

ELECTRON SCATTERING ON MAGNETIC IMPURITIES IN METALS AND ANOMALOUS RESISTIVITY EFFECTS

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Abstract

A special quantum field theory technique for a system of spins was used to evaluate the resistivity of metals containing paramagnetic impurities, assuming $J/\epsilon_F \ll 1$, for an arbitrary value of $(J/\epsilon_F) \ln(\epsilon_F/T)$, where J is the exchange scattering amplitude and ϵ_F the Fermi energy. The first term of this series has been found earlier by Kondo [1]. It is shown that exchange and ordinary interactions give independent contributions to the resistivity. For a ferromagnetic sign of the exchange interaction between the electron and the impurity ($J > 0$), the exchange component of resistivity decreases with temperature and disappears at $T = 0$. In the reverse case ($J < 0$), the resistivity starts increasing when the temperature decreases. After going through a maximum (for $T = T_{\max}$), where the exchange resistivity, due to a local impurity atom, is of the same order of magnitude as the usual resistivity, the exchange resistivity, even in this case, goes to zero at $T = 0$. Such a behavior is related to the resonant nature of the scattering amplitude for $J < 0$. The calculation assumes that the impurity spins are completely disordered, i.e., the temperature is higher than the Curie temperature of the impurity ferromagnetism. Since the latter is proportional to the concentration, while T_{\max} does not depend on concentration, the results obtained are reasonable for sufficiently small concentration.

THE observation of a minimum in the resistivity of some metals as a function of temperature has attracted attention for a long time. A study of this effect has shown that it is probably related to the presence of impurity atoms with unfilled inner shells in the original non-magnetic metal. However, until recently there had been no satisfactory explanation of the interaction of electrons with such atoms and for the occurrence of a minimum. Such an explanation has been advanced by Kondo [1] who calculated the second order approximation (compared to Born's approximation) for the scattering probability of electrons by an impurity, caused by an exchange interaction J . It then turned out that in the scattering probability there is a correction of the order of $(J/\epsilon_F) \ln(\epsilon_F/\xi)$, where ξ is the electron energy measured from ϵ_F . This gives a correction to the resistivity of the order of $-(J/\epsilon_F) \ln(\epsilon_F/T)$, which results in an increase in resistivity with decreasing temperature for a negative sign of J (antiferromagnetic interaction between electrons and impurities).

Kondo's calculation is reasonable within two restrictions. First, the correction must be small. Second, the spins of the impurity atoms must not be correlated. In other words, the

temperature must be above the Curie point for impurity ferromagnetism (or antiferromagnetism). It is well known (see for example Ref. 2) that the Curie point is proportional to the impurity concentration. Therefore if the concentration is small enough, the relative correction in resistivity, which is proportional to $(J/\epsilon_F) \ln(\epsilon_F/T)$ becomes of the order of or greater than unity even before the onset of spin ordering. In this case the perturbation theory is not usable and one must sum up the whole series. This calculation is the subject of this paper.

I. Techniques for Spins

The main difficulty in the calculation comes from the non-commutative nature of the spin operators for the impurities. Let us note that it is this very point which, according to Ref. 1, leads to logarithmic terms. This means that the exchange interaction Hamiltonian

$$H_{int} = - (J/N) \sum_n \psi_{\alpha}^+(\vec{r}_n) \vec{\sigma}_{\alpha\alpha'} \hat{S}_n \psi_{\alpha'}(\vec{r}_n) \quad (1)$$

(we consider the interaction to be a δ -function) is in fact not equivalent to the interaction of the electron spin with an external magnetic field. On the other hand, the operator \hat{S}_n is not an ordinary field operator [3]. The fact is that every average of the type $\langle T [\hat{S}^i(t_1) \hat{S}^k(t_2), \dots, \hat{S}^l(t_n)] \rangle$, where all the \hat{S}^i refer to one atom, (i is the direction index x, y, z), $\langle \dots \rangle = S_p \langle \dots \rangle / (2S + 1)$ cannot at all be represented by a sum of products of the type $\langle T [\hat{S}^i(t_1) \hat{S}^k(t_2)] \rangle \langle T [\hat{S}^l(t_3) \hat{S}^m(t_4)] \rangle \dots + \dots$. Because of this, in a calculation of effects where the non-commutation of the components of \hat{S} is essential a special technique is needed.

A number of such techniques were derived for spin $S = 1/2$ a few years ago by I.E. Dzyaloshinsky and the author (unpublished). Here, we use a variation of these which we were able to generalize to the case of an arbitrary spin.

Let us introduce operators for a virtual Fermi field $a_{n\beta}, a_{n\beta}^+$ corresponding to the spin operator \hat{S}_n according to the formula:

$$a_{n\beta}^+ \vec{S}_{\beta\beta'} a_{n\beta'} = \hat{S}_n \quad (2)$$

where the $S_{\beta\beta'}$ are spin matrices. Each index β and β' assumes $2S + 1$ values: $-S, \dots, S$.

The operators \hat{S}^i , introduced according to (2), have the proper commutation relations when we assign the usual commutation relations to the operators $a_{n\beta}$:

$$\begin{aligned} \{ a_{n\beta}^+ a_{m\beta'} \} &= \delta_{\beta\beta'} \delta_{mn} \\ \{ a_{n\beta} a_{m\beta'} \} &= 0 \end{aligned} \quad (3)$$

However, they have one important inadequacy: The operators a_{β} introduce fictitious states. In fact, let us assume that spin \hat{S} has a projection M on a chosen axis. There is a corresponding state Φ_M for which $a_{M\Phi_M} \neq 0, a_{\beta \neq M\Phi_M} = 0$, i.e., there is only one "particle" corresponding to

operator a_M and there are no "particles" corresponding to operators $a_{\beta \neq M}$. However, it is quite clear that by introducing operators a_{β} we have also brought into consideration non-physical states in which either all the occupation numbers are zero or more than one of them is equal to one. These states must be excluded.

