Numerical Calculation of a Generalized Complete Elliptic Integral

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OCCASIONALLY the solution of a problem in physics involves the numerical evaluation of one or more complete elliptic integrals. Although tables of these integrals are available,[1] if the arguments and moduli involved do not coincide exactly with those of the table, the required interpolation is often laborious and uncertain. Then, again, the reduction to the tabulated standard forms may give rise to rather complicated formulae.

The purpose of this paper is to develop a direct method of calculating a very general type of integral, which includes, as a special case, the standard forms of complete elliptic integrals. The method is based on Landen's transformation and utilizes the rapidly convergent scale of arithmetico-geometrical means developed by Lagrange, Legendre, Gauss and others.[2]

Legendre's complete elliptic integrals of the first, second and third kind may be regarded as special cases of the integral

$$I(m,\,n)=\int_0^{\pi/2}\frac{F(R)d\phi}{R},\tag{1}$$

where $F(R)$ is a continuous function of R and

$$R^2=m^2\cos^2\phi+n^2\sin^2\phi,\tag{2}$$

in which the parameters m and n are real positive numbers. For example, if

$$m=1,\quad n=(1-k^2)^{\frac{1}{2}},$$
$$F(R)=k^2(k^2+\eta-\eta R^2)^{-1},$$

the integral (1) becomes Legendre's standard complete elliptic integral of the third kind, namely

$$\Pi_3=\int_0^{\pi/2}\frac{d\phi}{(1+\eta\sin^2\phi)(1-k^2\sin^2\phi)^{\frac{1}{2}}}.$$

The definite integral (1), when expressed in terms of R alone is

$$I(m,\,n)=\int_n^m\frac{F(R)dR}{\Delta},\tag{3}$$

where

$$\Delta^2=(R^2-n^2)(m^2-R^2).\tag{4}$$

Let a new variable of integration be defined as follows[3]

$$R_1=\tfrac{1}{2}(R+nmR^{-1}).\tag{5}$$

As R varies from n to m, R_1 diminishes from the arithmetical mean

$$m_1=\tfrac{1}{2}(m+n),\tag{6}$$

to a minimum at the geometrical mean,

$$n_1=(mn)^{\frac{1}{2}},\tag{7}$$

and then increases to m_1. Furthermore,

$$\begin{aligned}R&=R_1\pm(R_1^2-n_1^2)^{\frac{1}{2}},\\ \Delta^2&=4R^2(m_1^2-R_1^2),\\ dR_1&=\tfrac{1}{2}(1-n_1^2R^{-2})dR\\ &=\pm R^{-1}(R_1^2-n_1^2)^{\frac{1}{2}}dR,\end{aligned}\tag{8}$$

and, consequently,

$$dR/\Delta=\pm dR_1/2\Delta_1,\tag{9}$$

where

$$\Delta_1^2=(R_1^2-n_1^2)(m_1^2-R_1^2).\tag{10}$$

[1] Smithsonian Mathematical Formulae and Tables of Elliptic Formulae.

[2] An excellent collection of formulae is given in Louis V. King's monograph "On the Direct Numerical Calculation of Elliptic Functions and Integrals."

[3] Landen's transformation.

From these relations it follows that the definite integral (3) equals

$$I_1(m_1, n_1) = \int_{n_1}^{m_1} \frac{F_1(R_1)dR_1}{\Delta_1}, \quad (11)$$

where

$$F_1(R_1) = \tfrac{1}{2}\{F[R_1 + (R_1^2 - n_1^2)^{\frac{1}{2}}] + F[R_1 - (R_1^2 - n_1^2)^{\frac{1}{2}}]\}. \quad (12)$$

