mathbf{x}). \quad (22'')$$

For $\kappa = 0$ we find

$$D(x, x_0) = \frac{1}{4\pi r}[\delta(r-r_0) - \delta(r+r_0)]. \quad (23)$$

In the general case of $\kappa \neq 0$ the singularity on the light cone is still given by this expression; however, D is no longer different from zero inside the cone. In fact, one finds[10]

$$D(x, x_0) = -\frac{1}{4\pi}\frac{1}{r}\frac{\partial}{\partial r}F(r, x_0), \quad (24)$$

$$F(r, x_0) = \begin{cases} J_0(\kappa(x_0^2 - r^2)^{\frac{1}{2}}) & \text{for } x_0 > r \\ 0 & \text{for } r > x_0 > -r \\ -J_0(\kappa(x_0^2 - r^2)^{\frac{1}{2}}) & \text{for } -r > x_0. \end{cases} \quad (25)$$

The change of the value of the function F by ± 1 on the light cone corresponds to the δ-like singularity of D which occurs there. It is of particular importance for what follows that D vanishes outside of the light cone (i.e., for $r > x_0 > -r$).

From Eqs. (21) and (21′) the known commutation relations may be obtained by first differentiating with respect to $\mathbf{x}$ and then substituting $x_0 = x_0'$:

$$i[U(x, x_0), U^*(x', x_0)] = 0,$$

$$i\left[\frac{\partial U(x, x_0)}{\partial x_0}, U^*(x', x_0)\right] = i\left[\frac{\partial U^*(x, x_0)}{\partial x_0}, U(x', x_0)\right] = \delta(\mathbf{x}-\mathbf{x}').$$

We turn now to a consideration of the quantization when the exclusion principle is assumed to be operative. Using the Hamiltonian function (8) or (12) as a basis for our procedure, we must require first of all that the relations (18) continue to hold where the bracket symbols have their previous meaning; on the other hand the expressions

$$[U(k), U(k')]_+, \quad [U^*(k), U^*(k')]_+, \quad [U(k), U^*(k')]_+$$

are c numbers. (The bracket symbol used here is defined by $[A, B]_+ = AB + BA$.) From (18) it follows as before that the first two of these brackets are always zero while the last is different from zero only if $\mathbf{k} = \mathbf{k}'$ and the $+$ or $-$ indices on the two quantities occurring in the brackets are the same. Furthermore we have

$$U_+^*(k)U_+(k) + U_+(k)U_+^*(k) = 1,$$
$$U_-^*(k)U_-(k) + U_-(k)U_-^*(k) = -1.$$

The last of the two equations exhibits a contradiction of the assumption that U^* shall be the Hermitian conjugate of U, since if the assumption is satisfied, the left side of the equation is essentially positive. This assumption, however, is required in order that physical quantities, as for example the charge density s_0 shall have real eigenvalues.

We can also show that the scalar field theory cannot be quantized in accord with the exclusion principle without reference to the special Hamiltonian and Eqs. (16). Besides the function D

[10] See P. A. M. Dirac, Proc. Camb. Phil. Soc. **30**, 100 (1934).

there is another which is invariant and which satisfies the wave Eq. (1), namely,

$$D_1(x, x_0)=\frac{1}{(2\pi)^3}\int\frac{d^3k}{k_0}e^{i\mathbf{k}\cdot\mathbf{x}}\cos(k_0x_0). \quad (22')$$

For $\kappa=0$ we have

$$D_1(x, x_0)=\frac{1}{(2\pi)^2}\frac{1}{r^2-x_0^2}. \quad (23')$$

In general we may write

$$D_1(x, x_0)=\frac{1}{4\pi}\frac{1}{r}\frac{\partial}{\partial r}F_1(r, x_0), \quad (24')$$

where

$$F_1(r, x_0)=\begin{cases} N_0[\kappa(x_0^2-r^2)^{\frac{1}{2}}] & \text{for } x_0>r \text{ or } -r<x_0 \\ -iH_0^{(1)}[i\kappa(r^2-x_0^2)^{\frac{1}{2}}] & \text{for } r>x_0>-r. \end{cases} \quad (25')$$

N_0 is the Neumann function; $H_0^{(1)}$ is the first Hankel cylinder function. The strongest singularity of D_1 which occurs on the light cone is defined by (23) even in the general case of $\kappa\neq 0$.

Since the scalar field $U(x, x_0)$ must satisfy the wave equation (1) and must be relativistically invariant, we have as the only possibilities the relations

$$[U(x, x_0), U^*(x', x_0')]_\pm=cD(x-x', x_0-x_0')+c_1D_1(x-x', x_0-x_0'), \quad (26)$$

where c and c_1 are constants. We shall therefore expressly postulate for the following that any two physical quantities, the relative coordinates of which lie outside the light cone, commute. As a consequence the left side of (26) must vanish for such points if we use the plus sign. Otherwise the non-commutativity, in the ordinary sense, of the gauge invariant quantities which are bilinear in U and U^*, e.g., the charge density, would follow.[11]

The justification of our postulate lies in the fact that measurements at space-time points which have a space-like connection line can never perturb one another since signals cannot be propagated with a velocity greater than that of light. In any event theories which employ the D_1 function in the quantization, instead of or in addition to the D function, have very different consequences from those which are known at present.

Thus if our commutativity postulate is fulfilled, the constant c_1 in (26) must vanish so that we have

$$[U(x, x_0), U^*(x', x_0')]=\text{const.}\, D(x-x', x_0-x_0'). \quad \langle 21\rangle$$

If, however, the bracket with the plus sign is introduced, the left side is intrinsically positive for $\mathbf{x}=\mathbf{x}'$, $x_0=x_0'$, while the right side vanishes for $x_0=x_0'$. Thus we arrive at a contradiction similar to the one obtained above.[12]

The result of our arguments is that a relativistic theory for particles without spin based on general postulates must necessarily be quantized in accord with the Einstein-Bose statistics.

(*d*) *A real field.*—In this case we always have $U=U^*$; the current vector vanishes identically; the associated particles cannot generate an electromagnetic field. For the Lagrange function and energy momentum tensor we write:

$$L=\tfrac{1}{2}\textstyle\sum_i(\partial V/\partial x_i)^2+\tfrac{1}{2}\kappa^2V^2, \quad (27)$$

$$T_{ik}=\frac{\partial V}{\partial x_i}\frac{\partial V}{\partial x_k}-L\delta_{ik}. \quad (27')$$

An additional relation exists between the coefficients of the Fourier expansion (11)

$$V_-(k)=V_+(k); \quad V_-^*(k)=V_-^*(k).$$

This enables us to write (11) in a simpler form:

$$V(x, x_0)=(V)^{-\frac{1}{2}}\textstyle\sum_k(2k_0)^{-\frac{1}{2}}\times[V(k)\exp[i(\mathbf{k}\cdot\mathbf{x}-k_0x_0)]+V^*(k)\exp[i(-\mathbf{k}\cdot\mathbf{x}+k_0x_0)]. \quad (28)$$

For the energy and momentum we have

$$E=\textstyle\sum_k\tfrac{1}{2}k_0[V^*(k)V(k)+V(k)V^*(k)], \quad (29)$$

$$\mathbf{G}=\textstyle\sum_k\mathbf{k}\tfrac{1}{2}[V^*(k)V(k)+V(k)V^*(k)]. \quad (30)$$

[11] Compare W. Pauli, Inst. H. Poincaré Ann. **6**, 137 (1936). For the generalization of these considerations to the case of any integral spin, see W. Pauli, Phys. Rev. **58**, 716 (1940).

[12] This also becomes clear through a spatial Fourier expansion of U and U^*. If $u(k)=U_+(k)+U_-(k)$, $u^*(k)=U_+^*(k)+U_-^*(k)$ [see Eq. (17)] are the associated amplitudes (21′) required for $x_0=x_0'$ for the case of the exclusion principle $[u(k), u^*(k)]_+=0$ for every eigenstate; this has as a consequence $U_+(k)=U_-(k)=0$.

The commutation relations (11) remain valid

$$[V(k),\ V^*(k)]=1, \tag{31}$$

while the brackets $[V(k),\ V(k')]$, $[V^*(k),\ V^*(k')]$ and, if $\mathbf{k}\neq\mathbf{k}'$, $[V(k),\ V^*(k')]$ vanish. From Eq. (29) we see that the zero-point energy of the vacuum is again a half quantum, $\frac{1}{2}k_0$, per eigenstate. In place of (21) we have

$$i[V(x,x_0),\ V(x',x_0')]=D(x-x',\ x_0-x_0'). \tag{32}$$

In the quantization in accord with the exclusion principle there are two possibilities: either the energy becomes a constant c number, which is impossible, or the function D_1 appears in the right of (32), which contradicts our previous postulates.

It may be noted that the original form of the theory with a complex function U is clearly equivalent to one with two real fields $V=V^*$, $W=W^*$ which correspond to the real and imaginary parts of U. In this connection it is useful to introduce a factor $1/\sqrt{2}$ so that no numerical factor occurs in the commutation relations. We put, therefore,

$$U=(2)^{-\frac{1}{2}}(V+iW);\quad U^*=(2)^{-\frac{1}{2}}(V-iW). \tag{33}$$

It then follows from (21) that

$$\begin{aligned} i[V(x,x_0),\ V(x',x_0')] &= i[W(x,x_0),\ W(x',x_0')] \\ &= D(x-x',\ x_0-x_0'), \\ i[V(x,x_0),\ W(x',x_0')] &= 0. \end{aligned} \tag{34}$$

The energy tensor (7) becomes

$$\begin{aligned} T_{ik} &= \tfrac{1}{2}\left[\frac{\partial V}{\partial x_i}\frac{\partial V}{\partial x_k}+\frac{\partial W}{\partial x_i}\frac{\partial W}{\partial x_k}\right]-L\delta_{ik}, \\ L &= \tfrac{1}{2}\sum\left[\left(\frac{\partial V}{\partial x_0}\right)^2+\left(\frac{\partial W}{\partial x_0}\right)^2\right]+\tfrac{1}{2}\kappa^2(V^2+W^2). \end{aligned} \tag{35}$$

The current vector, however, is

$$s_k=\epsilon\left[\frac{\partial W}{\partial x_k}V-\frac{\partial V}{\partial x_k}W\right]. \tag{36}$$

If the V and W are expanded in terms of the eigenstates according to (28) with the commutation relations of the form (31), we obtain for the energy E and the momentum $\mathbf{G}$ the sum of two expressions, one in V and one in W, of the form (29) and (30), respectively; for the charge we find on the other hand

$$e=\epsilon i\sum_k[W(k)V^*(k)-V(k)W^*(k)]. \tag{37}$$

The "abbreviated" theory with a single real field V can be obtained from the above by striking out W and setting the current vector equal to zero.

2. Wave fields for particles of spin 1

(*a*) *The c number theory for case of no external fields.*—This case holds the center of current interest since Yukawa supposed the meson to have spin 1 in order to explain the spin dependence of the force between proton and neutron. The theory for this case has been given by Proca.

The simplest possibility for a generalization of the theory formulated in §1 seems to be that which is obtained by introducing a vector field U_k which satisfies the wave equation

$$\square U_k-\kappa^2U_k=0 \tag{38}$$

and the components of which are treated as independent scalars. It is easy to see, however, that with such a formulation the component $U_4=iU_0$ gives rise to negative terms in the energy when the signs are so taken that the space-like components of the vector U_k are associated with positive energy terms. This difficulty can be removed by requiring, in addition to (38), the supplementary condition

$$\partial U_k/\partial x_k=0. \tag{39}$$

The meaning of this becomes especially clear in the rest system of the particle where the wave field depends periodically on the time but not on space coordinates. In this system (39) requires the vanishing of U_4; from this it is clear that in this case, as a consequence of (39) the energy is necessarily positive. It follows from the Lorentz invariance of the theory that in general the total energy E (the volume integral of the energy density) is positive. It will appear later that we can also prove for this case that the energy density at every space point is positive definite.

