

Fermionic Spinon Theory of Square Lattice Spin Liquids near the Néel State

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Quantum fluctuations of the Néel state of the square lattice antiferromagnet are usually described by a $\mathbb{C}\mathbb{P}^1$ theory of bosonic spinons coupled to a $U(1)$ gauge field, and with a global $SU(2)$ spin rotation symmetry. Such a theory also has a confining phase with valence bond solid (VBS) order, and upon including spin-singlet charge-2 Higgs fields, deconfined phases with \mathbb{Z}_2 topological order possibly intertwined with discrete broken global symmetries. We present dual theories of the same phases starting from a mean-field theory of fermionic spinons moving in π flux in each square lattice plaquette. Fluctuations about this π -flux state are described by $(2+1)$ -dimensional quantum chromodynamics (QCD₃) with a $SU(2)$ gauge group and $N_f = 2$ flavors of massless Dirac fermions. It has recently been argued by Wang *et al.* [*Deconfined Quantum Critical Points: Symmetries and Dualities*, *Phys. Rev. X* **7**, 031051 (2017).] that this QCD₃ theory describes the Néel-VBS quantum phase transition. We introduce adjoint Higgs fields in QCD₃ and obtain fermionic dual descriptions of the phases with \mathbb{Z}_2 topological order obtained earlier using the bosonic $\mathbb{C}\mathbb{P}^1$ theory. We also present a fermionic spinon derivation of the monopole Berry phases in the $U(1)$ gauge theory of the VBS state. The global phase diagram of these phases contains multicritical points, and our results imply new boson-fermion dualities between critical gauge theories of these points.

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I. INTRODUCTION

Spin-liquid states of the square lattice antiferromagnet, with global $SU(2)$ spin rotation symmetry, have long been recognized as important ingredients in the theory of the cuprate high temperatures superconductors [1–4]. The earliest established examples of gapped states were “chiral spin liquids,” which were constructed by analogy to the fractional quantum Hall states [5,6]. These have a topological order which is not compatible with time-reversal symmetry. Soon after, “ \mathbb{Z}_2 spin liquids” were proposed [7–13]: their topological order is compatible with time-reversal symmetry, and exactly solvable examples were later found in Kitaev’s toric code and honeycomb lattice models [14–16]. Wen [17] used a fermionic spinon representation of the antiferromagnet to obtain a plethora of possible square lattice spin-liquid states, distinguished by different realizations of symmetry-enriched topological order [18,19]. Wen’s classification criterion was that the spin-liquid states preserve time-reversal, $SU(2)$ spin rotations, and all the square lattice space group symmetries.

However, in the application to the cuprates, there is no fundamental reason all such symmetries should be preserved. If we also allow for breaking of time-reversal and/or point group symmetries, then many more spin-liquid states are clearly possible, all of which preserve $SU(2)$ spin rotations and the square lattice translational symmetry [7,8,20,21]. This proliferation of possible spin liquids, intertwining with broken symmetries, sets up a daunting task of deciding which states, if any, are relevant for the pseudogap phase of the underdoped cuprates.

We need an energetic and physical criterion to focus on a smaller set of relevant spin-liquid states, rather than relying exclusively on symmetry and topology. In recent work, Chatterjee *et al.* [22] proposed examining spin liquids which are proximate to the magnetically ordered Néel state. These proximate states are reachable by continuous (or nearly continuous) quantum phase transitions involving the long-wavelength excitations of the antiferromagnet. Specifically, they used a $\mathbb{C}\mathbb{P}^1$ theory of quantum fluctuations of the Néel state, expressed in terms of bosonic spinons z_α to argue for the importance of three possible \mathbb{Z}_2 spin-liquid states. These three states are identified here as A_b , B_b , and C_b , and appear in Figs. 1(a) and 2(a). The state A_b preserves all symmetries [23], while B_b breaks lattice rotation symmetries and so has Ising-nematic order [7]. The state C_b breaks inversion and time-reversal symmetries, but not their product, and was argued to possess current-loop order.

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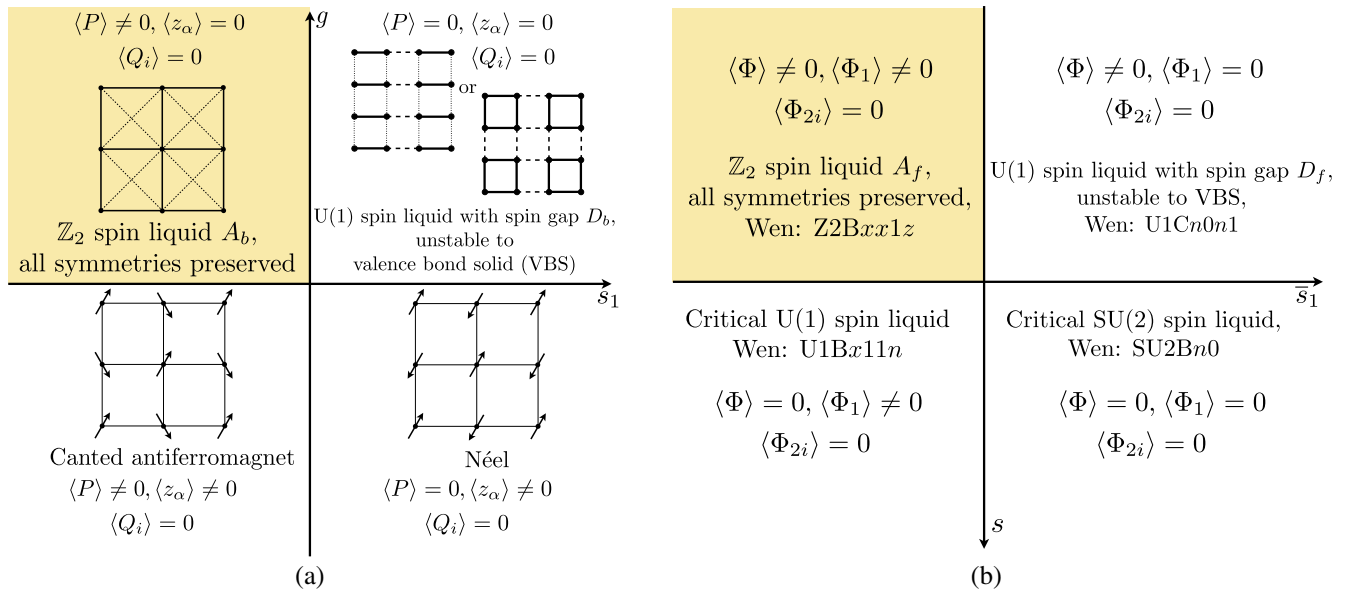


FIG. 1. (a) Schematic phase diagram of the $\mathbb{C}\mathbb{P}^1$ theory in Eq. (1.4) as a function of g and s_1 [s_2 in Eq. (1.7) is large and positive]; Eq. (1.2) describes the deconfined critical Néel-VBS transition at a critical $g = g_c$. (b) Schematic phase diagram of the $\text{SU}(2)$ QCD₃ theory with $N_f = 2$ flavors of massless Dirac fermions in Eq. (1.6) as a function of s and \bar{s}_1 [\bar{s}_2 in Eq. (1.8) is large and positive]. The “Wen” labels refer to the naming scheme in Ref. [17]. The \mathbb{Z}_2 spin liquids A_b and A_f in (a) and (b) are argued to be topologically identical, as are the confining states with VBS order. The critical spin liquids in (b) are argued to be unstable to the corresponding phases with magnetic order in (a), with the critical SU(2) spin liquid surviving only at the Néel-VBS transition. All \mathbb{Z}_2 spin liquids are shown shaded in all figures.

A related motivation for the physical importance of these states comes from an examination of the classical phase diagram of frustrated antiferromagnets on the square lattice. By examining models with two-spin near-neighbor and

four-spin ring exchange interactions, Ref. [22] found magnetically ordered states with canted, spiral, and conical spiral order near the Néel state. Quantum fluctuations about these classical ordered states can be described by

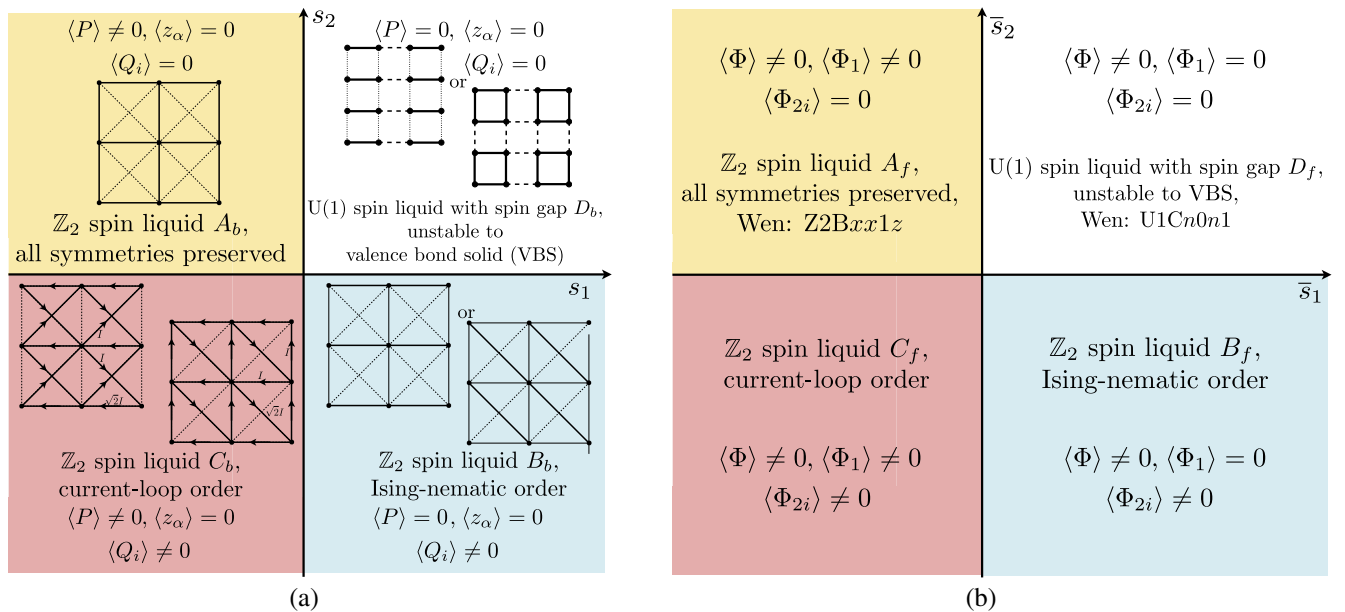


FIG. 2. (a) Schematic phase diagram of the $\mathbb{C}\mathbb{P}^1$ theory in Eqs. (1.4) and (1.7) as a function of s_1 and s_2 (for large g). (b) Schematic phase diagram of the $\text{SU}(2)$ QCD₃ theory with $N_f = 2$ flavors of massless Dirac fermions in Eqs. (1.6) and (1.8) as a function of \bar{s}_1 and \bar{s}_2 (for $s < 0$ and $|s|$ large). All four phases in (a) and (b) are argued to be topologically identical. So for the \mathbb{Z}_2 spin liquids, $A_b = A_f$, $B_b = B_f$, and $C_b = C_f$. Phases B_f and C_f do not appear in Wen’s classification [17] because they break global symmetries.

extensions of the $\mathbb{C}\mathbb{P}^1$ theory, and the quantum disordered states obtained across a continuous transition involving loss of magnetic order are precisely the three \mathbb{Z}_2 spin liquids, with the correspondence [22,24]

$$\begin{aligned} \text{canted order} &\rightarrow A_b, \\ \text{spiral order} &\rightarrow B_b, \\ \text{conical spiral order} &\rightarrow C_b. \end{aligned} \quad (1.1)$$

One of the purposes of the present paper is to present a unified theory of the three \mathbb{Z}_2 spin liquids noted above, but using the fermionic spinon approach [17,25,26]. For gapped \mathbb{Z}_2 spin-liquid states, a mapping between the fermionic and bosonic spinons approaches has been achieved for specific states on the kagome, triangular, square, and rectangular lattices [19,21,23,27–32]. This mapping relies on the fusion rules of the toric code [14]: the fusion of any two of the anyon species yields the third. In \mathbb{Z}_2 spin liquids, the three types of anyons are bosonic spinons, fermionic spinons, and a bosonic \mathbb{Z}_2 -flux spinless vison. We extend such mappings here to the states of interest on the square lattice, but using a method which allows us to treat the three \mathbb{Z}_2 spin liquids and the quantum phase transitions between them in a unified manner. We obtain a phase diagram of the states proximate to the Néel state using the fermionic spinon approach, and propose critical theories of the phase transitions involving massless Dirac fermions. The connection to the earlier analysis [22] using the bosonic spinons of the $\mathbb{C}\mathbb{P}^1$ model also leads us to propose new boson-fermion dualities of the strongly coupled, gapless, quantum field theories describing the (multi)critical points.

Our point of departure will be a boson-fermion duality of a conformal field theory (CFT) proposed by Wang *et al.* [33]. They examined the critical theory of the Néel-valence bond solid (VBS) transition in the $\mathbb{C}\mathbb{P}^1$ theory [34–37], and proposed that it was equivalent to quantum chromodynamics (QCD) with a $SU(2)$ gauge group and $N_f = 2$ flavors of massless, two-component Dirac fermions [note that the $SU(2)$ gauge group is not to be confused with the global $SU(2)$ spin rotation symmetry]. The latter theory can also be obtained from the fermionic spinon approach to the square lattice antiferromagnet: it describes fluctuations about a π -flux mean-field theory [17,25,26], which is labeled by Wen as $SU2Bn0n1$.