Thus the transformation (5) does not alter the form of the integral (3), but merely replaces m and n by m_1 and n_1, respectively, and the function F by the function F_1. Since (3) was equivalent to the form (1), it follows that (11) is equivalent to

$$I_1(m_1, n_1) = \int_0^{\pi/2} \frac{F_1(R_1)d\phi_1}{R_1}, \quad (13)$$

where

$$R_1^2 = m_1^2 \cos^2 \phi_1 + n_1^2 \sin^2 \phi_1. \quad (14)$$

The same argument made for the integral (1) may be applied to the integral (13) to yield another integral. By repetition of this process a sequence of integrals is obtained, all equal to the integral (1). These integrals may be written

$$I_i(m_i, n_i) = \int_0^{\pi/2} \frac{F_i(R_i)d\phi_i}{R_i} \quad i = 0, 1, 2, \cdots, \quad (15)$$

where

$$R_i^2 = m_i^2 \cos^2 \phi_i + n_i^2 \sin^2 \phi_i, \quad (16)$$

and

$$m_i = \tfrac{1}{2}(m_{i-1} + n_{i-1}), \quad n_i = (m_{i-1} n_{i-1})^{\frac{1}{2}}; \quad (17)$$

$$F_i(R_i) = \tfrac{1}{2}\{F_{i-1}[R_i + (R_i^2 - n_i^2)^{\frac{1}{2}}] + F_{i-1}[R_i - (R_i^2 - n_i^2)^{\frac{1}{2}}]\}; \quad (18)$$

while for $i = 0$,

$$m_0 = m, \quad n_0 = n, \quad R_0 = R, \quad \text{and} \quad F_0 = F.$$

The integral (1) may be obtained by evaluating any one of the integrals (15) or by finding the limit of the sequence as i increases without bound.

As i increases, the m_i and n_i rapidly approach a common limit, the arithmetico-geometrical mean, which will be denoted by m_L. Since by (16) R_i is in the interval n_i to m_i,

$$\text{limit } m_i = \text{limit } n_i = \text{limit } R_i = m_L.$$

Furthermore, it can be demonstrated that if $F(R)$ is a continuous function,

$$\text{limit } F_i(R_i) = \text{limit } F_i(m_L),$$

a definite constant, so that the integrands of (15) approach a constant and (18) becomes

$$I(m, n) = (\pi/2m_L)[\text{limit } F_i(m_L)]. \quad (19)$$

Of particular interest to the computer is the rapidity with which the limit is obtained. This is best illustrated by an example. For the case

$$m = 1, \quad n = 1/\sqrt{2} = 0.707106\ 78119,$$

the sequence is

$m_1 = 0.853553\ 39059$, $n_1 = 0.840896\ 41525$;
$m_2 = 0.847224\ 90292$, $n_2 = 0.847201\ 26674$;
$m_3 = 0.847213\ 08483$, $n_3 = 0.847213\ 08475$;
$m_4 = 0.847213\ 08479$,

and the error made in using m_3 for the limit is 4×10^{-11}, a negligible quantity in most problems. If, however, m_4 were computed to a sufficient number of places, the error could be reduced to 10^{-21}. Furthermore, if the ratio m/n is greater than 0.7 or less than 1.4, the convergence is more rapid, so that in this interval m_L may be taken equal to m_3. Suppose in addition R_3 is taken equal to m_3, a constant, then the integrand of (15) also reduces to a constant and all that is required for its evaluation is $F_3(m_3)$. Now by (18) and (17), taking $i = 3$ and $R_3 = m_3$,

$$F_3(m_3) = \tfrac{1}{2}\{F_2[m_3 + (m_3^2 - n_3^2)^{\frac{1}{2}}] + F_2[m_3 - (m_3^2 - n_3^2)^{\frac{1}{2}}]\} \quad (20)$$
$$= \tfrac{1}{2}\{F_2(m_2) + F_2(n_2)\}.$$

But the right members of this equation are by (18) expressible in terms of F_1 and hence, by (12), in terms of F. The result is

$$F_3(m_3) = \tfrac{1}{4}\{\tfrac{1}{2}[F(m) + F(n)] + F(n_1) + F(m') + F(n')\}, \quad (21)$$

where

$$m' = n_2 + (n_2^2 - n_1^2)^{\frac{1}{2}}, \quad n' = n_2 - (n_2^2 - n_1^2)^{\frac{1}{2}}. \quad (22)$$

