SATURATION OF CURRENT ALGEBRA EQUATIONS WITH HIGHER BARYONIC RESONANCES

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Abstract

The saturation of the commutation relations of the chiral $U(3) \otimes U(3)$ algebra with supermultiplets of SU_6 or $SU_6 \otimes O_3$ is considered. The general helicity dependence of the matrix elements of axial charges at $\mathbf{p} = \infty$ is calculated. A continuity postulate is introduced to select the physically significant solutions from the multiplicity of solutions of the non-linear set of saturated commutator equations. A linearized approach is developed to derive criteria of "connection" between two supermultiplets with respect to the chiral algebra and to calculate the renormalizations (at second order) of the couplings. The possible relevance of the connection criteria and of the compatibility with data of the coupling renormalizations to discriminate among various possibilities of classification of higher resonances is pointed out. Special cases discussed are classifications of negative parity resonances according to the representations and $\mathbf{56}$ of SU_6 and $(\mathbf{20}, L = 1)$ of $SU_6 \otimes O_3$. Only the last one gives consistent predictions.

1. Introduction

HE recent work [1] by Adler and Weissberger has led to a calculation of the ratio G_A/G_V from sum rule over pion-nucleon cross-sections derived from an equal time commutation relation beween non-strange axial generators of the chiral $U(3) \otimes U(3)$ algebra. Extension of their work o the strange generators has similarly led to sum-rules over K-nucleon cross-sections, allowng for a calculation of the D/F ratio [2]. These calculations are based on the knowledge of xtrapolated empirical values of the pion-nucleon and K-nucleon cross-sections, inserted into he dispersive integrals appearing in the sum rules. A different approach is that used by Lee 3] and by Dashen and Gell-Mann [4], by saturating the commutation relations among stable articles and low-lying resonances and searching for a consistent solution of the obtained lgebraic system of equations.

In a preceding paper [5] we have followed the latter approach. We have saturated the commutaion relations of the chiral $U(3) \otimes U(3)$ algebra among the octet and decuplet baryon states

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plus a set of resonant states selected on the basis of a previous classification of higher baryonic resonances [6]. The adopted classification of higher baryonic resonances of negative parity is based on the representation 20 of SU_6 with orbital angular momentum $L = 1$ [6]. The states of 20 with $L = 1$ (briefly $(20, L = 1)$) are three unitary singlets with $J = 1/2$, $3/2$, and $5/2$ and two unitary octets with $J = 1/2$ and $3/2$.

In the saturation of the commutation relations the following set of states are thus included among the initial, final and intermediate states:

the states of 56:

one octet with $J^P = 1/2^+$

one decuplet with $J^P = 3/2^+$

the states 20 with $L = 1$:

one singlet and one octet with $J^P = 1/2^-$

one singlet and one octet with $J^P = 3/2^-$

one singlet with $J^P = 5/2^-$.

In this note we present a more detailed discussion of the problem and consider its possible extensions. The extension of our results are made particularly simple by the use of the "linear ized approach" that we discuss in Section 4. The "linearized approach" consists in solving the commutator algebra equations perturbatively in the mixings between different supermultiplets. Only solutions that can be obtained with continuity from the symmetric solution corresponding to absence of mixing are obtained by this procedure. We postulate that only these solutions have physical meaning (continuity postulate). Comparison with the particular case discussed of the mixing among 56 and $(20, L = 1)$ shows that this is indeed the case. In Sections 2 and 3 we present a general procedure to write down the saturated commutator equations including states of any spin. We also consider in detail the solutions of the algebraic system in the case of 56 and $(20, L = 1)$ and show that only two solutions may be a priori considered as physically possible. Only one is however admissible on the basis of the "continuity postulate" mentioned above.

2. Saturation of the Commutation Rules

Let us consider the commutation relation

$$\left[F^5_\mu, F^5_\nu\right] = -\sqrt{3}\begin{pmatrix} 8 & 8 & 8_a \\ \mu & \nu & \lambda \end{pmatrix} F_\lambda \tag{1}$$

between states with momentum p_1 and p_2 in the limit $p_1 = p_2 = \infty$ [7]. In equation (1) F^5_λ is an axial generator (λ denotes the set of SU_3 quantum numbers I, I_3, Y) of the chiral $U(3) \otimes U(3)$ and the symbol $\begin{pmatrix} 8 & 8 & 8_a \\ \lambda & \mu & \nu \end{pmatrix}$ denotes the relevant SU_3 Clebsch-Gordan coefficient. The saturation of (1) requires the explicit evaluation of the limits for $p_1 = p_2 \to \infty$ of the matrix elements

$$< J_1^{p_1},\ h_1 | F^5 | J_2^{p_2},\ h_2 > \tag{2}$$

here $< J_1^{P_1}, h_1 |$ represents a baryon state with spin J_1 and parity P_1, with helicity h_1, taken at momentum $\mathbf{p}_1$, and similarly for the state $| J_2^{P_2}, h_2 >$. In (2) we have for the moment omitted SU_3 indices. The axial generator F^5 is defined as

$$F^5 = \int d^3x \, J_0^5(\mathbf{x}, t)$$

here $J_\mu^5(x)$ is the axial current. We note that the operator F^5 has the selection rule $\Delta h = 0$, where h is the helicity, and that the matrix elements between states of helicity h and states of opposite helicity are related by the equation

$$< J_1^{P_1}, h | F^5 | J_2^{P_2}, h > = - P_1 P_2 < J_1^{P_1}, -h | F^5 | J_2^{P_2}, -h > \tag{3}$$