In the case of spin $S = 1/2$, this happens automatically in the following way. All the physical quantities contain only the averages of products of operators \hat{S}^i . Other combinations of a_{β} are not involved. But the operators \hat{S}^i themselves, for $S = 1/2$, have the property of giving zero when acting on the non-physical states (0, 0) or (1, 1). Thus, one can average over all the states. However, one must introduce a normalization. Since in a complete average, the result is divided by 4^n , while physically only 2^n states are relevant, where n is the number of different atoms participating in the average, the result must be multiplied by 2^n .

This procedure can be easily generalized to the case $S > 1/2$. The operators (2), in this case, give zero when acting on the state (0, 0 ... 0), but the result may be different from zero when they operate on the states where more than one occupation number is equal to 1. Because of this, we shall consider that each "particle" has an energy $\lambda \gg T$. Therefore, the greatest contribution will come from those states where $\sum_{\beta} N_{\beta}$ is minimum (i.e. 1). This procedure also requires a correct normalization so that the average of the form $(2S + 1)^{-1} S_p(\hat{S}^i \hat{S}^k, \dots)$ (for one atom) has the correct value. The value calculated by our method must be multiplied by $e^{\lambda/T}/(2S + 1)$ and then we must let $\lambda \rightarrow \infty$. In the general case, the result must be multiplied by $[e^{\lambda/T}/(2S + 1)]^n$ where n is the number of participating impurity atoms.

But this normalization leads, in fact, to an essential inconvenience when the above technique is used. The fact is that, in contrast to the usual methods of field theory, the averages of the operators pertaining to a single atom and of the operators for different atoms do not have the same weight. This is very important in problems dealing with spin ordering. However, in the case considered here, each impurity atom scatters the electrons independently (this will be demonstrated later), and therefore one can consider a single atom. Then, the normalization causes no difficulty.

We use the finite temperature technique of field theory. The "free" Green's functions for the operators a_{β} (for one atom) have then the form:

$$\begin{aligned} \mathcal{G}_{\beta\beta'}(\tau - \tau') &= - \langle T [a_{\beta}(\tau) a_{\beta'}^+(\tau')] \rangle \\ \mathcal{G}_{\beta\beta'}(\tau) &= T \sum_{\omega} G_{\beta\beta'}(\omega) e^{-i\omega\tau} \\ \mathcal{G}_{\beta\beta'}(\omega) &= \delta_{\beta\beta'} / (i\omega - \lambda), \quad \omega = (2n + 1)\pi T \end{aligned} \quad (4)$$

(here, the average is over all the states). On the diagrams, the \mathcal{G} functions will be represented by dotted lines.

By way of example, let us calculate the simplest self energy diagram for the electron. Beside its illustrative value, this calculation will permit us to anticipate the most important portions of the complete analysis. Let us consider first the case of one impurity atom at

point \vec{r} . The interaction Hamiltonian is given by formula (1). The diagram of the first non-vanishing approximation is shown in Fig. 1. It represents the expression:

$$\sum_{\alpha\alpha'}^{(1)} \left(\vec{p}, \vec{p}', \omega \right) = - \frac{e\lambda/T}{2S+1} \left(\frac{J}{N} \right)^2 Sp \left(S_i S_k \right) \left(\sigma_i \sigma_k \right)_{\alpha\alpha'} e^{i(\vec{p}-\vec{p}')\vec{r}} \times$$

$$\times \int d\vec{p}'' / (2\pi)^3 T^2 \sum_{\omega_1 \omega_2} \frac{1}{i(\omega + \omega_1 - \omega_2) - \xi(\vec{p}'')} \times \frac{1}{i\omega_1 - \lambda} \times \frac{1}{i\omega_2 - \lambda}$$

where ξ is the electron energy with respect to the Fermi level.

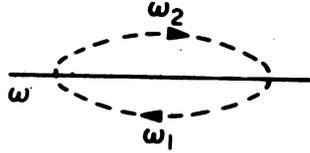


FIGURE 1

Having summed over ω_1 and ω_2 we get in place of $T^2 \sum_{\omega_1 \omega_2}$, the expression:

$$- \frac{1}{4} \frac{1}{i\omega - \xi(\vec{p}'')} \left(\tanh \xi/2T + \tanh \lambda/2T \right) \left(\coth \frac{\lambda + \xi}{2T} - \tanh \lambda/2T \right)$$

Assuming $\lambda \gg T$, we get simply $-e^{-\lambda/T} [i\omega - \xi(\vec{p}'')]^{-1}$. Averaging over the positions \vec{r} of the impurity atom, and summing over all the impurity atoms, integrating over \vec{p}'' and discarding the constant part of Σ which gives only a small renormalization of the chemical potential, we get:

$$\sum_{\alpha\alpha'} \left(\vec{p}, \vec{p}', \omega \right) = \delta_{\alpha\alpha'} \delta(\vec{p} - \vec{p}') \left(2\pi \right)^3 \Sigma(\omega)$$

$$\Sigma(\omega) = - (i/2\tau_0) \text{sign } \omega$$

$$\frac{1}{\tau_0} = (J/N)^2 S(S+1) N_i \frac{p_0^m}{\pi}$$

(5)

where N_i is the number of impurity atoms per unit volume.