On substituting these results in (15), taking $i=3$ and replacing R_3 by m_3, the following quadrature formula is derived:

$$\int_0^{\pi/2} \frac{F(R)d\phi}{R} = \frac{\pi}{8m_3}\{\tfrac{1}{2}[F(m)+F(n)] + F(n_1)+F(m')+F(n')\}. \quad (23)$$

For the case

$$m=1, \quad n=1/\sqrt{2},$$

the remaining arguments of the function F occurring in (23) are

$$n_1 = 0.840896\ 41525,$$
$$m' = 0.950367\ 17766,$$
$$n' = 0.744035\ 35581.$$

Many applications of formula (23) may be found in the calculation of electrostatic and electromagnetic fields. For example, the magnetic intensity due to an electrical current, i, in a circle of radius, a, for a point, P, in the plane of the circle and at a distance, H, from center, is given by the integral

$$Z = i\int_0^{2\pi} \frac{a \sin\lambda\, d\theta}{R^2}, \quad (24)$$

where λ is the angle between an element, $ad\theta$ of the circle and the line joining P to this element, R is the distance from P to the element, and θ is the angle at center of circle made by P and an element. In terms of the variable of integration

$$R^2 = a^2+H^2-2aH\cos\theta,$$
$$\sin\lambda = (R^2+a^2-H^2)/(2aR).$$

On replacing the variable of integration by

$$\begin{aligned} \phi &= (\theta+\pi)/2, \\ R^2 &= a^2+H^2+2aH\cos 2\phi \\ &= (a^2+H^2)(\cos^2\phi+\sin^2\phi) + 2aH(\cos^2\phi-\sin^2\phi) \\ &= (a+H)^2\cos^2\phi+(a-H)^2\sin^2\phi \\ &= m^2\cos^2\phi+n^2\sin^2\phi, \end{aligned} \quad (25)$$

where m and n are the maximum and minimum distances of the point P from the circle. The integral (24) then becomes for the case $a>H$

$$Z = 2i\int_0^{\pi/2} (R^{-1}+mnR^{-3})d\phi, \quad (26)$$

and is of the form (1) with

$$F(R) = 2i(1+mnR^{-2}). \quad (27)$$

If $i=1$, $m=1$ and $n=1/\sqrt{2}$, by formula (23)

$$Z = 7.528347\ 14.$$

The integral (26) is also expressible in terms of complete elliptic integrals of the first and second kind. The value of Z derived from tables of elliptic integrals agrees with that found from formula (23) to eight significant figures. But formula (23) applies equally well to integrals that are not exactly elliptical, but only approximately elliptical. For example, suppose the electrical circuit is not a circle but an ellipse of major and minor axes a and b, respectively. It can be demonstrated that the field on the major axis, at a distance H from the center of the ellipse, is

$$Z = \int_0^{\pi/2} \frac{F(R)d\phi}{R}, \quad (28)$$

where

$$R^2 = (a+H)^2\cos^2\phi+(a-H)^2\sin^2\phi,$$
$$F(R) = 16ibH^3a^{-1}\{1+R^{-2}(a^2-H^2)\}$$
$$\times\{4H^2+e^2R^{-2}(a^2-H^2)^2-2e^2(a^2+H^2)+e^2R^2\}^{-\frac{3}{2}},$$

and e is the eccentricity of the ellipse. If e is small, formula (23) gives satisfactory results. Thus, for $H=0.2$, $e=0.25$, $a=0.8$, and $i=1$ it is found that

$$m=1.0, \quad n_1=0.7745967, \quad m'=0.9274718,$$
$$n=0.6, \quad m_3=0.7872471, \quad n'=0.6469199;$$

and by formula (23),

$$Z = 8.377580,$$

a result that is correct to seven significant figures.