To obtain equation (3) one applies a parity operation followed by a 180° rotation around an axis orthogonal to the momentum. From translation and rotation invariance we can also write the matrix element (2) in the form

$$< J_1^{P_1}, h_1 | F^5 | J_2^{P_2}, h_2 > = (2\pi)^3 \delta^3(\mathbf{p}_1 - \mathbf{p}_2) \delta_{h_1 h_2} < J_1^{P_1}, h | J_0^5(\mathbf{o}, t) | J_2^{P_2}, h >$$

$$\tag{4}$$

where h is the common value of h_1 and h_2. We are thus led to the calculation of the matrix element of the fourth component of an axial vector between two states of same momentum $\mathbf{p}$ and spin-parity $J_1^{P_1}$ and $J_2^{P_2}$. Such a matrix element will be a linear combination of all fourth components of axial vectors that can be constructed starting from the available covariants.

We shall use the Rarita-Schwinger formalism for particles of spin J. We describe a particle of spin J by a spinor with $J - 1/2$ Lorentz indices $\mu, \nu, \ldots, \lambda$

$$\psi_{\mu\nu\ldots\lambda}(p) \tag{5}$$

subject to the conditions:

(1) symmetry in the indices $\mu, \nu, \ldots, \lambda$ and zero trace

(2) transversality

$$p^\mu \psi_{\mu\nu\ldots\lambda}(p) = 0 \tag{6}$$

(3) orthogonality to γ^μ

$$\gamma^\mu \psi_{\mu\nu\ldots\lambda}(p) = 0 \tag{7}$$

(4) Dirac equation

$$(\not{p} - m) \psi_{\mu\nu\ldots\lambda}(p) = 0 \tag{8}$$

From the transformation properties of $\psi_{\mu\nu\ldots\lambda}(p)$ under Lorentz transformations we can write in general

$$\psi_{\mu\ldots\lambda}(p) = \Lambda_\mu^k(p) \ldots \Lambda_\lambda^h(p) \, S(\Lambda) \, \psi_{k\ldots h}(o) \tag{9}$$

where: $\Lambda_\mu^{\;\nu}(p)$ is the special Lorentz transformation which brings the four-vector $\tilde{p} = (\mathbf{o}, m)$ into the four-vector $p = (\mathbf{p}, E)$ i.e.

$$p_\mu = \Lambda_\mu^{\;\nu}(p)\tilde{p}_\nu \tag{10}$$

while $S(\Lambda)$ is the representation of the Lorentz transformation Λ in spinor space. Explicitly

$$S(\Lambda) = \sqrt{\frac{E + m}{2m}} \begin{vmatrix} 1 & \dfrac{\boldsymbol{\sigma}\mathbf{p}}{E + m} \\ \\ \dfrac{\boldsymbol{\sigma}\mathbf{p}}{E + m} & 1 \end{vmatrix} \tag{11}$$

In equation (9) $\Psi_{k\ldots h}(o)$ (with $h, \ldots, k = 1, 2, 3$) is a four-component spinor of the form

$$\psi_{k\ldots h}(o) = \begin{bmatrix} \chi_{k\ldots h} \\ \\ 0 \end{bmatrix} \tag{12}$$

where $\chi_{k\ldots h}$ is a Pauli spinor and has the properties of being: symmetric and traceless in its indices $k \ldots h$ and orthogonal to the matrices σ, i.e. satisfying

$$\sigma^k \chi_{k\ldots h} = 0 \tag{13}$$

It is easy to verify that the representation (9) provides us with a wave function satisfying the required properties (1), (2), (3) and (4). In general any possible coupling takes the form

$$\bar{\psi}^{\mu\ldots\lambda}(p_1)\; \Gamma_{\mu\ldots\lambda,\,05}^{\rho\ldots\sigma}\; \varphi_{\rho\ldots\sigma}(p_2) \tag{14}$$

where $\psi_{\mu\ldots\lambda}(p_1)$ describes the particle of spin-parity $J_1^{P_1}$ and momentum $p_1 \equiv (\mathbf{p}, E_1)$, while $\varphi_{\rho\ldots\sigma}(p_2)$ describes the particle of spin-parity $J_2^{P_2}$ and momentum $p_2 \equiv (\mathbf{p}, E_2)$. The matrix $\Gamma_{\mu\ldots\lambda,\,05}^{\rho\ldots\sigma}$ is constructed from the Dirac matrices and from the available momenta in such a way that the covariant (14) behaves exactly as the fourth component of an axial vector. Explicit calculation of the covariants (14) involves the calculation of matrices of the kind

$$S^+(\Lambda_1)\gamma_4\gamma_R S(\Lambda_2) \tag{15}$$

where γ_R, with $R = 1, \ldots, 16$ belongs to the set of independent Dirac matrices. Because of the form (12) of $\psi_{k\ldots h}(o)$, only the diagonal portion of (15) corresponding to the first two spinor indices is of relevance here. For the cases of interest to us we can limit ourselves to the evaluation of (15) for the following γ_R: 1, γ_4, $-i\,\gamma_4\gamma_5$ and $i\,\gamma_5$. The form taken by the diagonal 2 x 2 portion of $S^+(\Lambda_1)\gamma_4\gamma_R S(\Lambda_2)$ mentioned before is:

$$\frac{m}{\sqrt{m_1 m_2}} \tag{16}$$

for $\gamma_R \equiv 1$, where $m = 1/2(m_1 + m_2)$ is the average of the mass of the particle 1 and of the mass of particle 2;

$$\xrightarrow{\dfrac{E}{\sqrt{m_1 m_2}}}\Big| \tag{17}$$

for $\gamma_R \equiv \gamma_4$, where $E = |\mathbf{p}_1| = |\mathbf{p}_2|$ in the limit when these momenta tend to infinity;

$$\frac{E}{\sqrt{m_1 m_2}} \quad \frac{\sigma \cdot \mathbf{p}}{|\mathbf{p}|} \tag{18}$$

for $\gamma_R \equiv -i\, \gamma_4\gamma_5$; and

$$\frac{\Delta m}{\sqrt{m_1 m_2}} \quad \frac{\sigma \cdot \mathbf{p}}{|\mathbf{p}|} \tag{19}$$

for $\gamma_R \equiv i\gamma_5$, with $\Delta m = m_1 - m_2$. Also of relevance are the contributions from expressions of the kind

$$q^\mu \psi_{\mu \ldots \nu}(p_1) = P^\mu \psi_{\mu \ldots \nu}(p_1) \tag{20}$$

where $q^\mu = (p_1 - p_2)^\mu$ and $P^\mu = (p_1 + p_2)^\mu$. Equation (20) holds by virtue of equation (6). Explicitly, choosing the space axis 3 in the direction of $\mathbf{p}$,

$$q^\mu \psi_{\mu\rho \ldots \nu}(p_1) = q^\mu \Lambda_\mu^m (p_1)\, \Lambda_\rho^r (p_1)\, \ldots\, \Lambda_\nu^n (p_1)\, S(\Lambda) \begin{bmatrix} X_{mr \ldots n} \\ 0 \end{bmatrix} \tag{21}$$

and, in the limit $\mathbf{p} \to \infty$, one has

$$q^\mu \psi_{\mu\rho \ldots \nu}(p_1) = \frac{m}{m_1} \Delta m\, \Lambda_\rho^r(p_1)\, \ldots\, \Lambda_\nu^n(p_1)\, S(\Lambda) \begin{bmatrix} X_{3r \ldots n} \\ 0 \end{bmatrix} \tag{22}$$

We also note that, in the limit $\mathbf{p} \to \infty$,

$$\Lambda_h^\lambda(p_1)\Lambda_\lambda^k(p_2) \to \delta_h^k + \frac{(\Delta m)^2}{2m_1 m_2} \delta_3^k \delta_h^3 \tag{23}$$

This result is useful whenever an index of one spinor is saturated with an index of the other spinor. We give now some examples of calculations of the couplings (14) (the spinors $X_{mr \ldots n}$ and $\xi_{mr \ldots n}$ refer to particles 1 and 2 respectively)

$$\frac{\pm}{2} \longleftrightarrow \frac{1}{2}^\pm:$$

$$-i\bar{\psi}\gamma_4\gamma_5\varphi \to -\frac{E}{\sqrt{m_1 m_2}} \chi^+\sigma_3\xi$$

$$-i\frac{P_4}{2m}\bar{\psi}\gamma_5\varphi \to -\frac{E}{\sqrt{m_1 m_2}}\left(\frac{\Delta m}{m}\right)\chi^+\sigma_3\xi$$

$\dfrac{1}{2}^+ \longleftrightarrow \dfrac{1}{2}^-$:

$$\bar{\psi}\gamma_4\varphi \;\rightarrow\; \frac{E}{\sqrt{m_1 m_2}}\, \chi^+\xi$$

$$\frac{P_4}{2m}\,\bar{\psi}\varphi \;\rightarrow\; \frac{E}{\sqrt{m_1 m_2}}\, \chi^+\xi$$

$\dfrac{1}{2}^\pm \longleftrightarrow \dfrac{3}{2}^\pm$:

$$\bar{\psi}_4\varphi \;\rightarrow\; \frac{E}{\sqrt{m_1 m_2}}\, \chi_3^+\xi$$

$$\frac{q^\mu}{m}\,\bar{\psi}_\mu\gamma_4\varphi \;\rightarrow\; \frac{E}{\sqrt{m_1 m_2}}\left(\frac{\Delta m}{m}\right)\chi_3^+\xi$$

$\dfrac{1}{2}^+ \longleftrightarrow \dfrac{3}{2}^\pm$:

$$-i\,\bar{\psi}_4\gamma_5\varphi \;\rightarrow\; \frac{E}{\sqrt{m_1 m_2}}\,\frac{\Delta m}{\sqrt{m_1 m_2}}\, \chi_3^+\sigma_3\xi$$

$$-i\,\frac{q^\mu}{m}\,\bar{\psi}_\mu\dot{\gamma}_4\gamma_5\varphi \quad \text{and} \quad -i\,\frac{P_4}{2m^2}\,q^\mu\psi_\mu\gamma_5\varphi \quad \text{giving the same contribution}$$