A skew-symmetric tensor $U_{ik}=-U_{ki}$, which is related to the U_k as the field strengths are related to the potential in electrodynamics can be formed from the U_k by a rotation:

$$U_{ik}=(\partial U_k/\partial x_i)-(\partial U_i/\partial x_k). \tag{40}$$

By means of (38) and (39) we get from (40)

$$(\partial U_{ik}/\partial x_k)+\kappa^2 U_i=0. \tag{41}$$

This relation is important in that it shows that U_i is uniquely defined by a given U_{ik}, just as the U_{ik} is defined by U_i from (40). As a consequence, for non-vanishing rest-mass, the addition of a gradient to U_i is not permitted. Hence no gauge transformations of the second kind exist for the U_i when $\kappa\neq 0$. It is worth noting that (38) and (39) follow from (40) and (41). When (41) is differentiated with respect to x_i and summed over i, the first term vanishes because of the skew symmetry of U_{ik} and we get (39). Equation (36) then follows upon the substitution of (40) into (41). Finally we have from (40) the relations

$$(\partial U_{ik}/\partial x_l)+(\partial U_{li}/\partial x_k)+(\partial U_{kl}/\partial x_i)=0. \tag{42}$$

There are certain advantages in considering (40) and (41) as the basic equations of the theory and the remaining equations as derived, since (40) and (41) can be obtained from a variation principle

$$\delta\int L d^4x=0,$$

if for L is taken

$$L=-\tfrac{1}{2}U_{ik}^*U_{ik}+\tfrac{1}{2}U_{ik}^*\left(\frac{\partial U_k}{\partial x_i}-\frac{\partial U_i}{\partial x_k}\right)+\tfrac{1}{2}\left(\frac{\partial U_k^*}{\partial x_i}-\frac{\partial U_i^*}{\partial x_k}\right)U_{ik}+\kappa^2 U_i^*U_i. \tag{43}$$

(In this expression the customary dummy index convention is employed.) In the variation process the quantities U_i, $U_{ik}=-U_{ki}$ and their conjugates are to be varied independently.

For the canonical energy tensor defined in accord with I (4) we have

$$T_{ik}=U_{kr}^*\frac{\partial U_r}{\partial x_i}+\frac{\partial U_r^*}{\partial x_i}U_{kr}-L\delta_{ik}. \tag{44}$$

This can be transformed into a symmetric tensor. Following the general formulation of Part I, Eqs. (14), (13c), we get upon using (40) and (41)

$$T_{ik}=\theta_{ik}+\frac{\partial}{\partial x_r}(U_{kr}^*U_i+U_i^*U_{kr}), \tag{45}$$

where

$$\theta_{ik}=\theta_{ki}=U_{ir}^*U_{kr}+U_{kr}^*U_{ir}+\kappa^2(U_i^*U_k+U_k^*U_i)-\delta_{ik}(\tfrac{1}{2}U_{rs}^*U_{rs}+\kappa^2U_r^*U_r). \tag{46}$$

From (37) it follows on the one hand that

$$\partial\theta_{ik}/\partial x_k=\partial T_{ik}/\partial x_k, \tag{45a}$$

so that from the vanishing of the second divergence we can conclude the vanishing of the first, and on the other hand that

$$\int\theta_{i4}dV=\int T_{i4}dV, \tag{45b}$$

from which it is clear that it makes no difference whether the total energy and total momentum is calculated by means of the canonical or the symmetric tensor.

From (41) we get for the energy density $-\theta_{44}$ on the introduction of the quantities U_0, U_{0k},[13]

$$U_4=iU_0,\quad U_{4k}=iU_{0k},\quad U_4^*=iU_0^*,\quad U_{4k}^*=iU_{0k}^*(k=1,2,3), \tag{47}$$

$$W=-\theta_{44}=\sum U_{0k}^*U_{0k}+\sum_{i\le k}^{3}U_{ik}^*U_{ik}+\kappa^2\left(U_0^*U_0+\sum_{k=1}^{3}U_k^*U_k\right). \tag{48}$$

The energy density, therefore, is positive-definite as in the scalar theory.

The current vector defined according to I (19) is

$$s_i=i(U_{ir}^*U_r-U_r^*U_{ir}). \tag{49}$$

We shall see, however, that this expression is not unique since additional terms in L proportional to f_{ik} can modify it even in the absence of external fields.

[13] In this connection it is important to note the meaning of $U_{4k}^*(k=1,2,3)$ and U^* as given in Part I. U_0^* and U_{0k}^* denote the actual conjugate complex values of U_0 and U_{0k}.

We shall sketch the dual theory to the above without going into details. The vector U_i is replaced by a tensor U_{lmn} which is skew-symmetric in all indices (pseudo-vector) and U_{ik} by a skew-symmetric tensor of the same type as the original U_{ik}. Equations (39) to (42) are replaced by

$$\frac{\partial U_{lmn}}{\partial x_k}-\frac{\partial U_{mnk}}{\partial x_l}+\frac{\partial U_{nkl}}{\partial x_m}-\frac{\partial U_{klm}}{\partial x_n}=0, \tag{31'}$$

$$U_{ik}=\partial U_{ikr}/\partial x_r, \tag{40'}$$

$$\frac{\partial U_{ik}}{\partial x_l}+\frac{\partial U_{kl}}{\partial x_i}+\frac{\partial U_{li}}{\partial x_k}-\kappa^2 U_{lik}=0. \tag{41'}$$

$$\partial U_{ik}/\partial x_i=0. \tag{42'}$$

(*b*) *Eigenstates in the momentum space.*—We first write the amplitudes of the three spatial components of U_k (without normalization)

$$\begin{aligned}\mathbf{U}^*(x, x_0)&=(V)^{-\frac{1}{2}}\textstyle\sum_k (2)^{-\frac{1}{2}}\{\mathbf{U}_+^*(k)\exp[i(-\mathbf{k}\cdot\mathbf{x}+k_0x_0)]+\mathbf{U}_-(k)\exp[i(\mathbf{k}\cdot\mathbf{x}-k_0x_0)]\},\\ \mathbf{U}(x, x_0)&=(V)^{-\frac{1}{2}}\textstyle\sum_k (2)^{-\frac{1}{2}}\{\mathbf{U}_+(k)\exp[i(\mathbf{k}\cdot\mathbf{x}-k_0x_0)]+\mathbf{U}_-^*(k)\exp[i(-\mathbf{k}\cdot\mathbf{x}+k_0x_0)]\},\end{aligned} \tag{50}$$

in which k_0 is given by ⟨10⟩. The supplementary condition (39) then requires for the fourth components of U_k^* and U_k

$$\begin{aligned}U_0^*(x, x_0)=(V)^{-\frac{1}{2}}\textstyle\sum_k (2)^{-\frac{1}{2}}[&(1/k_0)\mathbf{k}\cdot\mathbf{U}_+^*(k)\exp[i(-\mathbf{k}\cdot\mathbf{x}+k_0x_0)]\\ &+(1/k_0)\mathbf{k}\cdot\mathbf{U}_-(k)\exp[i(\mathbf{k}\cdot\mathbf{x}-k_0x_0)]],\\ U_0(x, x_0)=(V)^{-\frac{1}{2}}\textstyle\sum_k (2)^{-\frac{1}{2}}[&(1/k_0)\mathbf{k}\cdot\mathbf{U}_+(k)\exp[i(\mathbf{k}\cdot\mathbf{x}-k_0x_0)]\\ &+(1/k_0)\mathbf{k}\cdot\mathbf{U}_-^*(k)\exp[i(-\mathbf{k}\cdot\mathbf{x}+k_0x_0)]].\end{aligned} \tag{51}$$

If we define a spatial vector $\mathbf{V}_0$ with components U_{0k} ($k=1, 2, 3$) and a second vector $\mathbf{V}$ with components U_{23}, U_{31}, U_{21}, we find from (35)

$$\begin{aligned}\mathbf{V}_0^*(x, x_0)=(V)^{-\frac{1}{2}}\textstyle\sum_k \frac{i}{\sqrt{2}}[&\{-k_0\mathbf{U}_+^*(k)+(\mathbf{k}/k_0)(\mathbf{k}\cdot\mathbf{U}_+^*(k))\}\exp(i[-\mathbf{k}\cdot\mathbf{x}+k_0x_0])\\ &+\{k_0\mathbf{U}_-(k)-(\mathbf{k}/k_0)(\mathbf{k}\cdot\mathbf{U}_-(k))\}\exp(i[\mathbf{k}\cdot\mathbf{x}-k_0x_0])],\\ \mathbf{V}_0(x, x_0)=(V)^{-\frac{1}{2}}\textstyle\sum_k \frac{i}{\sqrt{2}}[&\{k_0\mathbf{U}_+(k)-(\mathbf{k}/k_0)(\mathbf{k}\cdot\mathbf{U}_+(k)\}\exp(i[\mathbf{k}\cdot\mathbf{x}-k_0x_0])\\ &+\{-k_0\mathbf{U}_-^*(k)+(\mathbf{k}/k_0)(\mathbf{k}\cdot\mathbf{U}_-^*(k))\}\exp(i[-\mathbf{k}\cdot\mathbf{x}+k_0x_0])],\end{aligned} \tag{52}$$

$$\begin{aligned}\mathbf{V}^*(x, x_0)=V^{-\frac{1}{2}}\textstyle\sum_k \frac{i}{\sqrt{2}}[&-[\mathbf{k}\times\mathbf{U}_+^*(k)]\exp(i[-\mathbf{k}\cdot\mathbf{x}+k_0x_0])\\ &+[\mathbf{k}\times\mathbf{U}_-(k)]\exp(i[\mathbf{k}\cdot\mathbf{x}-k_0x_0])],\\ \mathbf{V}(x, x_0)=(V)^{-\frac{1}{2}}\textstyle\sum_k \frac{i}{\sqrt{2}}[&[\mathbf{k}\times\mathbf{U}_+(k)]\exp[i(\mathbf{k}\cdot\mathbf{x}-k_0x_0)]-[\mathbf{k}\times\mathbf{U}_-^*(k)]\exp(i[-\mathbf{k}\cdot\mathbf{x}+k_0x_0])].\end{aligned} \tag{53}$$

For the energy, momentum, and charge, we obtain from (41), (42), (43), with the abbreviations

$$N_+(k)=k_0(\mathbf{U}_+^*(k)\cdot\mathbf{U}_+(k))-(1/k_0)(\mathbf{k}\cdot\mathbf{U}_+^*(k))(\mathbf{k}\cdot\mathbf{U}_+(k)), \tag{54}$$

$$N_-(k)=k_0(\mathbf{U}_-^*(k)\cdot\mathbf{U}_-(k))-(1/k_0)(\mathbf{k}\cdot\mathbf{U}_-^*(k))(\mathbf{k}\cdot\mathbf{U}_-(k)), \tag{55}$$

the results,

$$E=\sum_k k_0[N_+(k)+N_-(k)], \tag{56}$$

$$\mathbf{G}=\sum_k \mathbf{k}[N_+(k)+N_-(k)], \tag{57}$$

$$e=\epsilon \sum_k [N_+(k)-N_-(k)]. \tag{58}$$

The expressions $N_+(k)$ and $N_-(k)$ are bilinear forms in the three components of $\mathbf{U}$ and $\mathbf{U}^*$. They can be brought into diagonal form and normalized if the $\mathbf{U}$ and $\mathbf{U}^*$ are divided into a component parallel to k (longitudinal vibration) and two components perpendicular to k (transverse vibration). Let e_1 and e_2 be two complex orthogonal unit vectors normal to k

$$(\mathbf{e}_r\cdot\mathbf{e}_s)=\delta_{rs}, \quad (\mathbf{e}_r\cdot\mathbf{k})=(\mathbf{e}_r^*\cdot\mathbf{k})=0 \quad (r, s=1, 2).$$

If we set

$$\begin{aligned} \mathbf{U}_\pm(k)&=(k_0)^{-\frac{1}{2}}\sum_{r=1,2}\mathbf{e}_r U_{r,\pm}(k)+\frac{(k_0)^{\frac{1}{2}}}{\kappa}\frac{\mathbf{k}}{|k|}U_{3,\pm}(k),\\ \mathbf{U}_\pm^*(k)&=(k_0)^{-\frac{1}{2}}\sum_{r=1,2}\overset{*}{\mathbf{e}}_r \overset{*}{U}_{r,\pm}(k)+\frac{(k_0)^{\frac{1}{2}}}{\kappa}\frac{\mathbf{k}}{|k|}\overset{*}{U}_{3,\pm}(k), \end{aligned} \tag{59}$$

the $N_+(k)$ and $N_-(k)$ appear in normal form

$$N_+(k)=\sum_{r=1}^{3}\overset{*}{\mathbf{U}}_{r,+}(k)\cdot\mathbf{U}_{r,+}(k); \quad N_-(k)=\sum_{r=1}^{3}\overset{*}{U}_{r,-}(k)\cdot U_{r,-}(k). \tag{60}$$

This is simply the transformation to the principal axis.

(*c*) *Quantization.*—Before we formulate the commutation relations we shall point out a difference between the special case $\kappa=0$—the electrodynamics—and the one in which we are interested. In the electrodynamics it is usual to quantize the vector components U_i as independent scalars in accord with an immediate generalization of (21)

$$i[U_i(x, x_0), U_k^*(x', x_0')]=i[U_i^*(x, x_0), U_k(x', x_0')]=\delta_{ik}D(x-x', x_0-x_0').$$

However, the relation (39) must then be introduced as an auxiliary condition in the form

$$(\partial U_k/\partial x_k)\Psi=0.$$

The operator $\partial U_k/\partial x_k$ on the left of this relation need not commute with all other quantities but gives zero when applied to the Schrödinger function Ψ. But it is required of an auxiliary condition that it commute with its conjugate complex at different space-time points. In our case a simple calculation shows that

$$i[\sum_k (\partial U_k/\partial x_k)_{x,x_0}, \sum_k (\partial U_k^*/\partial x_k)_{x',x_0'}]=-\Box D(x-x', x_0-x_0').$$

However, $\Box D=\kappa^2 D$, and for $\kappa\neq 0$ the right side is not zero. We have the result, therefore, that for non-vanishing rest-mass the commutation relations for the U_i cannot be the same as those for independent scalars.