One method of estimating the accuracy of formula (23) is to compare the result found from (23) with that obtained by carrying the process defined by Eqs. (17) and (18) one step further, that is, to m_4. The quadrature formula so derived resembles formula (23) but involves four

TABLE I. *Values of m_3, n_1, m' and n' for use in formula* (23). ($m=1$, $n=0.1$ *to* 1.0)

n	m_3	n_1	m'	n'	n	m_3	n_1	m'	n'
0.10	0.4250	0.3162	0.6889	0.1452	0.55	0.75822	0.74162	0.91546	0.60079
.11	.4362	.3317	.7012	.1569	.56	.76408	.74833	.91794	.61006
.12	.4468	.3464	.7125	.1684	.57	.76992	.75498	.92038	.61931
.13	.4570	.3606	.7228	.1798	.58	.77572	.76158	.92278	.62853
.14	.4669	.3742	.7325	.1911	.59	.78150	.76811	.92515	.63774
.15	.4765	.3873	.7415	.2023					
.16	.4858	.4000	.7500	.2133	0.60	0.78725	0.77460	0.92747	0.64692
.17	.4949	.4123	.7580	.2243	.61	.79297	.78102	.92976	.65608
.18	.5037	.4243	.7655	.2351	.62	.79866	.78740	.93202	.66522
.19	.5124	.4359	.7726	.2459	.63	.80433	.79373	.93424	.67435
					.64	.80997	.80000	.93643	.68345
0.20	0.5208	0.4472	0.7794	0.2566	.65	.81559	.80623	.93859	.69253
.21	.5291	.4583	.7859	.2672	.66	.82118	.81240	.94072	.70159
.22	.5372	.4690	.7920	.2778	.67	.82675	.81854	.94282	.71064
.23	.5452	.4796	.7979	.2882	.68	.83229	.82462	.94489	.71966
.24	.5530	.4899	.8036	.2987	.69	.83782	.83066	.94693	.72867
.25	.5608	.5000	.8090	.3090					
.26	.5684	.5099	.8142	.3193	0.70	0.84332	0.83666	0.94895	0.73766
.27	.5759	.5196	.8193	.3296	.71	.84880	.84261	.95094	.74663
.28	.5833	.5292	.8241	.3398	.72	.85425	.84853	.95291	.75558
.29	.5906	.5385	.8288	.3499	.73	.85969	.85440	.95485	.76452
					.74	.86511	.86023	.95677	.77344
0.30	0.5978	0.5477	0.8334	0.3600	.75	.87051	.86603	.95866	.78234
.31	.6049	.5568	.8378	.3700	.76	.87589	.87178	.96053	.79123
.32	.6119	.5657	.8420	.3800	.77	.88124	.87750	.96238	.80010
.33	.6189	.5745	.8461	.3900	.78	.88658	.88318	.96421	.80895
.34	.6258	.5831	.8501	.3999	.79	.89191	.88882	.96602	.81779
.35	.6326	.5916	.8540	.4098					
.36	.6394	.6000	.8578	.4197	0.80	0.89721	0.89443	0.96781	0.82661
.37	.6461	.6083	.8615	.4295	.81	.90250	.90000	.96958	.83541
.38	.6527	.6164	.8651	.4392	.82	.90777	.90554	.97133	.84421
.39	.6593	.6245	.8686	.4490	.83	.91302	.91104	.97306	.85298
					.84	.91826	.91652	.97477	.86174
0.40	0.6658	0.6325	0.8721	0.4587	.85	.92348	.92195	.97647	.87049
.41	.6723	.6403	.8754	.4684	.86	.92868	.92736	.97814	.87922
.42	.6787	.6481	.8787	.4780	.87	.93387	.93274	.97980	.88793
.43	.6851	.6557	.8819	.4876	.88	.93904	.93808	.98145	.89664
.44	.6914	.6633	.8850	.4972	.89	.94420	.94340	.98307	.90532
.45	.6976	.6708	.8880	.5067					
.46	.7039	.6782	.8910	.5163	0.90	0.94934	0.94868	0.98468	0.91400
.47	.7101	.6856	.8939	.5258	.91	.95447	.95394	.98628	.92266
.48	.7162	.6928	.8968	.5352	.92	.95958	.95917	.98786	.93130
.49	.7223	.7000	.8996	.5447	.93	.96468	.96437	.98943	.93994
					.94	.96977	.96954	.99098	.94856
0.50	0.72840	0.70711	0.90239	0.55408	.95	.97484	.97468	.99252	.95716
.51	.73443	.71414	.90510	.56347	.96	.97990	.97980	.99404	.96576
.52	.74043	.72111	.90776	.57284	.97	.98494	.98489	.99555	.97434
.53	.74639	.72801	.91038	.58218	.98	.98997	.98995	.99705	.98290
.54	.75232	.73485	.91294	.59149	.99	.99499	.99500	.99853	.99146

