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FALLING CHARGES*

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Abstract

The radiative damping force on an electrically charged particle falling freely in a static weak gravitational field is computed in the nonrelativistic limit of small velocities. It is shown that, in this limit, the force separates naturally into two components, a conservative part which arises from the fact that the mass of the particle is not concentrated at a point but is partly distributed as electric field energy in the space surrounding the particle, and a nonconservative part which depends linearly on both the velocity and the Riemann tensor. The conservative force is shown to correspond to a repulsive inverse square potential and to make a retrograde contribution to the perihelion precession. The nonconservative part is shown to produce an average energy loss identical with that of the traditional formula which is used for accelerations caused by nongravitational forces. Because the nonconservative force depends on the velocity rather than its second derivative, however, the phenomenon of preacceleration does not occur with gravitational forces. The questions answered by this investigation are of conceptual interest only, since the forces involved are far too small to be detected experimentally.

1. Introduction

THE problem of an electrically charged particle falling in a gravitational field raises some of the most delicate issues in classical particle physics. On the basis of flat space-time intuition one expects that an accelerated charge should emit radiation, and hence suffer a reactive damping force, regardless of the nature of the acceleration, whether produced by gravity or by other forces. However, the equivalence principle, as dramatized by the falling elevator concept, injects an element of uncertainty and confusion into the picture. It is the purpose of this paper to remove the confusion by deriving the detailed law of motion in the simple case of a radiating charge moving at nonrelativistic velocities in a weak static (but otherwise arbitrary) gravitational field. In the course of the derivation the limits of validity of flat space-time ideas will become apparent, and we shall see that, carefully applied, they need not be totally abandoned but, on the contrary, agree with the rigorous result in this special case.

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In order to proceed in a fundamentally sound manner one must, from the outset, insist upon two things: (1) The basic equations must be derived completely within the framework of general relativity. (2) The nonlocality of the physical processes involved, which is responsible for the failure of naive applications of the equivalence principle, must be taken into account through a study of the global behavior of the electromagnetic field.

Fortunately, most of the requisite work has already been done. DeWitt and Brehme [1], with the aid of the theory of covariant Green's functions in curved space-time, were able to obtain the following equation of motion for a particle of charge e and mass m, moving in a Riemannian space-time of arbitrary (hyperbolic) metric and subjected to an incoming (externally imposed) electromagnetic field $F_{\mu\nu}{}^{in}$:

$$m\ddot{z}^{\alpha} = eF^{in\ \alpha}{}_{\beta}\dot{z}^{\beta} + \frac{2}{3}e^2(\dddot{z}^{\alpha} - \dot{z}^{\alpha}\ddot{z}^2) + e^2\dot{z}^{\beta}\int_{-\infty}^{\tau} f^{\alpha}{}_{\beta\ \gamma'}\dot{z}^{\gamma'}d\tau'. \tag{1.1}$$

The notation here is the same as in reference 1 [2], except that we have chosen units in which c = 1. In particular,

$$\left.\begin{aligned} \dot{z}^{\alpha} &\equiv dz^{\alpha}/d\tau, \\ \ddot{z}^{\alpha} &\equiv d\dot{z}^{\alpha}/d\tau + \Gamma_{\beta\gamma}{}^{\alpha}\dot{z}^{\beta}\dot{z}^{\gamma}, \\ \dddot{z}^{\alpha} &\equiv d\ddot{z}^{\alpha}/d\tau + \Gamma_{\beta\gamma}{}^{\alpha}\ddot{z}^{\beta}\dot{z}^{\gamma}, \text{ etc.,} \end{aligned}\right\} \tag{1.2}$$

so that $\ddot{z}^{\alpha} = 0$, $\dddot{z}^{\alpha} = 0$, etc., for geodetic motion

The calculation leading to equation (1.1) was patterned directly on Dirac's Lorentz invariant treatment of the classical electron [3], and reduces to it in the limit of flat space-time [4]. Equation (1.1) differs, however, in two important respects from Dirac's equation. First, it possesses a nonlocal term, involving an integral over the past history of the particle, which the Dirac equation does not possess. Second, ordinary derivatives with respect to the proper time are replaced by the covariant derivatives (1.2). The latter difference has the consequence that in the absence of an incoming electromagnetic field equation (1.1) would be solved by geodetic motion if it were not for the nonlocal term. This is a reflection of the fact that the particle tries its best to satisfy the naive equivalence principle, and is only prevented from doing so by the nonlocal nature of its own electromagnetic field.

When a charged particle is accelerated by means of nongravitational forces, the electric field lines which emanate from it bend and redistribute themselves in the vicinity of the particle (i.e., within a distance of the order of the classical radius) in such a way as to exert, on the average, a net retarding force, over and above the force of inertial reaction [5]. With purely gravitational forces, however, this is not the way things happen. The field in the immediate vicinity of the particle tends to fall freely with the particle, and although it suffers a local tidal distortion characteristic of an explicit occurrence of the Riemann tensor [as equation (5.12) of ref. 1 shows], the net retarding force due to this distortion is zero when integrated over solid angle. *The deviation of the particle motion from geodetic when* $F_{\mu\nu}{}^{in} = 0$ *is caused not by the local field of the particle but by a field which originates well outside the classical radius and which is manifested by the nonlocal term of equation (1.1).* This means that, to order e^2, the term of equation (1.1) which *looks* like the usual radiation damping term may be completely ignored [6], and hence our job in this paper is to analyze the nonlocal term and to derive explicit expressions for it which, in the limiting case of nonrelativistic velocities, permit a comparison of the exact equations of motion with those derived from flat space-time concepts.

Physically the nonlocal term arises from a back-scatter process in which the Coulomb field of the particle, as it sweeps over the "bumps" in space-time, receives "jolts" which are propagated back to the particle. The process has its mathematical origin in the fact that the retarded Green's functions for fields of zero rest mass do not vanish inside the light cone in a curved space-time as they do in a flat one.

Since this process differs conceptually so much from the normal radiation damping process occurring for nongravitationally induced accelerations, there is no reason to expect that the precise motion of the particle will coincide with that which would be computed by naive application of the instantaneous energy loss formula

$$dE/dt = \frac{2}{3}e^2(d\ddot{z}^0/dt - \ddot{z}^2), \tag{1.3}$$

the dots now denoting ordinary differentiation with respect to the proper time rather than the covariant differentiation of equations (1.2). Indeed we shall discover that there are quite real differences between the two motions. Nevertheless, it is not difficult to see that the *total* energy loss, or the loss averaged over one period if the motion is that of a bound orbit, will to order e^2, be correctly given by the traditional formula if the gravitational field is weak and vanishes asymptotically. This is because a quasi-Minkowskian coordinate system can then be set up, in which the Green's functions themselves go over asymptotically to those of flat-space time. In such a coordinate system the integrated radiation flux at infinity depends only on the "coordinate motion" of the particle. Moreover, since the expression for the asymptotic field [e.g., equation (3.45) of ref. 1] is invariant under coordinate transformations which are confined to the orbital region of the particle, the computed energy loss cannot depend on which quasi-Minkowskian coordinate system is chosen [7].

This suggests that the force exerted on the particle by its self-field can, in the nonrelativistic limit, be separated in some natural way into a nonconservative part which corresponds immediately to equation (1.3), and a conservative part which gives rise to no net energy loss. We shall find, remarkably enough, that this is indeed the case and that the "anomalous" conservative part has a very simple physical interpretation. It arises from the fact that the total mass of the particle is not concentrated at a point but is partly distributed as electric field energy in the space around the particle. The energy of interaction of this distributed mass with the gravitational field therefore does not correspond to a simple attractive $1/r$ potential but contains, in addition, an "anomalous" repulsive $1/r^2$ component [8].