Starting from the $SU(2)$ QCD₃ theory, we explore routes to condensing Higgs fields for fermionic bilinears, so that the $SU(2)$ gauge group is ultimately broken down to \mathbb{Z}_2 and we obtain gapped spin liquids with \mathbb{Z}_2 topological order. Our main results are contained in the phase diagrams in Fig. 2. These phase diagrams contain the phases A_f , B_f , and C_f , which are fermionic counterparts of the A_b , B_b , and C_b states obtained from the bosonic $\mathbb{C}\mathbb{P}^1$ theory.

One important feature of the fermionic phase diagram in Fig. 1(b) is that it does not contain the counterparts of the magnetically ordered Néel and canted states in the bosonic phase diagram in Fig. 1(a). Instead, Fig. 1(b) contains two critical phases, with massless Dirac fermions interacting with gapless $SU(2)$ and $U(1)$ gauge bosons. Building on the fermion-boson equivalence of Wang *et al.* [33], we argue here that these critical phases of Fig. 1(b) are unstable to the corresponding magnetically ordered phases in Fig. 1(a); the instability is assumed to be driven by relevant operators which are allowed by the symmetries of the underlying square lattice antiferromagnet. However, given the strongly coupled nature of the critical theories, this conclusion is based upon circumstantial, rather than firm, evidence.

A. Summary of results

Let us first recall the bosonic spinon approach [22,38] to the phases in Fig. 1(a). This is obtained by extending the Lagrangian for the theory of deconfined criticality for the Néel-VBS transition [39]:

$$\mathcal{L}_{\text{dcp}} = \frac{1}{g} |(\partial_\mu - ib_\mu)z_\alpha|^2 + S_B. \quad (1.2)$$

The Lagrangian is in three spacetime dimensions with μ a spacetime index in Minkowski signature $(+, -, -)$, and $\alpha, \beta = \uparrow, \downarrow$, so there is global $SU(2)$ spin rotation symmetry. The Néel order parameter is $z_\alpha^* \sigma_\alpha^a z_\beta$, where σ^a are the Pauli matrices. The $U(1)$ gauge field b_μ is compact, and monopole tunneling events are permitted, and associated with a Berry phase S_B [35,40]. The spinons are represented by the bosonic complex scalar z_α , which is of unit length

$$\sum_\alpha |z_\alpha|^2 = 1, \quad (1.3)$$

and carries unit $U(1)$ charge. For small g , z_α is condensed, and this yields the Néel phase with broken spin rotation symmetry. For large g , z_α is not condensed, and we appear to obtain a $U(1)$ spin liquid (which we call D_b) with a gapless photon b_μ , and gapped z_α spinons. However, the condensation of monopoles yields the confinement of spinons and the appearance of VBS order [34,35]. The transition from the Néel state to the VBS is described by a deconfined critical theory [36,37] at $g = g_c$ in which monopoles are suppressed.

We now extend \mathcal{L}_{dcp} by including complex, charge-2 Higgs fields whose condensation can induce phases with \mathbb{Z}_2 topological order, while preserving $SU(2)$ spin rotation symmetry. We can construct such Higgs fields by pairing spinons, but the simplest possibility, $\varepsilon_{\alpha\beta} z_\alpha z_\beta$, vanishes identically. Any such spinon pair Higgs field must involve gradients, and the simplest nonvanishing cases involve a single temporal or spatial gradient. We consider first the Higgs field P conjugate to a pair of spinons with a single

temporal gradient, and include the spatial gradient Higgs field Q_i later. The Lagrangian for z_α , b_μ , and P is

$$\mathcal{L}_b = \mathcal{L}_{\text{dcp}} + |(\partial_\mu - 2ib_\mu)P|^2 - s_1|P|^2 + \lambda_1 P^* \varepsilon_{\alpha\beta} z_\alpha \partial_0 z_\beta + \lambda_1 P \varepsilon_{\alpha\beta} z_\alpha^* \partial_0 z_\beta^* + \dots, \quad (1.4)$$

where $\varepsilon_{\alpha\beta}$ is the unit antisymmetric tensor, and so SU(2) spin rotation symmetry is maintained. For s_1 large and positive, when there is no P condensate, we obtain the phases of \mathcal{L}_{dcp} already described. For smaller s_1 , when there is a P Higgs condensate, we obtain the canted antiferromagnet and the symmetric \mathbb{Z}_2 spin liquid A_b for small and large g , respectively, as shown in Fig. 1(a). The \mathbb{Z}_2 spin liquid A_b was first obtained in Ref. [23], where it was called $\mathbb{Z}_2[0, 0]$.

Now we turn to our results for the fermionic counterpart of Fig. 1(a), which is shown in Fig. 1(b). We start with the fermionic equivalent of the deconfined Néel-VBS critical theory, which was identified by Wang *et al.* [33] as SU(2) QCD₃ with $N_f = 2$, described by the Lagrangian

$$\mathcal{L}_{\text{QCD}_3} = \text{itr}[\bar{X}\gamma^\mu(\partial_\mu X + iXa_\mu)]. \quad (1.5)$$

Here, X represents the massless Dirac fermions, γ^μ are Dirac matrices, and the details of the index structure are specified in Sec. II B. The SU(2) gauge field is represented by a_μ . The fermion kinetic term in Eq. (1.5) has a global SO(5) symmetry, which is an enlargement of the global SU(2) spin rotation and \mathbb{Z}_4 lattice rotation symmetries of the lattice Hamiltonian [33]. To obtain Fig. 1(b), we extend Eq. (1.5) in Sec. III B by adding two real Higgs fields, $\Phi = \Phi^a \sigma^a$ and $\Phi_1 = \Phi_1^a \sigma^a$, both of which transform as adjoints of the gauge SU(2). So we have the Lagrangian

$$\mathcal{L}_f = \mathcal{L}_{\text{QCD}_3} + (D_\mu \Phi^a)^2 - s(\Phi^a)^2 + \lambda_2 \Phi^a \text{tr}(\sigma^a \bar{X} \mu^\nu X) + (D_\mu \Phi_1^a)^2 - \bar{s}_1 (\Phi_1^a)^2 + i\lambda_3 \Phi_1^a \text{tr}(\sigma^a \bar{X} \partial_0 X) + \dots \quad (1.6)$$

Here, D_μ is a covariant derivative, a is SU(2) gauge index, σ^a are Pauli matrices, while μ^ν is a Pauli matrix which acts on the flavor space. We assume the higher order terms are such that when both Higgs condensates are present, $\langle \Phi \rangle$ and $\langle \Phi_1 \rangle$ will be oriented perpendicular to each other in SU(2) gauge space. For instance, the topological order would be stabilized by the presence of a term like $-\mu(\Phi^a \Phi_1^a)^2$ when $\mu > 0$. By varying s and \bar{s}_1 we can obtain four phases in which the two Higgs condensates are either present or absent, as shown in Fig. 1(b). We show in Sec. IV A 1 that the gapped \mathbb{Z}_2 spin liquid A_f , so obtained, is topologically identical to the \mathbb{Z}_2 spin liquid A_b in Fig. 1(a).

We also examine the U(1) spin liquid with a spin gap D_f obtained when there is only a Φ condensate. We compute

the monopole Berry phases in this state in Sec. IV C, and find that they are identical to those indicated by S_B in the bosonic theory in Eq. (1.4). As monopoles are eventually expected to proliferate in this U(1) spin liquid [34], we expect VBS order to appear, just as in the corresponding phase in Fig. 1(a).

Now we turn our attention to the critical U(1) and SU(2) spin liquids in Fig. 1(b). As we noted earlier, we expect that in the absence of fine-tuning, there are relevant perturbations to \mathcal{L}_f which will drive these critical phases to the corresponding magnetically ordered phases in Fig. 1(a). These perturbations will break the SO(5) flavor symmetry of $\mathcal{L}_{\text{QCD}_3}$ down to the symmetries of the underlying lattice Hamiltonian [33].

Finally, we note that both Figs. 1(a) and 1(b) contain multicritical points accessed by tuning two couplings where all four phases meet. A natural conjecture is that these multicritical points are identical to each other. On the bosonic side, this is the theory obtained by tuning g and s_1 , so that both the matter fields z_α and P are critical. On the fermionic side, this is the theory obtained by tuning s and \bar{s}_1 , so that the bosonic matter fields Φ and Φ_1 are critical, while the fermionic matter X remains critical. A further conjecture is that the Yukawa couplings λ_1 and λ_3 renormalize to zero at the multicritical point, then both the bosonic and fermionic theories will represent CFTs.

We also extend our results to include additional Higgs fields which lead to phases with \mathbb{Z}_2 topological order and broken lattice rotation and/or time-reversal symmetries. On the bosonic side, we introduce the complex, charge-2 Higgs field Q_i , where $i = x, y$ is a spatial index, leading to the Lagrangian [8]

$$\mathcal{L}'_b = \mathcal{L}_b + |(\partial_\mu - 2ib_\mu)Q_i|^2 - s_2|Q_i|^2 + \lambda_4 Q_i^* \varepsilon_{\alpha\beta} z_\alpha \partial_i z_\beta + \lambda_4 Q_i \varepsilon_{\alpha\beta} z_\alpha^* \partial_i z_\beta^* + \dots \quad (1.7)$$

In the absence of magnetic order, so that g is large, the phase diagram obtained by varying s_1 and s_2 , with possible condensates of P and Q_i , is shown in Fig. 2(a). There are now three \mathbb{Z}_2 spin liquids, and these meet at a possible multicritical point with the VBS state.

On the fermionic side, in Sec. III C, we add another real Higgs field Φ_{2i} , which transforms as the adjoint of SU(2). We now extend \mathcal{L}_f in Eq. (1.6) to

$$\mathcal{L}'_f = \mathcal{L}_f + (D_\mu \Phi_{2i}^a)^2 - \bar{s}_2 (\Phi_{2i}^a)^2 + i\lambda_5 \Phi_{2i}^a \text{tr}(\sigma^a \bar{X} \partial_i X) + \dots \quad (1.8)$$

The phase diagram obtained by varying \bar{s}_1 and \bar{s}_2 , to obtain possible Higgs condensates of Φ_1 and Φ_{2i} , is shown in Fig. 2(b). We assume that s is negative, so that a Higgs condensate Φ is always present in Fig. 2(b). We obtain three \mathbb{Z}_2 spin liquids in Fig. 2(b), and one of our main results is that these are topologically identical to the

corresponding \mathbb{Z}_2 spin liquids in Fig. 2(a). The relative orientations of the condensates of Φ , Φ_1 , and Φ_2 in gauge space are discussed in Sec. III D. Note that the spin liquids B_f and C_f do not appear in Wen's classification: this is because they break global symmetries associated with the appearance of Ising-nematic and current-loop order, respectively.

Again, the multicritical points in Figs. 2(a) and 2(b), if present, are expected to map to each other, setting up possible dualities of critical fermionic and bosonic gauge theories.

The paper is organized as follows. In Sec. II, we provide the background information necessary for our analysis. We begin by discussing the relevant symmetries and reviewing the π -flux phase, showing that its low-energy dynamics are described by $N_f = 2$ QCD. The section finishes with a brief summary of the boson-fermion duality proposed by Wang *et al.* [33]. Section III explains our procedure for finding spin liquids and how these phases are classified. Using this, we next list all gapped spin liquids accessible using our methods and which are either fully symmetric or have Ising-nematic order. We also describe how spin liquids breaking additional discrete symmetries can be realized, with particular focus given to the \mathbb{Z}_2 spin liquid C_f with current-loop order. These spin liquids, both symmetric and ordered, are subsequently identified in Sec. IV. We start by using the symmetry fractionalization technique to verify the correspondence between the \mathbb{Z}_2 spin liquids we study and those realized using Schwinger bosons. This allows us to verify the equivalence of A_f , B_f , and C_f with A_b , B_b , and C_b . A comparison with Wen's [17] lattice classification scheme is also provided before we turn to the unstable U(1) spin liquid D_f and demonstrate that the proliferation of monopoles necessarily results in a confined phase with VBS order. We conclude in Sec. V with some discussion.

We note a related paper [41], which appeared recently, describing phases of antiferromagnets with only a U(1), easy-plane, global spin rotation symmetry.

II. π -FLUX PHASE AND $N_f = 2$ QCD

A. Model and symmetries

We are interested in this paper in spin-liquid states of the spin-1/2 Heisenberg model on the square lattice, with Hamiltonian of the form

$$H_H = J \sum_{\langle ij \rangle} \mathbf{S}_i \cdot \mathbf{S}_j + \dots, \quad (2.1)$$

where the summation is over nearest neighbors and the ellipsis indicates interactions over further distances or terms which are composed of three or more spin operators. In the absence of these higher order terms, the ground state is known to have Néel order; nonetheless, we operate under

the assumption that the terms contained in the ellipsis provide enough frustration that the ground state loses long-range magnetic order.

It has been shown that a fully symmetric phase describing spin 1/2's on a square lattice must have topological order [42,43]. It turns out that there are many possible such symmetric spin liquids, and a large body of work has been directed at classifying these phases. One such scheme is provided by Wen in Ref. [17]. He extended the physical symmetry group to include gauge transformations, and showed that distinct spin liquids can be differentiated based on the behavior of the gauge degrees of freedom. We take this approach and apply it to a continuum formulation of the phases in question. However, as discussed, the true hallmark of a spin liquid is topological order, not the absence of broken symmetries, and there is no *a priori* reason to restrict to fully symmetric spin liquids. We therefore also consider phases in which certain discrete symmetries are broken.