Among the terms of further approximations to the self-energy there may be terms depending on only one impurity atom and terms depending on a great number of such atoms. Let us consider first of all the second order term, which depends only on a single impurity atom. There are two diagrams corresponding to this case (Fig. 2a and b). For diagram 2a, after averaging over the positions and summing over all the atoms, we have:

$$\sum_{\alpha\alpha'}^{(2a)} (\vec{p}, \omega) = -\frac{e\lambda/T}{2S+1} (J/N)^3 S_p(S_i S_k S_l) (\sigma_i \sigma_k \sigma_l)_{\alpha\alpha'} \times$$

$$\times T^3 \sum_{\omega_1 \omega_2 \omega_3} \int \frac{d\vec{p}'}{(2\pi)^3} \frac{d\vec{p}''}{(2\pi)^3} \frac{1}{i(\omega + \omega_3 - \omega_1) - \xi(\vec{p}')} \times \frac{1}{i(\omega + \omega_3 - \omega_2) - \xi(\vec{p}'')} \times$$

$$\times \frac{1}{i\omega_1 - \lambda} \times \frac{1}{i\omega_2 - \lambda} \times \frac{1}{i\omega_3 - \lambda}$$

In this expression, one can easily see the rules for writing the formula corresponding to a diagram. (a) Each dotted line has its own frequency. The electron frequencies are determined by conservation rules. (b) The electron momentum on every internal line is independent, and one



FIGURE 2.

should integrate over the momenta. (c) One takes the $\alpha\alpha'$ component of the product of electron spin operators in the order of their position on the electron line. (d) All the dotted lines corresponding to one atom form together a closed loop. A trace is taken of the product of the impurity spin operators in the order of their position along the loop.

Summing over $\omega_1, \omega_2,$ and ω_3 and assuming $\lambda \gg T$, we get:

$$e^{-\lambda/T} \left[\frac{1}{e^{-\xi(\vec{p}'')/T} + 1} \times \frac{1}{\xi(\vec{p}') - \xi(\vec{p}'')} \times \frac{1}{i\omega - \xi(\vec{p}')} + \right.$$

$$\left. + \frac{1}{e^{-\xi(\vec{p}')/T} + 1} \times \frac{1}{\xi(\vec{p}'') - \xi(\vec{p}')} \times \frac{1}{i\omega - \xi(\vec{p}'')} \right]$$

These are two completely symmetrical expressions. One can readily see that the first term gives a logarithmic integral, for positive values of $\xi(\vec{p}'')$, in the region $T, \xi(\vec{p}') \ll \xi(\vec{p}'') \ll \epsilon_F$. In principle, the integral over \vec{p}'' extends over a larger range. In fact, its upper limit is a momentum of the order of the reciprocal radius of interaction; the latter can be considerably lower than $1/p_0$ for d or f orbitals. However, as will be shown in Section 3, this inconvenience disappears in a renormalization of the scattering amplitude J . As to the remaining integral, in that we can take $\xi(\vec{p}') \sim \omega$. Thus, in diagram 2a one of the integrals over $\xi(\vec{p}')$, can be logarithmic, but then the other one must necessarily be non-logarithmic. Since either integral may

be logarithmic. two identical terms appear.

As a whole, we get, with logarithmic accuracy:

$$\sum^{(2a)} = \frac{i}{2\tau_0} \text{sign } \omega \frac{2J}{N} \frac{\rho_0^m}{2\pi^2} \ln \frac{\epsilon_F}{|\omega|}$$

In the same way we find that the Σ^{2b} corresponding to diagram (2b) is equal to Σ^{2a} . The result can be interpreted as a modification of the quantity $1/\tau_0$:

$$\frac{1}{\tau_0} \rightarrow \frac{1}{\tau_0} \left(1 - \frac{4J}{N} \frac{\rho_0^m}{2\pi^2} \ln \frac{\epsilon_F}{|\omega|} \right) = \frac{1}{\tau_0} \left(1 - \frac{3zJ}{\epsilon_F} \ln \frac{\epsilon_F}{|\omega|} \right) \quad (6)$$

(z is the number of electrons per atom). This is in fact the result obtained by Kondo [1] by a different method.

Having analysed the above calculations, one can easily see that they can be described as follows: In each of the diagrams 2a and 2b we can draw two vertical sections through one electron line and two spin lines. Let us make such a section. In two summations and one integral corresponding to such a section, the essential values of the variables are of the order of ω . The remaining sum and integral are of logarithmic type, in which the essential values of the variables are:

$$|\omega| \ll |\omega_1| \sim |\xi_1| \ll \epsilon_F.$$

In each diagram one must draw all the possible sections across three lines. The diagrams may be analysed in the following way. If one retains those terms which have the highest power of the log for a given value of J [these are the terms $J^{n+2}(\ln \epsilon_F/|\omega|)^n$], then we must conclude that these are diagrams in which the integral along the section is taken for $|\xi|, |\omega_i| \sim |\omega|$, while all the other integrals are logarithmic and the important values of the variables are $|\xi_i|, |\omega_i| \gg |\omega|$. This situation strongly reminds one of the study of electron damping in a Fermi liquid by G.M. Eliashberg [4], and this is why we use his method.

However, before doing so, let us consider the diagrams of the next order in which not one, but two impurity atoms participate. Elementary diagrams are shown in Figs. 3a and 3b. If one



FIGURE 3

averages over impurity positions, one can readily see that the contribution of diagram 3a compared to diagram 1 is $1/\tau\epsilon_F$, i.e. very small.