What is perhaps more remarkable, however, is that the nonconservative force will be found, in the nonrelativistic limit, to depend neither on the acceleration nor on the derivative of the acceleration but simply on the *velocity* of the particle (relative to the static gravitational field) and on the Riemann tensor, the dependence on each being linear [see equation (3.37)].

It is only the close relation which exists between the Riemann tensor and the particle motion which, in the case of free fall, permits the nonconservative force to be recast in the form

$$\underline{F}_{NC} = \frac{2}{3}e^2\,\dddot{\underline{r}}, \tag{1.4}$$

giving rise to

$$dE/dt = \frac{2}{3}e^2\,\dot{\underline{r}}\cdot\dddot{\underline{r}}, \tag{1.5}$$

which is the nonrelativistic limit of (1.3). Herein lies a significant difference between the rigorous theory and the flat space-time theory. Since the primary expression for the nonconservative force involves $\dot{\underline{r}}$ rather than $\dddot{\underline{r}}$, and since equation (1.4) is obtained only by an iteration which approximates the damped motion by undamped motion, the phenomenon of *pre-acceleration* does not occur with gravitational forces.

The calculation of the damping integral will be broken down into several steps. In the next section explicit expressions will be obtained for the function $f^{\alpha}{}_{\beta\gamma'}$, and attention will be focused on the special case in which the gravitational field is produced by a point-like central mass. The integral itself will be computed in section 3 under the assumption of slow motion (i.e., nonrelativistic velocities), and the separation of the damping force into a conservative and a nonconservative part will be explicitly given. The results will then be generalized to the case of an arbitrary weak static gravitational field, and the possibility of expressing the nonconservative force in the form (1.4) will be explicitly shown. Finally, the interpretation stated above for the conservative component of the force will be demonstrated in section 4. An appendix follows at the end.

2. Computation of the "Tail Function"

The function $f^{\alpha}{}_{\beta\gamma'}$ appearing in the nonlocal term of equation (1.1) is called the "tail function" in ref. 1. It is the curl of the nonvanishing component of the retarded vector Green's function inside the light cone.

Explicitly [9],

$$f^{\alpha}{}_{\beta\gamma'} = v^{\alpha}{}_{\gamma';\beta} - v_{\beta\gamma';}{}^{\alpha} \tag{2.1}$$

where

$$G^{-}_{\mu\nu'} = (4\pi)^{-1}\,\theta(x,x')\,[\Delta^{1/2}\,\bar{g}_{\mu\nu'}\delta(\sigma) - v_{\mu\nu'}\theta(-\sigma)], \tag{2.2}$$

$G^{-}_{\mu\nu'}$ being the retarded Green's function, $\theta(x,x')$ the temporal step function, Δ the scalarized Van Vleck determinant, $\bar{g}_{\mu\nu'}$ the parallel displacement bivector, and σ one-half the square of the geodetic distance between x and x'. For points z, z' on the world-line of the particle, with z lying to the future of z', equation (2.2) reduces to

$$v_{\alpha\gamma'} = -4\pi G^{-}_{\alpha\gamma'}. \tag{2.3}$$

A means of computing $v_{\alpha\gamma'}$ in power series in σ is given in ref. 1. This, however, is of no use to us here, since we need the values of $v_{\alpha\gamma'}$ well inside the light cone (σ large and negative). We adopt instead an alternative expansion procedure based on our assumption that the gravitational field is weak. We start from the basic defining equation [10],

$$g^{1/2}(G^{-\mu}{}_{\nu';\sigma}{}^{\sigma} + R^{\mu}{}_{\sigma}G^{-\sigma}{}_{\nu'}) = -\delta^{\mu}{}_{\nu'}, \tag{2.4}$$

and consider how $G^{-\mu}{}_{\nu'}$ changes as the metric is made to suffer variations about a given arbitrary value.

A general method for treating a wide class of problems of this type consists of first suppressing the indices in equation (2.4) and then using a trick due to Schwinger [11] which expresses the Green's function as the matrix element of an abstract operator G^{-} in a fictitious Hilbert space:

$$G^{-}(x,x') = \langle x|G|x'\rangle. \tag{2.5}$$

The basis vectors $|x'\rangle$ are eigenvectors of a set of commuting Hermitian operators x^{μ}:

$$x^{\mu}|x'\rangle = x'^{\mu}|x'\rangle, \tag{2.6}$$

and equation (2.4) may be rewritten in the form

$$FG^{-} = -1, \tag{2.7}$$

$$F \equiv -(\pi_{\mu}\, g^{1/2}\, g^{\mu\nu}\,\pi_{\nu} - \mathcal{R}),$$

where the 1 on the right hand side of (2.7) is really the product of the Kronecker delta and the identity operator in the Hilbert space, and where

$$\langle x|\mathcal{R}^{\mu}{}_{\nu}|x'\rangle = g^{1/2}\,R^{\mu}{}_{\nu}\,\delta(x,x'). \tag{2.9}$$

The operators π_{μ} are defined by

$$\pi_{\mu} \equiv p_{\mu} - iS^{\nu}{}_{\sigma}\Gamma_{\mu\nu}{}^{\sigma}, \tag{2.10}$$

where the p_{μ} are Hermitian operators satisfying the commutation relations

$$[x^{\mu}, p_{\nu}] = i\delta^{\mu}{}_{\nu}, \qquad [p_{\mu}, p_{\nu}] = 0, \tag{2.11}$$

and the $S^{\nu}{}_{\sigma}$ are the Lie-algebra generators of the linear group for the representation to which the Green's function in question corresponds. In the present case the pertinent representation is that of a contravariant vector, and when all indices are displayed we have

$$(S_\sigma{}^\nu)^\rho = \delta^\nu \delta^\rho{}_\sigma. \tag{2.12}$$

Under an infinitesimal variation in the metric the operator F suffers the change [12]

$$\begin{aligned}\delta F = &\frac{1}{2}\pi_\mu g^{1/2}(g^{\mu\sigma}g^{\nu\tau} + g^{\mu\tau}g^{\nu\sigma} - g^{\mu\nu}g^{\sigma\tau})\delta g_{\sigma\tau}\pi_\nu \\ &+ \frac{1}{2}iS^{\sigma\tau}(\delta g_{\mu\tau\cdot\sigma} + \delta g_{\sigma\tau\cdot\mu} - \delta g_{\mu\sigma\cdot\tau})g^{1/2}g^{\mu\nu}\pi_\nu \\ &+ \frac{1}{2}i\pi_\mu g^{1/2}g^{\mu\nu}S^{\sigma\tau}(\delta g_{\nu\tau\cdot\sigma} + \delta g_{\sigma\tau\cdot\nu} - \delta g_{\nu\sigma\cdot\tau}) + \delta\mathcal{R}.\end{aligned} \tag{2.13}$$

in which use has been made of the well known variational equation

$$\delta\Gamma_{\mu\nu}{}^\sigma = \frac{1}{2}g^{\sigma\tau}(\delta g_{\mu\tau\cdot\nu} + \delta g_{\nu\tau\cdot\mu} - \delta g_{\mu\nu\cdot\tau}). \tag{2.14}$$