The simplest method for getting a consistent second quantization in the case $\kappa\neq 0$ (which we now expressly assume) is to formulate the commutation relations in such a way that not only the wave Eq. (38) but also the supplementary Eq. (39) is identically satisfied as equations in q numbers. Such a formulation is the following:

$$i[U_i(x, x_0), U_k^*(x', x_0')]=i[U_i^*(x, x_0), U_k(x', x_0')]=\left(\delta_{ik}-\frac{1}{\kappa^2}\frac{\partial^2}{\partial x_i\partial x_k}\right)D(x-x', x_0-x_0'). \tag{61}$$

From this it follows that

$$[\partial U_i/\partial x_i,\ U_k^*(x', x_0')]=0 \quad \text{as} \quad (\Box-\kappa^2)D=0.$$

The bracket symbols which have not been explicitly written $[U_i(x, x_0),\ U_k(x', x_0')]$ and $[U_i^*(x, x_0),\ U_k^*(x', x_0')]$ must vanish. From (61) we get furthermore

$$[U_{ik}(x, x_0),\ U_r^*(x', x_0')]=i[U_{ik}^*(x, x_0),\ U_r(x', x_0')]=\left(\delta_{kr}\frac{\partial}{\partial x_i}-\delta_{ir}\frac{\partial}{\partial x_k}\right)D(x-x',\ x_0-x_0'), \quad (62)$$

$$i[U_{ik}(x, x_0),\ U_{rs}^*(x', x_0')]=i[U_{ik}^*(x, x_0),\ U_{rs}(x', x_0')] \quad (63)$$

$$=\left[\delta_{kr}\frac{\partial^2}{\partial x_i\partial x_s}-\delta_{ir}\frac{\partial^2}{\partial x_k\partial x_s}-\delta_{ks}\frac{\partial^2}{\partial x_i\partial x_r}+\delta_{is}\frac{\partial^2}{\partial x_k\partial x_r}\right]D(x-x',\ x_0-x_0'). \quad (63)$$

Note that (61) leads to expressions for $[U_4(x),\ \mathbf{U}^*(x')]$ and for $\{U_4(x),\ [\partial U_4^*(x')]/\partial x_4\}$ with $x_0=x_0'$ which are different from zero in contradistinction to the results obtained from the "canonical" commutation relations.

Stuckelberg[14] has given a variant of the above formulation. He introduces two auxiliary fields—a vector A_i and a scalar B_0—which satisfy the supplementary condition

$$(\partial A_i/\partial x_i+\kappa B)\Psi=0.$$

If we treat the A_i and B as independent scalars with respect to their commutation relations, we get

$$i[A_i(x, x_0),\ A_k^*(x', x_0')]=\delta_{ik}D(x-x',\ x_0-x_0'),$$

$$i[B(x, x_0),\ B^*(x', x_0')]=D(x-x',\ x_0-x_0').$$

From this we have

$$i[(\partial A_i/\partial x_i+\kappa B)_{x,\, x_0},\ (\partial A_i^*/\partial x_i+\kappa B^*)_{x',\, x_0'}]=0.$$

Thus the supplementary condition is consistent. Furthermore this makes the total energy positive provided the Lagrange function consists of a sum of contributions from the independent field components A_i, B. Then the U_i which satisfy the relation

$$(\partial U_i/\partial x_i)\Psi=0$$

are given by

$$U_i=A_i+\frac{1}{\kappa}\frac{\partial B}{\partial x_i}.$$

This leads us back to the commutation relations (61) for the U_i. For the A_i and B there exist gauge-tra nsformations of the second type

$$A_i'=A_i+\partial f/\partial x_i;\quad B'=B-\kappa f,\quad \text{with}\quad \Box f-\kappa^2 f=0.$$

The U_i are invariants with respect to this transformation group.

The advantage of this method, as Stuckelberg shows, is that the interaction between the mesons described by such fields and protons and neutrons can be handled by a formalism which is completely analogous to that employed by Dirac[15] for the treatment of the interaction between light and electrons.

In the following, however, we shall not introduce the auxiliary fields, and shall treat the supplementary Eq. (39) simply as an identity.

[14] E. C. G. Stuckelberg, Helv. Phys. Acta **11**, 225–299 (1938).

[15] See P. A. M. Dirac, *Quantum Mechanics* (Oxford, 1935), second edition. In the case of mesons the supplementary conditions on A_i and B remain homogeneous even in the interaction with the heavy particles (this is not so in the analogous case for light) while on the other hand additional terms occur in U_0 which arise from the difference between differentiation with respect to the time of the meson field and with respect to general time.

The only arbitrariness which yet remains in (61) is associated with the possible introduction of a numerical factor on the right side. This is connected through Eq. (16) with the corresponding normalization of the numerical factor in the Hamiltonian operator. We show here that the normalization of (61) is in agreement with the use of (44) for the energy momentum tensor. For this purpose the decomposition of the fields into eigenstates is most suitable. In the calculation of the energy expression one must take into consideration the order of the factors. As can be seen from Eq. (12), the factors in the terms arising from the $\mathbf{U}(k)$ and $\mathbf{U}^*(k)$ appear in an order which is the reverse of that in Eq. (55). As is shown further by the comparison with (18), Eqs. (16) require that the commutation relations for the $U_r(k)$ and $U_r^*(k)$ be

$$[U_{r,+}(k),\ U_{s,+}^*(k)]=[U_{r,-}(k),\ U_{s,-}^*(k)]=\delta_{rs}\quad (r, s=1, 2, 3),\tag{64}$$

while all other brackets in the given quantities must vanish. From this, however, it follows that the $U(k)$ and $U^*(k)$ [see Eq. (50)] satisfy

$$[U_{i,+}(k),\ U_{k,r}^*(k)]=[U_{i,-}(k),\ U_{k,-}^*(k)]=\frac{1}{k_0}\left(\delta_{ik}+\frac{1}{\kappa^2}k_ik_k\right).\tag{65}$$

Introducing these results into Eqs. (50), (51), we get agreement with (61) for D given by (22).

We see, therefore, that the quantities $N_+(k)$ *and* $N_-(k)$ *defined through (65) give the number of particles with charge +1 and −1, respectively, and with momentum* $\mathbf{k}$. The sequence of factors in the energy expression leads, as in §1 (6) to a zero point energy of a half quantum per eigenstate.

As in the scalar theory it is impossible to quantize in accord with the exclusion principle for the function $[1-(\partial^2/\partial x_i^2)]D$ vanishes for $x_0=x_0'$ just as D does.

The distinction between longitudinal and transverse vibrations is lost in the rest system for the particle, i.e., for the case $k=0$. The introduction of the normal vibrations according to (59) is superfluous as well as singular since the second part of (65) vanishes. Moreover, as follows from (51) and as has already been mentioned, $U_0=0$. For a given sign of the frequency, therefore, we have in the rest system three characteristic solutions which can be transformed into one another by spatial rotations of the coordinate systems. The statement that the field theory under discussion, when quantized, describes particles of spin 1, and only such particles, is thus justified.

(*d*) *The c number theory when there is an external electromagnetic field.*—The general rule of I §2(*b*) for the extension of the field equations when an external field is present may be immediately applied if we begin with the variation principle. By means of the operators

$$D_k=\partial/\partial x_k-i\epsilon\varphi_k;\quad D_k^*=\partial/\partial x_k+i\epsilon\varphi_k,$$

we may now write the generalized Lagrange function (43) as

$$L=-\tfrac{1}{2}U_{ik}^*U_{ik}+\tfrac{1}{2}U_{ik}^*(D_iU_k-D_kU_i)+\tfrac{1}{2}U_{ik}(D_i^*U_k^*-D_k^*U_i^*)+\kappa^2U_i^*U_i.\tag{66}$$

We get from the variation principle

$$U_{ik}=D_iU_k-D_kU_i,\tag{67}$$

$$D_kU_{ik}+\kappa^2U_i=0,\tag{68}$$

in place of (40) and (41). Using the fact that

$$D_iD_k-D_kD_i=-i\epsilon f_{ik},$$

we now get, on applying the operator D_i to (68) and summing over i,

$$\kappa^2D_iU_i-\tfrac{1}{2}i\epsilon f_{ik}U_{ik}=0;\tag{69}$$

furthermore, we get in place of (42),

$$D_lU_{ik}+D_iU_{kl}+D_kU_{li}=-i\epsilon(f_{li}U_k+f_{ik}U_l+f_{kl}U_i).\tag{70}$$

Finally, the substitution of (67) into (68) and use of (69) leads to

$$\sum_k D_k^2U_i-\kappa^2U_i-i\epsilon f_{ik}U_k-\frac{i}{2}\frac{\epsilon}{\kappa^2}\frac{\partial f_{rs}}{\partial x_i}U_{rs}-\frac{i}{2}\frac{\epsilon}{\kappa^2}f_{rs}D_iU_{rs}=0.\tag{71}$$

It is to be noted that Eqs. (67), (68), which can be derived directly from the variation principle, are distinguished from (69), (70) and (71) through the fact that they contain no terms in which f_{ik} occurs explicitly.

Expression (49) for the current vector is unchanged in this theory except for the altered meaning of the U_{ik}; in accord with I, §2(b), we have for the energy tensor, instead of (44)

$$T_{ik}=U_{kr}^*(D_iU_r)+(D_i^*U_r^*)U_{kr}-L\delta_{ik}, \quad \langle 44\rangle$$

this can be transformed into the symmetric energy tensor θ_{ik} of Eq. (46) which, therefore, remains valid.

The discussion of the non-relativistic limiting case shows[16] that the particle described by the field possesses a magnetic moment which has the same ratio to the mechanical moment as for the classical rotating charge, namely $eh/2m_0c$. However, this result is not unique. It is possible to introduce new terms into the Lagrange function of the form

$$L'-L=\epsilon K\tfrac{1}{2}f_{ik}i(U_i^*U_k-U_k^*U_i), \quad (66')$$

in which K is a dimensionless factor. This leads to no change in (67), but (68) must be replaced by

$$D_kU_{ik}+\kappa^2U_i+Ki\epsilon f_{ik}U_k=0, \quad (68')$$

and the current vector according to I (26) becomes

$$s_k'=s_k-i\epsilon K\frac{\partial}{\partial x_l}(U_l^*U_k-U_k^*U_l). \quad \langle 49\rangle$$

For small velocities of the particle this change introduces a factor $(1+K)$ into the original expression for the magnetic moment; the moment, therefore, can be given an arbitrary value.

(e) *Remarks on real fields and the special case of vanishing rest-mass.*—The transition to real fields is accomplished by means of the scheme of §1(d); we put $U_k=(1/\sqrt{2})(V_k+iW_k)$ where V_k and W_k are real. The theory with a single real vector is obtained by identifying U_k and U_k^* in a position space and in the momentum space by identifying the $\mathbf{U}_+(k)$ and $\mathbf{U}_-(k)$. Instead of $N_+(k)$ and $N_-(k)$ one obtains in the expression for the energy and momentum only the one number

$$N(k)=k_0(\mathbf{U}^*(k)\cdot\mathbf{U}(k))-\frac{1}{k_0}(\mathbf{k}\cdot\mathbf{U}^*(k)(\mathbf{k}\cdot\mathbf{U}(k)).$$

[16] A. Proca, J. de phys. et rad. [7] **9**, 61 (1938).

The current vector and the charge vanish. Recently this possibility has been introduced for the description of neutral mesons.[17]

An important and in a certain sense a singular special case is that of vanishing rest-mass $m=0$. As is well known, this case includes the quantum electrodynamics. The Lagrange function and the energy tensor depend only on the U_{ik}. The equations derived from the action principle

$$U_{ik}=(\partial U_k/\partial x_i)-(\partial U_i/\partial x_k), \quad (40')$$

$$\partial U_{ik}/\partial x_i=0, \quad (41')$$

together with the U_{ik} remain invariant if an arbitrary gradient is added to U_k; or, in other words, in gauge transformations of the second kind

$$U_k\rightarrow U_k+(\partial f/\partial x_k).$$

This transformation seems to establish a fundamental and qualitative difference between the cases $\kappa=0$ and $\kappa\neq 0$. On this basis an assumption to the effect that the photons have a very small but finite rest mass seems to be physically unsatisfactory. Since a gauge transformation of the first kind could not be applied to the photon field if $\kappa=0$, the gauge transformation of the second kind with phase factor which depends arbitrarily on space and time, $\exp[i\alpha(x, x_0)]$ would no longer be possible for electron and proton fields.