This pertains to all other diagrams with intersecting lines from various atoms. As to diagrams with internal inclusions (Type 3b), one can reason as follows. In accord with the foregoing, the role of the internal inclusion (see Ref. 5) consists in replacing $G = (i\omega - \xi)^{-1}$ by $G' = (i\omega + i/2\tau \text{sign } \omega - \xi)^{-1}$. Let us introduce in diagram 3b the substitution $G' = G + (G' - G)$

and in the term with $G' - G$ we shall integrate again over ξ . This yields zero. This means that only the integral with G remains. This happens in all the diagrams with internal inclusions. Therefore, it is sufficient to consider only diagrams for a single impurity atom.

2. The Self-Energy Part

As already stated, we shall use the method of Reference 4. In any of the diagrams which may be of interest to us, we can make a section across three lines. The sums and integrals over these variables must be calculate accurately since the values of the arguments are quite small. As to the other portions of the diagram, their variables have quite large values, and, with logarithmic accuracy, we can replace all the sums over frequencies by integrals. Thus, we must calculate the full sum of diagrams having the form shown in Fig. 4, where the cross-hatched

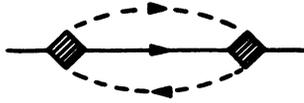


FIGURE 4.

squares represent the sum of all "four tails" calculated with logarithmic accuracy. We shall designate this by the symbol:

$$\Gamma_{\alpha\beta, \alpha'\beta'} \left(\omega, \vec{p}, \omega_1; \omega + \omega_1 - \omega_2, \vec{q}, \omega_2 \right)$$

The following expression corresponds to the diagram of Fig. 4:

$$\begin{aligned} & - \frac{e\lambda/T}{2S + 1} N_i T^2 \sum_{\omega_1 \omega_2} \int \frac{d\vec{q}}{(2\pi)^3} \Gamma_{\alpha\beta, \alpha'\beta'} \left(\omega, \vec{p}, \omega_1; \omega + \omega_1 - \omega_2, \vec{q}, \omega_2 \right) \times \\ & \times \Gamma_{\alpha'\beta', \alpha\beta} \left(\omega + \omega_1 - \omega_2, \vec{q}, \omega_2; \omega, \vec{p}, \omega_1 \right) G \left(\omega + \omega_1 - \omega_2, \vec{q} \right) \mathcal{G}(\omega_1) \mathcal{G}(\omega_2) \end{aligned} \quad (7)$$

where $G(\omega, \vec{p}) = [i\omega - \xi(\vec{p})]^{-1}$, $\mathcal{G}(\omega) = (i\omega - \lambda)^{-1}$. We shall see later that Γ will depend only on $\ln \epsilon_F/|\omega|$. Therefore, we can make the transition from a summation over imaginary frequencies to an integration over real values and place ω on the real axis. From symmetry considerations, it follows that the spin dependence of Γ has the form:

$$\Gamma = \Gamma^{(0)} + \Gamma^{(\sigma)}(\vec{\sigma}, \vec{S})$$

As in Reference (4) we have:

$$\begin{aligned} & \frac{1}{2S + 1} \Gamma_{\alpha\beta, \alpha'\beta'}(1, 2; 3, 4) \Gamma_{\alpha'\beta', \alpha\beta}(3, 4; 1, 2) = \\ & = \left[|\Gamma^{(0)}|^2 + S(S + 1) |\Gamma^{(\sigma)}|^2 \right] \delta_{\alpha\alpha'} \end{aligned}$$

Having transformed the sum over the frequencies ω_1 and ω_2 as in reference (4) we get:

$$\begin{aligned}
 T^2 \sum_{\omega_1 \omega_2} G(\omega + \omega_1 - \omega_2, \vec{q}) \mathcal{G}(\omega_1) \mathcal{G}(\omega_2) &= \\
 &= \frac{i}{(2\pi)^2} \int \text{Im } \mathcal{G}_R(\omega_1) \text{Im } \mathcal{G}_R(\omega_2) \text{Im } G_R(\omega + \omega_1 - \omega_2) \times \\
 &\times \left(\tanh \frac{\omega_2}{2T} - \coth \frac{\omega_2 - \omega}{2T} \right) \left(\tanh \frac{\omega_1}{2T} - \tanh \frac{\omega + \omega_1 - \omega_2}{2T} \right) d\omega_1 d\omega_2
 \end{aligned} \tag{8}$$

Here, we note that in the integrals small values of ω_1 , ω_2 are essential (considering that $T \ll \lambda \ll \epsilon_F$). Setting $\text{Im } G_R(\omega) = -\pi\delta(\omega - \xi)$, $\text{Im } \mathcal{G}_R(\omega) = -\pi\delta(\omega - \lambda)$ and taking into account the condition $\lambda \gg T$, we get:

$$\sum = -i \text{sign } \omega N_i \frac{p_0 m}{2\pi} \left[|\Gamma^{(0)}(\omega, \lambda; \omega, \lambda)|^2 + S(S+1) |\Gamma^{(\sigma)}(\omega, \lambda; \omega, \lambda)|^2 \right] \tag{9}$$

which differs from the first approximation (see eq. 5) by the substitution of the square bracket for $(J/N)^2 S(S+1)$.

Now let us examine Γ . As already mentioned, Γ can be calculated as if it were at $T = 0$, replacing all the summations over frequency by integrals. But these are still integrals along the imaginary axis. The author of this paper has demonstrated (unpublished) that all these integrals can be transformed to the real axis, together with all the external arguments. Thereupon, all the temperature Green functions may be replaced by time functions at $T = 0$ via the transformation $i\omega \rightarrow \omega + i\delta \text{sign } \omega$. Then we shall do the following. The integrals over ω_i involve only the frequencies on the spin lines. Since all these integrals are taken from $-\infty$ to $+\infty$, we can substitute $\omega - \lambda \rightarrow \omega$. Since at every vertex there is a spin line entering and one leaving, conservation laws are not violated. We must only make a transformation from $\omega - \lambda$ to ω in the external frequencies of the lines.