Taking the variation of equation (2.7), we obtain

$$F\delta G^- = -\delta F G^-, \tag{2.15}$$

which, in view of the kinematics of the retarded Green's function, has the solution

$$\delta G^- = G^-\delta F G^-. \tag{2.16}$$

Inserting (2.13) into (2.16) and taking the matrix element between the vectors $\langle x|$ and $|x'\rangle$, we then obtain

$$\begin{aligned}\delta G^-(x,x') = \int\Big[&\frac{1}{2}G^-_{\cdot\mu''}(x,x'')g''^{1/2}(g^{\mu''\sigma''}g^{\nu''\tau''} + g^{\mu''\tau''}g^{\nu''\sigma''} - g^{\mu''\nu''}g^{\sigma''\tau''})\delta g_{\sigma''\tau''}G^-_{\cdot\nu''}(x'',x') \\ &+ \frac{1}{2}G^-(x,x'')S^{\sigma''\tau''}(\delta g_{\mu''\tau''\cdot\sigma''} + \delta g_{\sigma''\tau''\cdot\mu''} - \delta g_{\mu''\sigma''\cdot\tau''})g''^{1/2}g^{\mu''\nu''}G^-_{\cdot\nu''}(x'',x') \\ &- \frac{1}{2}G^-_{\cdot\mu''}(x,x'')g''^{1/2}g^{\mu''\nu''}S^{\sigma''\tau''}(\delta g_{\nu''\tau''\cdot\sigma''} + \delta g_{\sigma''\tau''\cdot\nu''} - \delta g_{\nu''\sigma''\cdot\tau''})G^-(x'',x') \\ &+ G^-(x,x'')\delta\mathcal{R}''G^-(x'',x')\Big]d^4x'',\end{aligned} \tag{2.17}$$

where $g^{\mu''\sigma''}$ is an abbreviation for $g^{\mu\sigma}(x'')$, etc.

When the gravitational field is weak a coordinate system may be introduced for which the metric tensor takes the form

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}, \tag{2.18}$$

where $\eta_{\mu\nu}$ = diag (–1, 1, 1, 1) and the components of $h_{\mu\nu}$ are small compared to unity. To first order in $h_{\mu\nu}$ we may write

$$G^- = {}^oG^- + {}^oG^-\Delta F\,{}^oG^-, \tag{2.19}$$

where ΔF is given by equation (2.13) with $\delta g_{\mu\nu}$ replaced by $h_{\mu\nu}$ and $g_{\mu\nu}$ by $\eta_{\mu\nu}$, and where $^{\circ}G^{-}$ is the retarded Green's function in a Lorentz frame in flat space-time:

$$^{\circ}G^{-}(x,x'') = (4\pi)^{-1}\theta(x,x'')\delta(\sigma), \tag{2.20}$$

$$\sigma \equiv \frac{1}{2}(x - x'')^2. \tag{2.21}$$

Furthermore, if x and x' lie inside each other's light cones [13], with x to the future of x', then, since $^{\circ}G^{-}$ vanishes off the light cone, we may write

$$v(x,x') = -4\pi\Delta G^{-}(x,x'), \tag{2.22}$$

where ΔG^{-} is given by equation (2.17) with G^{-} replaced by $^{\circ}G^{-}$ and the other quantities taken as in ΔF. Making use of the fact that $^{\circ}G^{-}(x,x'')$ depends only on the difference $x - x''$ we find, upon restoring all indices and integrating by parts in equation (2.17), the following equation, valid to first order:

$$\begin{aligned} v_{\mu\nu'} = -(8\pi)^{-1}\int [&\eta_{\mu\nu}\delta_{,\sigma}(\sigma)(2h^{\sigma''\tau''} - \eta^{\sigma\tau}h_{\rho''}{}^{\rho''})\delta_{,\tau'}(\sigma') \\ &-\delta_{,\mu}{}^{\sigma}(\sigma)h_{\sigma''\nu''}\delta(\sigma') - \delta_{,}{}^{\sigma}(\sigma)h_{\sigma''\nu''}\delta_{,\mu'}(\sigma') \\ &+\delta_{,\nu}{}^{\sigma}(\sigma)h_{\sigma''\mu''}\delta(\sigma') + \delta_{,}{}^{\sigma}(\sigma)h_{\sigma''\mu''}\delta_{,\nu'}(\sigma') \\ &+\delta_{,\mu}(\sigma)h_{\sigma''\nu''}\delta_{,}{}^{\sigma'}(\sigma') + \delta(\sigma)h_{\sigma''\nu''}\delta_{,\mu'}{}^{\sigma'}(\sigma') \\ &-\delta_{,\nu}(\sigma)h_{\sigma''\mu''}\delta_{,}{}^{\sigma'}(\sigma') + \delta(\sigma)h_{\sigma''\mu''}\delta_{,\nu'}{}^{\sigma'}(\sigma') \\ &+2\delta(\sigma)R_{\mu''\nu''}\delta(\sigma')]\,d^4x'', \end{aligned} \tag{2.23}$$

where

$$\sigma' \equiv \frac{1}{2}(x'' - x')^2 \tag{2.24}$$

and

$$R_{\mu\nu} \equiv \frac{1}{2}(h_{\mu\nu,\sigma}{}^{\sigma} + h_{\sigma}{}^{\sigma}{}_{,\mu\nu} - h_{\mu\sigma,\nu}{}^{\sigma} - h_{\nu\sigma,\mu}{}^{\sigma}), \tag{2.25}$$

the comma denoting ordinary differentiation.

We may record here also the corresponding expression for the scalar Green's function $G^{-}(x,x')$ which satisfies the equation

$$g^{1/2}G^{-}{}_{;\mu}{}^{\mu}(x,x') = -\delta(x,x'), \tag{2.26}$$

namely,

$$v(x,x') = -(8\pi)^{-1}\int \delta_{,\mu}(\sigma)(2h^{\mu''\nu''} - \eta^{\mu\nu}h_{\sigma''}{}^{\sigma''})\delta_{,\nu'}(\sigma')\,d^4x'' \tag{2.27}$$

This expression may be used as a check on equation (2.23) through the relation

$$v^{\mu}{}_{\nu',\mu} = -v_{,\nu'}(x,x') \tag{2.28}$$

which is a necessary consequence of the more general relation

$$G^{-\mu}{}_{\nu',\mu} = -G^{-}{}_{\cdot\nu'}(x,x'), \tag{2.29}$$

[see ref. 1, equation (2.75)].

Let us now suppose that the gravitational field is produced by a mass M located at the origin of spatial coordinates. Then, if harmonic coordinates are employed [14], the gravitational "potentials" $h_{\mu\nu}$ at a point $x = (t, \underline{r})$ are given, in the weak field approximation, by

$$h_{\mu\nu} = \frac{2GM}{r}\delta_{\mu\nu}, \tag{2.30}$$

and the Ricci tensor takes the form

$$R_{\mu\nu} = GM\,\delta_{\mu\nu}\,\nabla^2\frac{1}{r} = -4\pi GM\,\delta_{\mu\nu}\delta(\underline{r}), \tag{2.31}$$

where G is the gravitation constant. It should be remarked that the singularity of expressions (2.30) and (2.31) at $r = 0$ does not represent a violation of the weak field approximation. The mass M is here understood to have a radius r_M which satisfies

$$r_M \gg GM \tag{2.32}$$

and these expressions really hold only for $r > r_M$. Nevertheless, since the internal structure of M is of no consequence at the present level of approximation, the mass is adequately represented by a delta function, and we shall see that no unwanted divergences are subsequently engendered by this idealization.