The physical symmetries relevant to the problem are the SU(2) spin symmetry, time-reversal \mathcal{T} , and the space group symmetries. The space group of the lattice is generated by the two translation operators, T_x and T_y , the inversion operator P_y , and the rotation operator $R_{\pi/2}$. These act on the lattice sites as

$$\begin{aligned} T_x: (i_x, i_y) &\mapsto (i_x + 1, i_y), & T_y: (i_x, i_y) &\mapsto (i_x, i_y + 1), \\ P_y: (i_x, i_y) &\mapsto (i_x, -i_y), & R_{\pi/2}: (i_x, i_y) &\mapsto (-i_y, i_x). \end{aligned} \quad (2.2)$$

In addition, these generators imply a symmetry under inversion of the x coordinate, $P_x = R_{\pi/2} P_y R_{\pi/2}^{-1}$, as well as reflection about the $x = y$ axis, $P_{xy} = P_y R_{\pi/2}^{-1}$. An equivalent definition of the space group is given through its commutation relations:

$$\begin{aligned} T_y^{-1} T_x T_y T_x^{-1} &= \mathbb{1}, & P_y^{-1} R_{\pi/2} P_y R_{\pi/2} &= \mathbb{1}, \\ P_y^{-1} T_x P_y T_x^{-1} &= \mathbb{1}, & R_{\pi/2}^4 &= \mathbb{1}, \\ P_y^{-1} T_y P_y T_y^{-1} &= \mathbb{1}, & R_{\pi/2}^{-1} T_x R_{\pi/2} T_x &= \mathbb{1}, \\ P_y^2 &= \mathbb{1}, & R_{\pi/2}^{-1} T_y R_{\pi/2} T_x^{-1} &= \mathbb{1}. \end{aligned} \quad (2.3)$$

The generators all commute with time reversal, $G^{-1} \mathcal{T}^{-1} G \mathcal{T} = \mathbb{1}$, $G = \{T_x, T_y, P_y, R_{\pi/2}\}$. Because the fundamental degrees of freedom are bosonic spins, we have $\mathcal{T}^2 = \mathbb{1}$.

Naturally, a different set of commutation relations is required to describe the space group in a symmetry-broken phase, and these will be presented as needed. To make contact with these phases, we will often describe the action of P_x independently from the other symmetries even when considering fully symmetric spin liquids.

B. Heisenberg antiferromagnet and the π -flux state

We now present a lattice derivation of the π -flux model. We begin by rewriting the spin operators in terms of so-called slave fermions [17]:

$$\mathbf{S}_i = \frac{1}{2} f_{i\alpha}^\dagger \boldsymbol{\sigma}_{\alpha\beta} f_{i\beta}, \quad (2.4)$$

where $\boldsymbol{\sigma} = (\sigma^x, \sigma^y, \sigma^z)$ are the Pauli matrices. This expression introduces additional degrees of freedom and therefore cannot reproduce the Hilbert space of the spin operators without being supplemented by a constraint. It can easily be verified that provided $\sum_a f_{i\alpha}^\dagger f_{i\alpha} = 1$ on every site, the representation in Eq. (2.4) is correct. This further implies that $\sum_{\alpha,\beta} \epsilon_{\alpha\beta} f_{i\alpha}^\dagger f_{i\beta} = \sum_{\alpha,\beta} \epsilon_{\alpha\beta} f_{i\alpha}^\dagger f_{i\beta}^\dagger = 0$, where $\epsilon_{\alpha\beta}$ is the fully antisymmetric 2-index tensor. By defining a matrix

$$\mathcal{X}_i = \begin{pmatrix} f_{i\uparrow} & -f_{i\downarrow}^\dagger \\ f_{i\downarrow} & f_{i\uparrow}^\dagger \end{pmatrix}, \quad (2.5)$$

we see that these constraints generate a SU(2) gauge symmetry which acts on \mathcal{X}_i as

$$\text{SU}(2)_g: \mathcal{X}_i \rightarrow \mathcal{X}_i U_{g,i}^\dagger. \quad (2.6)$$

The physical spin symmetry acts on \mathcal{X}_i on the left:

$$\text{SU}(2)_s: \mathcal{X}_i \rightarrow U_s \mathcal{X}_i. \quad (2.7)$$

The absence of a charge degree of freedom suggests that a more natural fermionic representation may be obtained by replacing the complex f fermions with Majoranas:

$$f_{i\uparrow} = \frac{1}{\sqrt{2}}(\chi_{i,0} + i\chi_{i,z}), \quad f_{i\downarrow} = \frac{1}{\sqrt{2}}(-\chi_{i,y} + i\chi_{i,x}), \quad (2.8)$$

where $\chi_{i,a}^\dagger = \chi_{i,a}$ and $\{\chi_{i,a}, \chi_{j,b}\} = \delta_{ab} \delta_{ij}$. In this notation, the matrix \mathcal{X}_i is written $\mathcal{X}_i = (1/\sqrt{2})(\chi_{i,0} + i\chi_{i,a} \sigma^a)$ and the local constraints can be expressed as the conditions

$$\text{tr}(\sigma^a \mathcal{X}_i^\dagger \mathcal{X}_i) = 0. \quad (2.9)$$

The first step to an approximate solution to H_H is to loosen the local constraint on the fermions to

$$\langle \text{tr}(\sigma^a \mathcal{X}_i^\dagger \mathcal{X}_i) \rangle = 0. \quad (2.10)$$

Next, we decouple the four-fermion interaction through a Hubbard-Stratonovich transformation, leaving a quadratic mean-field Hamiltonian. The most general such Hamiltonian which can be made symmetric under spin rotation symmetry is [17,44]

$$H_{\text{MF}} = \sum_{\langle ij \rangle} [i\alpha_{ij} \text{tr}(\mathcal{X}_i^\dagger \mathcal{X}_j) + \beta_{ij}^a \text{tr}(\sigma^a \mathcal{X}_i^\dagger \mathcal{X}_j) + i\gamma_{ij} \text{tr}(\sigma^a \mathcal{X}_i^\dagger \sigma^a \mathcal{X}_j)], \quad (2.11)$$

where α_{ij} , β_{ij}^a , and γ_{ij} are real numbers. In accordance with its name, the π -flux state is obtained by threading a π flux through every plaquette: we take $\beta_{ij}^a = \gamma_{ij} = 0$ and

$$\alpha_{ij} = -\alpha_{ji}, \quad \alpha_{i+\hat{x},i} = \alpha, \quad \alpha_{i+\hat{y},i} = (-1)^{i_x} \alpha. \quad (2.12)$$

This gives

$$H_\pi = -i\alpha \sum_i [\text{tr}(\mathcal{X}_i \mathcal{X}_{i+\hat{x}}) + (-1)^{i_x} \text{tr}(\mathcal{X}_i \mathcal{X}_{i+\hat{y}})]. \quad (2.13)$$

While it is clear that this *Ansatz* preserves the full SU(2) gauge and spin symmetries, the invariance of the π -flux Hamiltonian under the space group symmetries may be less clear. In particular, translations in the x direction do not preserve the form of H_π . However, the original Hamiltonian can be recovered through a gauge transformation, implying that the symmetry transformed state is (gauge) equivalent to the original. Wen [17] termed this extended symmetry group the ‘‘projective symmetry group’’ (PSG) and used it to show the existence of eight distinct fully symmetric SU(2) spin liquids on the square lattice. In his scheme, the Hamiltonian H_π describes the SU(2) $B\pi$ 0 state (this is shown in Appendix C 1). We discuss the PSG extensively in subsequent sections, albeit in a slightly different context than originally formulated. His scheme is briefly reviewed in Appendix B.

The band structure of H_π has two Dirac cones. We expand about these cones, labeling them by a valley index $v = 1, 2$. A convenient expression for the resulting theory is achieved by defining the 4×2 matrix operator

$$X_{\alpha,v;\beta} = \frac{1}{\sqrt{2}}(\chi_{0,v} \delta_{\alpha\beta} + i\chi_{\alpha,v} \sigma_{\alpha\beta}^a), \quad (2.14)$$

where α, β , and v are spin, gauge, and valley indices, respectively. The low-energy excitation of H_{MF} are described by the relativistic Dirac Lagrangian

$$\mathcal{L}_{\text{MF}} = i \text{tr}(\bar{X} \gamma^\mu \partial_\mu X), \quad (2.15)$$

where $\bar{X} = \chi^T \gamma^0$, $(\gamma^0, \gamma^x, \gamma^y) = (\tau^y, i\tau^z, i\tau^x)$. Here and in what follows, we express operators in *real* time.

While Eq. (2.10) may hold in the ground state of H_π , the full constraint in Eq. (2.9) does not, and gauge fluctuations must be included to take this into account. The SU(2) gauge transformation in Eq. (2.6) becomes

$$\text{SU}(2)_g: X \rightarrow XU_g^\dagger, \quad a_\mu \rightarrow U_g a_\mu U_g^\dagger + i\partial_\mu U_g U_g^\dagger, \quad (2.16)$$

in the continuum. As before, global spin rotations act on the Majorana X on the left:

$$\text{SU}(2)_s: X \rightarrow U_s X. \quad (2.17)$$

Letting $D_\mu^a X = \partial_\mu X + iX a_\mu$, the inclusion of quantum fluctuations results in the following Lagrangian:

$$\mathcal{L}_{\text{QCD}_3} = i\text{tr}(\bar{X}\gamma^\mu D_\mu^a X). \quad (2.18)$$

$\mathcal{L}_{\text{QCD}_3}$ can be expressed in a more familiar form by defining Dirac fermions:

$$\psi_{1,v} = \frac{i}{\sqrt{2}}(\chi_{x,v} - i\chi_{y,v}), \quad \psi_{2,v} = -\frac{1}{\sqrt{2}}(\chi_{0,v} + i\chi_{z,v}). \quad (2.19)$$

In terms of these operators, the Lagrangian becomes

$$\mathcal{L}_{\text{QCD}_3} = \sum_{v=1,2} i\bar{\psi}_v \gamma^\mu (\partial_\mu - i a_\mu^a \sigma^a) \psi_v. \quad (2.20)$$

That is, the low-energy physics of the π -flux state is described by QCD_3 with $N_f = 2$ fermions. The Dirac representation is not nearly as useful as the Majorana representation of $\mathcal{L}_{\text{QCD}_3}$: while gauge transformations act on the ψ fermions in the usual fashion, the action of the spin symmetry is nontrivial. We therefore primarily use the form given in Eq. (2.18).

A side effect of the expansion about the Dirac cones is that the χ fermions transform nontrivially under time-reversal and the space group symmetries:

$$\begin{aligned} T_x: \chi &\rightarrow \mu^x \chi, & R_{\pi/2}: \chi &\rightarrow e^{i\pi\tau^y/4} e^{-i\pi\mu^y/4} \chi(-y, x), \\ T_y: \chi &\rightarrow \mu^y \chi, & P_x: \chi &\rightarrow \tau^z \mu^z \chi(-x, y), \\ T: \chi &\rightarrow \tau^y \mu^y \chi, \quad i \rightarrow -i, & P_y: \chi &\rightarrow -\tau^x \mu^x \chi(x, -y). \end{aligned} \quad (2.21)$$

In addition, the spin and space group symmetries of the model are significantly enlarged at this fixed point. Not only is $\mathcal{L}_{\text{QCD}_3}$ Lorentz invariant, but it is symmetric under rotations mixing the spin and valley indices of X : $X \rightarrow LX$, where L is a 4×4 unitary matrix. Because X is composed of Majorana fermions, there is an important reality condition,

$$X^* = \sigma^y X \sigma^y, \quad (2.22)$$

and therefore only L such that $L^T \sigma^y L = \sigma^y$ are allowed. This reduces what would have been a $\text{U}(4)$ symmetry to $\text{Sp}(4)$. Finally, since both $\text{SU}(2)_g$ and $\text{Sp}(4)$ share the nontrivial element -1 , the true global symmetry is obtained by taking the quotient: $\text{Sp}(4)/\mathbb{Z}_2 \cong \text{SO}(5)$.

C. Dual description

As with any mean-field approach involving a continuous gauge group, the existence of $\mathcal{L}_{\text{QCD}_3}$ is by no means guaranteed once gauge fluctuations have been taken into account. However, in spite of some of the terminology, in this paper we do not view the π -flux ‘‘phase’’ as a stable state of matter existing over a finite region in parameter space. Instead, we treat it as a parent theory with instabilities potentially leading to $\text{U}(1)$ and \mathbb{Z}_2 spin liquids, as well as to ordered phases like Néel and VBS. This approach is motivated by a duality between $\mathcal{L}_{\text{QCD}_3}$ and \mathbb{CP}^1 proposed by Wang *et al.* [33] to describe the Néel-VBS transition. We discuss the relation between \mathbb{CP}^1 and QCD_3 in this context.

One of the key components to their proposal is the $\text{SO}(5)$ symmetry we just discussed. On the QCD_3 side of the duality, an order parameter for this symmetry is

$$n^j = \text{tr}(\bar{X}\Gamma^j X), \quad \Gamma^j = \{\mu^x, \mu^z, \mu^y \sigma^x, \mu^y \sigma^y, \mu^y \sigma^z\}. \quad (2.23)$$

The symmetry transformations in Eq. (2.21) indicate that n^1 and n^2 are the VBS order parameters, while n^3 , n^4 , and n^5 correspond to the Néel order parameter. Using this, Refs. [45–47] showed that taking $\mathcal{L}_{\text{QCD}_3}$ to

$$\mathcal{L}_{\text{QCD}_3, \phi} = \mathcal{L}_{\text{QCD}_3} + m\phi^j \text{tr}(\bar{X}\Gamma^j X), \quad (2.24)$$

and subsequently integrating out the fermions, yields a nonlinear sigma model for ϕ with a Wess-Zumino-Witten term. This topological term manifests itself physically by making the defects of the order parameter of one symmetry transform nontrivially under the action of the other symmetry. These nontrivial correlations prompted Tanaka and Hu [45] and Senthil and Fisher [46] to propose this nonlinear sigma model as a description of the critical theory describing the Landau-forbidden continuous phase transition between Néel and VBS.