We mention here the possibility of a complex field for the case $\kappa=0$; a current vector s_k is then defined by (49). The current vector would not then be invariant with respect to the substitution (72) but the volume integrated total charge would be. (This can be seen by use of Eq. ⟨49⟩.) However, we know of no case of $\kappa=0$ and integral spin which requires for its description a complex field (or *two* real fields). We shall, therefore, assume that the (U_i, U_{ik}) field is real and identify it with the photon field (φ_i, f_{ik}).

The singularity of the case $\kappa=0$ exhibits itself in the c number theory through the fact that for this case Eq. (39) and the wave equation of the second order (38) for U_i are no longer consequences of Eqs. (40′) and (41′). In the q number theory the commutation relations (61) become singular. There are two methods for

[17] N. Kemmer, Proc. Camb. Phil. Soc. **34**, 354 (1938) (Part III).

formulating the theory when $\kappa=0$. One consists in introducing no commutation relations for quantities which are not invariant with respect to the substitution (72) and retaining the commutation relations (63) for the field strengths with unrestricted validity of the gauge group. The other method was developed by Fermi[18] and has certain advantages in calculations on the interaction between light and charged particles. In this theory the supplementary condition (already mentioned in (*c*))

$$(\partial\varphi_k/\partial x_k)\Psi=0 \qquad \langle 39\rangle$$

for the state under consideration, and the wave equation of the second order for the vector potential

$$\Box\varphi_k=0 \qquad \langle 38\rangle$$

are introduced as q number relations. The latter equations introduce a limitation on the gauge group to such f as satisfy the second order wave equation

$$\Box f=0. \qquad (72)$$

However, this limitation makes it possible to require the following simple commutation relations for the φ_i:

$$i[\varphi_i(x,x_0),\varphi_k(x',x_0')] =\delta_{ik}D(x-x',x_0-x_0'). \qquad (61\text{ bis})$$

It should be noted that in analysis into eigenvibrations which was made in *b*) we now have $k_0=|k|$ and

$$N(k)=|k|(U(k))^2-\frac{1}{|k|}(\mathbf{k}\cdot\mathbf{U}(k))^2=|k|(U_\perp(k))^2,$$

where

$$U_\perp(k)=\mathbf{U}(k)-\frac{\mathbf{k}}{|k|}(\mathbf{k}\cdot\mathbf{U}(k))$$

is the component of $\mathbf{U}$ normal to $\mathbf{k}$. Only the two transverse vibrations associated with $r=1, 2$, appear in the energy expression, and for a given $\mathbf{k}$ there are only two physically different states.

As mentioned in Part I the spin of the photon is exhibited through the fact that the lowest eigenvalue of the square of the total angular momentum $j(j+1)$ for the state of a single photon is given by $j=1$ rather than $j=0$.[19]

3. Dirac's positron theory (spin $\frac{1}{2}$)

(*a*) *The c number theory.*—In the Dirac wave equation of the electron

$$\gamma_k(\partial u/\partial x_k)+\kappa u=0, \qquad (73)$$

there occur the familiar 4-rowed matrices γ_k ($k=1, \cdots, 4$) which satisfy the relations

$$\tfrac{1}{2}(\gamma_i\gamma_k+\gamma_k\gamma_i)=\delta_{ik}. \qquad (74)$$

As is well known, the u defined by Eq. (73) satisfies the wave equation

$$\Box u-\kappa^2u=0. \qquad (73\text{a})$$

We introduce in addition the adjoint functions $u^\dagger$ which satisfy

$$(\partial u^\dagger/\partial x_k)\gamma_k-\kappa u^\dagger=0. \qquad (73^\dagger)$$

The functions u have four components u_ρ ($\rho=1, \cdots, 4$). We shall use $(\gamma_k u)_\rho$ and $(u^\dagger\gamma_k)_\sigma$ as abbreviations for $\sum_\sigma \gamma_{k,\rho\sigma}u_\sigma$ and $\sum_\rho u_\rho^\dagger\gamma_{k,\rho\sigma}$, respectively. The Lorentz invariance of the system of Eqs. (73) with given γ_k requires that for the orthogonal substitution

$$x_i'=\textstyle\sum_k a_{ik}x_k,$$

there exists a similarity transformation of the γ_i which has the property

$$\Lambda^{-1}\gamma_i\Lambda=\textstyle\sum_k a_{ik}\gamma_k. \qquad (75)$$

The u_ρ and $u_\rho^\dagger$ are transformed according to

$$u'=\Lambda u, \qquad (76)$$

$$u^{\dagger\prime}=\pm u^\dagger\Lambda^{-1}. \qquad (76^\dagger)$$

We shall not give a proof of the existence of Λ for Lorentz transformations.[20] We note simply that the matrix Λ is defined by (75) only to within a constant factor. We shall limit this factor to the four roots of unity ±1; $\pm i$ by the additional requirement

$$\text{Det}\,\Lambda=1. \qquad (75\text{a})$$

For the proper (continuous) Lorentz group the

[18] E. Fermi, Rev. Mod. Phys. **4**, 125 (1932); P. A. M. Dirac, *Quantum Mechanics* (Oxford, 1935), second edition.

[19] W. Pauli, *Handbuch der Physik*, Vol. 24/1, p. 259.

[20] For proofs see P. A. M. Dirac, *Quantum Mechanics* (Oxford, 1935), second edition. W. Pauli, Inst. H. Poincaré, Ann **6**, 109 (1936).

+ sign is now uniquely determined in (76^+) as can be shown by a continuity argument with (75) and (75^+); for the reflections of the space coordinates or the time the factor ± 1 or $\pm i$ in Λ and the sign in (76^+) remains undefined. In the following it will prove to be useful to take the Λ as follows:

$$\Lambda = i\gamma_4 \qquad \text{for } x' = -x,\ x_4' = x_4. \tag{77a}$$

$$\Lambda = \gamma_1\gamma_2\gamma_3 \qquad \text{for } x' = x,\ x_4' = -x_4. \tag{77b}$$

$$\Lambda = i\gamma_1\gamma_2\gamma_3\gamma_4 \quad \text{for } x' = -x,\ x_4' = -x_4. \tag{77c}$$

This establishment—proposed by Racah[21]—is in agreement with (75).

The form (73) of the wave equation is useful for the discussion of the Lorentz invariance. Reality relations, however, become clearer when the equation is put into the form

$$(\partial u/\partial x_0) + \boldsymbol{\alpha}\cdot(\partial u/\partial \mathbf{x}) + i\kappa\beta u = 0, \tag{78}$$

where

$$\alpha_k = i\gamma_4\gamma_k \text{ for } k = 1, 2, 3 \quad \text{and} \quad \beta = \gamma_4. \tag{79}$$

From (74) we see that it is permissible to take the γ_k and therefore the α and β as Hermitian matrices: we shall always so take them in the following. The complex conjugate of Eq. (78) is therefore

$$\frac{\partial u^*}{\partial x_0} + \frac{\partial u^*}{\partial \mathbf{x}}\cdot\boldsymbol{\alpha} - i\kappa u^*\beta = 0. \tag{74*}$$

Comparison with (73^+) shows that we may put[22]

$$u^\dagger = u^*\gamma_4. \tag{80}$$

From $u^{\dagger\prime} = u^{*\prime}\gamma_4$ and (76) it follows now that

$$u^{\dagger\prime} = u^\dagger(\gamma_4\Lambda^\dagger\gamma_4),$$

where by $\Lambda^\dagger$ we understand the Hermitian conjugate of Λ. After comparing this with (76^+) and (77) the sign in (76^+) can be made more precise:

$$u^{\dagger\prime} = +u^\dagger\Lambda^{-1} \quad \text{for fixed direction of time,} \qquad \langle 72\rangle$$

$$u^{\dagger\prime} = -u^\dagger\Lambda^{-1} \quad \text{for reversal of direction of time,} \qquad \langle 72^+\rangle$$

if (76) and (80) hold with the + sign without exception.

[21] G. Racah, Il Nuovo Cimento **14**, 322 (1937).

[22] Usually a factor i is employed on the right side of (80). We prefer, however, not to follow this procedure in order to make the i appear in the expression for the current vector.

The wave Eqs. (73) and (78) can be derived from a Lagrange function

$$\begin{aligned} L &= \frac{1}{2}\left(u^\dagger\gamma_k\frac{\partial u}{\partial x_k} - \frac{\partial u^\dagger}{\partial x_k}\gamma_k u\right) + \kappa u^\dagger u \\ &= \frac{1}{2i}\left[\left(u^*\frac{\partial u}{\partial x_0} - \frac{\partial u^*}{\partial x_0}u\right)\right. \\ &\qquad \left. + \left(u^*\boldsymbol{\alpha}\cdot\frac{\partial u}{\partial \mathbf{x}} - \frac{\partial u^*}{\partial \mathbf{x}}\cdot\boldsymbol{\alpha}u\right)\right] + \kappa u^*\beta u. \end{aligned} \tag{81}$$

Incidentally L vanishes if the wave equations are satisfied. Using Eq. I (19) we get for the current vector

$$s_k = \epsilon i u^\dagger\gamma_k u \tag{82}$$

or

$$s_0 = \epsilon u^* u, \quad \mathbf{s} = \epsilon u^\dagger\boldsymbol{\alpha}u, \tag{82a}$$

and for the canonical energy tensor, remembering that $L = 0$,

$$T_{ik} = \frac{1}{2}\left(u^\dagger\gamma_k\frac{\partial u}{\partial x_i} - \frac{\partial u^\dagger}{\partial x_i}\gamma_k u\right). \tag{83}$$

This tensor is not symmetric but by a familiar transformation,[23] which is contained as a special case in Eqs. I, (13c), (14), we get

$$\theta_{ik} = \tfrac{1}{2}(T_{ik} + T_{ki}).$$

This satisfies the continuity equation and leads to the same results for the volume integrated total momentum as does the canonical tensor.

For the energy density and momentum density we obtain from (83) and (80)

$$W = -T_{44} = \frac{1}{2i}\left(-u^*\frac{\partial u}{\partial x_0} + \frac{\partial u^*}{\partial x_0}u\right), \tag{83a}$$

$$\mathbf{G} = \frac{1}{2i}\left(u^*\frac{\partial u}{\partial \mathbf{x}} - \frac{\partial u^*}{\partial \mathbf{x}}u\right). \tag{83b}$$

It is important to note that the charge density is positive definite while the energy exhibits two different signs.

As a consequence of the substitutions (77) the current vector behaves like an ordinary vector with respect to spatial reflections; the component s_4 does not change sign, however, when

[23] W. Pauli, *Handbuch der Physik*, Vol. 24/1, p. 235.

the sign of the time coordinate is changed whereas the space components s_k do then experience a change in sign. A reversal of the signs of all coordinates, therefore, leaves the s_k unchanged. On the other hand the energy tensor changes sign, as can also be seen from $\langle 76^+\rangle$ when the signs of all coordinates are changed.

We consider now the u_ρ which are plane waves with a definite propagation vector $(\mathbf{k}, k_0)$. From (73a) it again follows that $k_0^2=k^2+\kappa^2$. It is seen that for a given $\mathbf{k}$ and a given sign of k_0 there are two solutions of the wave Eqs. (73) or (78) which, in the rest system $\mathbf{k}=0$, may be transformed into one another by spatial rotations. The particles associated with these waves thus have the spin $\frac{1}{2}$.

We want now to investigate more closely the connection between the solutions

$$u_\rho=a_\rho^r(k)\exp(i[\mathbf{k}\cdot\mathbf{x}-k_0x_0])\text{ for } r=1,2 \tag{84a}$$

and

$$u_\rho=b_\rho^r(k)\exp(i[-\mathbf{k}\cdot\mathbf{x}-k_0x_0])\quad\text{for } r=1,2, \tag{84b}$$

where by k_0 we mean, as in §§1 and 2, the positive quantity,

$$k_0=+(k^2+\kappa^2)^{\frac{1}{2}}.$$

If the a_ρ^r are suitably normalized we may write

$$\textstyle\sum_\rho a_\rho^{*r}a_\rho^s=\delta_{rs};\quad \sum_\rho b_\rho^{*r}b_\rho^s=\delta_{rs}. \tag{85}$$

By means of the method of the annihilation operators, and with the help of the wave equation we find

$$\sum_{r=1,2} a_\rho^r a_\sigma^{*r}=\frac{1}{2k_0}(k_0+(\boldsymbol{\alpha}\cdot\mathbf{k})+\kappa\beta);$$

$$\sum_{r=1,2} b_\rho^r b_\sigma^{*r}=\frac{1}{2k_0}(k_0-(\boldsymbol{\alpha}\cdot\mathbf{k})-\kappa\beta). \tag{86}$$

Furthermore, there exists a Lorentz invariant ordering between the solutions with positive and negative frequency.[24] In order to show this we note that u_- satisfies the same wave equation as u_+, where

$$u_-^*=Cu_+,\quad u_+=C^{-1}u_-^*, \tag{87}$$

if

$$\beta^*=-C\beta C^{-1},\quad \alpha^*=C\alpha C^{-1}. \tag{88}$$

[24] See reference 11, Inst. H. Poincaré Ann., p. 130.