Inserting (2.30) and (2.31) into (2.23) we obtain [15]

$$\begin{aligned} v_{\mu\nu'} = -\frac{1}{2}GM[2\eta_{\mu\nu}(I_{,\sigma\sigma'} - I_{,\sigma}{}^{\sigma'}) - I_{,\mu}{}^{\nu} - I_{,}{}^{\nu}{}_{\mu'} + I_{,\mu}{}^{\nu'} + I_{,\mu'}{}^{\nu'} \\ + I_{,\nu}{}^{\mu} + I_{,}{}^{\mu}{}_{\nu'} - I_{,\nu}{}^{\mu'} - I_{,\nu'}{}^{\mu'} + \delta_{\mu\nu}(I_{,ii} + 2I_{,ii'} + I_{,i'i'})], \end{aligned} \tag{2.33}$$

where

$$I \equiv \frac{1}{2\pi}\int \delta(\sigma)\frac{1}{r''}\delta(\sigma')\,d^4x''. \tag{2.34}$$

The evaluation of this integral is carried out in the appendix. The result is

$$\begin{aligned} I = \frac{1}{|\underline{r} - \underline{r}'|}\Bigg[\theta(r + r' - t + t')\log\frac{r + r' + |\underline{r} - \underline{r}'|}{r + r' - |\underline{r} - \underline{r}'|} \\ + \theta(t - t' - r - r')\log\frac{t - t' + |\underline{r} - \underline{r}'|}{t - t' - |\underline{r} - \underline{r}'|}\Bigg], \end{aligned} \tag{2.35}$$

which shows an abrupt change in functional form according as the time interval between t and t' is less than or greater than the time it takes a signal to propagate from the point $\underline{r}'$ to the central mass M and thence to the point $\underline{r}$. In the next section we shall examine the derivatives of I which appear in equation (2.33).

Component by component we find, from (2.33) [15]

$$v_{oo'} = GM(3I_{,oo'} - I_{,ii'}), \tag{2.36}$$

$$v_{oi'} = -v_{io'} = GM(I_{,oi'} - I_{,io'}), \tag{2.37}$$

$$v_{ij'} = -GM[\delta_{ij}(I_{,oo'} + I_{,kk'}) + I_{,ij'} - I_{,ji'}], \tag{2.38}$$

in which use has been made of

$$I_{,o} = -I_{,o'} \tag{2.39}$$

and

$$I_{,\mu}{}^{\mu} = I_{,\mu'}{}^{\mu'} = 0, \tag{2.40}$$

which hold as long as x and x′ lie inside each other's light cones. For the scalar function (2.27) we likewise find

$$v(x,x') = -2\,GM I_{,oo'}, \tag{2.41}$$

which may be used as a check on equations (2.36), (2.37) and (2.38) through the relation (2.28).

3. Evaluation of the "Tail Integral"

Inserting expressions (2.36), (2.37) and (2.38) into the integral of equation (1.1), we obtain for the force exerted on the particle by its self-field,

$$\begin{aligned}
F_i &= e^2\int_{-\infty}^{t} f_{i\alpha\beta'}\dot z^{\alpha}\dot z^{\beta'}dt' = e^2\int_{-\infty}^{t}(v_{i\beta',\alpha} - v_{\alpha\beta',i})\dot z^{\alpha}\dot z^{\beta'}dt' \\
&= e^2\int_{-\infty}^{t}[v_{io',o} - v_{oo',i} + (v_{io',j} - v_{jo',i})\dot z^{j} \\
&\qquad + (v_{ij',o} - v_{oj',i})\dot z^{j'} + (v_{ik',j} - v_{jk',i})\dot z^{j}\dot z^{k'}]dt' \\
&= e^2 GM\int_{-\infty}^{t}[-2(I_{,ioo'} + I_{,ioj'}\dot z^{j'}) - I_{,ooi'} + I_{,ijj'} + (I_{,oij'} - I_{,oji'})\dot z^{j} \\
&\qquad + (-\delta_{ij}I_{,ooo'} - \delta_{ij}I_{,okk'} + I_{,oji'} + I_{,ijo'})\dot z^{j'} \\
&\qquad + (-\delta_{ik}I_{,joo'} - \delta_{ik}I_{,jll'} - I_{,kji'} + \delta_{jk}I_{,ioo'} + \delta_{jk}I_{,ill'} + I_{,kij'})\dot z^{j}\dot z^{k'}]dt',
\end{aligned} \tag{3.1}$$

where the dots now denote differentiation with respect to coordinate time. Similarly, we obtain for the rate of work done by this force,

$$\begin{aligned}
dE/dt &= e^2\int_{-\infty}^{t} f^{o}{}_{\alpha\beta'}\dot z^{\alpha}\dot z^{\beta'}dt' = -e^2\int_{-\infty}^{t}(v_{o\beta',\alpha} - v_{\alpha\beta',o})\dot z^{\alpha}\dot z^{\beta'}dt' \\
&= -e^2\int_{-\infty}^{t}[(v_{oo',i} - v_{io',o})\dot z^{i} + (v_{oj',i} - v_{ij',o})\dot z^{i}\dot z^{j'}]dt' \\
&= F_i\dot z^{i}.
\end{aligned} \tag{3.2}$$

The first term in the integrand of the final expression for F_i in (3.1) may be expressed as a total time derivative:

$$I_{,ioo'} + I_{,ioj'}\dot z^{j'} = \frac{d}{dt'}I_{,io} \tag{3.3}$$

Since $I_{,io}$ vanishes both when $t' = t$ and when $t' = -\infty$ [see equation (3.7) below] this term makes no contribution to the integral and may be dropped. We shall also drop the term in $\dot z^{j}\dot z^{k'}$. The latter term arises from the magnetic component of the self-field and not only does no work on the particle but, in the nonrelativistic limit

$$\dot{z}^i \ll 1, \tag{3.4}$$

is also much smaller in order of magnitude than the other terms. In fact, when the particle is in a bound orbit it is inconsistent to keep this term. This is because, for a bound orbit,

$$\dot{z}^i \sim (GM/r)^{1/2}, \tag{3.5}$$

and the term becomes of the same order as those which arise only in the *second* order of approximation in $h_{\mu\nu}$. We therefore work, from now on, with the reduced expression

$$F_i = e^2 GM \int_{-\infty}^{t} [-I_{,ooi'} + I_{,ijj'} + (I_{,oij'} - I_{,oji'})\dot{x}_j + (-\delta_{ij} I_{,ooo'} - \delta_{ij} I_{,okk'} + I_{,oji'} + I_{,ijo'})\dot{x}'_j] dt' \tag{3.6}$$

in which the symbols $\dot{z}^j$ and $\dot{z}^{j'}$ have for convenience been replaced by $\dot{x}_j$ and $\dot{x}'_j$ respectively.