$\dfrac{1}{2}^+ \longleftrightarrow \dfrac{5}{2}^-$:

$$-i\,\frac{q^\mu}{m}\,\bar{\psi}_{\mu4}\gamma_5\varphi \;\rightarrow\; \frac{E}{\sqrt{m_1 m_2}}\left(\frac{\Delta m}{m}\right)^2\chi_{33}^+\sigma_3\xi$$

the other possible couplings giving the same contribution

$\pm \longleftrightarrow \dfrac{3^{\pm}}{2}:$

$$-i\,\overline{\psi}^{\lambda}\gamma_4\gamma_5\varphi\lambda \;\rightarrow\; \frac{E}{\sqrt{m_1 m_2}}\left[\chi_h^{+}\sigma_3\xi_h + \frac{1}{2}\frac{(\Delta m)^2}{m_1 m_2}\,\chi_3^{+}\sigma_3\xi_3\right]$$

$$-i\,\frac{q^{\lambda}}{m}\,\overline{\psi}\lambda\gamma_5\varphi_4 \;\rightarrow\; \frac{E}{\sqrt{m_1 m_2}}\;\frac{(\Delta m)^2}{m_1 m_2}\,\chi_3^{+}\sigma_3\xi_3$$

$+ \longleftrightarrow \dfrac{3^{-}}{2}:$

$$\overline{\psi}^{\lambda}\gamma_4\varphi\lambda \;\rightarrow\; \frac{E}{\sqrt{m_1 m_2}}\left[\chi_h^{+}\xi_h + \frac{1}{2}\frac{(\Delta m)^2}{m_1 m_2}\,\chi_3^{+}\xi_3\right]$$

$$\frac{q^{\lambda}}{m}\,\overline{\psi}\lambda\varphi_4 \;\rightarrow\; \frac{E}{\sqrt{m_1 m_2}}\;\frac{\Delta m}{\sqrt{m_1 m_2}}\,\chi_3^{+}\xi_3$$

$- \longleftrightarrow \dfrac{5^{-}}{2}:$

$$\overline{\psi}_4\lambda\varphi^{\lambda} \;\rightarrow\; \frac{E}{\sqrt{m_1 m_2}}\left[\chi_{3h}^{+}\xi_h + \frac{1}{2}\frac{(\Delta m)^2}{m_1 m_2}\,\chi_{33}^{+}\xi_3\right]$$

$$\frac{q^{\mu}q^{\nu}}{m^2}\,\overline{\psi}_{\mu\nu}\varphi_4 \;\rightarrow\; \frac{E}{\sqrt{m_1 m_2}}\left(\frac{\Delta m}{m}\right)^2\chi_{33}^{+}\xi_3$$

ot all the possible covariants have been reported above. However the ones reported give all hose covariants which are linearly independent in the limit **p** → ∞. To make the calculations easible we shall make two approximations:

(i) Inside each supermultiplet, in our case **56** and **20** with $L = 1$, $\Delta m/m$ is neglected;

(ii) In the couplings between components of the different supermultiplets, i.e. in our case, in the couplings between **56** and (**20**, $L = 1$), only the lowest order couplings in $\Delta m/m$ are con- sidered. Under such assumptions we have the following dependences from the helicity of the matrix elements of F_5

$$< \frac{1^{\pm}}{2} \; |F_5| \; \frac{1^{\pm}}{2} > \; \propto \; \chi^{+}\sigma_3\xi \qquad\qquad < \frac{3^{\pm}}{2} \; |F_5| \; \frac{1^{\mp}}{2} > \; \propto \; \chi_3^{+}\sigma_3\xi$$

$$< \frac{1^{\pm}}{2} \; |F_5| \; \frac{1^{\mp}}{2} > \; \propto \; \chi^{+}\xi \qquad\qquad < \frac{3^{\pm}}{2} \; |F_5| \; \frac{3^{\pm}}{2} > \; \propto \; \chi_k^{+}\sigma_3\xi_k$$

$$< \frac{1^{+}}{2} \; |F_5| \; \frac{5^{-}}{2} > \; \propto \; \chi^{+}\sigma_3\xi_{33} \qquad\qquad < \frac{3^{\pm}}{2} \; |F_5| \; \frac{3^{\mp}}{2} > \; \propto \; \chi_k^{+}\xi_k$$

$$< \frac{3^{\pm}}{2} \; |F_5| \; \frac{1^{\pm}}{2} > \; \propto \; \chi_3^{+}\xi \qquad\qquad < \frac{5^{-}}{2} \; |F_5| \; \frac{3^{-}}{2} > \; \propto \; \chi_{i3}^{+}\xi_i$$

Including now the SU_3 indices we write the states as $|R\rho, \; J^P \, h >$, where the additional labels R and ρ denote respectively the dimensionality of the SU_3 representation and the set of quantum numbers I, I_3 and Y. The matrix elements of $J_{0\lambda}^5(\mathbf{o}, \; t)$ can be written as

$$< R_1\rho_1, \; J_1^{P_1}h_1 |J_{o\lambda}^5(\mathbf{o}, \; t)|R_2\rho_2, \; J_2^{P_2}h_2 > \; = \; \frac{1}{(2\pi)^3} \; \delta_{h_1h_2} C(J_1^{P_1}J_2^{P_2}, \; h) \; \times$$