Conversely, the \mathbb{CP}^1 formulation of the phase transition circumvents the obstruction to continuity by eschewing the traditional notion of an order parameter. While the Néel phase is entered through the condensation of $N^a = z^\dagger \sigma^a z$, the VBS phase is described by the proliferation of monopoles, events which change the flux of the gauge field by 2π (or, equivalently, change the global Skyrmion number by one). Not only do these monopoles confine the $\text{U}(1)$ gauge field, but, because they transform nontrivially under the space group, this symmetry is necessarily broken in the condensate. In spite of the very different forms the Néel and VBS order parameters take, numerics [48] have observed an emergent $\text{SO}(5)$ symmetry between the two, implying that $\text{SO}(5)$ emerges as a symmetry in the IR. In this version, the VBS portion of $\text{SO}(5)$ order parameter is given by $(\phi_1, \phi_2) = 2(\text{Re}\mathcal{M}, \text{Im}\mathcal{M})$, where \mathcal{M} denotes the monopole operator, while the remaining pieces are simply $(\phi_3, \phi_4, \phi_5) = (z^\dagger \sigma^x z, z^\dagger \sigma^y z, z^\dagger \sigma^z z)$.

Wang *et al.* [33] suggest that both of these models flow to the same SO(5) symmetric CFT in the IR. An important feature of this CFT is the absence of a relevant singlet operator. The critical point is instead obtained by tuning the coupling μ of a relevant, anisotropic operator to zero:

$$\mathcal{L} = \mathcal{L}_{\text{SO}(5)} + \mu \mathcal{O}_{an},$$

$$\mathcal{O}_{an} \sim \frac{2}{5}(\phi_3^2 + \phi_4^2 + \phi_5^2) - \frac{3}{5}(\phi_1^2 + \phi_2^2). \quad (2.25)$$

When $\mu > 0$, the system has VBS order, while when $\mu < 0$, it orders along the Néel directions. The approach we take is slightly different in spirit to this proposal, and we discuss this further in Sec. IV C.

III. SPIN LIQUIDS PROXIMATE TO THE π -FLUX PHASE

In this section, we describe the Higgs descendants of QCD₃ and our approach to their classification. We start by discussing which operators can couple to the Higgs field, before turning to a more complete discussion of the projective symmetry group than what was provided in the previous section. Given a set of criteria described below, we conclude that there exists a single (spin) gapped U(1) spin liquid among the Higgs descendants of QCD₃. We next list all gapped and fully symmetry \mathbb{Z}_2 spin liquids, as well as all gapped \mathbb{Z}_2 spin liquids with Ising-nematic order. Special note is taken of the spin liquids A_f and B_f , though we wait until Sec. IV A 1 to prove their equivalence to A_b and B_b . The section finishes with a description of the gapped \mathbb{Z}_2 spin liquid with current-loop order we call C_f .

A. Higgs fields

We begin by examining the set of operators we will be coupling to the Higgs field. QCD₃ is strongly coupled in the IR, and so very little can be said with certainty regarding the operators and their scaling dimensions in the IR. We focus on fermion bilinears since these are the most relevant gauge-invariant bosonic operators of the UV theory. Nonperturbative operators such as monopoles are not considered.

We consider interaction terms of the form

$$\text{tr}(\varphi \bar{X} M X) = \varphi^a \text{tr}(\sigma^a \bar{X} M X), \quad (3.1)$$

where $\varphi = \varphi^a \sigma^a$ is a generic Higgs field transforming in the adjoint representation of SU(2)_g and M is a matrix acting on the sublattice, color, and/or flavor space of the fermions and which may or may not contain derivatives. The physical properties of the various possible Higgs phases are defined primarily by the bilinear it couples to.

Restricting for the moment to bilinears without derivatives, those which are charged under the gauge group are

$$\text{tr}(\sigma^a \bar{X} \gamma^\mu X), \quad \text{tr}(\sigma^a \bar{X} \Gamma^j \gamma^\mu X), \quad \text{tr}(\sigma^a \bar{X} T^j X), \quad (3.2)$$

where $\Gamma^j = \{\mu^z, -\mu^x, \mu^y \sigma^a\}$ and $T^j = \{\mu^y, \sigma^a, \mu^x \sigma^a, \mu^z \sigma^a\}$ are the vector and adjoint representations of SO(5), respectively. The first set of operators are the gauge currents $J^{a,\mu}$. These cannot couple a Higgs field since the gauge theory description of the Heisenberg model is predicated on the requirement that these currents vanish. In fact, the gauge fields can be interpreted as Lagrange multipliers which have been added to $\mathcal{L}_{\text{QCD}_3}$ in order to impose the $J^{a,\mu} = 0$ constraint.

No such obstacles exist for the other two sets of bilinears. The second group of operators, $\text{tr}(\sigma^a \bar{X} \Gamma^a \gamma^\mu X)$, are SO(5) and spacetime vectors in addition to gauge adjoints. The presence of the gamma matrices γ^μ indicates that the fermions will remain massless upon coupling these bilinears to a condensed φ .

On the other hand, should the Higgs field couple to one of the final operators in Eq. (3.2), $\langle \varphi \rangle \neq 0$ will act as a mass for the fermions. The only other bilinears which act as masses to the fermions are the singlet and SO(5) vector, neither of which are fully symmetric. Therefore, given the aforementioned restriction on which operators we consider, we conclude that an operator of the form $\text{tr}(\sigma^a \bar{X} T^j X)$ must couple to a condensed Higgs field in A_f and D_f . (It can also be verified that these color-singlet mass terms cannot provide a spin gap to the ordered spin liquids, B_f or C_f .)

We will see shortly that the operators in Eq. (3.2) are not sufficient to reproduce the phase diagram in Figs. 1(b) and 2(b). Consequently, we also allow the Higgs field to couple to bilinears which contain a single derivative:

$$\text{tr}(\sigma^a \bar{X} i \partial_\mu X), \quad \text{tr}(\sigma^a \bar{X} \Gamma^j i \partial_\mu X), \quad \text{tr}(\sigma^a \bar{X} T^j \gamma^\mu i \partial_\nu X). \quad (3.3)$$

We now discuss how symmetries manifest in Higgs phases. The action of the space group and time reversal on the bilinears listed above is given in Figs. 3 and 4; the spin symmetry rotates operators with spin indices among themselves in the usual way. It naively appears that a Higgs field coupling to any of these bilinears will necessarily break one or more symmetries upon condensing. As

T^j	\mathcal{T}	P_x	P_y	T_x	T_y	$R_{\pi/2}$
μ^y	−	+	+	−	−	μ^y
σ^a	−	−	−	+	+	σ^a
$\mu^x \sigma^a$	+	+	−	+	−	$\mu^z \sigma^a$
$\mu^z \sigma^a$	+	−	+	−	+	$-\mu^x \sigma^a$

FIG. 3. How $\text{tr}(\sigma^a \bar{X} T^j X)$ transform under the physical symmetries. $T^j = \{\mu^y, \sigma^a, \mu^x \sigma^a, \mu^z \sigma^a\}$ are the ten generators of SO(5).

Γ^j	\mathcal{T}	P_x	P_y	T_x	T_y	$R_{\pi/2}$
$\mu^x \gamma^0$	+	-	+	+	-	$\mu^z \gamma^0$
$\mu^x \gamma^x$	-	+	+	+	-	$\mu^z \gamma^y$
$\mu^x \gamma^y$	-	-	-	+	-	$-\mu^z \gamma^x$
$\mu^z \gamma^0$	+	+	-	-	+	$-\mu^x \gamma^0$
$\mu^z \gamma^x$	-	-	-	-	+	$-\mu^x \gamma^y$
$\mu^z \gamma^y$	-	+	+	-	+	$\mu^x \gamma^x$
$\mu^y \sigma^a \gamma^0$	+	-	-	-	-	$\mu^y \sigma^a \gamma^0$
$\mu^y \sigma^a \gamma^x$	-	+	-	-	-	$\mu^y \sigma^a \gamma^y$
$\mu^y \sigma^a \gamma^y$	-	-	+	-	-	$-\mu^y \sigma^a \gamma^x$

FIG. 4. How $\text{tr}(\sigma^a \bar{X} \Gamma^j \gamma^\mu X)$ transform under the physical symmetries. $\Gamma^j = \{\mu^x, \mu^z, \mu^y \sigma^a\}$ transform under the vector representation of the emergent SO(5).

with the π -flux Hamiltonian in Eq. (2.13), H_π , this intuition does not account for the fact that the Higgs field is not a gauge-invariant operator. A symmetry is only truly broken if the original and symmetry transformed actions are *not* gauge equivalent.

For instance, in Eq. (1.6), $\text{tr}(\sigma^a \bar{X} \mu^y X)$ couples to the Higgs field Φ . Since $\text{tr}(\sigma^a \bar{X} \mu^y X)$ maps to minus itself under \mathcal{T} , T_x , and T_y , the naive argument would suggest that these symmetries are broken when $\langle \Phi^a \rangle \neq 0$. However, it is not difficult to find a gauge transformation capable of “undoing” the action of these symmetries. In particular, supposing that only $\langle \Phi^x \rangle \neq 0$, we see that the gauge transformation $V = i\sigma^z$ takes $\text{tr}(\sigma^x \bar{X} \mu^y X)$ to minus itself, thereby proving the equivalence of the original and symmetry transformed actions.

This set of gauge transformations comprises the PSG and is what we use to characterize the Higgs descendants. More generally, when a group element acts on a bilinear as

$$G: \text{tr}(\sigma^a \bar{X} M X) \rightarrow \text{tr}(\sigma^a \bar{X} \bar{U}_G M U_G X), \quad (3.4)$$

where $\bar{U}_G = \gamma^0 U^\dagger \gamma^0$, the projective symmetry group is defined as

$$\mathcal{P}G: \text{tr}(\sigma^a \bar{X} M X) \rightarrow \text{tr}(V_G^\dagger \sigma^a V_G \bar{X} \bar{U}_G M U_G X), \quad (3.5)$$

where

$$\text{tr}(V_G^\dagger \sigma^a V_G \bar{X} \bar{U}_G M U_G X) = \text{tr}(\sigma^a \bar{X} M X). \quad (3.6)$$

We will see that requiring the existence of a V_G for every U_G places stringent conditions on which operators can couple to a Higgs field while preserving certain symmetries in the condensed phase.

B. Symmetric spin liquids

In this section, we focus on fully symmetric and gapped spin liquids (by “gapped,” we are referring specifically to the matter content). As mentioned, in order to simultaneously gap the fermions and Higgs the gauge boson, an operator of the form $\text{tr}(\sigma^a \bar{X} T^j X)$, where T^j is a generator of SO(5), must couple to a Higgs field. These are listed in Fig. 3. Of the ten generators of SO(5), nine transform as vectors under the spin symmetry, and we show in Appendix A that a fully symmetry spin liquid cannot be formed by coupling a Higgs field to any of these bilinears. Roughly, the argument relies on the fact that in order to preserve the spin symmetry, a linear combination of the form $\sim \sum_a \text{tr}(\sigma^a \bar{X} M \sigma^a X)$ for $M = 1, \mu^x, \mu^z$ must couple to the Higgs, which then makes it impossible to preserve all of the discrete symmetries.

This observation establishes $\text{tr}(\sigma^a \bar{X} \mu^y X)$ as the only fermion bilinear capable of both giving the fermions a mass and coupling to a Higgs field. As indicated in Eq. (1.6) and Sec. III A, we denote the Higgs field coupling to this bilinear as Φ^a . Since the action remains invariant under all gauge transformations about the direction of the condensate, Φ^a cannot fully Higgs the SU(2) gauge symmetry down to \mathbb{Z}_2 . For instance, if we assume that only $\langle \Phi^x \rangle \neq 0$, U(1) operations of the form $X \rightarrow X e^{-i\theta \sigma^x}$ remain a gauge symmetry. We label this U(1) spin liquid D_f .

It is well known [49] that without gapless degrees of freedom, a U(1) gauge theory is unstable to the proliferation of monopoles and confinement [50]. We ignore the ultimate fate of D_f until Sec. IV C, where we show that the true ground state is a VBS.

With this caveat in mind, we deduce the projective symmetry group of the gapped U(1) spin liquid from Fig. 3:

$$\begin{aligned} V_t &= e^{i\theta_t \sigma^x} i\sigma^z, & V_{tx} &= e^{i\theta_{tx} \sigma^x} i\sigma^z, \\ V_{py} &= e^{i\theta_{py} \sigma^x}, & V_{ty} &= e^{i\theta_{ty} \sigma^x} i\sigma^z, \\ V_{px} &= e^{i\theta_{px} \sigma^x}, & V_r &= e^{i\theta_r \sigma^x}, \end{aligned} \quad (3.7)$$

where the θ_G are arbitrary angles parametrizing the residual U(1) gauge degree of freedom. Here, the subscripts t , px , py , tx , ty , and r indicate that these gauge transformations accompany the action of \mathcal{T} , P_x , P_y , T_x , T_y , and $R_{\pi/2}$, respectively.

We note that while the physical symmetries are all preserved in D_f , the emergent SO(5) symmetry of QCD₃ has been broken. Of the SO(5) generators, $T^j = \{\mu^y, \sigma^a, \mu^x \sigma^a, \mu^z \sigma^a\}$, the U(1) gauge theory is only invariant under $\{\mu^y\} \times \{\sigma^a\}$, indicating that the SO(5) is broken to U(1) \times SU(2). From the perspective of the SO(5) order parameter, $n^j = \text{tr}(\bar{X} \Gamma^j X)$, $\Gamma^j = \{\mu^x, \mu^z, \mu^y \sigma^a\}$, the VBS order parameters, n^1 and n^2 , can no longer be rotated into the Néel order parameters, n^3 , n^4 , and n^5 .