By straightforward computation starting from equation (2.35), one finds [16]

$$I_{,io} = -\delta(t - t' - r - r')\sigma^{-1} r_{,i} - \theta(t - t' - r - r')\sigma^{-2}(x_i - x'_i), \tag{3.7}$$

$$I_{,ooi'} = -\delta'(t - t' - r - r')\sigma^{-1} r'_{,i'} + \delta(t - t' - r - r')\sigma^{-2}[-(t - t')r'_{,i'} + x_i - x'_i] + 2\theta(t - t' - r - r')\sigma^{-3}(t - t')(x_i - x'_i), \tag{3.8}$$

$$I_{,ijj'} = -\delta'(t - t' - r - r')\sigma^{-1} r_{,i} r_{,j} r'_{,j'} + \delta(t - t' - r - r')\sigma^{-2}[\sigma r_{,ij} r'_{,j'} - (x_i - x'_i) r_{,j} r'_{,j'} + (t - t') r_{,i}] + 2\theta(t - t' - r - r')\sigma^{-3}(t - t')(x_i - x'_i), \tag{3.9}$$

$$I_{,oij'} = \delta'(t - t' - r - r')\sigma^{-1} r_{,i} r'_{,j'} + \delta(t - t' - r - r')\sigma^{-2}[(x_i - x'_i) r'_{,j'} - (x_j - x'_j) r_{,i}] + \theta(t - t' - r - r')\sigma^{-3}[\sigma\delta_{ij} - 2(x_i - x'_i)(x_j - x'_j)], \tag{3.10}$$

$$I_{,ooo'} = -\delta'(t - t' - r - r')\sigma^{-1} - 2\delta(t - t' - r - r')\sigma^{-2}(t - t') - \theta(t - t' - r - r')\sigma^{-3}[\sigma + 2(t - t')^2], \tag{3.11}$$

$$I_{,okk'} = \delta'(t - t' - r - r')\sigma^{-1} r_{,k} r'_{,k'} + \delta(t - t' - r - r')\sigma^{-2}(t - t')(r_{,k} r'_{,k'} - 1) - \theta(t - t' - r - r')\sigma^{-3}[\sigma + 2(t - t')^2], \tag{3.12}$$

$$I_{,ijo'} = -\delta'(t - t' - r - r')\sigma^{-1} r_{,i} r_{,j} + \delta(t - t' - r - r')\sigma^{-2}[\sigma r_{,ij} - r_{,i}(x_j - x'_j) - r_{,j}(x_i - x'_i)] + \theta(t - t' - r - r')\sigma^{-3}[\sigma\delta_{ij} - 2(x_i - x'_i)(x_j - x'_j)] \tag{3.13}$$

where, in the present section, we define

$$\sigma \equiv \frac{1}{2}(x - x')^2 = \frac{1}{2}(\underline{r} - \underline{r}')^2 - \frac{1}{2}(t - t')^2, \tag{3.14}$$

$$\underline{r} = (x_i), \qquad \underline{r}' = (x'_i). \tag{3.15}$$

Expressions (3.7) to (3.13) all have the remarkable property that they vanish for $t' > t - r - r'$. In fact, if we start at t and proceed into the past, we see that the integrand of equation (3.6) vanishes until the particle reaches that point at which a signal could have emanated from it and travelled to the central mass M and thence back again, whereupon the integrand switches on with a bang (delta function and its derivative) and then quickly decays according to a $1/t'^4$ law.

This striking behavior of the integrand illustrates the essential nonlocality of the radiation damping process when gravitational accelerations are involved. The damping forces which the particle experiences at any instant t are determined by what the particle was doing at times $t' < t - r - r'$. This means that, in general, the evaluation of the integral (3.6) and the solution of the equations of motion (1.1) is a rather difficult business, and that the solution will differ more or less widely from what would be expected according to flat space-time ideas [17]. In the nonrelativistic limit, however, the particle is moving so slowly and the integrand is "on" for such an effectively brief period that we may use

$$\underline{r}' \approx \underline{r} + (t' - t)\dot{\underline{r}} \tag{3.16}$$

as an adequate approximation in evaluating the integral [18].

Inserting expressions (3.8) to (3.13) into equation (3.6), and using (3.16) together with its corollaries

$$r' \approx r + (t' - t)r^{-1}\underline{r} \cdot \dot{\underline{r}}, \tag{3.17}$$

$$\nabla' r' = r'^{-1}\underline{r}' \approx r^{-1}\underline{r} + (t' - t)r^{-3}[r^2\dot{\underline{r}} - \underline{r}(\underline{r} \cdot \dot{\underline{r}})], \tag{3.18}$$

$$r_{,i}r'_{,i'} = (\nabla r) \cdot (\nabla' r') \approx 1, \tag{3.19}$$

$$\sigma \approx -\frac{1}{2}(t - t')^2 \tag{3.20}$$

and the identity

$$r_{,ij} = r^{-1}(\delta_{ij} - r^{-2}x_i x_j), \tag{3.21}$$

we find, on keeping terms of no higher than the first degree in the velocity,

$$\begin{aligned} \underline{F} = e^2GM\int_{-\infty}^{t} \{ & 2\delta'(t - t' - r - r')(t - t')^{-1}r^{-3}[r^2\dot{\underline{r}} - \underline{r}(\underline{r} \cdot \dot{\underline{r}})] \\ & + 2\delta(t - t' - r - r')(t - t')^{-2}r^{-4}(t - t' - 3r)[r^2\dot{\underline{r}} - \underline{r}(\underline{r} \cdot \dot{\underline{r}})] \\ & + 8\delta(t - t' - r - r')(t - t')^{-3}r^{-1}\underline{r} \\ & - 16\theta(t - t' - r - r')(t - t')^{-4}\dot{\underline{r}}\}dt'. \end{aligned} \tag{3.22}$$

Consider now the third term in the above integrand (i.e., the term in $r^{-1}\underline{r}$). To first order in the velocity the argument of the delta function in this term may be replaced by $(1 + r^{-1}\underline{r} \cdot \dot{\underline{r}})(t - t') - 2r$. This gives

$$\begin{aligned} 8\int_{-\infty}^{t} \delta(t - t' - r - r')(t - t')^{-3}dt' &\approx r^{-3}(1 + r^{-1}\underline{r} \cdot \dot{\underline{r}})^2 \\ &\approx r^{-3} + 2r^{-4}\underline{r} \cdot \dot{\underline{r}}. \end{aligned} \tag{3.23}$$

The remaining terms are already of first order in the velocity and hence the arguments of the delta and step functions in these terms may be replaced simply by $t - t' - 2r$. The corresponding integrations are then trivial. The first and second terms are found to cancel each other, and we obtain finally

$$\underline{F} = \underline{F}_C + \underline{F}_{NC}, \tag{3.24}$$

where

$$\underline{F}_C = e^2 GM \frac{\underline{r}}{r^4}, \tag{3.25}$$

$$\underline{F}_{NC} = -\frac{2}{3} e^2 GM \left(\frac{\underline{1}}{r^3} - 3 \frac{\underline{r}\,\underline{r}}{r^5} \right) \cdot \dot{\underline{r}} = \frac{2}{3} e^2 \dot{\underline{r}} \cdot \nabla\nabla \frac{GM}{r}. \tag{3.26}$$

$\underline{F}_{NC}$ is the nonconservative force which gives rise to radiation damping [19]. Owing to its dependence on velocity it is small in magnitude compared to the force $\underline{F}_C$ which is conservative. The latter force corresponds to a repulsive inverse square potential

$$V_C = \frac{e^2 GM}{2r^2}. \tag{3.27}$$

It is easy to show that this potential makes the following (retrograde) contribution to the perihelion precession:

$$\delta\phi_C = -\frac{2\pi r_e}{a(1 - \epsilon^2)}, \tag{3.28}$$

where a is the semi-major axis of the orbit, ϵ is its eccentricity and r_e is the classical radius of the particle:

$$r_e = e^2/m. \tag{3.29}$$

The total relativistic contribution to the precession of the perihelion of a charged particle is therefore

$$\delta\phi_{\text{tot}} = \frac{2\pi(3GM - r_e)}{a(1 - \epsilon^2)}. \tag{3.30}$$