Since we must find $\Gamma(\omega, \lambda; \omega, \lambda)$, after all the transformations the task is reduced to a calculation of $\Gamma(\omega, 0; \omega, 0)$ in the usual method of time diagrams at $T = 0$, whereupon the G function becomes $(\omega - \xi + i\delta \text{sign } \xi)^{-1}$ and the \mathcal{G} function $(\omega + i\delta)^{-1}$ since $\lambda > 0$.

The first diagrams for the vertex part are shown in Fig. 5. We can express them as follows (corrections to J/N):

$$\begin{aligned}
 -i(J/N)^2 (\vec{\sigma} \cdot \vec{S})^2 \frac{p_0 m}{2\pi^2} \int \frac{d\omega_1}{2\pi} \int d\xi_1 \frac{1}{\omega_1 + i\delta} \times \frac{1}{\omega - \omega_1 - \xi_1 + i\delta \text{sign } \xi_1} &= \\
 = (J/N)^2 \frac{p_0 m}{2\pi^2} \left[S(S+1) - \vec{\sigma} \cdot \vec{S} \right] \left[\ln \frac{\epsilon_F}{|\omega|} + i\pi\theta(\omega) \right] - & \\
 -i(J/N)^2 \sigma^i \sigma^k S^k S^i \frac{p_0 m}{2\pi^2} \int \frac{d\omega_1}{2\pi} \int d\xi_1 \frac{1}{\omega_1 + i\delta} \times \frac{1}{\omega + \omega_1 - \xi_1 + i\delta \text{sign } \xi_1} &=
 \end{aligned} \tag{10a}$$

$$= - (J/N)^2 \frac{\rho_0 m}{2\pi^2} \left[S(S + 1) + \vec{\sigma} \vec{S} \right] \left[\ln \frac{\epsilon_F}{|\omega|} + i\pi\theta(-\omega) \right] \tag{10b}$$

where $\theta(x) = \begin{cases} 1, & x > 0 \\ 0, & x < 0 \end{cases}$ We left the non-logarithmic imaginary term here because it will be needed for further reasoning.

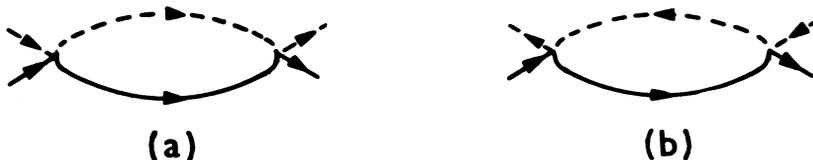


FIGURE 5

By examining higher order diagrams (Fig. 6) one can easily see that the diagrams with the highest power of the logarithm for each order are only those diagrams which can subsequently be reduced by two lines, an electron line and a spin line, in such a way that the result is a

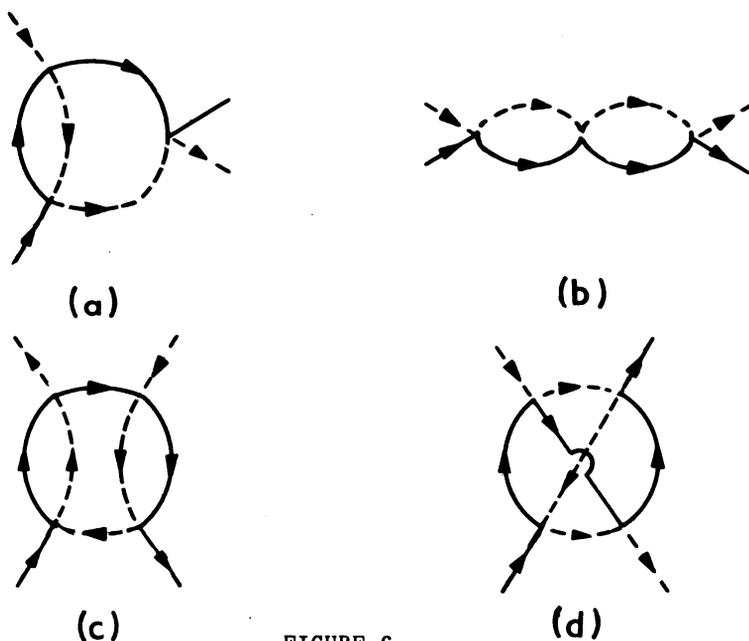


FIGURE 6

simple vertex. For example, by simplifying Fig. 6c. one can first obtain a diagram of the 6a type, then 6b, then a simple vertex. These properties are exhibited by so-called "parquet" diagrams 5a and b, 6a, b, c and are not exhibited by, for example, diagram 6d.

We make a summation of the "parquet" by the method of V.V. Sudakov [5]. Let us point out that there are two different elements of a "parquet", as shown in Fig. 5a and 5b. We can introduce two different summations of the diagrams as shown in Fig. 7a and 7b. Let us designate as Λ_1 the diagram which can be cut through two parallel lines (Fig. 7a) and Λ_2 the diagram which

can be cut through two antiparallel lines (Fig. 7b). It is obvious that $\Gamma = \Gamma_0 + \Lambda_1 + \Lambda_2$, where Γ_0 is the original interaction $J/N(\vec{\sigma} \vec{S})$. Let us point out that in the logarithmic integration inside the internal Λ_1, Λ_2 of Fig. 7a and 7b, the essential values of arguments are larger than in the external lines, i.e. than in corresponding pairs of two parallel or two antiparallel lines.