The matrices α^*, β^*, the complex conjugates of α, β, are defined by

$$(\alpha^*)_{\rho\sigma}=(\alpha_{\rho\sigma})^*=\alpha_{\sigma\rho};\quad (\beta^*)_{\rho\sigma}=(\beta_{\rho\sigma})^*=\beta_{\sigma\rho}.$$

Such a matrix as C actually exists since $-\beta^*$, α^* satisfy the same relation (74) as β, α and the γ_k. From (87) it is clear that C^*C commutes with all γ_k: it is thus a constant. We note without proof that the matrix C is symmetric where the γ_k are Hermitian[25]

$$C_{\sigma\rho}=C_{\rho\sigma}. \tag{88a}$$

It follows from this that the constant C^*C is positive. It is therefore possible to obtain

$$C^*C=1 \tag{88b}$$

by suitably choosing the arbitrary constant factor in C. There is a special representation of the γ_k for which α and $i\beta$ are real; the C is then the unit matrix.

There is an invariance of the ordering expressed in (87) with respect to Lorentz transformations if by virtue of (76) we can get from (87) to the corresponding equation for the primed functions, i.e., if

$$\Lambda^*C=C\Lambda\quad\text{or}\quad \Lambda^*=C\Lambda C^{-1}. \tag{89}$$

The proof of the validity of this relation has been given for proper Lorentz transformations in reference 25. As can be seen from (77) Λ is defined for all reflections so that (89) continues to be valid. Therefore the ordering indicated by (87) remains invariant with respect to all reflections.

Clearly a permitted specialization of the a_ρ^r is obtained by setting

$$b_\rho^{*r}(k)=\textstyle\sum_\sigma C_{\rho\sigma}a_\sigma^r(k);\quad a_\rho^{*r}(k)=\sum C_{\rho\sigma}b_\sigma^r(k). \tag{90}$$

Following Kramers,[26] we can speak of the two solutions associated through (87) as charge conjugate solutions. The terminology is justified by a consideration of the effects of an external electromagnetic field. This can be done according to Part I, §2(d) by the substitution of $(\partial/\partial x_k)\rightarrow D_k$ in the wave Eq. (73) or (78). If u_+

[25] Inst. H. Poincaré Ann. **6**, 109 (1936), p. 121ff., and p. 130.

[26] H. A. Kramers, Proc. Amst. Akad. Sci. **40**, 814 (1937). The concept of the charge conjugate solutions may be generalized for higher arbitrary spin values. We cannot, however, go into this matter here.

satisfies the wave equation with the charge $+e$, then u_- satisfies it with the charge $-e$. It appears, on the other hand, that the current vector (82) retains its sign in the charge conjugate states. However, we shall see that this defect is removed in the q number theory.

In this connection it is of interest, in analogy to the procedure of Section 2(d) to add terms of the form

$$L'-L=l\tfrac{1}{2}u^\dagger\gamma_i\gamma_k u f_{ik}$$

to the Lagrange function; the f_{ik} are the external field strengths in natural units (I, §1) and l is of the dimension of a length. Without the additional terms we have, as is well known, the value $e/2\kappa$ for the magnetic moment. We then get in an external field the modified wave equation

$$\gamma_k D_k u+\kappa u+\tfrac{1}{2}il f_{ik}\gamma_i\gamma_k u=0. \qquad (91)$$

For small particle velocity we are led to an additional term in the magnetic moment which has the form $-l(\hbar c)^{\frac{1}{2}}$. By Eq. I (27) we see that the additional term in the wave equation requires a new term in the current. For the new current we have

$$s_k'=\epsilon i u^\dagger\gamma_k u+l\frac{\partial}{\partial x_l}(u^\dagger i\gamma_k\gamma_l u). \qquad (92)$$

It is noteworthy that for the electron the magnetic moment is just $\frac{1}{2}(e/\kappa)$ so that the additional term is unnecessary. The situation is different, however, for both proton and neutron. The magnetic moment of the latter must be obtained from the new term alone for, as $\epsilon=0$, the substitution of D_k for $\partial/\partial x_k$ is unnecessary. It is also important that in going to the charge conjugate solution the sign of l must change together with that of ϵ; these solutions are thus also conjugate with respect to the magnetic moment of the particle (see I, §2). It should be noted that the additional term introduces a new constant with the dimension of a length into the theory. For a discussion of the consequences, for the case of spin 1, of additional terms in the Lagrangian and the resulting terms in the current vector, see §2(d).

(b) *Quantization in accord with the exclusion principle.*—We have seen that for spin $\frac{1}{2}$ the energy in the c number theory is not positive definite; there are just as many negative as positive energy eigenvalues. This condition would be unchanged through the introduction of Einstein-Bose quantization. Dirac has pointed out, however, that the difficulties of the negative energy states are eliminated through a change in the definition of the vacuum if quantization in accord with the exclusion principle is introduced. The vacuum is defined as the state with the smallest energy among those having values for the occupation numbers of the states which are compatible with the exclusion principle. This limits the values of the occupation numbers of a non-degenerate eigenstate to 0 and 1. The vacuum therefore is defined as that total state in which all individual negative energy states are occupied. The absence of a particle from a negative energy state, a so-called hole, then behaves, relative to the newly defined vacuum, as a particle with positive energy and opposite charge to the original.

This formulation of the Dirac theory of holes is not entirely symmetric with respect to the two particles of opposite charge. We shall follow below a formalism proposed by Heisenberg[27] which expresses the same physical content in more symmetric manner.

For this purpose we begin with the formulation of the commutation relation for the wave functions. When the exclusion principle applies, we must consider, according to Jordan and Wigner, the bracket

$$[u_\rho(x,x_0),\,u_\sigma^*(x',x_0')]_+\equiv u_\rho(x,x_0)_\rho u_\sigma^*(x',x_0')+u_\sigma^*(x',x_0')u_\rho(x,x_0).$$

It is to be noted that not only the wave equation of the second order (73a) but also the wave equation of the first order (73) or (78) must be fulfilled by the right side of this expression. This is the case if

$$[u_\rho(x,x_0),\,u_\sigma^*(x',x_0')]_+=\left(\frac{\partial}{\partial x_0}I-\boldsymbol{\alpha}\cdot\frac{\partial}{\partial\mathbf{x}}-i\kappa\beta\right)_{\rho\sigma}D(x-x',\,x_0-x_0'), \quad (93)$$

in which D is the function belonging to the wave equation of the second order with rest-mass κ

[27] W. Heisenberg, Zeits. f. Physik **90**, 209, and **92**, 692 (1934).

which is defined in Eqs. (24), (25). By making use of the wave function $u^\dagger$ the matrices γ_k and Eqs. (79) and (80), we may rewrite (93) in the form

$$i[u_\rho(x, x_0), u_\sigma^\dagger(x', x_0')]_+ = \left(-\gamma_k\frac{\partial}{\partial x_k}+\kappa I\right)_{\rho\sigma} D(x-x', x_0-x_0'). \quad (94)$$

The wave equation of the first order is fulfilled since the operator $(\partial/\partial x_0)I+\boldsymbol{\alpha}\cdot(\partial/\partial\mathbf{x})+i\kappa\beta$ applied to (93) or the operator $\gamma_k(\partial/\partial x_k)+\kappa I$ to (94) produces the operator $-\Box+\kappa^2$, which annihilates the D function. The form (94) shows the relativistic co-variance of the expression which has been set up while the reality relations are most readily seen from (93). The consistency of expression (93) with the plus sign in the bracket is based essentially on the fact that first derivatives (more generally, derivatives of odd order would be possible) of the D function occur on the right. As a consequence the right side becomes an even function of $x-x'$, x_0-x_0', and for $x=x'$ and $x_0=x_0'$ the algebraic requirement that the left side be positive is satisfied. In fact we have for $x_0=x_0'$ according to (22″)

$$[u_\rho(x, x_0), u_\sigma^*(x', x_0')]_+=\delta_{\rho\sigma}\delta(x-x'). \quad (93a)$$

We now introduce the decomposition of the $u_\rho(x)$ into eigenvibrations.

$$u_\rho(x, x_0)=(V)^{-\frac{1}{2}}\textstyle\sum_k \sum_{r=1,2} \times\{u_+^r(k)a_\rho^r(k)\exp(i[\mathbf{k}\cdot\mathbf{x}-k_0x_0] + u_-^{*r}(k)b_\rho^r(k)\exp(i[-\mathbf{k}\cdot\mathbf{x}+k_0x_0)\},$$

$$u_\rho^*(x, x_0)=(V)^{-\frac{1}{2}}\textstyle\sum_k \sum_{r=1,2} \times\{u_+^{*r}(k)a_\rho^{*r}(k)\exp(i[-\mathbf{k}\cdot\mathbf{x}+k_0x_0] + u_-^{r}(k)b_\rho^{*r}(k)\exp(i[\mathbf{k}\cdot\mathbf{x}-k_0x_0])\},$$

in which $u_+^r(k)$ and $u_-^r(k)$ and their conjugates are q numbers, while the c number factors $a_\rho^r(k)$ and $b_\rho^r(k)$ are defined and normalized through (84) and (85). From (85) and the definition of the D function (22), the equivalence of (93a) with the bracket relation

$$[u_+^r(k), u_+^{*s}(k)]_+=[u_-^r(k), u_-^{*s}(k)]_+=\delta_{rs} \quad (95)$$

follows; all remaining brackets with the plus sign in these quantities vanish.

We carry further the idea of the Dirac theory of holes by introducing the following rule of Heisenberg for the sequence of factors which is to be used in transforming calculations in the c number theory to the q number theory. Let F be any Hermitian operator of the c number theory; then

$$u_\rho^*F_{\rho\sigma}u_\sigma=u^*Fu$$

is to be replaced by

$$\tfrac{1}{2}(u_\rho^*F_{\rho\sigma}u_\sigma-u_\sigma F_{\rho\sigma}u_\rho^*)=\tfrac{1}{2}(u^*Fu-uF^*u^*). \quad (96)$$

The last form is also correct in the sense of the operator calculus if F contains an Hermitian differential operator. For the associated operator *density* terms must eventually be added in which the differential operator acts on the first term and which in the integration over the volume gives the same as the terms in the expression. The application of this rule to the expressions (82a), (83a, b) for the energy momentum and charge leads at once to

$$E=\textstyle\sum_k k_0 \sum_{r=1,2}[\tfrac{1}{2}(u_+^{*r}u_+^r-u_+^ru_+^{*r}) + \tfrac{1}{2}(-u_-^ru_-^{*r}+u_-^{*r}u_-^r)],$$

$$\mathbf{G}=\textstyle\sum_k \mathbf{k} \sum_{r=1,2}[\tfrac{1}{2}(u_+^{*r}u_+^r-u_+^ru_+^{*r}) + \tfrac{1}{2}(-u_-^ru_-^{*r}+u_-^{*r}u_-^r)],$$

$$e=\epsilon\textstyle\sum_k \sum_{r=1,2}[\tfrac{1}{2}(u_+^{*r}u_+^r-u_+^ru_+^{*r}) + \tfrac{1}{2}(u_-^ru_-^{*r}-u_-^{*r}u_-^r)],$$

or by making use of (95) and the definitions

$$N_r^+(k)=u_+^{*r}(k)u_+^r(k), \quad N_r^-(k)=u_-^{*r}(k)u_-^r(k), \quad (97)$$

$$E=\textstyle\sum_k k_0 \sum_{r=1,2}[N_r^+(k)+N_r^-(k)-1]; \quad \mathbf{G}=\sum_k \mathbf{k}\sum_{r=1,2}[N_r^+(k)+N_r^-(k)-1], \quad (98a)$$

$$e=\epsilon\textstyle\sum_k \sum_{r=1,2}[N_r^+(k)-N_r^-(k)]. \quad (98b)$$

It is easy to see, furthermore, that on the basis of the Heisenberg rule, relations (16) with the ordinary bracket are fulfilled for all quantities which do not explicitly contain the time.