$$\tag{24}$$

$$\sum_{\xi} \; G_{\xi}(R_1R_2 \; | \; J_1^{P_1}J_2^{P_2}) \; \begin{pmatrix} R_2 & 8 & R_1 \, \xi \\ \rho_2 & \lambda & \rho_1 \end{pmatrix}$$

where: $C(J_1^{P_1}J_2^{P_2}, \; h)$ are the values assumed by the couplings of our preceding discussion for the various cases of $J_1^{P_1}$ and $J_2^{P_2}$; $G_{\xi}(R_1R_2 \; | \; J_1^{P_1}J_2^{P_2})$ are the unknown coupling coefficients and the sum is extended over all the representations R_{ξ} connected through the Clebsch-Gordan coefficient in equation (24). For $J_1^{P_1} = J_2^{P_2} = \frac{1^+}{2}$ our normalization coincides with that of Lee [4]. We note that from the property expressed by equation (3) the couplings $C(J_1^{P_1}, \; J_2^{P_2}, \; h)$ will satisfy

$$C(J_1^{P_1}J_2^{P_2}, \; h) \; = \; - \; P_1P_2 C(J_1^{P_1}J_2^{P_2}, \; -h) \tag{25}$$

Also one has the symmetry relation

$$C(J_1^{P_1}J_2^{P_2}, \; h) \; = \; C(J_2^{P_2}J_1^{P_1}, \; h) \tag{26}$$

For the vector generators F_v one can similarly write

$$< R_1 \rho_1, \; J_1^{P_1} h_1 | F_v | R_2 \rho_2, \; J_2^{P_2} h_2 > = \delta^3(\mathbf{p}_1 - \mathbf{p}_2) \delta_{J_1^{P_1} J_2^{P_2}} \delta_{R_1 R_2} \delta_{h_1 h_2} f(R_1) \begin{pmatrix} R_2 & 8 & R_{1a} \\ \rho_2 & v & \rho_1 \end{pmatrix}$$

(27)

where for the octet representation $f(8) = \sqrt{3}$ and for the decuplet $f(10) = \sqrt{6}$. Substituting into equation (1), taken between states $|R_1, \; J_1^{P_1} >$ and $|R_2, \; J_2^{P_2} >$, we obtain the equation

$$\sqrt{\frac{R_1}{Q}} \; \xi_1 (R_2^* R_1 Q_\beta) \xi_2 (R_2 Q R_{1\beta}) \xi_3 (R_2 R_1^* Q_\beta^*) \quad [1 - \xi_1 (88 Q_\alpha)] \; \times$$

$$\sum_{T \xi \xi' j^\pi} T \, G_\xi (T R_1 | j^\pi J_1^{P_1}) G_{\xi'} (T R_2 | j^\pi J_2^{P_2}) (T \xi' \xi | \beta_{II} (R_1 8 R_2 8) | Q \alpha \beta) \; \times$$

(28)

$$C(J_1^{P_1} j^\pi, \; h) C(J_2^{P_2} j^\pi, \; h) = -\sqrt{3} \, \delta_{Q8} \delta_{\alpha a} \delta_{\beta a} \delta_{R_1 R_2} \delta_{J_1^{P_1} J_2^{P_2}} R_1 f(R_1)$$

In equation (28) the notations are the same as those used by de Swart [8]: in particular $(T \xi' \xi | \beta_{II} (R_1 8 R_2 8) | Q \alpha \beta)$ is a crossing matrix, the symbols $\xi_1 (R_2^* R_1 Q_\beta)$, $\xi_2 (R_2 Q \, R_{1\beta})$, etc. are phase factors, $Q \alpha \beta$ stands for 1, 8_{aa}, 8_{as}, 8_{sa}, 8_{ss}, 10, 10^*, 27. The values of $(R_1 J_1^{P_1})$, $(R_2, J_2^{P_2})$, $(T, \; j^\pi)$ vary over the set of baryon states selected to saturate the commutation relations.

3. Solution of the Saturated Current Algebra Equations

The compact form, equation (28), is general and applies to any problem of the kind we are discussing. Specialized to our case, with the particular choice of saturating states we have made, it leads to a set of 49 non-linear equations to be satisfied by the unknown strengths $G_\xi (R_1 R_2 | J_1^{P_1} J_2^{P_2})$. In spite of their apparent complication it is possible to solve the system of equations. We find essentially (i.e. apart from some choices of signs) two significant solutions (on the requirements that $D/F > o$ and $G(8, 10 | 1/2^+, 3/2^+) \neq o$). The two solutions are reported in Table 1. We recall that the strengths G are all in principle susceptible of physical interpretations in terms of extrapolations of matrix elements of high energy neutrino processes, and in terms of their connection to strong couplings implied by Goldberger-Treiman arguments.

TABLE 1

Solutions of the current-algebra equations, equation (28). Solution I implies the

relation $-\dfrac{G_A}{G_V} = \dfrac{1}{3}\dfrac{D+F}{D-F}$, well satisfied experimentally.

Solution II implies instead the relation $-\dfrac{G_A}{G_V} = \dfrac{1}{3}\dfrac{D+F}{2F-D}$.