The above results admit of immediate generalization to arbitrary weak static gravitational fields. Since we are working in the linear approximation in $h_{\mu\nu}$ the superposition principle holds, and the force $\underline{F}$ is expressible as a sum of contributions arising from all the elementary sources of the gravitational field. We may therefore write

$$\underline{F}_C = -m\nabla\psi, \tag{3.31}$$

$$\underline{F}_{NC} = -\frac{2}{3} e^2 \dot{\underline{r}} \cdot \nabla\nabla\phi = -\frac{2}{3} e^2 \frac{d}{dt}(\nabla\phi), \tag{3.32}$$

where

$$\psi(\underline{r}) \equiv \frac{1}{2} r_e G \int \frac{\rho(\underline{r}')}{(\underline{r} - \underline{r}')^2} d^3 r', \tag{3.33}$$

$$\phi(r) \equiv -G \int \frac{\rho(\underline{r}')}{|\underline{r} - \underline{r}'|} d^3 r', \tag{3.34}$$

ρ being the mass density producing the field. The function ϕ is the ordinary gravitational potential (divided by m) and is related to $h_{\mu\nu}$ by

$$\phi = -\frac{1}{2} h_{oo}, \tag{3.35}$$

and to the Riemann tensor by

$$R_{iojo} = -\phi_{,ij}. \tag{3.36}$$

The latter relation permits equation (3.32) to be rewritten in the form

$$\underline{F}_{NCi} = \frac{2}{3} e^2 R_{iojo} \dot{x}_j, \tag{3.37}$$

which exhibits the damping effect of the "bumps" in space-time in a very direct fashion, and shows that the damping vanishes whenever the particle is in a flat region of space-time even though its coordinate acceleration there may be different from zero [20].

Equation (3.32) may be written in still another form by making use of the undamped equation of motion

$$\ddot{\underline{r}} = -\nabla\phi \tag{3.38}$$

as a first approximation [21]. This gives

$$\underline{F}_{NC} = \frac{2}{3} e^2 \dddot{\underline{r}} \tag{3.39}$$

in agreement with equation (1.4) and the flat space-time theory. From this it is but a step to equation (1.5) and an integration by parts to obtain

$$\Delta E_{\text{orbit}} = -\frac{2}{3} e^2 \int_{\text{orbit}} \ddot{\underline{r}}^2 \, dt, \tag{3.40}$$

which expresses either the total energy loss for an unbound orbit or the loss in one period for a bound orbit. When nongravitational forces are present the particle acceleration is no longer given by (3.38). Nevertheless, equations (3.39) and (3.40) continue to hold since the traditional radiation damping term of equation (1.1) then makes its own contribution to the damping force. When only gravitational forces are present, however, it is important to remember that equation (3.39) cannot be used to argue that preacceleration occurs, since equation (3.37) shows that it does not.

4. Interpretation of $\underline{F}_C$

In this section we shall show that the "anomalous" force $\underline{F}_C$ arises from the fact that the total mass of the particle is not concentrated at a point but is partly distributed as field energy surrounding the particle. To do this we must compute the interaction energy with the gravitational field, namely,

$$V_{\text{int}} = -\frac{1}{2}\int h_{\mu'\nu'} T^{\mu'\nu'} d^3 r', \tag{4.1}$$

where $T^{\mu\nu}$ is the combined stress-energy tensor for the particle and its self-field.

Now the total energy in the electric field is well known to be infinite for a point particle. Hence, in order to avoid divergences we must introduce a cut-off radius ϵ. Here, however, we must be particularly careful. In order that the interaction energy be coordinate invariant we must make sure that the stress-energy tensor satisfies the conservation law

$$T^{\mu\nu}{}_{;\nu} = 0. \tag{4.2}$$

This requires that we include also the self-stress of the particle, which produces the forces which prevent the particle from blowing itself apart.

For simplicity we assume the charge of the particle to be distributed uniformly in a spherical shell, and we ignore its magnetic field, since the latter contributes a term which is quadratic in the velocity. The stress-energy tensor at a point $\underline{r}'$ is then given by

$$T_{oo}(\underline{r}') = m_o\delta(\underline{r}' - \underline{r}) + (8\pi)^{-1}\underline{E}'^2, \tag{4.3}$$

$$T_{oi}(\underline{r}') = m_o\dot{x}_i\delta(\underline{r}' - \underline{r}), \tag{4.4}$$

$$T_{ij}(\underline{r}') = -(4\pi)^{-1}E'_iE'_j + (8\pi)^{-1}\delta_{ij}\underline{E}'^2 + T_{ij}{}^{\text{self}}(\underline{r}'), \tag{4.5}$$

where $\underline{r}$ is the position of the particle, $T_{ij}{}^{\text{self}}$ is its self-stress, m_o is its "bare" mass, and the electric field $\underline{E}'$ has the form

$$\underline{E}' = e\,\theta(|\underline{r}' - \underline{r}| - \epsilon)\frac{\underline{r}' - \underline{r}}{|\underline{r}' - \underline{r}|^3}. \tag{4.6}$$

In view of the law of motion (3.38) it is readily seen that equation (4.2) reduces (in dyadic notation) to

$$\nabla \cdot \underline{T} = 0. \tag{4.7}$$

From equations (4.5), (4.6) and (4.7) it is then a simple matter to infer the structure of $\underline{T}^{\text{self}}$, namely,

$$\underline{T}^{\text{self}}(\underline{r}') = -\frac{e^2}{16\pi}\left[\frac{1}{|\underline{r}' - \underline{r}|^3} - \frac{(\underline{r}' - \underline{r})(\underline{r}' - \underline{r})}{|\underline{r}' - \underline{r}|^5}\right]\delta(|\underline{r}' - \underline{r}| - \epsilon), \tag{4.8}$$

which corresponds to an isotropic tension tangential to the surface of the spherical shell.

When the gravitational field is produced by a central mass M we obtain, on substituting equations (2.30), (4.3), (4.4), (4.5) and (4.8) into equation (4.1),

$$\begin{aligned} V_{\text{int}} &= -GM\int\frac{1}{r'}[T_{oo}(\underline{r}') + tr\,\underline{T}(\underline{r}')]\,d^3r' \\ &= -\frac{GMm_o}{r} - \frac{e^2GM}{4\pi}\int\frac{1}{|\underline{r} + \underline{r}''|}\left[\theta(r'' - \epsilon)\frac{1}{r''^4} - \frac{1}{2}\delta(r'' - \epsilon)\frac{1}{r''^3}\right]d^3r'', \end{aligned} \tag{4.9}$$

where, in the second line, we have replaced the integration variable $\underline{r}'$ by

$$\underline{r}'' = \underline{r}' - \underline{r}. \tag{4.10}$$

The integration of (4.9) is elementary, and we immediately obtain

$$V_{\text{int}} = -\frac{GMm}{r} + V_C, \tag{4.11}$$

where m is the "observed" mass of the particle,

$$m = m_o + \frac{e^2}{2\epsilon}, \tag{4.12}$$

[cf. equation (5.24) of ref. 1] and V_C is the potential defined in equation (3.27). More generally, for an arbitrary static gravitational field, we have

$$V_{\text{int}} = m(\phi + \psi), \tag{4.13}$$

where ψ and ϕ are given by equations (3.33) and (3.34).

We note that the mass renormalization (4.12) is an unavoidable part of the attempt to give the force $\underline{F}_C$ a simple physical interpretation. It is a virtue of the rigorous theory that when equation (1.1) is used the mass renormalization is already automatically taken into account, no matter what the motion or how strong the gravitational field, and need not be considered a second time.

One should be cautioned not to assume, from the above results, that the Coulomb field surrounding the particle is a rigid structure which transmits information on the ambient gravitational field instantaneously back to the particle. These results hold only for nonrelativistic velocities, for only in this case is the propagation of information so rapid, in comparison to the motion, that it *is* effectively instantaneous.