To break the gauge group down to \mathbb{Z}_2 , an additional Higgs field Φ_1 is needed. However, there are strict constraints on which bilinears can couple to Φ_1 in order for the resultant \mathbb{Z}_2 spin liquid to preserve all physical symmetries. We approach this problem from a vector representation by associating an $\text{SO}(3)$ matrix Q to each $\text{SU}(2)_g$ gauge transformation V . That is, instead of looking at V such that $\text{tr}(\varphi \bar{X} M X) \rightarrow \text{tr}(V^\dagger \varphi V \bar{X} M X)$, we consider Q such that $\varphi^a \text{tr}(\sigma^a \bar{X} M X) \rightarrow (Q\varphi)^a \text{tr}(\sigma^a \bar{X} M X)$. In this notation, when $\langle \Phi^x \rangle \neq 0$, we must have

$$Q_G = \begin{pmatrix} 1 & \mathbf{0} \\ \mathbf{0} & R_G \end{pmatrix}, \quad G = px, py, r, \quad (3.8)$$

and

$$Q_G = \begin{pmatrix} -1 & \mathbf{0} \\ \mathbf{0} & \tilde{R}_G \end{pmatrix}, \quad G = t, tx, ty, \quad (3.9)$$

where R_G and \tilde{R}_G are determined by the bilinear coupling to Φ_1 . The constraints on this bilinear arise from the fact that Q_G must be special orthogonal, therefore implying that R_G and \tilde{R}_G must be 2×2 orthogonal matrices with determinants $+1$ and -1 , respectively.

We now argue that none of the operators in Fig. 4 satisfy these requirements. First, all bilinears with spin indices can be excluded by the same reasoning given above and in Appendix A. Next, we note that all remaining operators still transform differently than $\text{tr}(\sigma^a \bar{X} \mu^y X)$ under at least one of the symmetries, and therefore the Φ_1 condensate *must* be perpendicular to x in color space. For the remaining six operators, the obstruction to forming a spin liquid may be understood by studying the action of a 90° rotation. The last column of the table indicates that

$R_{\pi/2}$ maps each bilinear to plus or minus another bilinear in the table; e.g., $R_{\pi/2} : \text{tr}(\sigma^a \bar{X} \mu^x \gamma^0 X) \rightarrow \text{tr}(\sigma^a \bar{X} \mu^z \gamma^0 X)$. In order for this to describe a rotationally symmetric phase, both bilinears must couple to a Higgs field. We might imagine that Φ_1 couples to both operators in a pair, but this is not a viable option because the other discrete symmetries do not act on the members of each pair in the same way. For instance, no gauge transformation can preserve the form of $\langle \Phi_1^a \rangle \text{tr}(\sigma^a \bar{X} \gamma^0 [\mu^x \pm \mu^z] X)$ under P_x , P_y , T_x , and T_y since $\text{tr}(\sigma^a \bar{X} \gamma^0 \mu^x X)$ and $\text{tr}(\sigma^a \bar{X} \gamma^0 \mu^z X)$ behave differently under these symmetries. We might try coupling each of these operators to *different* Higgs fields, Φ_1 and Φ_1' , and require that they condense in mutually perpendicular channels; e.g., $\langle \Phi_1^y \rangle \neq 0$ and $\langle \Phi_1'^z \rangle \neq 0$. However, the matrix required to undo the action of the time-reversal symmetry is then $Q_t = \text{diag}(-1, 1, 1)$, which is not an element of $\text{SO}(3)$. We conclude that this does not work either.

We next perform the same analysis on bilinears containing a single derivative. Once again, the arguments in Appendix A are valid, and we immediately exclude all operators in Eq. (3.3) which transform nontrivially under spin rotations. The action of the space group and time-reversal symmetries on the remaining operators is provided in Fig. 5. Again, $R_{\pi/2}$ maps many of the operators to plus or minus a different operator in the table. As discussed in the previous paragraph, only bilinears which transform in the same way under \mathcal{T} , P_x , P_y , T_x , and T_y as their partner under $R_{\pi/2}$ are suitable candidates, and these are highlighted in different colors. In Fig. 6, we list the PSGs of all gapped and symmetric \mathbb{Z}_2 spin liquids which can be formed using this set of operators.

In Sec. IV A 1, we determine which (if any) bosonic *Ansatz* these PSGs correspond to. We find that sPSG5

$\mu^y \gamma^\mu i \partial_\nu$	\mathcal{T}	P_x	P_y	T_x	T_y	$R_{\pi/2}$
$\mu^y \gamma^0 i \partial_0$	+	-	-	-	-	$\mu^y \gamma^0 \partial_0$
$\mu^y \gamma^0 i \partial_x$	-	+	-	-	-	$\mu^y \gamma^0 \partial_y$
$\mu^y \gamma^0 i \partial_y$	-	-	+	-	-	$-\gamma^0 \mu^y \partial_x$
$\mu^y \gamma^x i \partial_0$	-	+	-	-	-	$\mu^y \gamma^y \mu_0^0$
$\mu^y \gamma^x i \partial_x$	+	-	-	-	-	$\mu^y \gamma^y i \partial_y$
$\mu^y \gamma^x i \partial_y$	+	+	+	-	-	$-\mu^y \gamma^y i \partial_x$
$\mu^y \gamma^y i \partial_0$	-	-	+	-	-	$-\mu^y \gamma^x i \partial_0$
$\mu^y \gamma^y i \partial_x$	+	+	+	-	-	$-\mu^y \gamma^x i \partial_y$
$\mu^y \gamma^y i \partial_y$	+	-	-	-	-	$\mu^y \gamma^x i \partial_x$

$\mu^{0,x,y} i \partial_\mu$	\mathcal{T}	P_x	P_y	T_x	T_y	$R_{\pi/2}$
$i \partial_0$	+	-	-	+	+	$i \partial_0$
$i \partial_x$	-	+	-	+	+	$i \partial_y$
$i \partial_y$	-	-	+	+	+	$-i \partial_x$
$\mu^x i \partial_0$	-	+	-	+	-	$\mu^z i \partial_0$
$\mu^x i \partial_x$	+	-	-	+	-	$\mu^z i \partial_y$
$\mu^x i \partial_y$	+	+	+	+	-	$-\mu^z i \partial_x$
$\mu^z i \partial_0$	-	-	+	-	+	$-\mu^x i \partial_0$
$\mu^z i \partial_x$	+	+	+	-	+	$-\mu^x i \partial_y$
$\mu^z i \partial_y$	+	-	-	-	+	$\mu^x i \partial_x$

FIG. 5. Symmetry transformation properties of bilinears of the form $\text{tr}(\sigma^a \bar{X} i \partial_\mu X)$, $\text{tr}(\sigma^a \bar{X} \Gamma^j i \partial_\mu X)$, and $\text{tr}(\sigma^a \bar{X} T^j \gamma^\mu i \partial_\nu X)$, which do *not* transform under spin. The operators that can couple to a Higgs field in a gapped symmetric spin \mathbb{Z}_2 spin liquid are colored; entries with the same color transform into one another under $R_{\pi/2}$.

$s\text{PSG}$	M	Q_t/V_t	$Q_{px,py}/V_{px,py}$	$Q_{tx,ty}/V_{tx,ty}$	Q_r/V_r
1	$\mu^y \gamma^0 i \partial_0, \mu^y (\gamma^x i \partial_x + \gamma^y i \partial_y)$	diag(-1, 1, 1)	diag(1, -1, -1)	diag(-1, -1, 1)	$\mathbb{1}$
		$i\sigma^y$	$i\sigma^x$	$i\sigma^z$	$\mathbb{1}$
2	$\mu^y (\gamma^x i \partial_x - \gamma^y i \partial_y)$	diag(-1, 1, 1)	diag(1, -1, -1)	diag(-1, -1, 1)	diag(1, -1, -1)
		$i\sigma^y$	$i\sigma^x$	$i\sigma^z$	$i\sigma^x$
3	$\mu^y (\gamma^x i \partial_y + \gamma^y i \partial_x)$	diag(-1, 1, 1)	$\mathbb{1}$	diag(-1, -1, 1)	diag(1, -1, -1)
		$i\sigma^y$	$\mathbb{1}$	$i\sigma^z$	$i\sigma^x$
4	$\mu^y (\gamma^x i \partial_y - \gamma^y i \partial_x)$	diag(-1, 1, 1)	$\mathbb{1}$	diag(-1, -1, 1)	$\mathbb{1}$
		$i\sigma^y$	$\mathbb{1}$	$i\sigma^z$	$\mathbb{1}$
5	$i \partial_0$	diag(-1, 1, 1)	diag(1, -1, -1)	diag(-1, 1, -1)	$\mathbb{1}$
		$i\sigma^y$	$i\sigma^x$	$i\sigma^y$	$\mathbb{1}$

FIG. 6. All symmetric PSGs associated with symmetric \mathbb{Z}_2 spin liquids in which $\langle \Phi^x \rangle \neq 0$, where Φ couples to $\text{tr}(\sigma^a \bar{X} \mu^y X)$. These are listed as a function of the operator $\text{tr}(\sigma^a \bar{X} M X)$ which Φ_1 couples to. We assume that only $\langle \Phi_1^y \rangle \neq 0$.

corresponds to the fully symmetric spin liquid A_b , and for this reason we denote it A_f .

C. \mathbb{Z}_2 spin liquids with Ising-nematic order

As emphasized in Sec. I, it is not necessary to restrict to fully symmetric spin liquids. We therefore also study gapped, nematic \mathbb{Z}_2 spin liquids proximate to the gapped U(1) spin liquid D_f . In particular, we investigate spin liquids which are obtained by coupling a third Higgs field, Φ_{2i} , to the operators in Figs. 3, 4 and 5, and which preserve the continuous spin symmetry, \mathcal{T} , P_x , P_y , T_x , and T_y , but break the 90° rotation symmetry $R_{\pi/2}$. The absence of rotation symmetry makes it possible to couple any of the operators in Figs. 3 and 4 to the Higgs field, and the ten candidates we find are listed in Fig. 7.

We note that $n\text{PSG}5$ and $n\text{PSG}6$ are continuously connected to $s\text{PSG}1-2$ and $s\text{PSG}3-4$, respectively. For instance, in the case of $n\text{PSG}5$, if the Higgs field couples as $\sum_{i=x,y} \Phi_{2i}^a \text{tr}(\sigma^z \bar{X} \gamma^i i \partial_i X)$, then phases where the condensate satisfies $\langle \Phi_{2x}^a \rangle = \pm \langle \Phi_{2y}^a \rangle$ do not break $R_{\pi/2}$ and are precisely $s\text{PSG}1$ and $s\text{PSG}2$. The same considerations hold for $n\text{PSG}6$ in relation to $s\text{PSG}3$ and $s\text{PSG}4$.

In all cases, the phase with $\langle \Phi \rangle = 0$ and $\langle \Phi_1 \rangle \neq 0$ is a fully symmetric U(1) spin liquid. However, unlike D_f , the matter sector is gapless.

In the next section we find that $n\text{PSG}7$ is the fermionic version of the bosonic phase B_b , leading us to label it B_f .

D. \mathbb{Z}_2 spin liquid with current-loop order

Thus far, we have defined three separate Higgs fields. To ensure that the condensed phases had a spin gap, Φ and

$\text{tr}(\sigma^a \bar{X} \mu^y X)$ were required to couple. We then identified which bilinears could couple to a second Higgs field Φ_1 , such that the phase with $\langle \Phi \rangle \neq 0$, $\langle \Phi_1 \rangle \neq 0$, and $\langle \Phi \rangle \perp \langle \Phi_1 \rangle$ was a fully symmetric spin \mathbb{Z}_2 liquid. Similarly, we determined in the previous section which bilinears could couple to a Higgs field Φ_{2i} such that the phase with $\langle \Phi \rangle \neq 0$ and $\langle \Phi_{2i} \rangle \neq 0$ was a \mathbb{Z}_2 spin liquid with Ising-nematic order, again provided $\langle \Phi \rangle \perp \langle \Phi_{2i} \rangle$.

A natural extension is to ask which phases result when all three Higgs fields have condensed: $\langle \Phi \rangle \neq 0$, $\langle \Phi_1 \rangle \neq 0$, and $\langle \Phi_{2i} \rangle \neq 0$. However, there are clearly a large number of possibilities. Not only have we identified many candidate $s\text{PSGs}$ and $n\text{PSGs}$, but different symmetries will be broken depending on the relative orientation of the Higgs fields. Therefore, we focus on producing the phase diagram in Fig. 2(b) and restrict our study to the situation where the symmetric and nematic spin liquids are A_f and B_f , the phases described by $s\text{PSG}5$ and $n\text{PSG}7$.

This scenario describes the following four different patterns of symmetry breaking.

- (1) $\langle \Phi \rangle \perp \langle \Phi_1 \rangle$, $\langle \Phi \rangle \perp \langle \Phi_{2i} \rangle$, & $\langle \Phi_1 \rangle \parallel \langle \Phi_{2i} \rangle$.
- (2) $\langle \Phi \rangle \perp \langle \Phi_1 \rangle$, $\langle \Phi_1 \rangle \perp \langle \Phi_{2i} \rangle$, & $\langle \Phi \rangle \parallel \langle \Phi_{2i} \rangle$.
- (3) $\langle \Phi_{2i} \rangle \perp \langle \Phi \rangle$, $\langle \Phi_{2i} \rangle \perp \langle \Phi_1 \rangle$, & $\langle \Phi \rangle \parallel \langle \Phi_1 \rangle$.
- (4) $\langle \Phi \rangle \perp \langle \Phi_1 \rangle \perp \langle \Phi_{2i} \rangle$.

In Fig. 8 we list which symmetries are broken for each of these cases.