The strength for the transition between the states $J_1^{P_1}$ of the SU_3 representation R_1 and the states $J_2^{P_2}$ of the representation R_2 (for $R_1 = R_2 = 8$ the index ξ distinguishes between symmetric (s) and antisymmetric (a) coupling is called $G_\xi(R_1 R_2 \mid J_1^{P_1} J_2^{P_2})$. The solutions depend on two parameters, a and d, subject to the inequalities $0 \leqslant a \leqslant 1$ and $4/5\sqrt{3} \leqslant |d| \leqslant 4\sqrt{3}$. The symbols $\varepsilon_1\varepsilon_2\varepsilon_3$, $\eta_1\eta_2\eta_3$ denote arbitrary factors ± 1. For $a = 1$ the solutions reproduce well-known SU_6 results for the $1/2^+$ octet and the $3/2^+$ decuplet.

	I	II
$G_a(88 \mid \frac{1^+}{2} \frac{1^+}{2})$	$\dfrac{2}{\sqrt{3}}\,a$	$\dfrac{2}{\sqrt{3}}\,a$
$G_s(88 \mid \frac{1^+}{2} \frac{1^+}{2})$	$\sqrt{\dfrac{5}{3}}\,\dfrac{1}{3}(1+2a)$	$\sqrt{\dfrac{5}{3}}\,\dfrac{4a-1}{3}$
$G(8\ 10 \mid \frac{1^+}{2} \frac{3^+}{2})$	$\varepsilon_2\, 2\sqrt{15a}$	$\varepsilon_2\sqrt{60a}$
$G_a(88 \mid \frac{1^+}{2} \frac{3^-}{2})$	0	0
$G_s(88 \mid \frac{1^+}{2} \frac{3^-}{2})$	$\dfrac{2}{3}\sqrt{10(1-a)}\,(-\varepsilon_1\varepsilon_3)$	$\dfrac{2}{3}\sqrt{10(1-a)}\,(-\varepsilon_1\varepsilon_3)$
$G_a(88 \mid \frac{1^+}{2} \frac{1^-}{2})$	$\varepsilon_1\dfrac{2}{\sqrt{3}}\sqrt{a(1-a)}$	$\varepsilon_1\dfrac{2}{\sqrt{3}}\sqrt{a(1-a)}$
$G_s(88 \mid \frac{1^+}{2} \frac{1^-}{2})$	$\varepsilon_1\dfrac{2}{3}\sqrt{\dfrac{5a(1-a)}{3}}$	$\varepsilon_1\dfrac{4}{3}\sqrt{\dfrac{5a(1-a)}{3}}$
$G(18 \mid \frac{3^-}{2} \frac{1^+}{2})$	$\varepsilon_1\eta_1\eta_2\, 4\sqrt{\dfrac{(1-a)(25d^2-48)}{27d^2-48}}$	$\varepsilon_1\eta_1\eta_2\, 4\sqrt{\dfrac{(1-a)(25d^2-48)}{27d^2-48}}$
$G(18 \mid \frac{1^-}{2} \frac{1^+}{2})$	$-\eta_2\varepsilon_1\, 4d\sqrt{\dfrac{2(1-a)}{3(27d^2-48)}}$	$-\eta_2\varepsilon_1\, 4d\sqrt{\dfrac{2(1-a)}{3(27d^2-48)}}$
$G(10\ 10 \mid \frac{3^+}{2} \frac{3^+}{2})$	$\sqrt{6}$	$\sqrt{6}$

	I	II
$G\ (8\ 10\ \vert\ \dfrac{3^-}{2}\ \dfrac{3^+}{2})$	0	0
$G\ (8\ 10\ \vert\ \dfrac{1^-}{2}\ \dfrac{3^+}{2})$	$\epsilon_1\epsilon_2\ \sqrt{10(1-a)}$	$\epsilon_1\epsilon_2\ \sqrt{10(1-a)}$
$G_a(88\ \vert\ \dfrac{3^-}{2}\ \dfrac{3^-}{2})$	0	0
$G_s(88\ \vert\ \dfrac{3^-}{2}\ \dfrac{3^-}{2})$	$-\sqrt{\dfrac{5}{3}}$	$+\sqrt{\dfrac{5}{3}}$
$G_a(88\ \vert\ \dfrac{3^-}{2}\ \dfrac{1^-}{2})$	0	0
$G_s(88\ \vert\ \dfrac{3^-}{2}\ \dfrac{1^-}{2})$	$\epsilon_3\ 4\sqrt{\dfrac{5}{3}}\ a$	$\epsilon_3\ 4\sqrt{\dfrac{5}{3}}\ a$
$G\ (18\ \vert\ \dfrac{1^-}{2}\ \dfrac{3^-}{2})$	$\eta_1\ \sqrt{25d^2-48}$	$\eta_1\ \sqrt{25d^2-48}$
$G\ (18\ \vert\ \dfrac{3^-}{2}\ \dfrac{3^-}{2})$	d	d
$G\ (18\ \vert\ \dfrac{5^-}{2}\ \dfrac{3^-}{2})$	$\eta_3\ \sqrt{16-3d^2}$	$\eta_3\ \sqrt{16-3d^2}$
$G\ (18\ \vert\ \dfrac{5^-}{2}\ \dfrac{1^+}{2})$	0	0
$G_a(88\ \vert\ \dfrac{1^-}{2}\ \dfrac{1^-}{2})$	$\dfrac{2}{\sqrt{3}}\ (1-a)$	$\dfrac{2}{\sqrt{3}}\ (1-a)$
$G_s(88\ \vert\ \dfrac{1^-}{2}\ \dfrac{1^-}{2})$	$\sqrt{\dfrac{5}{3}}\ (1-\dfrac{2}{3}\ a)$	$\sqrt{\dfrac{5}{3}}\ (1-\dfrac{4}{3}\ a)$
$G\ (18\ \vert\ \dfrac{1^-}{2}\ \dfrac{1^-}{2})$	$\eta_2\ 4d\ \sqrt{\dfrac{2a}{3(27d^2-48)}}$	$\eta_2\ 4d\ \sqrt{\dfrac{2a}{3(27d^2-48)}}$
$G\ (18\ \vert\ \dfrac{3^-}{2}\ \dfrac{1^-}{2}$	$-\eta_1\eta_2\ 4\sqrt{\dfrac{6a(25d^2-48)}{27d^2-48}}$	$-\eta_1\eta_2\ 4\sqrt{\dfrac{6a(25d^2-48)}{27d^2-48}}$