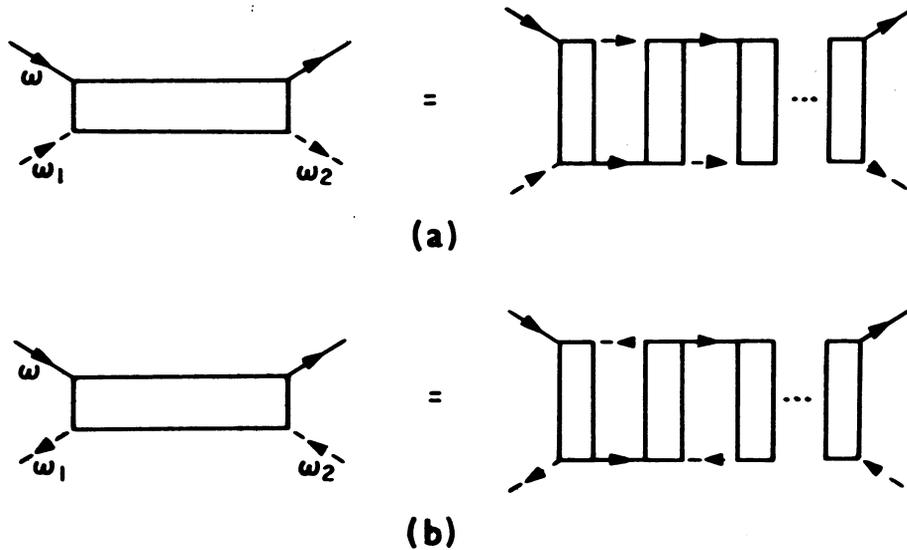


FIGURE 7

Let us examine now the diagram 7a in order to determine Λ_1 . It is obvious that the lower limit for the logarithmic integrals in this diagram is ω (if the external $\omega_1 = \omega_2 = 0$). We shall select that internal dotted line in which ω_i is minimum. To the left and to the right of this element there can be any element of Γ , i.e. in other words, in the sum there is a full Γ both on the left and on the right sides. The quantity ω_i plays the role of ω for these Γ . Thus one can write (with logarithmic accuracy) the equation:

$$\Lambda_2 \alpha\beta, \alpha'\beta', (\omega) = \frac{p_0 m}{2\pi^2} \int_{|\omega|}^{\epsilon_F} \frac{d\omega_1}{\omega_1} \Gamma_{\alpha\beta', \alpha''\beta''}(\omega_1) \Gamma_{\alpha''\beta'', \alpha'\beta}(\omega_1) \quad (11a)$$

and similarly:

$$\Lambda_1 \alpha\beta, \alpha'\beta', (\omega) = \frac{p_0 m}{2\pi^2} \int_{|\omega|}^{\epsilon_F} \frac{d\omega_1}{\omega_1} \Gamma_{\alpha\beta'', \alpha''\beta''}(\omega_1) \Gamma_{\alpha''\beta, \alpha'\beta''}(\omega_1) \quad (11b)$$

Adding these expression we get:

$$\Gamma_{\alpha\beta, \alpha'\beta', (\omega)} = \frac{J}{N} (\vec{\sigma} \vec{S}) + \frac{p_0 m}{2\pi^2} \int_{|\omega|}^{\epsilon_F} \frac{d\omega_1}{\omega_1} \left[\Gamma_{\alpha\beta, \alpha''\beta''}(\omega_1) \Gamma_{\alpha''\beta'', \alpha'\beta}(\omega_1) - \right.$$

$$-\Gamma_{\alpha\beta'', \alpha''\beta'}(\omega_1) \Gamma_{\alpha''\beta, \alpha'\beta''}(\omega_1) \quad (12)$$

Let us introduce the transformation $\ln \varepsilon_F/\omega = x$ and let us look for a solution in the form $\Gamma = \Gamma^{(0)} + (\vec{\sigma} \vec{S})\Gamma^{(\sigma)}$ Then we get:

$$\Gamma^{(\sigma)}(x) = \frac{J}{N} - \frac{p_0 m}{\pi^2} \int_0^x dy \left[\Gamma^{(\sigma)}(y) \right]^2$$

$$\Gamma^{(0)} = 0 \quad (13)$$

Solving the integral equation, we obtain:

$$\Gamma = \frac{(J/N) \vec{\sigma} \vec{S}}{1 + \frac{J p_0 m}{N \pi^2} \ln \frac{\varepsilon_F}{|\omega|}} \quad (14)$$

Substituting into (9), we get:

$$\Sigma = -\frac{i}{2\tau_0} \text{sign } \omega \left[1 + \frac{3Jz}{2\varepsilon_F} \ln \frac{\varepsilon_F}{|\omega|} \right]^{-2} \quad (15)$$

Expressions (14) and (15) have an important property. If $J < 0$, for some value of $|\omega|$ these expressions have a pole. The presence of such a pole indicates that there is a resonance in the scattering. When taking into account the final resonance width one must add to the logarithmic term in the denominator of formula (14) an imaginary component proportional in $i \frac{J p_0 m}{N \pi^2}$. This can already be seen from formula (10). Our approximation is not applicable for finding such terms. Hence, we shall use expressions (14) and (15) far from the pole; but in the vicinity of the pole we shall evaluate only orders of magnitude, on the basis that a term proportional to $i\pi$ is added to $\ln \varepsilon_F/|\omega|$ in the denominator.

3. Ordinary Interaction. Transition to the Scattering Amplitude

Until now we have limited ourselves to the study of the exchange interaction. However, the interaction of the electron with the impurity atom involves necessarily the usual non-exchange term which is of considerably larger magnitude. Naturally, this raises the question of possible interference effects.

The usual interaction, in the absence of exchange interaction, can be considered as an external field, and it is designated by a cross on the electron line.