We may finally point out that agreement with the rigorous results is obtained only by using the full stress-energy tensor in equation (4.1). If the energy density alone is used, a value for V_C is obtained which is too small by a factor 2. This is the well known factor by which the electromagnetic field characteristically differs from ordinary matter in its interaction with gravity.

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Appendix

Evaluation of I

The integrand of equation (2.34) is nonvanishing only on the ellipsoid of intersection of the light cones emanating from x and x'. It is evidently convenient to replace x''^0, x''^1, x''^2, x''^3 as integration variables by the variables σ, σ', θ, ϕ, where θ and ϕ are angles defining a point on the surface of the ellipsoid. These angles may be related to the direction cosines of a perpendicular dropped from the point x'' to the straight line joining x and x'. This perpendicular will meet the line at a point which is a fraction ξ, say, of the distance from x to x'. We may write

$$x'' = \xi x + (1 - \xi)x' + \rho\Omega, \tag{A.1}$$

where the vector $\rho\Omega^\mu$ describes the perpendicular and the Ω^μ are the direction cosines. More precisely,

$$\Omega^\mu = \Omega_i n_i{}^\mu, \tag{A.2}$$

where the $n_i{}^\mu$ are a set of three unit vectors satisfying

$$n_i{}^\mu n_{j\mu} = \delta_{ij}, \qquad (x^\mu - x'^\mu)n_{i\mu} = 0, \tag{A.3}$$

and where

$$\Omega_i\Omega_i = 1. \tag{A.4}$$

We now have

$$2\sigma = (x - x'')^2 = (1 - \xi)^2(x - x')^2 + \rho^2 \tag{A.5}$$

$$2\sigma' = (x' - x'')^2 = \xi^2(x - x')^2 + \rho^2 \tag{A.6}$$

which may be solved to yield

$$\xi = \frac{1}{2} + \frac{\sigma - \sigma'}{-(x - x')^2}, \tag{A.7}$$

$$\rho^2 = -\frac{1}{4}(x - x')^2 + \sigma + \sigma' + \frac{(\sigma - \sigma')^2}{-(x - x')^2}. \tag{A.8}$$

From this the Jacobian of the variable transformation may be computed:

$$\begin{aligned}\frac{\partial(x'')}{\partial(\sigma, \sigma', \theta, \phi)} &= \det\begin{pmatrix}(x^\mu - x'^\mu)\frac{\partial\xi}{\partial\sigma} + \Omega^\mu\frac{\partial\rho}{\partial\sigma} & (x^\mu - x'^\mu)\frac{\partial\xi}{\partial\sigma'} + \Omega^\mu\frac{\partial\rho}{\partial\sigma'} & \rho\frac{\partial\Omega^\mu}{\partial\theta} & \rho\frac{\partial\Omega^\mu}{\partial\phi}\end{pmatrix} \\ &= \rho\det\begin{pmatrix}\Omega^\mu & -\frac{x^\mu - x'^\mu}{-(x - x')^2} & \frac{\partial\Omega^\mu}{\partial\theta} & \frac{\partial\Omega^\mu}{\partial\phi}\end{pmatrix}\end{aligned} \tag{A.9}$$

Since the Jacobian of a proper Lorentz transformation is +1 we may conveniently evaluate this determinant in the special coordinate system in which $(n_1{}^\mu) = (0, 1, 0, 0)$, $(n_2{}^\mu) = (0, 0, 1, 0)$, $(n_3{}^\mu) = (0, 0, 0, 1)$. Setting

$$\begin{aligned}\Omega_1 &= \sin\theta\cos\phi, \\ \Omega_2 &= \sin\theta\sin\phi, \\ \Omega_3 &= \cos\theta,\end{aligned} \tag{A.10}$$

one then finds

$$\frac{\partial(x'')}{\partial(\sigma, \sigma', \theta, \phi)} = \frac{\rho}{\sqrt{-(x - x')^2}}\sin\theta. \tag{A.11}$$

Because of the delta functions in equation (2.34), this Jacobian need be evaluated only for $\sigma = \sigma' = 0$, at which points we have $\xi = \frac{1}{2}$, $\rho = \frac{1}{2}\sqrt{-(x - x')^2}$,

$$\left[\frac{\partial(x'')}{\partial(\sigma, \sigma', \theta, \phi)}\right]_{\sigma=\sigma'=0} = \frac{1}{2}\sin\theta. \tag{A.12}$$

The integral (2.34) therefore takes the form

$$I = \frac{1}{4\pi}\int_0^{2\pi} d\phi\int_0^{\pi}\sin\theta\, d\theta\frac{1}{r''}, \tag{A.13}$$

where r'' is the magnitude of the 3-vector component of

$$x''^\mu = \frac{1}{2}(x^\mu + x'^\mu) + \frac{1}{2}\sqrt{-(x - x')^2}\;\Omega^\mu, \tag{A.14}$$

namely of

$$\underline{r}'' = \frac{1}{2}(\underline{r} + \underline{r}') + \frac{1}{2}\sqrt{-(x - x')^2}\;(\underline{n}_1\sin\theta\cos\phi + \underline{n}_2\sin\theta\sin\phi + \underline{n}_3\cos\theta). \tag{A.15}$$

As θ and ϕ vary the vector $\underline{r}''$ describes a prolate ellipsoid of revolution with foci at $\underline{r}$ and $\underline{r}'$, having semi-major axis $\frac{1}{2}(t - t')$ and semi-minor axis $\frac{1}{2}\sqrt{-(x - x')^2}$. This ellipsoid is just the projection of the light cone intersection on any hypersurface t = constant, and it is not difficult to see that the evaluation of the integral (A.13) is equivalent to the calculation of the potential which would be produced by this ellipsoid if it were a perfect conductor carrying a unit charge.

In the general case the points $\underline{r}$ and $\underline{r}'$ will not be colinear with the origin, and it is then convenient to choose the unit vectors $n_i{}^{\mu}$ in the form

$$\left.\begin{aligned}(n_1{}^{\mu}) &= (0,\ (\underline{r}-\underline{r}')(\underline{r}\times\underline{r}')/|\underline{r}-\underline{r}'||\underline{r}\times\underline{r}'|),\\ (n_2{}^{\mu}) &= (0,\ -\underline{r}\times\underline{r}'/|\underline{r}\times\underline{r}'|),\\ (n_3{}^{\mu}) &= [-(x-x')^2]^{-\frac{1}{2}}(|\underline{r}-\underline{r}'|,\ (t-t')(\underline{r}-\underline{r}')/|\underline{r}-\underline{r}'|).\end{aligned}\right\} \tag{A.16}$$

In a coordinate system with origin at the center of the ellipsoid and with axes directed along the 3-vectors $\underline{n}_1, \underline{n}_2, \underline{n}_3$, the coordinates ξ, η, ζ of a point on the surface of the ellipsoid are given by

$$\left.\begin{aligned}\xi &= \sqrt{\bar{r}^2-\bar{\rho}^2}\ \sin\theta\cos\phi,\\ \eta &= \sqrt{\bar{r}^2-\bar{\rho}^2}\ \sin\theta\sin\phi,\\ \zeta &= \bar{r}\cos\theta,\end{aligned}\right\} \tag{A.17}$$

where

$$\bar{r} = \frac{1}{2}(t-t'), \tag{A.18}$$

$$\bar{\rho} = \frac{1}{2}|\underline{r}-\underline{r}'|, \tag{A.19}$$

while the coordinates of the original origin (i.e., of the mass M) are given by

$$\left.\begin{aligned}\xi_0 &= \sqrt{\bar{r}_0^2-\bar{\rho}^2}\ \sin\theta_0\cos\phi_0,\\ \eta_0 &= \sqrt{\bar{r}_0^2-\bar{\rho}^2}\ \sin\theta_0\sin\phi_0,\\ \zeta_0 &= \bar{r}_0\cos\theta_0,\end{aligned}\right\} \tag{A.20}$$

where

$$\bar{r}_0 = \frac{1}{2}(r+r'), \tag{A.21}$$

$$\cos\theta_0 = -(r-r')/|\underline{r}-\underline{r}'|, \tag{A.22}$$

$$\phi_0 = 0. \tag{A.23}$$

It should be noted that the vector $\underline{n}_3$ is *not* a unit 3-vector but the spatial component of a unit 4-vector.