Of the two significant solutions, solution I agrees with experiment, while solution II disagrees. We note that in both solutions the important strengths $G_a(88 \mid 1/2^+ 1/2^+)$ and $G_s(88 \mid 1/2^+ 1/2^+)$, which determine all octet-octet axial transitions depend on the single parameter a (the other parameter d only occurs in higher transitions). Eliminating this parameter and expressing the strengths in terms of the usual parameters G_A/G_V and D/F leads to the relations:

for solution I:

$$-\frac{G_A}{G_V} = \frac{1}{3}\frac{D + F}{D - F} \qquad (29)$$

for solution II:

$$-\frac{G_A}{G_V} = \frac{1}{3}\frac{D + F}{2F - D} \qquad (30)$$

Furthermore the inequalities $0 \leqslant a \leqslant 1$ that must be satisfied by the parameter a give

for solution I:

$$\frac{D}{F} \geqslant \frac{3}{2} \qquad\qquad -\frac{G_A}{G_V} \leqslant \frac{5}{3} \qquad (31)$$

for solution II:

$$\frac{D}{F} \leqslant \frac{3}{2} \qquad\qquad -\frac{G_A}{G_V} \leqslant \frac{5}{3} \qquad (32)$$

We note that both solutions (29) and (30) contain the SU_6 limit

$$\frac{D}{F} = \frac{3}{2} \qquad\qquad -\frac{G_A}{G_V} = \frac{5}{3}$$

However only solution I allows for $D/F \geqslant 3/2$, as suggested by the experiment. With $-G_A/G_V = 1.2$ one obtains $D/F = 1.76$, in agreement with experiment [9].

From Table 1 we can also obtain a relation between the $N^* N$ axial weak coupling at zero momentum transfer and G_A. Defining the $N^* N$ weak axial current as

$$\frac{G}{\sqrt{2}} F_1^A \; \bar{\psi}_\mu \psi \; + \; \ldots$$

where ψ_μ denotes the N^* and ψ the nucleon, and we have neglected terms vanishing in the static

limit, we find, for both solutions I and II

$$F_1^A(o)^2 = -\frac{4}{g}\left(3\,\frac{G_A}{G_V} + 1\right)$$

which gives for $-G_A/G_V = 1.2$

$$F_1^A(o)^2 = 1.16$$

in good agreement with experiment.

4. The Linearized Problem

Both solutions I and II reproduce for $a = 1$ the SU_6 results for the strengths G limited to the multiplets of 56, i.e. for $(8, 1/2)$ and for $(10, 3/2)$. However only solution I is capable, for limiting values of the parameters a and d, to reproduce the complete SU_6 solution both for 56 and for 20 with $L = 1$, and, of course, with zero mixing between the two supermultiplets. The selection in favor of solution I can thus be decided, on purely theoretical basis, from the requisite (continuity postulate), that it must continuously develop from the SU_6 symmetric solution by turning on the mixing parameters. Once we assume that the physical significant solution must satisfy such a requisite we can think of an iterative scheme based on linearized equations, which allows us to avoid the mathematical complexity due to the non-linearity of the problem. In general we shall proceed as follows. Suppose we saturate the current-algebra equations by a set of supermultiplets of SU_6 or $SU_6 \otimes O_3$. We denote by S_1, S_2, ... the different super-multiplets. The left-hand side of a commutation relation like (1) can be written in block-form

$$\left|\begin{array}{ll} < S_1|\,\mathscr{C}\,|S_1 > & < S_1|\,\mathscr{C}\,|S_2 > \,\ldots \\[2em] < S_2|\,\mathscr{C}\,|S_1 > & < S_2|\,\mathscr{C}\,|S_2 > \,\ldots \end{array}\right| \qquad (33)$$

where

$$< S_i \,|\,\mathscr{C}\,|\, S_j >$$

denotes the submatrix obtained by taking the matrix element of the commutator $\mathscr{C}$ between the multiplets of S_i and those of S_j. We now look for a solution infinitesimally close to the SU_6 solution. The non-diagonal elements S_{ij} with $i \neq j$ of (33) are linear in the supposedly in-finitesimal transition strengths between S_i and S_j, as long as we neglect quadratic terms (in-cluding those leading from S_i to S_k with $k \neq i, j$). From the commutator equation we thus obtain, for the $i \longleftrightarrow j$ amplitudes a finite homogeneous linear set of equations, in the unknown strengths $i \longleftrightarrow j$. Such a system may have non-trivial solution only if its rank is less than the number of unknowns. If this is the case we say that S_i and S_j are "connected", otherwise we say that they are "unconnected".