First of all let us mention that, for the same reasons as earlier, it is sufficient to

examine the interaction with only one atom. Now let us assume that in diagram 5a we have placed on the electron line a number of crosses (Fig. 8). Considering for simplicity's sake that the interaction is isotropic, we get

$$-\frac{i}{2\pi} \left(\frac{J}{N}\right)^2 \left[S(S+1) - \vec{\sigma} \vec{S} \right] \left(\frac{p_0^m}{2\pi^2}\right)^2 \varphi \int d\omega_1 d\xi_1 d\xi_2 \frac{1}{\omega_1 + i\delta} \frac{1}{\omega - \omega_1 - \xi_1 + i\delta \operatorname{sign} \xi_1} \times \\ \times \frac{1}{\omega - \omega_1 - \xi_2 + i\delta \operatorname{sign} \xi_2}$$

where φ designates the whole insert from the first to the last cross. An integration will convince us that the logarithmic expression is not obtained in this case. Taking into account the fact that the sum of all the possible φ has a magnitude ε_F/N , we find that the diagram having the insert is $(\ln \varepsilon_F/\omega)^{-1}$ times smaller than the diagram without insert.

However, a cross can be placed also on that electron line which does not participate in logarithmic integrals. The simplest is to place crosses in the very first diagram 1a. Then, one can be convinced that the diagram with the insert has a magnitude ω/ε_F . From this it follows that the usual interaction gives a separate component in S and cannot interfere with the exchange part. The usual part has been studied in a whole series of papers (see for example, ref.

3, § 39.2). It has the form $-i/2\tau_{\text{ord}} \operatorname{sign} \omega$, where $\frac{1}{\tau_{\text{ord}}} = N_i \frac{mp_0}{\pi} |a|^2$; a is the scattering amplitude corresponding to the non-exchange interaction.

Now, let us examine the second relevant problem. In all of the foregoing calculations, the integrals over ξ were taken in the range $-\varepsilon_F < \xi < \varepsilon_F$. As a matter of fact, these integrals are limited, for positive values of ξ only by the condition $|\vec{p}| < 1/r_{\text{int}}$ where r_{int} is the interaction radius, which may be much smaller than $1/p_0$. Therefore a renormalization of the interactions must be made.

Instead of the Born scattering amplitude, we shall introduce the true amplitude of scattering of an electron by an atom. If \hat{f} is the Born amplitude and \hat{f}_1 is the actual amplitude, then by analogy of ref. 3 § 25.4 we have:

$$\hat{f} = \hat{f}_1 - \hat{f}\hat{f}_1 \frac{p_0^m}{2\pi^1} \int \frac{d\xi_1}{\xi - \xi_1 + i\delta} \quad (16)$$

Until now, the role of \hat{f} was played by the quantity $-\frac{J}{N} \vec{\sigma} \vec{S}$. But now we shall take $\hat{f} = -\frac{J}{N} \vec{\sigma} \vec{S} + f_0$ where f_0 does not depend on spins, and we shall choose f_0 in such a way that equation (16) will be satisfied for $\hat{f}_1 = -\frac{J_1}{N} (\vec{\sigma} \vec{S})$. This means that in the interaction Hamiltonian we have included a part of the usual interaction term. The diagram-method does not suffer from this since the operator $f_0 a_\beta^\dagger a_\beta$ also gives zero when it operates on the state with zero occupation numbers, while the other improper states are excluded by our procedure. This situation, generally speaking, permits a study of the usual interaction together with the exchange interaction. However, since the usual term, except for the renormalization, makes no contribution to the logarithmic integral (inserting the usual interaction in integrals 10a and 10b

gives a mutual cancellation), we have considered it separately. This has the advantage that the usual interaction U is expected to be stronger than the exchange interaction, and therefore a simultaneous study would have imposed an unnecessary limitation: $J/N \ln \epsilon_F/T \gg U$. The inserted term f_0 , as we shall soon see, has a magnitude of $(J/N)(J/\epsilon_F)$ and does not impose new limitations.



FIGURE 8

Solving equation (16) to second order accuracy we have:

$$\hat{f} = \frac{J_1}{N} \vec{\sigma} \vec{S} - \left(\frac{J_1}{N}\right)^2 \frac{p_0^m}{2\pi^2} \left[S(S+1) - \vec{\sigma} \vec{S} \right] \int \frac{d\xi_1}{\xi - \xi_1 + i\delta} \tag{17}$$

When we find second order terms (formula 10), we renormalize in the following way. We replace J by J_1 in (10); furthermore, in first order terms, at the pole in Γ we change to J_1/N using (17). If the integration over ξ is not performed, we get in the sum:

$$\left(\frac{J_1}{N}\right)^2 \left[S(S+1) - \vec{\sigma} \vec{S} \right] \frac{p_0^m}{2\pi^2} \int d\xi_1 \left[\frac{\theta(\xi_1)}{\xi_1 - \omega - i\delta} - \frac{1}{\xi_1 - \xi + i\delta} \right] \tag{10a'}$$

Later on we shall be convinced that to find the conductivity, we shall need the case $\xi \ll \epsilon_F$. In this case we obtain with logarithmic accuracy:

$$-\left(\frac{J_1}{N}\right)^2 \left[S(S+1) - \vec{\sigma} \vec{S} \right] \frac{p_0^m}{2\pi^2} \int d\xi_1 \frac{\theta(-\xi_1)}{\xi_1 - \omega} = \left(\frac{J_1}{N}\right)^2 \frac{p_0^m}{2\pi^2} \left[S(S+1) - \vec{\sigma} \vec{S} \right] \ln \frac{\epsilon_F}{|\omega|}$$

Since now the integration is performed over negative ξ_1 ($|p| < p_0$), there is no question about the range of integration.