additional ordinates. It may be written

$$\int_0^{\pi/2} \frac{F(R)d\phi}{R} = \frac{\pi}{16m_4}\{\tfrac{1}{2}[F(m)+F(n)] + F(n_1)+F(m')+F(n')+F(m'') + F(n'')+F(m''')+F(n''')\}, \quad (29)$$

where the four additional ordinates are given by

$$m'',\ n'',\ m''',\ n''' = S \pm (S^2 - n_1^2)^{\frac{1}{2}}, \quad (30)$$

where S has the two values

$$S = n_3 \pm (n_3^2 - n_2^2)^{\frac{1}{2}} \quad (31)$$

and the n_i are given by (17).

On comparing formula (23) with (29) it is seen that, if m_3 and m_4 are essentially equal, then formula (29) is the arithmetical mean of (23) and an analogous formula involving the four new ordinates. Consequently the sum

$$\tfrac{1}{2}[F(m)+F(n)]+F(n_1)+F(m')+F(n') \quad (32)$$

should be approximately equal to the sum

$$F(m'')+F(n'')+F(m''')+F(n'''). \qquad (33)$$

The number of significant figures to which these two sums agree is an indication of the accuracy of formula (23).

As another example take

$$F(R)=R^{\frac{1}{2}}, \quad m=1, \quad n=1/\sqrt{2},$$

then

$$S=0.851687\,908748 \quad \text{or} \quad 0.842738\,198622$$

and

$$m''=0.986838\,326207, \qquad n''=0.716537\,615543,$$
$$m'''=0.898423\,836184, \qquad n'''=0.787052\,561061.$$

The sums (32) and (33) are found to be

$$3.674894\,86598,$$
$$3.674894\,86593,$$

so that, for this example, formula (23) should give results correct to eleven significant figures. The accuracy of formula (29) is, of course, far greater.

The integral of this example belongs also to the type

$$\int_0^{\pi/2}(m^2\cos^2\phi+n^2\sin^2\phi)^{-s-1}d\phi, \qquad (34)$$

and arises in the theory of zonal harmonics. In fact, the zonal harmonic[4]

$$P_s(x)=\frac{2}{\pi}\int_0^{\pi/2}\{[x+(x^2-1)^{\frac{1}{2}}]\cos^2\phi + [x-(x^2-1)^{\frac{1}{2}}]\sin^2\phi\}^{-s-1}d\phi.$$

If s is not too large, formula (23) gives satisfactory results for the integral (34). In fact, if m/n is between 0.7 and 1.4 and s between -10 and $+10$, formula (23) gives a result that is correct to at least five significant figures.

Tables may be readily constructed to facilitate the computation of integrals by formula (23). In the first place, by replacing ϕ by $\pi/2-\phi$ in (1), the form of (1) remains unchanged, except that

$$R^2=n^2\cos^2\phi+m^2\sin^2\phi.$$

[4] Byerly, *Fourier's Series and Spherical Harmonics*, p. 167.

That is, m and n may always be interchanged, and the integral written so that m is greater than n. Secondly, the function F may always be so modified that m is equal to unity, so that for tabular purposes $m=1$ and n varies from 0 to 1. In the belief that such a table may be useful, a four to five place table of m_3, n_1, m' and n' for $m=1$ and n between 0.1 and 1.0 is included in this article. See Table I.