We now have

$$r''^2 = (\xi-\xi_0)^2 + (\eta-\eta_0)^2 + (\zeta-\zeta_0)^2, \tag{A.24}$$

and the quantity $1/r''$ can be expanded in spherical harmonics by standard formulas [22], with the result that for $\bar{r}_0 > \bar{r}$,

$$\frac{1}{r''} = \frac{1}{\bar{\rho}}\sum_{n=0}^{\infty}(2n+1)\left\{Q_n(\bar{r}_0/\bar{\rho})P_n(\bar{r}/\bar{\rho})P_n(\cos\theta_0)P_n(\cos\theta) + 2\sum_{m=1}^{n}\left[\frac{(n-m)!}{(n+m)!}\right]^2 Q_n^m(\bar{r}_0/\bar{\rho})P_n^m(\bar{r}/\bar{\rho})P_n^m(\cos\theta_0)P_n^m(\cos\theta)\cos m(\phi-\phi_0)\right\} \tag{A.25}$$

while for $\bar{r}_0 < \bar{r}$,

$$\frac{1}{r''} = \frac{1}{\bar{\rho}}\sum_{n=0}^{\infty}(2n+1)\left\{Q_n(\bar{r}/\bar{\rho})P_n(\bar{r}_0/\bar{\rho})P_n(\cos\theta_0)P_n(\cos\theta) + 2\sum_{m=1}^{n}\left[\frac{(n-m)!}{(n+m)!}\right]^2 Q_n^m(\bar{r}/\bar{\rho})P_n^m(\bar{r}_0/\bar{\rho})P_n^m(\cos\theta_0)P_n^m(\cos\theta)\cos m(\phi-\phi_0)\right\} \tag{A.26}$$

where P_n and Q_n are the Legendre functions of the first and second kinds respectively, and P_n^m and Q_n^m are the corresponding associated functions. Inserting these expansions into the integral (A.13), we obtain

$$I = \bar{\rho}^{-1}[\theta(\bar{r}_0 - \bar{r})Q_0(\bar{r}_0/\bar{\rho}) + \theta(\bar{r} - \bar{r}_0)Q_0(\bar{r}/\bar{\rho})], \tag{A.27}$$

which in virtue of equations (A.18), (A.19), (A.21) and the explicit form

$$Q_0(x) = \frac{1}{2}\log\frac{x+1}{x-1}, \tag{A.28}$$

leads to equation (2.35) of the text.

References

1. B. S. DEWITT and R. W. BREHME, *Ann. Phys.* **9**, 220 (1960).
2. The functions $z^\alpha(\tau)$ define the world line of the particle, the parameter τ being the proper time. Indices from the first part of the Greek alphabet are always associated with a world-line point z, while those from the middle of the alphabet are associated with arbitrary points x, x', etc. Abbreviations such as $\dot{z}^2 \equiv -\dot{z}_\alpha\dot{z}^\alpha$, $\ddot{z}^2 \equiv \ddot{z}_\alpha\ddot{z}^\alpha$, are employed.
3. P. A. M. DIRAC, *Proc. Roy. Soc.* A **167**, 148 (1938).
4. Equation (1.1) makes no allowance for *gravitational* radiation, i.e., for dynamical behavior of the gravitational field itself. The space-time metric is here externally imposed. The inclusion of gravitational radiative reaction effects raises a host of new problems and will not be considered in this paper. This neglect is perfectly justified if $e^2 \gg Gm^2$, where G is the gravitation constant.
5. We obviously exclude the well known anomalous case of eternally constant acceleration.
6. This term, of course, regains its usual significance as soon as $F_{\mu\nu}{}^{in}$ (or any external force other than that of the gravitational field) becomes the primary source of field distortion at the location of the particle.
7. This does *not* mean that the difference between the exact orbit and the naively computed one is merely a matter of coordinate definition. The exact orbit is well defined and quite unambiguous, and the question at issue is not how it *looks* in a given coordinate system but rather how it *differs* from a geodetic orbit which is also well defined and unambiguous. It is not possible to account for the difference solely on the basis of equation (1.3), no matter what quasi-Minkowskian coordinate system is used.
8. The authors are indebted to Professor J. A. Wheeler for suggesting this interpretation.
9. Covariant differentiation is denoted by a dot rather than a semi-colon (see ref. 1).
10. Here we simplify the notation of ref. 1 slightly. $\delta^\mu{}_{\nu'}$ stands for product of the Kronecker delta with a delta function which is assumed to have unit weight at x and zero weight at x'.

11. J. SCHWINGER, *Phys. Rev.* **82**, 664 (1951).
12. The symbol $\delta g_{\mu\nu\cdot\sigma}$ is effectively unambiguous. It means, of course, $(\delta g_{\mu\nu})_{\cdot\sigma}$ rather than $\delta(g_{\mu\nu\cdot\sigma})$, since the latter would be trivial.
13. To the order of accuracy employed here, no distinction need be made between the true light cone and that of flat space-time.
14. See V. FOCK, *The Theory of Space Time and Gravitation.* Pergamon Press, New York (1959).
15. Latin indices run from 1 to 3.
16. Note that for equation (2.35) to be usable we must have $r, r' \gg r_M \gg GM$, which is consistent with equations (3.4) and (3.5).
17. There is another type of "nonlocality" inherent in any attempt to project a curved space-time into a flat one, namely, the arbitrariness in the definition of the quasi-Minkowskian coordinates. In the present case the coordinates can be shifted through displacements of as much as GM without destroying the weak field approximation. The nonlocality inherent in the radiation damping process, however, is much greater than this and hence is quite real. For it involves time lags of the order $2r$ and hence differences in particle position of at least $(GMr)^{1/2}$ [see equation (3.5)] which, in view of the remark made in footnote 16, is much larger than GM.
18. Higher terms in the series expansion of $\underline{r}'$ – for example, $\frac{1}{2}(t' - t)^2 \ddot{\underline{r}}$ – are of higher order in GM (and hence in $h_{\mu\nu}$), when the particle is in free fall. A careful investigation of both elliptic and hyperbolic orbits shows that these terms may be neglected notwithstanding the fact that the series itself cannot, of course, be integrated term by term.
19. The number of different ways in which the well known factor 2/3 associated with the damping force can be derived is astonishing. In the present case it arises from the integral of $(t - t')^{-4}$.
20. Thus a charged particle falling in a so-called "uniform gravitational field" suffers no retardation force.
21. It would be more correct to use the equation $\ddot{\underline{r}} = -\nabla(\phi + \psi)$ which includes the "anomalous" potential (3.33).
22. See, for example, T. M. MacROBERT, *Spherical Harmonics* Chap. XI. Methuen, London (1927).