It is also to be noted in connection with the definitions (97) that for quantization with brackets with the plus sign in (95) the eigenvalues of both $u_r^* u_r$ and $u_r u_r^*$ are 0 and 1 and therefore both expressions can serve as definitions of the number of particles. The choice (97) is made so that the energy is smallest when all N vanish; thus this corresponds to the case of the vacuum. As a consequence we obtain a negative zero-point energy of the vacuum which amounts to a half quantum per eigenvibration.[28]

(c) *Decomposition with respect to charge conjugate functions. The case of a non-electric particle of spin* $\frac{1}{2}$.[29]—We first make a decomposition of our spinor field which corresponds exactly to the decomposition (33) of the scalar field U into its real and imaginary parts: we get

$$u=(1/\sqrt{2})(v+iw), \quad u^*=(1/\sqrt{2})C(v-iw), \quad (99)$$

where in analogy with (87) v and w fulfill the Lorentz invariant reality conditions

$$v^*=Cv, \quad w^*=Cw \quad (100)$$

and satisfy the same wave equations as u does. The inverse of (99) is

$$v=(1/\sqrt{2})(u+C^*u^*), \quad w=(1/\sqrt{2})\frac{1}{i}(u-C^*u^*). \quad (99a)$$

The plus brackets ($[\ ]_+$) between v and w vanish and we have

$$[v_\rho(x,x_0), v_\sigma(x',x_\sigma')]_+=[w_\rho(x,x_0), w_\sigma(x',x_0')]_+ = C_{\sigma\tau}^*[u_\rho(x,x_0), u_\tau^*(x',x_0')]_+ \quad (101)$$

for the right side the value (93) can be introduced. (Because of the properties (88a, c) of C, the right side is in fact symmetric with respect to the interchange of x, x_0 with x', x_0' and of ρ with σ.) In particular, for $x_0=x_0'$ we have

$$[v_\rho(x,x_0), v_\sigma(x',x_\sigma')]_+=[w_\rho(x,x_0), w_\sigma(x',x_0')] = C_{\sigma\rho}^*\delta(x-x'). \quad (101a)$$

The Heisenberg rule now replaces the $u_\rho^* F_{\rho\sigma} u_\sigma$ with Hermitian F of the c number theory by the expression

$$\tfrac{1}{4}v(CF-F^*C)v+\tfrac{1}{4}w(CF-F^*C)w + \frac{i}{4}v(CF+F^*C)w-\frac{i}{4}w(F^*C+CF)v. \quad (102)$$

The application of this to the vector current where F is I, α leads as a consequence of (88) to the vanishing of the (v, v) and (w, w) terms; thus

$$s_0=\tfrac{1}{2}(vCw-wCv); \quad \mathbf{s}=\frac{i}{2}(vC\boldsymbol{\alpha}w-w\boldsymbol{\alpha}Cv).$$

For the energy and momentum densities we have $F=-(1/i)(\partial/\partial x_0)$ and $F=(1/i)(\partial/\partial \mathbf{x})$, respectively. In these cases the mixed terms vanish and we are left with

$$W=\frac{1}{i}\frac{1}{4}\left[\left(vC\frac{\partial v}{\partial x_0}-\frac{\partial v}{\partial x_0}Cv\right)+\left(wC\frac{\partial w}{\partial x_0}-\frac{\partial w}{\partial x_0}Cw\right)\right], \quad (103)$$

$$\mathbf{G}=\frac{1}{i}\frac{1}{4}\left[\left(vC\frac{\partial v}{\partial \mathbf{x}}-\frac{\partial v}{\partial \mathbf{x}}Cv\right)+\left(wC\frac{\partial w}{\partial \mathbf{x}}-\frac{\partial w}{\partial \mathbf{x}}Cw\right)\right]. \quad (104)$$

The transition to the charge conjugate state is realized by the substitution

$$v\rightarrow v, \quad w\rightarrow -w. \quad (105)$$

The current vector changes its sign properly for this transformation while the energy and momentum are unchanged. In the c number theory just the reverse would have happened since there the current vector would have the (v, v) and (w, w) terms and the energy and momentum the (v, w) terms.

The decomposition into eigenstates is simply performed if we require the condition (90) for the a_ρ^{*r}, b_ρ^{*r} and decompose the quantities $u_-^r(k)$

[28] The concept of the energy density seems to be more problematic in this theory than that of the volume integrated total energy. The energy density is no longer positive definite for the theory of holes, in contradistinction to the case for the theories discussed in §§1 and 2. This is also shown in the c number theory; even if limitation is made to wave packets in which the partial waves all have the same sign of the frequency in the phase exp $i[\mathbf{k}\cdot\mathbf{x}-k_0x_0]$ the energy density (as distinguished from the total energy) cannot be made positive definite.

[29] This theory was first developed by E. Majorana [Il Nuovo Cimento **14**, 171 (1937)] in which use is made of the special representation of the Dirac matrices with α real and $C=I$ which is mentioned above. For the general case see G. Racah, reference 21, and H. A. Kramers, reference 26.

and $u_+^r(k)$ in accordance with

$$\left.\begin{aligned} u_+^r&=\frac{1}{\sqrt{2}}(v^r+iw^r)\,; & u_-^r&=\frac{1}{\sqrt{2}}(v^r-iw^r)\\ u_+^{*r}&=\frac{1}{\sqrt{2}}(v^{*r}-iw^{*r})\,; & u_-^{*r}&=\frac{1}{\sqrt{2}}(v^{*r}+iw^{*r})\end{aligned}\right\}\quad(106)$$

$$\left.\begin{aligned}[v^r, v^{*s}]_+&=[w^r, w^{*s}]_+=\delta_{rs}\\ [v^r, w^{*s}]_+&=[w^r, v^{*s}]_+=0.\end{aligned}\right\}\quad(107)$$

We then have

$$\begin{aligned} v_\rho(x)&=(V)^{-\frac{1}{2}}\textstyle\sum_k\sum_{r=1,2}\\ &\quad\times v^r(k)a_\rho{}^r(k)\exp\left[i(\mathbf{k}\cdot\mathbf{x}-k_0x_0)\right]\\ &\quad+v^{*r}(k)b_\rho{}^r(k)\exp\left(-i[\mathbf{k}\cdot\mathbf{x}-k_0x_0]\right)\},\quad(108)\\ w_\rho(x)&=(V)^{-\frac{1}{2}}\textstyle\sum_k\sum_{r=1,2}\\ &\quad\times\{w^r(k)a_\rho{}^r(k)\exp\left[i(\mathbf{k}\cdot\mathbf{x}-k_0x_0)\right]\\ &\quad+w^{*r}(k)b_\rho{}^r(k)\exp\left(-i[k\cdot\mathbf{x}-k_0x_0]\right)\}.\end{aligned}$$

The energy, momentum and charge become

$$E=\textstyle\sum_k\sum_{r=1,2}k_0(v^{*r}v^r+w^{*r}w^r-1),\quad(109\text{a})$$

$$\mathbf{G}=\textstyle\sum_k\sum_{r=1,2}\mathbf{k}(v^{*r}v^r+w^*w^r-1),\quad(109\text{b})$$

$$e=\epsilon\textstyle\sum_k\sum_{r=1,2}i(v^rw^{*r}-w^rv^{*r}).\quad(109\text{c})$$

The Majorana abbreviation of the theory through the identification of the charge conjugate states is obtained by striking out the part $w(x)$ and its bracket relations, retaining therefore only the first half of the relations (100), (101a). Thus

$$W=\frac{1}{4i}\left(vC\frac{\partial v}{\partial x_0}-\frac{\partial v}{\partial x_0}Cv\right),\quad\langle 103\rangle$$

$$E=\textstyle\sum_k\sum_{r=1,2}k_0(v^{*r}(k)v^r(k)-\tfrac{1}{2}).\quad\langle 109\rangle$$

The current vector vanishes identically as does also the magnetic moment; thus the particle cannot be a source of an electromagnetic field at all. Obviously this possibility exists only in the q number theory with quantization in accord with the exclusion principle. It is not yet known whether the neutrino which plays a role in the theory of β-decay should be described by the abbreviated or unabbreviated theory.

The case $\kappa=0$ permits no gauge transformation of the second type for the case of spin $\frac{1}{2}$, just as for spin 0. Such transformations occur for $\kappa=0$ for spin 1 and higher spins.

4. A special synthesis of the theories for spin 1 and spin 0

We write the Eqs. (1) for the wave field of a particle without spin in a form which is analogous to that of (73).

$$\beta_k(\partial u/\partial x_k)+\kappa u=0.\quad(110)$$

The β_k in this equation are 5-rowed matrices; four rows operate on the vector U_k, the fifth on the scalar U. We shall represent the field components, $(\kappa)^{-\frac{1}{2}}U_k$, $(\kappa)^{\frac{1}{2}}U$ as U_ρ with $\rho=1, 2, \cdots, 5$, and use $\beta_k u$ as an abbreviation for $\sum\beta_{k,\rho\sigma}u_\sigma$. The factor $(\kappa)^{-\frac{1}{2}}$ before U_k and the $(\kappa)^{\frac{1}{2}}$ before U are introduced to make the equations more symmetric; in this notation u^*u has the dimensions of a reciprocal volume as in the Dirac theory.

Equations (41), (42) for the field in the case of spin 1 can also be written in the form (110). For this purpose the β_k must be four 10-rowed matrices four rows of which operate on the vector U_k and six on the skew-symmetric tensor U_{ik}. The field components $(\kappa)^{\frac{1}{2}}U_k$, $(\kappa)^{-\frac{1}{2}}U_{ik}$ are represented as u_ρ with $\rho=1, \cdots, 10$.

Duffin[30] has noticed the interesting fact that both the 5-rowed and the 10-rowed matrices fulfill the commutation relationships.

$$\beta_i\beta_k\beta_l+\beta_l\beta_k\beta_i=\delta_{ik}\beta_l+\delta_{kl}\beta_i.\quad(111)$$

The algebra generated by these relations can be studied independently of special representations of the hypercomplex numbers β just as was the case for the algebra based on Dirac's γ_i.

The four matrices β_k the unit matrix I, and all powers and products of the β_k generate 126 linearly independent quantities. [The number of independent powers and products is limited by (111).] A 5-rowed and a 10-rowed representation of this algebra have already been specified. Beside these there is a trivial 1-rowed representation in which the β_k are zero and I is just 1. These representations are irreducible in the

[30] R. J. Duffin, Phys. Rev. **54**, 1114 (1938).

algebraic sense; there are no other irreducible representations.[31]

If, therefore, we omit the trivial 1-rowed representation (to which, however, we shall return later in a special connection) Eqs. (111) contain nothing but the formulations of the heories for particles of spin 0 and 1 given in §§1 and 2. However, we shall, following Kemmer,[32] discuss the formalism a little further for it seems that it can be generalized to include higher values of the spins.

On forming the part of (111) which is skew-symmetric in k and l we get by making use of the definition,

$$s_{kl}=-s_{lk}=\beta_k\beta_l-\beta_l\beta_k, \tag{112}$$

the result

$$\beta_i s_{kl}-s_{kl}\beta_i=\delta_{ik}\beta_l-\delta_{il}\beta_k. \tag{113}$$

By means of this equation it ca i be shown that Eqs. (110) are Lorentz invariant. If the orthogonal transformation

$$x_i'=\sum a_{ik}x_k$$

for fixed β_k corresponds to the transformation

$$u'=\Lambda u,$$

this Λ must satisfy the relation

$$\Lambda^{-1}\beta_i\Lambda=\sum_k a_{ik}\beta_k,$$

which is analogous to (75). For infinitesimal transformations

$$x_k'=x_k+\sum_l \epsilon_{kl}x_l$$

$\epsilon_{kl}=-\epsilon_{lk}$ numerical coefficients

$\Lambda=I+\frac{1}{2}\sum_k\sum_l \epsilon_{kl}s_{kl}$ $\qquad s_{kl}=-s_{lk}$ matrices,

we obtain the relation (113).[33] We see that the s_{kl} defined through determine the behavior of the u_ρ in the infinitesimal transformations.

It is important for calculations with the β_k to note that these matrices have no reciprocals. From (111) we have as a special case

$$\beta_i\beta_k\beta_i=0 \text{ for } i\neq k \tag{111a}$$

and on the other hand

$$\beta_k^3=\beta_k. \tag{111b}$$

The matrices

$$\eta_k=2\beta_k^2-1 \tag{114}$$

have simple properties:

$$\eta_k^2=I, \quad \eta_i\eta_k=\eta_k\eta_i, \tag{115}$$

$$\beta_i\eta_k=-\eta_k\beta_i, \text{ for } i\neq k. \quad \beta_i\eta_i=\eta_i\beta_i. \tag{116}$$

If we assume, as is compatible with (111), that the β_k and therefore also the η_k are Hermitian (in analogy with the Dirac matrices) we can with the help of η_4 [in the Dirac theory (Eq. (80) γ_4 is used] define the functions $u^\dagger$ by

$$u^\dagger=u^*\eta_4, \tag{117}$$

which satisfy the equations

$$(\partial u^\dagger/\partial x_k)\beta_k-\kappa u^\dagger=0 \tag{110$^+$}$$

and which transform in proper Lorentz transformations as

$$u^{\dagger\prime}=u^\dagger\Lambda^{-1},$$

thus $(u^\dagger u)$ is an invariant with respect to these transformations. For the spatial reflections [see (77a)] $x'=-x'$, $x_4'=x_4$, there are two possibilities, namely $u'=\eta_4 u$ and $u'=-\eta_4 u$. The latter belongs to the dual theories mentioned [Eqs. (9), (37'), (38')] in §§1 and 2.