This is the situation for the lower approximation. We shall show that such a renormalization can also be performed for higher orders of approximation. It is easiest to do so by going from the n -th order to the order $n+1$. This transition can be accomplished by replacing in turn each simple vertex by the sum of the two vertices shown in Fig. 5a and 5b.

If in the n -th order all the vertices contained J_1/N , then in the new diagrams of order $n+1$ there will be a J_1/N everywhere after this transformation. Furthermore, we must perfect the n -th order diagram by substituting in turn instead of J_1/N the expression (17). In other words, such a transition from n to $n+1$ order of approximation changes the integrals of type (10a) into those of type (10a). If in addition, we take into account that in all the logarithmic

integrals $\omega \sim \xi$, then it practically amounts to having a J_1/N in all vertices and all the integrals are taken over $\xi < 0$, i. e., one can take ϵ_F as the range for integrals over ξ .

4. Resistivity

As we know (see for example ref. (3), § 39.3), the relation between current and vector potential, which is frequency dependent, has the following form in the absence of spatial dispersion:

$$J_i(\omega) = Q_{ik}(\omega)A_k(\omega) \quad (18)$$

here, in the case of isotropic scattering, $Q(\omega)$ represents an analytic continuation to real ω of the expression:

$$\frac{Ne^2}{mc} \delta_{ik} + \frac{2e^2}{(2\pi)^3 m^2 c} T \sum_{\omega'} \int d\vec{p} p_i p_k G(\vec{p}, \omega') G(\vec{p}, \omega' - \omega) \quad (19)$$

where G are temperature Green functions. Let us change from a summation over ω' to an integration over real frequencies (see Ref. (3), § 21.2); then, considering $\omega > 0$ in (19), we can continue this expression analytically to real values. We obtain then:

$$\begin{aligned} \frac{Ne^2}{mc} \delta_{ik} + \frac{e^2}{\pi(2\pi)^3 m^2 c} \int_{-\infty}^{\omega} d\omega' \int d\vec{p} p_i p_k \operatorname{Im} G_R(\vec{p}, \omega') \times \\ \times \left[G_A(\vec{p}, \omega' - \omega) + G_R(\vec{p}, \omega' + \omega) \right] \tanh \frac{\omega'}{2T} \end{aligned} \quad (20)$$

In view of the fact that the G functions do not depend on the direction of \vec{p} and that the integrals over \vec{p} are taken in the neighborhood of the Fermi surface, we can make the substitution:

$$\int \frac{d\vec{p}}{(2\pi)^3} p_i p_k \rightarrow \frac{1}{3} \delta_{ik} \frac{p_0^3 m}{(2\pi)^2} \int d\xi$$

The integration must first be performed over ω and then over ξ . However, as always [3], we shall use the following approach. We shall add and subtract from the expression under the integral the "free" term, i. e. the term corresponding to no interaction with impurities. Then, in the integral with the "free" expression, we shall first integrate over the frequency but, in the convergent term with the difference, we shall integrate first over ξ .

Instead of G_R we shall insert the expression ($G_A = G_R^*$):

$$\left(\omega - \xi + \frac{i}{2\tau(\omega)} \right)^{-1} \quad (21)$$

where:

$$\frac{1}{\tau(\omega)} = \frac{1}{\tau_{ord}} + \frac{1}{\tau_{ex}(\omega)} \quad (22)$$

where τ_{ord} is the non-exchange collision time, and $1/\tau_{ex}(\omega)$ corresponds to the expression (15). After all the operations, assuming $\omega \rightarrow 0$, we get:

$$\vec{j} = \frac{i\sigma\omega}{c} \vec{A}$$

where

$$\sigma = \frac{Ne^2}{m} \int_0^\infty \frac{d\omega\tau(\omega)}{2T \cosh^2(\omega/2T)}$$

With logarithmic accuracy, we obtain the resistivity:

$$\rho = \rho_{ord} + \rho_{ex}$$

where

$$\rho_{ex} = \rho_{0,ex} \left[1 + \frac{3J_1 z}{2\varepsilon_F} \ln \frac{\varepsilon_F}{T} \right]^2 \quad (23)$$

and where

$$\rho_{0,ex} = \frac{3\pi m J_1^2 S(S+1)c}{2N\varepsilon_F e^2 h} \quad (24)$$

and c is the atomic concentration.

If $J_1 > 0$, ρ decreases with decreasing temperatures and becomes zero when $T \rightarrow 0$. In the case of $J_1 < 0$, expression (23) becomes infinite when

$$T_{max} = q \varepsilon_F \exp \left[-\frac{2\varepsilon_F}{3|J_1|z} \right] \quad (25)$$

($q \sim 1$). According to the considerations of Section 2, in the vicinity of such a point the resistivity goes through a maximum. The maximum value is

$$\rho_{ex} \sim \rho_{0,ex} \left(\frac{\varepsilon_F}{J_1} \right)^2$$

According to (24), this expression is of the same order of magnitude as the ordinary resistivity due to the same impurities. The linewidth of the temperature dependent peak is $\Delta T \sim T_0$. With further decrease in temperature $\rho_{ex} \rightarrow 0$.

We have not cleared up the effect of ferromagnetic ordering. However, it probably liquidates the effect we have analysed. Since the ferromagnetic transition temperature $T_c \sim cJ_1^2/\varepsilon_F$ (see Ref. 2), where c is the impurity concentration, in principle the peak can be obtained with any combination of metals with $J_1 < 0$ and for a sufficiently small impurity concentration.

In a future publication we shall consider the effect of impurity ferromagnetism and of external magnetic field.

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