There is a special class of integrals of the type (1) that permit of even simpler formulae than those here given, namely that class of functions for which the $F_i(R_i)$ of (18) all have the same form. One member of this class is

$$F(R)=\log(A+BR),$$

where A and B are constants. For by (12)

$$F_1(R_1)=\tfrac{1}{2}\log(A^2+B^2n_1^2+2ABR_1)$$
$$=\tfrac{1}{2}\log(A_1+B_1R_1),$$

where

$$A_1=A^2+B^2n_1^2,$$
$$B_1=2AB.$$

In general

$$F_i(R_i)=2^{-i}\log(A_i+B_iR_i),$$

where

$$A_i=A_{i-1}^2+B_{i-1}^2n_i^2,$$
$$B_i=2A_{i-1}B_{i-1}.$$

By computing the sequences A_i and B_i along with m_i and n_i, this integral is then given by

$$\int_0^{\pi/2}\log(A+BR)\frac{d\phi}{R}$$
$$=\frac{\pi}{2}\operatorname{limit}\log(A_i+B_im_L)/(2^im_L). \qquad (35)$$

Such an integral was found to arise in the calculation of logarithmic potentials of ellipses.

The integral

$$E=\int_0^{\pi/2}\frac{am\cos^2\phi+brn\sin^2\phi}{m\cos^2\phi+rn\sin^2\phi}\cdot\frac{d\phi}{R}, \qquad (36)$$

where a, b and r are constants, when expressed entirely in terms of R is found to belong to this same special class. After applying the transformation (5), the integral becomes (13) with

$$F_1=\frac{a_1m_1\cos^2\phi_1+b_1r_1n_1\sin^2\phi_1}{m_1\cos^2\phi_1+r_1n_1\sin^2\phi_1},$$

where

$$a_1=\tfrac{1}{2}(a+b),\quad b_1=(a+br)/(1+r),$$
$$r_1=(n_1/4m_1)(r+r^{-1}+2).$$

The integral is given by

$$E=(\pi/2m_L)[\text{limit } a_i], \tag{37}$$

where

$$a_{i+1}=\tfrac{1}{2}(a_i+b_i),\quad b_{i+1}=\frac{a_i+b_ir_i}{1+r_i}, \tag{38}$$

$$r_{i+1}=(n_{i+1}/4m_{i+1})(r_i+r_i^{-1}+2),\qquad i=1,2,3,\cdots,$$

for it can be demonstrated that the limit a_i = limit b_i.

Complete elliptic integrals of the first, second and third kind are all expressible in the form (36). For example, the integral (26) may also be written

$$Z=2i(m+n)\int_0^{\pi/2}\frac{\cos^2\phi+nm^{-1}\sin^2\phi}{m\cos^2\phi+n^2m^{-1}\sin^2\phi}\cdot\frac{d\phi}{R}$$

and hence, on omitting the factor $2i(m+n)$, is of the form (36) with

$$a=m^{-1},\quad b=n^{-1},\quad r=n/m.$$

For $m=1$, $n=1/\sqrt{2}$, and hence

$$a=1,\quad b=\sqrt{2},\quad r=1/\sqrt{2},$$

the sequences are

$$\begin{aligned}
&m_1=0.8535534, && n_1=0.8408964, &&\\
&a_1=1.2071068, && b_1=1.1715729, && r_1=1.0150518;\\
&m_2=0.8472249, && n_2=0.8472013, &&\\
&a_2=1.1893398, && b_2=1.1892071, && r_2=1.0000279;\\
&m_3=0.8472131, && n_3=0.8472131, &&\\
&a_3=1.1892734, && b_3=1.1892734, && r_3=1.0000000.
\end{aligned}$$

Consequently at the third step the a_i and b_i, as well as the m_i and n_i, agree to seven significant figures. Hence

$$Z=7.528347.$$

In conclusion, it is noted that these formulae are not only rapidly convergent but ideally suited for machine calculation.