By means of the $u^\dagger$ it is easy to construct a Lagrange function

$$L=\tfrac{1}{2}\left(u^\dagger\beta_k\frac{\partial u}{\partial x_k}-\frac{\partial u^\dagger}{\partial x_k}\beta_k u\right)+\kappa u^\dagger u, \tag{118}$$

a current vector

$$s_k=iu^\dagger\beta_k u \tag{119}$$

and a canonical energy tensor

$$T_{ik}=\tfrac{1}{2}\left(u^\dagger\beta_k\frac{\partial u}{\partial x_i}-\frac{\partial u^\dagger}{\partial x_i}\beta_k u\right), \tag{120}$$

the s_k and T_{ik} satisfy the continuity equation.

A certain amount of calculation is required to pass from (110) to the wave equation of the second order. We multiply (110) by $\beta_l\beta_i(\partial/\partial x_l)$

[31] The system of the hypercomplex numbers β_i, I, and the powers and products of β_i is known to the algebraists as semi-simple. According to a general theorem relating to the dimensionality of the representations and the order of the system, we have $126=1^2+5^2+10^2$.

[32] N. Kemmer, Proc. Roy. Soc. **A173**, 91 (1939); F. Booth, A. H. Wilson, Proc. Roy. Soc. **A175**, 483 (1940); A. H. Wilson, Proc. Camb. Phil. Soc. **36**, 363 (1940).

[33] See in this connection, for example, the article on wave mechanics, *Handbuch der Physik*, Gl (A'), p. 222.

and obtain

$$\tfrac{1}{2}(\beta_l\beta_i\beta_k+\beta_k\beta_i\beta_l)\frac{\partial^2 u}{\partial x_k\partial x_l}+\kappa\beta_l\beta_i\frac{\partial u}{\partial x_l}=0,$$

or by means of (111),

$$\beta_k(\partial^2 u/\partial x_k\partial x_i)+\kappa\beta_l\beta_i(\partial u/\partial x_l)=0.$$

Using (110) again, therefore, gives

$$\partial u/\partial x_i=\beta_l\beta_i(\partial u/\partial x_l). \quad (121)$$

By differentiating this expression with respect to x_i and summing over i we get

$$\Box u-\kappa^2 u=0. \quad (122)$$

On the other hand we have from (110) and (121)

$$\frac{\partial u}{\partial x_i}+s_{ik}\frac{\partial u}{\partial x_k}+\kappa\beta_i u=0. \quad (123)$$

We wish to call attention here to the existence of the possibility of another formulation of the theory, which is obtained when we start with (123) rather than with (110). By multiplication with $1-\beta_i^2$ we get the result (121) and further

$$\beta_i[\beta_k(\partial u/\partial x_k)+\kappa u]=0. \quad \langle 110\rangle$$

This is a weaker relation than (110) since the matrices β_i have no reciprocals. For the 1-rowed representation of the β_i incidentally, where $\beta_i=0$, $I=1$, we get from $\langle 110\rangle$ and (121) the solution $u=\text{const}$.

Returning now to the original theory in which the constant solutions are excluded by (122) we note that with certain transformations (see I, §2) we can obtain, following Kemmer, the symmetric energy tensor

$$\theta_{ik}=\kappa[u^\dagger(\beta_i\beta_k+\beta_k\beta_i)u-\delta_{ik}u^\dagger u], \quad (124)$$

which also satisfies the continuity equation. According to (114), (117) the associated energy density is positive definite,

$$W=-\theta_{44}=\kappa u^* u.$$

The commutation relations read

$$i[u_\rho(x,x_0),u_\sigma^\dagger(x',x_0')]$$
$$=\left[\beta_k\frac{\partial}{\partial x_k}-\frac{1}{2\kappa}(\beta_k\beta_l+\beta_l\beta_k)\frac{\partial^2}{\partial x_k\partial x_l}\right]_{\rho\sigma}$$
$$\times[D(x-x',x_0-x_0')]. \quad (125)$$

It is easily verified that this is consistent with the wave Eq. (110).

A special reducible representation is obtained —as was noted by Duffin—by the relation

$$\beta_k=\tfrac{1}{2}(\gamma_k I'+\gamma_k' I). \quad (126)$$

The γ_k and γ_k' are Dirac matrices which operate on different groups of four indices. The I is the unit matrix in the first system of indices, the I' in the second system. The β_k therefore are made up of 16 rows and 16 columns and the associated wave functions $u_{\rho\rho'}$ thus have 16 components.[34]

It appears arbitrary, however, not to reduce the 16-rowed representation (126) of the β_k into its irreducible constituents. These are just the 5-rowed, the 10-rowed and the trivial 1-rowed representations.[35]

This reduction finds a counterpart in the decomposition of the 16-component quantities $u_{\rho\rho'}$. If the rule of Racah is introduced for the space-like reflections, the symmetric part of $u_{\rho\rho'}$ (for which $u_{\rho\rho'}=u_{\rho'\rho}$) which belongs to the 10-rowed representation, consists of a skew-symmetric tensor and an ordinary vector; the anti-symmetric part splits into a scalar which is associated with the 1-rowed representation and

[34] If the special representation (126) is used in the field equations (110), the equations of de Broglie's "Theory of photons" result. If the representation is used with (123) instead of (110) another formulation of de Broglie's theory is obtained which permits the constant solutions,—de Broglie's so-called "solutions d'annihilation." On the basis of the interpretation of this paper, however, the de Broglie theory does not describe photons at all, but rather is a unified description of two particles with equal non-vanishing rest-mass, with spin values 0 and 1.

We refer in this connection to the arguments, cf. §2(*e*), on the gauge-transformations of the second kind which oppose the assignment of a non-vanishing rest-mass to the photons.

[35] This decomposition was carried out in detail by J. Géhéniau, *L'electron et photon* (Paris, 1938). It arises naturally, moreover, when two interacting particles of spin $\frac{1}{2}$ are considered. For example, in the case of the deuteron which is composed of a proton and neutron (we may assume that the difference of their rest-masses may be neglected) the 5-rowed representation is associated with the singlet state, the 10-rowed with the triplet state. [Compare also the older work of N. Kemmer, Helv. Phys. Acta **10**, 47 (1937) where the relative motion of neutron and proton is discussed on the basis of various assumptions for the interactions between them.] In general the different representations belong to states of different energy (the degeneracy is removed by the interaction). The one-rowed representation belongs to the combination of a proton with a positive, and a neutron with a negative rest-mass, and has no direct meaning in the c number theory.

TABLE I. *Scattering of mesotrons by a Coulomb field.* E_0 = *initial energy of mesotron;* M = *mass of mesotron;* θ = *angle of scattering;* $\eta = E_0/Mc^2$.

	Type of Mesotron: Spin (units $\hbar$)	Type of Mesotron: Magnetic Moment (units $e\hbar/2Mc$)	Cross Section for Scattering
I	0	0	$\frac{1}{4}\left(\frac{e^2}{Mc^2}\right)^2\frac{\eta^2}{(\eta^2-1)^2}\frac{d\Omega}{\sin^4\theta/2}$
II	$\frac{1}{2}$	1	$\frac{1}{4}\left(\frac{e}{Mc^2}\right)^2\left[\frac{\eta^2}{(\eta^2-1)^2}-\frac{1}{(\eta^2-1)}\sin^2\frac{\theta}{2}\right]\frac{d\Omega}{\sin^4\theta/2}$
III	$\frac{1}{2}$	$\gamma\neq 1$	$\frac{(\gamma-1)^2}{4}\left(\frac{e^2}{Mc^2}\right)^2\frac{d\Omega}{\sin^2\theta/2}$ *
IV	1	1	$\frac{1}{4}\left(\frac{e^2}{Mc^2}\right)^2\left[\frac{\eta^2}{(\eta^2-1)^2}+\frac{1}{6}\sin^2\theta\right]\frac{d\Omega}{\sin^4\theta/2}$
V	1	$\gamma\neq 1$	$\frac{(\gamma-1)^2}{3}\left(\frac{e^2}{Mc^2}\right)^2\eta^2 d\Omega$*

* III and V hold only for $\eta\gg 1$. I, III, V. Corben and Schwinger, Phys. Rev. **58**, 953 (1940). II. C. Møller, Zeits. f. Physik **70**, 786 (1931); Ann. d. Physik (5) **14**, 531 (1932). IV. Laporte, Phys. Rev. **54**, 905 (1938); Massey and Corben, Proc. Camb. Phil. Soc. **35**, 463 (1939).

a pseudo-vector and pseudo-scalar for the 5-rowed representation.[36, 37]

5. Applications*

We conclude this report with some simple applications of the theories discussed in Part II, §§1, 2(d) and 3(a), of the interaction of particles of spin 0, 1, and $\frac{1}{2}$ with the electromagnetic field. In the last two cases we denote the value $eh/2Mc$ of the magnetic moment as the normal one, where M is the rest mass of the particle. The assumption of a more general value $\gamma(eh/2Mc)$ for the magnetic moment demands the introduction of additional terms, proportional to $\gamma-1$, in the Lagrangian or Hamiltonian. These terms are given, for spin 1, by Eq. (66′), with $\gamma=1+K$, and for spin $\frac{1}{2}$ by Eq. (9′), with $\gamma=1-(2Mc/h)[(hc)^{\frac{1}{2}}/e]l$. The applications which we consider are the radiationless collision of two charged particles, the Compton effect, bremsstrahlung, and pair generation. Since the cross sections for these processes may be used to understand the nature of the penetrating component of cosmic radiation, we shall for convenience use the word "mesotron" to denote the particle, of charge e, to which various values of the spin and the magnetic moment are attributed.

(a) *Radiationless collision of mesotrons with electrons.*—In Table I is given the cross sections for the scattering of mesotrons by a Coulomb field with a fixed center; in Table II the cross sections for a mesotron in electromagnetic interaction with an electron, in the coordinate system in which the electron is initially at rest. The cross sections are calculated by the well-known method of Møller, introducing the matrix elements of the interaction of the electron with the electromagnetic field produced, according to the different theories, by the mesotron. In both this process and the process of bremsstrahlung, discussed in (c), we are interested particularly in the case in which the initial energy E_0 of the mesotron is large compared with Mc^2. The results given in Table II and in rows III and V of Table I hold only for this case, giving the leading terms in $\eta=E_0/Mc^2$. In both tables, cases III and IV are one order of magnitude in η greater than I and II, and V is again one order of magnitude larger than III and IV. Both here and for all other processes discussed below the cross sections for a given value of the

[36] In this connection see the table of Eq. (31) of the author's work in the Inst. H. Poincaré Ann. **6**, 109 (1936); especially p. 129. As Racah has pointed out, the behavior of the quantities defined in the work quoted is just reversed if his rule for the reflections is used; thus the quantities designated by Ω_1, s_k, Ω_2 which belong to the skew-symmetric part of $u_{\rho\rho'}$ become respectively a pseudo-scalar, a pseudo-vector, and an ordinary scalar while the quantity $s[\lambda,\mu,\nu]$ which belongs to the symmetric part of $u_{\rho\rho'}$ becomes an ordinary vector.

[37] See especially the work of F. J. Belinfante, Nature **143**, 201 (1939); Physica **6**, 870 (1939), who proposes a description of the meson field in terms of a symmetric 'undor' $u_{\rho\rho'}=u_{\rho'\rho}$.

* I am indebted to Dr. H. C. Corben for discussions concerning the content of this section.

TABLE II. *Cross sections for elastic scattering of fast mesotrons by electrons, in coordinate systems in which electron is initially at rest. Terms of order $(M/m)(Mc^2/\epsilon E_0)$ or smaller have been neglected. M = mesotron mass; m = electron mass; E_0 = initial mesotron energy $\gg Mc^2$; ϵE_0 = energy transferred to the electron.*

	Type of Mesotron: Spin (units $\hbar$)	Magnetic Moment (units $e\hbar/2Mc$)	Cross Section per Collision
I	0	0	$2\pi\left(\frac{e^2}{Mc^2}\right)^2 \frac{M}{m}\frac{Mc^2}{E_0}\frac{d\epsilon}{\epsilon^2}(1-\epsilon)$
II	$\frac{1}{2}$	1	$2\pi\left(\frac{e^2}{Mc^2}\right)^2 \frac{M}{m}\frac{Mc^2}{E_0}\frac{d\epsilon}{\epsilon^2}\left(1-\epsilon+\frac{\epsilon^2}{2}\right)$
III	$\frac{1}{2}$	$\gamma\neq 1$	$\pi(\gamma-1)^2\left(\frac{e^2}{Mc^2}\right)^2 \frac{d\epsilon}{\epsilon}(1-\epsilon)$.
IV	1	1	$\frac{2\pi}{3}\left(\frac{e^2}{Mc^2}\right)^2 \frac{d\epsilon}{\epsilon}\left(1-\epsilon+\frac{\epsilon^2}{2}\right)$
V	1	$\gamma\neq 1$	$\frac{2\pi}{3}\left(\frac{e^2}{Mc^2}\right)^2 \frac{M}{m}\frac{E_0}{Mc^2}d\epsilon(1-\epsilon)$

I, III, V. Corben and Schwinger, Phys. Rev. **58**, 953 (1940). II. C. Møller, Ann. d. Physik (5) **14**, 531 (1932); Bhabha, Proc. Roy. Soc. **A164** 257 (1938). IV. Massey and Corben, Proc. Camb. Phil. Soc. **35**, 463 (1939); Oppenheimer, Snyder and Serber, Phys. Rev. **57**, 75 (1940).