Referring to the phase diagram in Fig. 1(b), it is natural to restrict to the case where A_f and B_f are accessible by taking $\langle \Phi_{2i} \rangle$ or $\langle \Phi_1 \rangle$ to zero. Since both the second and third cases have $\langle \Phi \rangle$ parallel to either $\langle \Phi_1 \rangle$ or $\langle \Phi_{2i} \rangle$, we eliminate these options. Of the remaining two phases, the resulting spin liquid only possesses current-loop order

n PSG		M_i	V_t	V_{px}	V_{py}	V_{tx}	V_{ty}
1	x	$\mu^x \gamma^0$	$i\sigma^y$	$i\sigma^x$	$\mathbb{1}$	$i\sigma^y$	$i\sigma^z$
	y	$\mu^z \gamma^0$	$i\sigma^y$	$\mathbb{1}$	$i\sigma^x$	$i\sigma^z$	$i\sigma^y$
2	x	$\mu^z \gamma^y$	$i\sigma^z$	$\mathbb{1}$	$\mathbb{1}$	$i\sigma^z$	$i\sigma^y$
	y	$\mu^x \gamma^x$	$i\sigma^z$	$\mathbb{1}$	$\mathbb{1}$	$i\sigma^y$	$i\sigma^z$
3	x	$\mu^x \gamma^y,$	$i\sigma^z$	$i\sigma^x$	$i\sigma^x$	$i\sigma^y$	$i\sigma^z$
	y	$\mu^z \gamma^x$	$i\sigma^z$	$i\sigma^x$	$i\sigma^x$	$i\sigma^z$	$i\sigma^y$
4	x	$\mu^y \gamma^0 i\partial_x, \mu^y \gamma^x i\partial_0$	$i\sigma^z$	$\mathbb{1}$	$i\sigma^x$	$i\sigma^z$	$i\sigma^z$
	y	$\mu^y \gamma^0 i\partial_y, \mu^y \gamma^y i\partial_0$	$i\sigma^z$	$i\sigma^x$	$\mathbb{1}$	$i\sigma^z$	$i\sigma^z$
5	x	$\mu^y \gamma^x i\partial_x$	$i\sigma^y$	$i\sigma^x$	$i\sigma^x$	$i\sigma^z$	$i\sigma^z$
	y	$\mu^y \gamma^y i\partial_y$	$i\sigma^y$	$i\sigma^x$	$i\sigma^x$	$i\sigma^z$	$i\sigma^z$
6	x	$\mu^y \gamma^x i\partial_y$	$i\sigma^y$	$\mathbb{1}$	$\mathbb{1}$	$i\sigma^z$	$i\sigma^z$
	y	$\mu^y \gamma^y i\partial_x$	$i\sigma^y$	$\mathbb{1}$	$\mathbb{1}$	$i\sigma^z$	$i\sigma^z$
7	x	$i\partial_x$	$i\sigma^z$	$\mathbb{1}$	$i\sigma^x$	$i\sigma^y$	$i\sigma^y$
	y	$i\partial_y$	$i\sigma^z$	$i\sigma^x$	$\mathbb{1}$	$i\sigma^y$	$i\sigma^y$
8	x	$\mu^x i\partial_0$	$i\sigma^z$	$\mathbb{1}$	$i\sigma^x$	$i\sigma^y$	$i\sigma^z$
	y	$\mu^z i\partial_0$	$i\sigma^z$	$i\sigma^x$	$\mathbb{1}$	$i\sigma^z$	$i\sigma^y$
9	x	$\mu^z i\partial_x$	$i\sigma^y$	$\mathbb{1}$	$\mathbb{1}$	$i\sigma^z$	$i\sigma^y$
	y	$\mu^x i\partial_y$	$i\sigma^y$	$\mathbb{1}$	$\mathbb{1}$	$i\sigma^y$	$i\sigma^z$
10	x	$\mu^x i\partial_x$	$i\sigma^y$	$i\sigma^x$	$i\sigma^x$	$i\sigma^y$	$i\sigma^z$
	y	$\mu^z i\partial_y$	$i\sigma^y$	$i\sigma^x$	$i\sigma^x$	$i\sigma^z$	$i\sigma^y$

FIG. 7. Nematic PSGs associated with order parameters of the form $\Phi^a \text{tr}(\sigma^a \bar{X} \mu^i X) + \Phi_{2i}^a \text{tr}(\sigma^a \bar{X} M^i X)$. We do not include $\text{tr}(\sigma^a \bar{X} \partial_0 X)$ since this operator is invariant under the action of $R_{\pi/2}$ and already accounted for as sPSG5. The labels x, y are simply a convenient notation and do not necessarily signify a physical direction.

when $\langle \Phi_1 \rangle \parallel \langle \Phi_{2i} \rangle$. This situation is further distinguished by breaking the fewest symmetries. We refer to this phase as C_f and later equate it and the bosonic phase C_b .

	Direction			broken
	$\langle \Phi \rangle$	$\langle \Phi_1 \rangle$	$\langle \Phi_{2x} \rangle$	
1	x	y	y	\mathcal{T}, P_x
2	x	y	x	P_y, T_x, T_y
3	x	x	y	$\mathcal{T}, P_x, P_y, T_x, T_y$
4	x	y	z	P_x, T_x, T_y

FIG. 8. Symmetries broken depending on the orientation in gauge space taken by the Higgs condensates. The fields couple to the bilinears as $\text{tr}(\Phi \bar{X} \mu^i X) + \text{tr}(\Phi_1 \bar{X} i\partial_0 X) + \text{tr}(\Phi_{2x} \bar{X} i\partial_x X)$.

IV. SPIN-LIQUID IDENTIFICATION

We now identify the phases examined above with previously studied spin liquids. On the lattice, Wen [17] showed that 58 distinct \mathbb{Z}_2 PSGs can be accessed from the π -flux state (SU2Bn0). However, his PSG classification gives no indication of the physical properties of these phases and, moreover, as we will see, it includes certain “anomalous” PSGs which cannot be obtained from a mean-field *Ansatz*. We therefore begin by discussing the “symmetry fractionalization” approach to spin-liquid classification, and relate it to Wen’s scheme. This will significantly simplify the process of relating the symmetric U(1) spin liquids and the phases in Fig. 6 to the spin liquids studied by Wen. Its greatest power, however, will be to treat fermionic and bosonic mean-field *Ansätze* on the same footing, allowing us to relate our results to phases described using Schwinger bosons, and prove our earlier claim that $A_f, B_f,$ and C_f are fermionic versions of $A_b, B_b,$ and C_b .

We next show that the gapped U(1) spin liquid D_f corresponds to D_b . The gapless gauge degrees of freedom invalidate the symmetry fractionalization approach to comparing spin liquids represented with bosons and fermions. Instead, we show through linear response that the proliferation of monopoles induces the condensation of the VBS order parameters given by the first two components of the vector in Eq. (2.23). We provide additional verification by demonstrating that the Berry phase of the monopole matches the calculation performed by Haldane [40] and Read and Sachdev [35].

A. Symmetry fractionalization and \mathbb{Z}_2 spin-liquid identification

In this section, we relate the gapped \mathbb{Z}_2 spin liquids determined in the previous section to spin liquids obtained using Schwinger bosons by Yang and Wang [23] and Chatterjee *et al.* [21]. Since these phases are gapped, they are completely defined via their symmetry fractionalization [19]. Of the PSGs listed in Figs. 6 and 7, we find that precisely one matches onto the spin liquid A_b , and one onto B_b of Fig. 2(a). We begin by briefly reviewing this classification scheme in the context of \mathbb{Z}_2 topological order. See Ref. [19] for more details.

One of the defining characteristics of topological order is the presence of anyonic excitations. For the \mathbb{Z}_2 case we consider here, there are two bosonic particles, typically denoted e and m , which are mutually semionic: the wave function picks up a minus sign upon the adiabatic motion of an e particle traveling around an m particle. A bound state of an e and m is a fermionic excitation, $\epsilon \sim em$, and it also satisfies mutual semionic statistics with e and m . We frequently refer to the m particle as the “vison” and the e and ϵ particles as the bosonic and fermionic “spinon,” respectively. These excitations carry \mathbb{Z}_2 gauge charge and therefore must appear in pairs. Nonetheless, the \mathbb{Z}_2 gauge

field is gapped and these phases are deconfined, meaning that e , m , and ε particles may be very far from one another.

A comparison of these particles with the excitations in the Higgs phases implies that the fermionic spinons ε should be identified with the excitations of the field operator X . In addition, in $(2+1)$ D the Abrikosov vortices of the condensate are pointlike, and we associate these with the vison excitations m . The remaining particle, the bosonic spinon e , is therefore described by a bound state of X and the vortex. In contrast, $\mathbb{C}\mathbb{P}^1$ is formulated in terms of the bosonic spinons. The vison is present as a vortex in the condensate as before, but now it is the fermionic spinon that is expressed as a bound state.

This representation of the degrees of freedom of a gapped \mathbb{Z}_2 spin liquid provides a means to compare phases described using fermionic and bosonic *Ansätze*. In a manner analogous to the classification of symmetries in terms of quantum numbers, these symmetry-enriched topological phases can be classified by what is known as symmetry fractionalization numbers. Independent of any formalism, suppose we create from the ground state two ε (or e or m) excitations and separate them so that they lie at very distant points \mathbf{r} and \mathbf{r}' : $|\mathbf{r}, \mathbf{r}'\rangle$. Since the rest of the system is indistinguishable from the ground state, the action of an unbroken symmetry G will exclusively affect these regions:

$$G|\mathbf{r}, \mathbf{r}'\rangle = \mathcal{G}_\varepsilon(\mathbf{r})\mathcal{G}_\varepsilon(\mathbf{r}')|\mathbf{r}, \mathbf{r}'\rangle, \quad (4.1)$$

where $\mathcal{G}_\varepsilon(\mathbf{r})$ has support only in the region immediately surrounding \mathbf{r} . As discussed in Sec. II A, the generators of a symmetry group satisfy certain commutation relations, and for the space group of the square lattice (plus time reversal), these relations are given in Eq. (2.3) and below. It follows that the action of any of these operations on *all* wave functions must be equivalent to the identity. For example, since $T_y^{-1}T_xT_yT_x^{-1} = \mathbb{1}$, it must map $|\mathbf{r}, \mathbf{r}'\rangle$ to itself:

$$|\mathbf{r}, \mathbf{r}'\rangle = T_y^{-1}T_xT_yT_x^{-1}|\mathbf{r}, \mathbf{r}'\rangle. \quad (4.2)$$

In terms of the local symmetry operations, this becomes

$$|\mathbf{r}, \mathbf{r}'\rangle = T_{\varepsilon,y}^{-1}(\mathbf{r})T_{\varepsilon,x}(\mathbf{r})T_{\varepsilon,y}(\mathbf{r})T_{\varepsilon,x}^{-1}(\mathbf{r}) \cdot T_{\varepsilon,y}^{-1}(\mathbf{r}')T_{\varepsilon,x}(\mathbf{r}')T_{\varepsilon,y}(\mathbf{r}')T_{\varepsilon,x}^{-1}(\mathbf{r}')|\mathbf{r}, \mathbf{r}'\rangle. \quad (4.3)$$

Since the transformations are localized at either \mathbf{r} and \mathbf{r}' , they must be independent from one another and therefore constant. However, because of the \mathbb{Z}_2 gauge degree of freedom, $\zeta_{lxy}^\varepsilon = T_{\varepsilon,y}^{-1}(\mathbf{r})T_{\varepsilon,x}(\mathbf{r})T_{\varepsilon,y}(\mathbf{r})T_{\varepsilon,x}^{-1}(\mathbf{r})$ need not necessarily equal unity: the symmetry can be *fractionalized* such that $\zeta_{lxy}^\varepsilon = -1$. The value of ζ_{lxy}^ε will be consistent for every excitation of that species within a phase

It is not difficult to connect this to the PSG classification of the previous section. The PSG is the set of gauge transformations required to preserve the form of the action following a symmetry transformation, as shown in

	Group relations	sPSG1	sPSG2	sPSG3	sPSG4	sPSG5	vison	twist	$\mathbb{Z}_2[0, 0]$	$\mathbb{Z}_2[0, \pi]$
1	$T_y^{-1}T_xT_yT_x^{-1}$	-1	-1	-1	-1	-1	-1	1	1	1
2	$P_y^{-1}T_xP_yT_x^{-1}$	-1	-1	1	1	-1	-1	1	1	1
3	$P_y^{-1}T_yP_yT_y$	-1	-1	1	1	-1	1	1	-1	-1
4	P_y^2	-1	-1	1	1	-1	1	-1	1	1
5	$P_y^{-1}R_{\pi/2}P_yR_{\pi/2}$	1	-1	-1	1	1	1	1	1	-1
6	$R_{\pi/2}^4$	1	1	1	1	1	-1	-1	1	1
7	$R_{\pi/2}^{-1}T_xR_{\pi/2}T_y$	-1	-1	-1	-1	-1	-1	1	1	1
8	$R_{\pi/2}^{-1}T_yR_{\pi/2}T_x^{-1}$	-1	-1	-1	-1	-1	1	1	-1	-1
9	$R_{\pi/2}^{-1}\mathcal{T}^{-1}R_{\pi/2}\mathcal{T}$	1	-1	-1	1	1	1	1	1	-1
10	$P_y^{-1}\mathcal{T}^{-1}P_y\mathcal{T}$	-1	-1	1	1	-1	1	-1	1	1
11	$T_x^{-1}\mathcal{T}^{-1}T_x\mathcal{T}$	1	1	1	1	-1	1	1	-1	-1
12	$T_y^{-1}\mathcal{T}^{-1}T_y\mathcal{T}$	1	1	1	1	-1	1	1	-1	-1
13	\mathcal{T}^2	-1	-1	-1	-1	-1	1	1	-1	-1

FIG. 9. The columns labeled “sPSG1-5” list the symmetry fractionalizations of the gapped, symmetric \mathbb{Z}_2 spin liquids given in Fig. 7. The corresponding bosonic symmetry fractionalization numbers are obtained by multiplying the sPSG numbers with the those given in the Vison and Twist columns. We see that sPSG5 corresponds to the $\mathbb{Z}_2[0, 0]$ state of Ref. [23]. No bosonic counterparts to sPSG1–4 are present in Ref. [23].