Consider now a diagonal submatrix $< S_i|\,\mathscr{C}\,|S_i >$ and keep only terms linear in the deviation

from the SU_6 solution (which we suppose to be well determined). The commutation equation imposes to these terms to be vanishing (this is a form of the non-renormalization theorem).

A deviation from SU_6 arises only if we include quadratic terms which arise in the saturation of the commutation relation with intermediate states of S_i itself and of those S_j "connected" to S_i. Inserting the solution for $i \longleftrightarrow j$ transitions, again we obtain a linear system of equations, which, when solved, yields the renormalized $i \longleftrightarrow i$ strengths.

If one applies this procedure restricting, as we have done above, to 56 and to $(20, L = 1)$ we find that the linearized non diagonal subsystem for the transitions $(56) \longleftrightarrow (20, L = 1)$ consists of 13 linear homogeneous equations in six unknowns. The rank of the system is five, leaving a single infinity of solutions depending on one parameter. Inserting these solutions in the diagonal subsystems $56 \longleftrightarrow 56$ and $(20, L = 1)$ one obtains the second order solution which is however sufficient to give the relation (29) between G_A/G_V and D/F.

It is also important to note that the assumption (11) can be dropped in the linearized calculation. The results are in fact unmodified when we allow for all couplings at any order in $\Delta m/m$. Application of the same procedure with restriction to 56 and to a negative parity 56 gives no non-trivial solution except SU_6. In fact these representations are easily seen to be "unconnected". Thus the mixing problems between 56 and another 56 multiplets of negative parity resonances has no solution, evolving with continuity from SU_6, except SU_6 itself. The simplicity of the linearized approach makes it easy to consider all similar cases.

Conclusions and Outlooks

A close analysis of the non-linear set of equations obtained from saturation of the chiral $U(3) \otimes U(3)$ commutation relations with the states of 56 and $(20, L = 1)$, shows the existence of two solutions (reported on Table 1) passible of physical interpretation. The solutions depend on two undetermined parameters. Solution I leads to the relation

$$- \frac{G_A}{G_V} = \frac{1}{3} \frac{D + F}{D - F} .$$

and to the inequality $D/F \geqslant 3/2$, both satisfied by experiment. Solution II does not agree with data. The choice of solution I determines (apart from some ambiguities in sign) all the axial transition strengths among the stable baryon states of 56 and the resonant states of 20 with $L = 1$, in terms of D/F (or G_A/G_V and another parameter).

The solutions depend on two undetermined parameters, related to the transition strengths between the two supermultiplets, in such a way that when these strengths are made vanishingly small one recovers the SU_6 solution for the 56 with no transition between the two supermultiplets. However only the empirically favored solution I reproduces, for vanishing mixings, the complete SU_6 solution both for 56 and $(20, L = 1)$. The choice in favor of solution I could thus be decided, purely theoretically, from the requisite that only solutions which develop continuously from the SU_6 solution, when the mixing parameters are increased from zero value, are acceptable solutions. We call this requirement "continuity postulate". The assumption of such a postulate suggests a <u>linearized approach</u> to the problem consisting in solving by iteration the commutator equations for small mixing parameters. The <u>linearized approach</u> can be formulated in general terms leading to the notions of <u>connected</u> and <u>unconnected</u> supermultiplets (with

respect to the chiral algebra). For two supermultiplets S_i, S_j the criterium of connectedness is given in terms of the rank of the determinant arising in the linearized set of equations obtained by restricting the saturated algebra to transitions from S_i to S_j. The two supermultiplets are connected if the rank is less than the number of SU_3 independent amplitudes $S_i \longleftrightarrow S_j$: otherwise they are unconnected. The transition strengths inside the same supermultiplets are only renormalized at the second iteration. The renormalization is obtained by saturation of the commutation algebra between initial and final S_i, by only inclusion of connected intermediate S_j (and of S_i itself) up to second order in the mixing strengths.

The linearized approach, applied to the mixing problem of 56 and (20, L = 1) immediately reproduces solution I, at second order, giving in particular the relation (29) between G_A/G_V and D/F. By the same procedure 56 is seen to be unconnected to a negative parity supermultiplet (56, L = 0). Classification of the negative parity baryonic resonances according to such a multiplet is therefore less appealing than our assumed classification based on (20, L = 1) [6]. This type of "indirect" application of current algebra may reveal itself very useful, and we are considering further extensions. The transition strengths between meson states can also be studied along the same lines. A discussion of higher meson resonances has led us to propose a classification according to the representation 35 with L = 1 [10]. By saturating the $U(3) \otimes U(3)$ chiral algebra among the mesons of 35 and the mesons of 35 with L = 1 one obtains verifiable predictions about the widths and branching ratios of the different modes of decay.

An interesting application concerns also the commutation relations of moments of local vector and axial currents satisfying the commutation relations of the chiral algebra.

The circumstance noted above, of the appearance of new parameters from the addition of a new set of saturating states, related to the transition strengths to such states, shows that no essential contradictions should arise when the set of saturating states is increased to include more and more states, unless the extension is made following wrong classification schemes. In principle one may then speculate of saturating the commutation relations in terms of representations of a non-compact group, except for the apparent mathematical complexities of such a problem.

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