TABLE III. *Cross sections for Compton scattering. k_0 = initial energy of quantum; k = final energy of quantum; scattering particle of mass M, and initially at rest; θ = angle of scattering.*

	Type of Scattering Particle: Spin	Magnetic Moment	Cross Sections for Scattering through Angle θ. Valid for all Energies, Except for Case III.
I	0	0	$d\Omega\left(\frac{e^2}{Mc^2}\right)^2 \frac{1}{2}\frac{k^2}{k_0^2}\cos^2\theta$
II	$\frac{1}{2}$	1	$d\Omega\left(\frac{e^2}{Mc^2}\right)^2 \frac{1}{2}\frac{k^2}{k_0^2}\left(\frac{k_0}{k}+\frac{k}{k_0}-\sin^2\theta\right)$
III	$\frac{1}{2}$	$\gamma\neq 1$	$d\Omega(\gamma-1)^4\left(\frac{e^2}{Mc^2}\right)^2 \frac{1}{4}\frac{k}{k_0}\left(\frac{k}{Mc^2}\right)^2+\cdots$
IV	1	1	$d\Omega\left(\frac{e^2}{Mc^2}\right)^2 \frac{1}{2}\frac{k^2}{k_0^2}\left[1+\cos^2\theta+\frac{1}{48(Mc^2)^2}\{kk_0(28-64\cos\theta+12\cos^2\theta)+(k^2+k_0^2)(29-16\cos\theta+\cos^2\theta)\}\right]$

TABLE IIIA. *Cross sections for Compton scattering. Notation the same as in Table III.*

	Type of Scattering Particle: Spin	Magnetic Moment	Total Scattering Cross Section $k_0\gg Mc^2$
IA	0	0	$\pi\left(\frac{e^2}{Mc^2}\right)^2 \frac{Mc^2}{k_0}$
IIA	$\frac{1}{2}$	1	$\pi\left(\frac{e^2}{Mc^2}\right)^2 \frac{Mc^2}{k_0}\left(\frac{1}{2}+\ln\frac{2k_0}{Mc^2}\right)$
IIIA	$\frac{1}{2}$	$\gamma\neq 1$	$\frac{\pi}{4}\left(\frac{e^2}{Mc^2}\right)^2(\gamma-1)^4\frac{k_0}{Mc^2}+\cdots$
IVA	1	1	$\frac{5\pi}{36}\left(\frac{e^2}{Mc^2}\right)^2 \frac{k_0}{Mc^2}$

References for Tables III, IIIA:
I, IA, IV. Booth and Wilson, Proc. Roy. Soc. **A175**, 483 (1940). II, IIA. Klein and Nishina, Zeits. f. Physik **52**, 853 (1929); Nishina, Zeits. f. Physik **52**, 869 (1929); Tamm, Zeits. f. Physik **62**, 545 (1930). III, IIIA, IVA. Corben (unpublished). III. S. B. Batdorf and R. Thomas, Phys. Rev. **59**, 621 (1941).

TABLE IV. *Cross sections for bremsstrahlung.* E_0 = *initial energy of mesotron,* $\gg Mc^2$; M = *mass of mesotron;* ϵE_0 = *energy of emitted* γ*-ray;* Z = *atomic number of material traversed;* $A = \frac{12(1-\epsilon)}{5Mc^2Z^{\frac{1}{3}}\epsilon}E_0$.

	Type of Mesotron: Spin	Magnetic Moment	Cross Section
I	0	0	$\left(\frac{e^2}{Mc^2}\right)^2 \alpha Z^2 d\epsilon \frac{16}{3}\left(\frac{1-\epsilon}{\epsilon}\right)(\ln A - \frac{1}{2}).$
II	$\frac{1}{2}$	1	$\left(\frac{e^2}{Mc^2}\right)^2 \alpha Z^2 d\epsilon \frac{16}{3}\left(\frac{3\epsilon}{4}+\frac{1-\epsilon}{\epsilon}\right)(\ln A - \frac{1}{2})$
III	$\frac{1}{2}$	$\gamma \neq 1$	$\left(\frac{e^2}{Mc^2}\right)^2 \alpha Z^2 (\gamma-1)^4 d\epsilon \left[\frac{(1-\epsilon)E_0}{Mc^2Z^{\frac{1}{3}}}+\frac{\epsilon}{2}\ln^2 A - \frac{3\epsilon}{2}\ln A + \cdots\right]$
IV	1	1	$\left(\frac{e^2}{Mc^2}\right)^2 dZ^2 d\epsilon \left[\frac{E_0}{Mc^2Z^{\frac{1}{3}}}\cdot\frac{\pi}{60}(2-2\epsilon+7\epsilon^2) + \frac{\epsilon}{12}\left(17+\frac{7\epsilon^2}{2(1-\epsilon)}\right)\ln^2 A + \left(\frac{16(1-\epsilon)}{3\epsilon}+\frac{13\epsilon}{12}-\frac{5\epsilon^3}{24(1-\epsilon)}\right)\ln A + \cdots\right]$

TABLE IVA. *Cross sections for pair-production.* E_0 = *initial energy of* γ*-ray;* ϵE_0 = *energy of positive mesotron created;* M = *mass of mesotron;* Z = *atomic number of material traversed;* $B = \frac{12\epsilon(1-\epsilon)}{5Mc^2Z^{\frac{1}{3}}}E_0$.

	Type of Mesotron: Spin	Magnetic Moment	Cross Section
I	0	0	$\left(\frac{e^2}{Mc^2}\right)^2 \alpha Z^2 d\epsilon \frac{16}{3}\epsilon(1-\epsilon)(\ln B - \frac{1}{2})$
II	$\frac{1}{2}$	1	$\left(\frac{e^2}{Mc^2}\right)^2 \alpha Z^2 d\epsilon \frac{16}{3}(\frac{3}{4}-\epsilon(1-\epsilon))(\ln B - \frac{1}{2})$
III	$\frac{1}{2}$	$\gamma \neq 1$	$\left(\frac{e^2}{Mc^2}\right)^2 \alpha Z^2 (\gamma-1)^4 d\epsilon \left[\frac{\epsilon(1-\epsilon)E_0}{Mc^2Z^{\frac{1}{3}}}+\cdots\right]+\cdots$
IV	1	1	$\left(\frac{e^2}{Mc^2}\right)^2 \alpha Z^2 d\epsilon \left[\frac{E_0}{Mc^2Z^{\frac{1}{3}}}\frac{\pi}{40}(7-2\epsilon+2\epsilon^2)+\cdots\right]$

References for Tables IV, IVA:
I, IV. R. F. Christy and S. Kusaka, Phys. Rev. **59**, 414 (1941). II. W. Heitler, *The Quantum Theory of Radiation* (Oxford, 1936), p. 168 (Bremsstrahlung), and p. 197 (pair-production). In each case appropriate modification has been made for the finite size of the nucleus. III. S. B. Batdorf and R. Thomas, Phys. Rev. **59**, 621 (1941). The author is indebted to Mr. Thomas for informing him of his results.

spin (except 0) are smallest when the magnetic moment assumes its normal value $eh/2\,Mc$.[38]

(*b*) *Scattering of a light quantum* (*Compton effect*).—For the calculation of the cross sections for this process and for the emission process discussed under (*c*), the use of the quantization of the electromagnetic field is not necessary. As is well known,[39] the results may be derived from ordinary wave mechanics with the aid of certain formal postulates, in accordance with the general correspondence with classical theory. The results of the perturbation theory are given in Table III, and are valid for all energies. The total scattering cross sections, as given in Table IIIA, hold only for values of k_0, the energy of the incident quantum, large compared with Mc^2, only leading terms in the ratio k_0/Mc^2 being given. For the differential cross section for scattering through a given angle we must bear in mind that the conservation of energy and momentum leads to the relation

$$k = k_0 \frac{1}{1+\frac{k_0}{Mc^2}(1-\cos\theta)}.$$

[38] It may be added that for the eigenvalue problem of the mesotron in a static Coulomb field no complete orthogonal system of eigenfunctions exists for cases III, IV and V (in distinction to cases I and II) since for these cases there occurs in the second order wave-equations a singularity at $r=0$ which is too strong. See I. Tamm, Comptes rendus U. S. S. R. (Doklady) **29**, 551 (1940); I. Tamm, Phys. Rev. **58**, 952 (1940); H. C. Corben and J. Schwinger, Phys. Rev. **58**, 953 (1940).

[39] The relation of the two methods to each other is discussed in detail by W. Pauli, *Handbuch der Physik*, article on wave mechanics, §15, pp. 201 *et seq.*

As for the radiationless collision cross sections, the results for III and IV are of higher order than for I and II.

(*c*) *Emission of a light quantum by a mesotron in the field of a nucleus* (*Bremsstrahlung*) *and pair generation.*—For the application of this process to cosmic radiation it is not permissible to idealize the nucleus as a point charge; rather one supposes that the nuclear charge Ze is distributed over a sphere of radius d. For this radius one assumes, according to the statistical nuclear model, the value $d=5/6z^{\frac{1}{3}}\hbar/Mc$.

For the case $E_0 \gg Mc^2$, the results of the various theories are given in Table IV. Again cases III and IV yield a cross section of higher order than I and II.

The transition probabilities for the Compton effect and the emission process are not independent of each other, for the latter may be also calculated by the method of virtual quanta[40] (although for case IV this method must be controlled by the direct method). The virtual quanta method consists in considering the coordinate system in which the mesotron is at rest and taking the Fourier expansion of the field of the quickly moving nucleus. The scattering of a light quantum out of this field, the cross section for which may be taken from the formulae for the Compton effect, then corresponds in the rest system of the nucleus to the emission of a quantum.

Very closely connected with the emission of a light quantum is the process of pair generation by a quantum. Indeed, according to the theory of holes, the process involved is of the same kind as the emission process, the only differences being that (in the case of pair production) the light quantum is present in the initial and absorbed in the final state, and that the electron in the initial state has a negative energy. The orders of magnitude of the cross sections for pair-production (Table IVA) are the same as for the corresponding bremsstrahlung processes.

Even if the fundamentals of the various theories are correct, some limitation of the validity of the derived results arises from the fact that the cross sections as calculated by the perturbation theory are only a first approximation. This is especially important in the cases of spin $\frac{1}{2}$, with an anomalous magnetic moment, and spin 1 for which particles the cross sections increase with increasing energy. The validity of the first approximation of the perturbation theory in these cases has been discussed by Oppenheimer[41] and by Landau.[42] The first author uses the criterion that in the coordinate system in which the mesotron and the light quantum have equal and opposite momenta the interaction energy between the mesotron and the quantum must be small compared with the unperturbed energy of either one. The second author employs the condition that the transition probabilities for all other processes that could arise from the same initial state (the emission of several pairs or light quanta, for example) must be small in comparison with the probability for the process in question. For bremsstrahlung, pair generation and the Compton effect, these two criteria give, for cases III and IV, the same condition $E_0<(\hbar c/e^2)Mc^2$ or $h\nu<(\hbar c/e^2)Mc^2$.[43] On the other hand, for spin 0 and for spin $\frac{1}{2}$, with normal magnetic moment these criteria are fulfilled for all energies if $(Ze^2/\hbar c)\ll 1$.

Although it is clear that these criteria are certainly necessary conditions for the validity of the first approximation of the perturbation theory, a closer investigation of the sufficiency of these conditions (particularly for spin 0) would be desirable. Such an investigation demands a discussion of the higher approximations of the perturbation theory. As these lead to infinities which must be avoided by a suitable cutting-off, the problem is, however, not a purely mathematical one, being connected with the physical problem of the region of applicability of the foundations of the underlying theory.

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[40] C. v. Weiszäcker, Zeits. f. Physik **88**, 612 (1934).

[41] J. R. Oppenheimer, Phys. Rev. **59**, 462 (1941). See also J. R. Oppenheimer, H. Snyder and R. Serber, IV of Table II.

[42] L. Landau, J. Phys. U. S. S. R. **2**, 483 (1940).

[43] For the validity of the cross section for radiationless scattering of a mesotron of spin 1, normal magnetic moment, by an electron initially at rest (Table II, case IV), Oppenheimer, Snyder and Serber give as an application of their criterion the condition $E_0<(M/m)(\hbar c/e^2)Mc^2$ (Phys. Rev. **57**, 75 (1940)).