Eq. (3.6). Now, however, we present the PSG action solely in terms of an operator which creates fermionic spinons, X :

$$\mathcal{P}G: X \rightarrow U_G X V_G^\dagger. \quad (4.4)$$

The same argument given above then requires that under the action of $T_y^{-1} T_x T_y T_x^{-1}$, X is mapped to plus or minus itself:

$$T_y^{-1} T_x T_y T_x^{-1}[X] = U_{tx}^\dagger U_{ty} U_{xy} U_{ty}^\dagger U_{tx}^\dagger X V_{ty} V_{tx}^\dagger V_{ty}^\dagger V_{tx} = \pm X. \quad (4.5)$$

This factor is precisely the fractionalization number of ε . When time reversal is involved, this is modified to

$$G^{-1} T^{-1} G T[X] = U_i^* \sigma^y U_G^* \sigma^y U_i^T U_G^\dagger X V_G V_i^* \sigma^y V_G^T \sigma^y V_i^T, \\ T^2[X] = U_i^* \sigma^y U_i \sigma^y X \sigma^y V_i^\dagger \sigma^y V_i^T, \quad (4.6)$$

where the reality condition in Eq. (2.22) has been used. Fig. 3 lists the numbers corresponding to each of the sPSGs in Fig. 9. (We note that the 7th group relation, $R_{\pi/2}^{-1} T_x R_{\pi/2} T_y = \mathbb{1}$, can be fixed by a gauge transformation on the relative sign of V_{tx} and V_{ty} . In keeping with the convention of Ref. [23], we require that the symmetry fractionalization number be -1 for the fermionic spinons.)

The argument also demonstrates a shortcoming of the PSG classification. While it immediately returns the symmetry fractionalization of the fermionic spinons, it provides no information regarding the symmetry fractionalization of the vison and bosonic spinon. However, it fortunately turns out that the vison's fractionalization numbers are independent of the precise \mathbb{Z}_2 spin liquid under study and can be obtained by examining a fully frustrated transverse-field Ising model [23,51–53]. We quote these results in the column labeled “Vison” in Fig. 9.

1. Correspondence between fermionic and bosonic Ansätze

Comparing fermionic and bosonic *Ansätze* may appear straightforward from this point: since $e \sim \varepsilon m$, it seems reasonable to assume that the symmetry fractionalization of the bosonic spinon is obtained through a simple multiplication of the symmetry fractionalization numbers of the fermionic spinon and the vison. However, the mutual statistics of ε and m occasionally change this relation. For instance, upon rotating e by 360° , $R_{\pi/2}^4$, either the vison will encircle the fermionic spinon or vice versa. In either case, an extra factor of -1 must be taken into account. These additional multiplication factors were worked out in Ref. [23], and we quote them under the column labeled “Twist” in Fig. 9.

The comparison with the bosonic symmetry fractionalization allows us to identify sPSG5 with $\mathbb{Z}_2[0,0]$, showing

that $A_f = A_b$ as promised. We do not find fermionic counterparts to the remaining four spin liquids in Ref. [23].

Using a slightly altered set of commutation relations to account for the symmetry breaking, the exact same analysis can be performed for the nematic spin liquids. These symmetry fractionalization numbers are shown in Fig. 11, and, as claimed, by comparing with the analysis of Ref. [21] we positively identify B_f (nPSG7) with the Ising-nematic \mathbb{Z}_2 spin liquid B_b .

Finally, the equivalence of A_f and B_f with A_b and B_b indicates the equivalence of C_f and C_b . In Appendix D, we provide additional verification of this result using the symmetry fractionalization technique.

B. Lattice classification of fermionic PSGs

The data compiled in Fig. 9 can also be used to compare the phases we find against fermionic spin liquids described on the lattice. In Appendix B, we review Wen's classification scheme [17] and identify the lattice PSGs corresponding to the two U(1) spin liquids as well as the five symmetric \mathbb{Z}_2 spin liquids. This classification is useful since it allows us to express the phases we have studied on the lattice without having to reverse-engineer the bilinears.

We identify the gapped U(1) spin liquid D_f with U1Cn0n1 and the gapless U(1) spin liquid ($\langle \Phi_1 \rangle \neq 0$) with U1Bx11n. The lattice PSGs corresponding to the five symmetric \mathbb{Z}_2 spin liquids we obtain are shown in Fig. 10.

Both sPSG1 and sPSG5 seemingly correspond to multiple lattice PSGs. However, in Appendix B 4, we prove that while the spin liquids have the same symmetry fractionalizations, of the two shown, only one of each pair actually corresponds to the spin liquids we consider. In the case of sPSG5, it is not difficult to show that Z2Bxx2z is always gapless, immediately ruling it out as a description of the gapped phase A_f . Further, we show that Z2Bxx2z is not proximate to either the gapped or gapless U(1) spin liquids U1Cn0n1 and U1Bx11n. Similarly, we find that Z2Bxx23 is not proximate to U1Cn0n1, leaving Z2Bxx13 as the sole realizable lattice PSG capable of reproducing sPSG1.

These statements can be verified explicitly by comparing our continuum theory with mean-field Hamiltonians on the

	sPSG1	sPSG2	sPSG3	sPSG4	sPSG5 (A_f)
Lattice PSG	Z2Bxx13 Z2Bxx23	Z2Bxx03	Z2B0013	Z2B0001	Z2Bxx1z Z2Bxx2z

FIG. 10. Spin liquids according to the labeling scheme given in Ref. [17] and reviewed in Appendix B 4. All of the spin liquids listed are found to be proximate to the π -flux phase SU2n0 though not necessarily U1Cn0n1. While the symmetry fractionalization of sPSG1 and sPSG5 corresponds to multiple fermionic PSGs, the two that are italicized (Z2Bxx23 and Z2Bxx2z) are not proximate to U1Cn0n1 and therefore cannot represent the Higgs phases we obtain (see Appendix B 4).

lattice which have been constructed using only information provided by the lattice PSG. In Appendix C, we study the lattice Hamiltonians for the gapped and gapless U(1) spin liquids, as well as A_f . We find that a low-energy expansion of the mean-field Hamiltonian describing U1Cn0n1 corresponds to adding $\text{tr}(\sigma^x \bar{X} \mu^y X)$ to the π -flux Hamiltonian as expected, but that no analogous statement can be made for either U1Bx11n or Z2Bxx1z. In particular, we demonstrate that no mean-field *Ansatz* on the lattice can realize the U(1) spin liquid U1Bx11n. This should not be too surprising as the continuum realization of this phase is the product of condensing Φ_1 when coupled to $\text{tr}(\sigma^a \bar{X} \partial_0 X)$, the time component of a vector. This description is manifestly dependent on the presence of temporal fluctuations in contrast to the purely static mean-field analysis.

Conversely, a lattice Hamiltonian describing the \mathbb{Z}_2 phase A_f does exist. However, upon expanding the resulting Hamiltonian about its Dirac cones, the hopping term which breaks the U(1) symmetry down to \mathbb{Z}_2 appears to arise from coupling $\text{tr}(\sigma^a \bar{X} \mu^y \partial_x \partial_y [\partial_x^2 - \partial_y^2] X)$ to a condensed Higgs field. We can see why this may be the case by observing how symmetries act on $\Xi = \text{tr}(\sigma^a \bar{X} \mu^y \partial_x \partial_y [\partial_x^2 - \partial_y^2] X)$:

$$\begin{aligned} \mathcal{T}[\Xi] &= -\Xi, & P_{x,y}[\Xi] &= -\Xi, \\ \mathcal{T}_{x,y}[\Xi] &= -\Xi, & R_{\pi/2}[\Xi] &= \Xi. \end{aligned} \quad (4.7)$$

It follows that a Higgs field Φ'_1 which couples to Ξ may have a nonzero expectation value in the A_f phase provided it is perpendicular in color space to both Φ and Φ_1 . That is, supposing $\langle \Phi^x \rangle \neq 0$ and $\langle \Phi_1^z \rangle \neq 0$, having $\langle \Phi_1'^z \rangle \neq 0$ will not break any of the symmetries.

It can also be shown that the Ising-nematic spin liquid B_f is not ‘‘anomalous’’ in the manner just discussed.

C. Identification of U(1) spin liquid

The arguments that allow us to compare \mathbb{Z}_2 spin liquids expressed using bosonic and fermionic spinons breaks down in the presence of gapless degrees of freedom. In both cases, these phases are unstable to the proliferation of monopoles, and their true ground states will break any symmetries under which the monopoles transform non-trivially. In order to ensure that D_f actually corresponds to the massive phase of the $\mathbb{C}\mathbb{P}^1$ theory D_b , we verify that the two spin liquids share the same fate and ultimately realize a VBS. We approach this problem from two perspectives. We first follow the method outlined in Ref. [54] and determine which bilinear operators *respond* to a weakly varying flux and, consequently, the monopoles' presence. We complement this analysis by calculating the Berry phase of the monopole in a certain limit and show that it agrees with the analogous calculation performed using Schwinger bosons in Ref. [35].

1. Flux response

The effective Lagrangian describing D_f is

$$\mathcal{L}_{\text{U}(1)} = i\text{tr}(\bar{X} \gamma^\mu [\partial_\mu X + iX \sigma^x a_\mu^x]) + \lambda_2 \langle \Phi^x \rangle \text{tr}(\sigma^x \bar{X} \mu^y X). \quad (4.8)$$

Because both a_μ^y and a_μ^z are gapped, only a_μ^x is included in $\mathcal{L}_{\text{U}(1)}$. In what follows, we drop the x index, taking $a_\mu^x \rightarrow a_\mu$ [this a_μ should not be confused with the gauge field of the original SU(2) gauge field]. Finally, at this point in the discussion, it is more convenient to express the Lagrangian in terms of Dirac spinors. Using Eq. (2.19), we find

$$\mathcal{L}_{\text{U}(1)} = \bar{\psi} i \gamma^\mu (\partial_\mu - i a_\mu \sigma^x) \psi + m \bar{\psi} \sigma^x \mu^y \psi, \quad (4.9)$$

where $m = \lambda_2 \langle \Phi^x \rangle$.

In the context of a U(1) gauge theory, a monopole is a topologically nontrivial field configuration of a_μ . In imaginary time, this configuration corresponds exactly to a (stationary) Dirac monopole in (3 + 1)D electromagnetism. However, instead of behaving as a particle, in (2 + 1)D the monopole is actually an instanton: it describes tunneling between different vacua or topological sectors labeled by their total flux, $\int dS_\mu \epsilon^{\mu\nu\lambda} \partial_\nu a_\lambda = 2\pi n$, where n is an integer. This number is zero in the deconfined phase of the gauge theory whereas it fluctuates and ceases to take a definite value once the monopoles proliferate.

A complete treatment of the monopole proceeds by first expanding the gauge field into a classical background piece A_μ and a quantum fluctuation piece \tilde{a}_μ , $a_\mu = A_\mu + \tilde{a}_\mu$, and quantizing the theory about this background. Because the monopole background breaks translational symmetry, this is quite an involved calculation which we do not perform. Instead, we investigate the impact a nonzero flux has on the other operators of the theory. That is, we assume that the classical monopole configuration described by A_μ varies very slowly and, through linear response, determine which operators \mathcal{O} the flux couples to at leading order: $\langle \mathcal{O}(x) \rangle = \int d^3 x' \chi_{\mathcal{O}}^\mu(x, x') A_\mu(x')$, where $\chi_{\mathcal{O}}^\mu = \langle \mathcal{O}(x) \bar{\psi} \gamma^\mu \sigma^x \psi(x') \rangle$. This calculation is outlined in Appendix E, and at low energies yields

$$\langle \bar{\psi} \gamma^\mu \mu^y \psi \rangle = \frac{1}{\pi} \epsilon^{\mu\alpha\beta} \partial_\alpha A_\beta. \quad (4.10)$$

Consequently, whenever there is a net flux, $\int d^2 x (\partial_x A_y - \partial_y A_x) \neq 0$, we expect $\langle \bar{\psi} \gamma^0 \mu^y \psi \rangle \neq 0$ as well. This allows us to identify $\bar{\psi} \gamma^\mu \mu^y \psi$ with the topological current. The topological charge is then obtained by integrating the zeroth component of the current over space:

$$Q = \frac{1}{2} \int d^2 r \bar{\psi} \gamma^0 \mu^y \psi. \quad (4.11)$$

The factor of 1/2 is chosen to ensure that Q is always an integer, as follows from Eq. (4.10).

A conserved charge is the generator of the associated symmetry, meaning that Q generates the flux conservation symmetry. However, this operator should be familiar from Sec. III A, where it is observed to be the generator of the U(1) VBS symmetry. This can be confirmed by checking that

$$[Q, V_x] = iV_y, \quad [Q, V_y] = -iV_x, \quad (4.12)$$

where $V_x = \frac{1}{2}\bar{\psi}\mu^x\psi$, $V_y = \frac{1}{2}\bar{\psi}\mu^y\psi$. It follows that Q is conjugate to the VBS order parameters.

When the gapped U(1) gauge theory confines, the monopole proliferation induces large fluctuations in Q . This in turn suppresses the fluctuations of the operators conjugate to Q , ultimately resulting in long-range order. We conclude from the analysis above that the proliferation of monopoles triggers the condensation of one of the VBS order parameters, proving that D_f is unstable to a VBS and therefore equivalent to D_b . This mechanism should be contrasted with the scenario outlined in Sec. II C, where VBS order is achieved by tuning $\mu > 0$ in Eq. (2.25).

2. Berry phase

A separate argument for the identification of the U(1) spin liquid proceeds by a computation of the monopole Berry phase, along the lines of the original arguments using the semiclassical quantization of the antiferromagnet [40], or the Schwinger boson theory of the U(1) spin liquid [35]. Here, this argument starts from a lattice Hamiltonian for the U(1) spin liquid, which we obtain from Eq. (C21) for a generic direction of the Higgs field Φ :

$$\begin{aligned} H = & -\sum_i \{i\alpha[\psi_i^\dagger\psi_{i+\hat{x}} + (-)^{i_x}\psi_i^\dagger\psi_{i+\hat{y}} + \text{H.c.}] \\ & + \Phi^a(-)^{i_x+i_y}\beta(\psi_i^\dagger\tau^a\psi_{i+2\hat{x}} + \psi_i^\dagger\tau^a\psi_{i+2\hat{y}} + \text{H.c.}) \\ & - \Phi^a a_0(-)^{i_x+i_y}\psi_i^\dagger\tau^a\psi_i\}. \end{aligned} \quad (4.13)$$

We are interested in saddle points of the associated action where the Φ^a Higgs field, and the associated SU(2) gauge field [not written explicitly in Eq. (4.13)], take the configuration of 't Hooft–Polyakov monopoles [55,56] in (2 + 1)-dimensional spacetime. After obtaining such saddle points, we have to compute the fermion determinant in such a background, and the phase of this determinant will yield the needed monopole Berry phase. This is clearly a demanding computation, and we will not attempt to carry it out in any generality. However, assuming the topological invariance of the needed quantized phase, we can compute it by distorting the saddle point Lagrangian, without closing the fermion gap, to a regime where the phase is easily computable. Specifically, consider the limit where the parameter a_0 in Eq. (4.13) is much larger than all other parameters, including α and β . For the 't Hooft–Polyakov monopole at the origin of spacetime, $\Phi^a \sim \hat{r}^a$, where \hat{r}^a is a unit radial vector in

spacetime. Ignoring all but the a_0 term in Eq. (4.13), we then have to compute the Berry phases of single fermions, each localized on a single site, in the presence of a staggered field $\propto \Phi^a$. However, this Berry phase is precisely that computed by Haldane [40]; in his case, the staggered field was the antiferromagnetic order parameter which acts in the spin SU(2) space [in contrast to the staggered field in the gauge SU(2) space in our case], and the Berry phase arose from that of a quantized $S = 1/2$ spin. As the Berry phase of a spin-1/2 localized fermion is equal to the spin Berry phase, we conclude that the 't Hooft–Polyakov monopole Berry phase is equal to that obtained by Haldane [40] for $S = 1/2$. Therefore, the monopole Berry phases in the fermionic spinon U(1) spin liquid U1Cn0n1 are equal to those of the U(1) spin liquid of the bosonic $\mathbb{C}\mathbb{P}^1$ theory of the square lattice antiferromagnet [38,39].

V. CONCLUSIONS

Two distinct classes of (2 + 1)-dimensional fermion-boson dualities have recently seen much discussion in the literature.

One class concerns gapped \mathbb{Z}_2 spin liquids which have both bosonic and fermionic spinon excitations. Binding with a vison transmutes a spinon from a boson to a fermion, or vice versa [57], and this allows one to map between \mathbb{Z}_2 spin liquids obtained in mean-field theory using fermionic or bosonic *Ansätze*. Specific examples of such dualities have been described on a variety of lattices [19,21,23,27–32], and our results for such dualities appear in Figs. 1 and 2. We describe the dualities between the bosonic \mathbb{Z}_2 spin liquids A_b, B_b, C_b and the fermionic spin liquids A_f, B_f, C_f , respectively. The first two of these dualities have been obtained earlier [21,23], but we obtain all three in a unified manner with reference to continuum theories.

The second class of dualities concerns conformal gauge theories with fermionic and bosonic matter [58–63]. Most relevant to our considerations is the proposed duality [33] between the critical bosonic $\mathbb{C}\mathbb{P}^1$ U(1) gauge theory and fermionic SU(2) QCD₃ with $N_f = 2$ flavors of Dirac fermions.

Among our results is a demonstration of the compatibility between these two classes of dualities. We Higgs the critical bosonic $\mathbb{C}\mathbb{P}^1$ and fermionic QCD₃ theories, and find nontrivial consistency between the gapped \mathbb{Z}_2 spin liquids so obtained. We also obtain a fermionic counterpart to the U(1) spin liquid with gapped bosonic spinons on the square lattice originally obtained by Arovas and Auerbach [38] (which is equivalent to the gapped z_α phase of the $\mathbb{C}\mathbb{P}^1$ theory [34,35]): the U(1) spin liquid with gapped fermionic spinons was identified as U1Cn0n1 (in Wen's notation). Both the bosonic and fermionic U(1) spin liquids are eventually unstable to monopole proliferation, confinement, and VBS order, and have identical monopole Berry phases (as shown in Sec. IV C).

Our analysis also led us to propose new fermion-boson dualities between multicritical theories. One example is the duality between (i) the U(1) gauge theory in Eq. (1.4) with two unit charge bosons z_a , a doubly charged Higgs field P , and the masses of both fields tuned to criticality, and (ii) the SU(2) gauge theory in Eq. (1.6) with $N_f = 2$ massless fundamental Dirac fermions ψ , and two adjoint Higgs fields Φ, Φ_1 , and the masses of both Higgs fields tuned to criticality.

The fermionic approach to square lattice spin liquids [17,25,26] yields a variety of critical spin liquids coupled to U(1) and SU(2) gauge fields. Two examples are in Fig. 1(b), the states labeled by Wen as U1Bx11n and SU2Bn0. The results of Wang *et al.* [33] indicate that the SU(2) critical state SU2Bn0 cannot appear as an extended critical phase in a square lattice antiferromagnet, and it is only realized as a critical point between the Néel and VBS states. From our comparison of Figs. 1(b) and 1(a), we obtain evidence that the critical U(1) spin liquid U1Bx11n also cannot be realized as an extended phase on the square lattice: it is unstable to the appearance of canted antiferromagnetic order.

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APPENDIX A: SPIN LIQUIDS WITH PROJECTIVE SPIN SYMMETRY

We expand upon our assertion in Sec. III B that a fully symmetric, gapped spin liquid cannot be obtained through the condensation of a Higgs field Φ coupling to a bilinear which transforms in a nontrivial manner under the SU(2) spin symmetry. As discussed, in order for the resulting spin liquid to have a spin gap, Φ must couple to one of the operators in Fig. 3. We start by studying $N^{ab} = \text{tr}(\sigma^a \bar{X} \sigma^b X)$ and couple it to a Higgs field as $\sum_{a,b} \Phi^{ab} N^{ba} = \tilde{\text{tr}}(\Phi N)$, where “ $\tilde{\text{tr}}$ ” refers to a trace over the spin and color *vector* labels (as opposed to the usual trace “ tr ” over spin and color *spinor* indices). In the Higgs phase, we write $\tilde{\Phi} = \langle \Phi \rangle \neq 0$.

Naturally, having the Higgs couple to N^{ab} implies that spin symmetry is realized projectively in the condensed phase, if at all. We associate SO(3) matrices to both the SU(2) gauge and spin transformations. That is, instead of studying the action of gauge and spin transformations U_g and U_s , we consider matrices $Q, R \in \text{SO}(3)$, such that

$$\begin{aligned} \text{SU}(2)_s: N^{ab} &\rightarrow \text{tr}(\sigma^a \bar{X} U_s^\dagger \sigma^b U_s X) = N^{ac} (R^T)^{cb}, \\ \text{SU}(2)_g: N^{ab} &\rightarrow \text{tr}(U_g \sigma^a U_g^\dagger \bar{X} \sigma^b X) = Q^{ac} N^{cb}. \end{aligned} \quad (\text{A1})$$

Under a projective spin transformation,

$$\mathcal{P}\text{SU}(2)_s: \tilde{\text{tr}}(\tilde{\Phi} N) \rightarrow \tilde{\text{tr}}(\tilde{\Phi} QNR^T) = \tilde{\text{tr}}(\tilde{\Phi} N), \quad (\text{A2})$$

implying that $Q = \tilde{\Phi}^{-1} R$. The requirement that $Q \in \text{SO}(3)$ implies that $\tilde{\Phi} \in \text{SO}(3)$ as well, for example, $\tilde{\Phi}^{ab} = |\Phi| \delta^{ab}$.

The obstruction to forming a fully symmetric spin liquid is then apparent. Since $N^{ab} \rightarrow -N^{ab}$ under \mathcal{T}, P_x , and P_y , the equivalence of the original and symmetry transformed states requires that $\tilde{\Phi}$ be gauge equivalent to $-\tilde{\Phi}$. This is only possible if $Q_{t,px,py} = -\mathbb{1} \notin \text{SO}(3)$.

These considerations apply equally to $\text{tr}(\sigma^a \bar{X} \mu^{x,z} \sigma^b X)$, as indicated in Sec. III B.

APPENDIX B: WEN’S LATTICE PSG CLASSIFICATION SCHEME

In this appendix, we relate our results to the spin-liquid classification scheme proposed in Ref. [17] by Wen. We begin by reviewing his conventions and formalism before explaining what it means for two spin liquids to be “proximate” in this language. We then discuss how we determined that the gapped and gapless U(1) spin liquids in Fig. 1(b) correspond to U1Cn0n1 and U1Bx11n, respectively. We subsequently consider the \mathbb{Z}_2 sPSGs and explain how the identification in Fig. 10 was obtained.

We note that frequent reference will be made to information that is present only in the arXiv version of Ref. [17].

1. Conventions and formalism

Here, we briefly review the spin-liquid classification scheme proposed in Ref. [17]; for a complete discussion, see to the original paper. In keeping with these conventions, we express the mean-field Hamiltonian of Eq. (2.11) in terms of fermions $\psi = (\psi_1, \psi_2)^T = (f_\uparrow, f_\downarrow)^T$. The mean-field *Ansatz* is written in terms of the matrix

$$u_{ij} = \frac{3}{8} J \begin{pmatrix} \alpha_{ij}^\dagger & \beta_{ij} \\ \beta_{ij} & -\alpha_{ij} \end{pmatrix} = u_{ji}^\dagger. \quad (\text{B1})$$

The average constraint in Eq. (2.10) becomes

$$\langle \psi_i^\dagger \tau^\ell \psi_i \rangle = 0, \quad (\text{B2})$$

where τ^ℓ are Pauli matrices (with $\tau^0 = \mathbb{1}$) and the mean-field Hamiltonian can then be written as

$$H_{\text{MF}} = \sum_{\langle ij \rangle} \left[\frac{4}{3J_{ij}} \text{tr}(u_{ij}^\dagger u_{ij}) - (\psi_i^\dagger u_{ij} \psi_j + \text{H.c.}) \right] + \sum_i a_0^\ell \psi_i^\dagger \tau^\ell \psi_i. \quad (\text{B3})$$

Here, u_{ij} is the analogue to α_{ij} , β_{ij}^a (when $\gamma_{ij} \neq 0$, the spin symmetry is realized projectively, a possibility this formalism does not take into account [44]). a_0^ℓ are Lagrange multipliers enforcing the constraint in Eq. (B2). In order for H_{MF} to preserve spin, we must choose $iu_{ij} \in \text{SU}(2)$. Finally, the $\text{SU}(2)$ gauge symmetry acts on the ψ fermions and *Ansatz* as

$$\psi_i \rightarrow W(\mathbf{i})\psi_i, \quad u_{ij} \rightarrow W(\mathbf{i})u_{ij}W^\dagger(\mathbf{j}). \quad (\text{B4})$$

The projective symmetry group in this context is expressed as the invariance of the *Ansatz* u_{ij} under the joint action of a symmetry transformation G and a gauge transformation W_G :

$$W_G G[u_{ij}] = u_{ij}, \quad (\text{B5})$$

where

$$\begin{aligned} G[u_{ij}] &= u_{G(\mathbf{i}, G(\mathbf{j}))}, \\ W_G[u_{ij}] &= W_G(\mathbf{i})u_{ij}W_G^\dagger(\mathbf{j}), \\ W_G(\mathbf{i}) &\in \text{SU}(2). \end{aligned} \quad (\text{B6})$$

Here, we assume that G is a space group operation; for time reversal, we have $\mathcal{T}[u_{ij}] = -u_{ij}$. The invariant gauge group is the set of gauge transformations which do not alter the *Ansatz*,

$$\mathfrak{B} = \{W(\mathbf{i})|W(\mathbf{i})u_{ij}W(\mathbf{j})^\dagger, W(\mathbf{i}) \in \text{SU}(2)\}, \quad (\text{B7})$$

and, therefore, \mathfrak{B} can be either $\text{SU}(2)$, $\text{U}(1)$, or \mathbb{Z}_2 . In the main body of the text, this is what we simply refer to as the gauge group or, sometimes in a Higgs phase, the ‘‘residual gauge group.’’

In order to make use of the symmetry fractionalization technique, we translate the commutation relations in Eq. (2.3) and below to the lattice case:

1. $W_{ty}^{-1}(i_x, i_y + 1)W_{tx}(i_x, i_y + 1)W_{ty}(i_x - 1, i_y + 1)W_{tx}^{-1}(i_x, i_y) \in \mathfrak{B}$,
2. $W_{py}^{-1}(i_x, -i_y)W_{tx}(i_x, -i_y)W_{py}(i_x - 1, -i_y)W_{tx}^{-1}(i_x, i_y) \in \mathfrak{B}$,
3. $W_{py}^{-1}(i_x, -i_y)W_{ty}(i_x, -i_y)W_{py}(i_x, -i_y - 1)W_{ty}(i_x, i_y + 1) \in \mathfrak{B}$,
4. $W_{py}(i_x, i_y)W_{py}(i_x, -i_y) \in \mathfrak{B}$,
5. $W_{py}^{-1}(i_x, -i_y)W_r(i_x, -i_y)W_{py}(-i_y, -i_x)W_r(-i_y, i_x) \in \mathfrak{B}$,
6. $W_r(i_x, i_y)W_r(i_y, -i_x)W_r(-i_x, i_y)W_r(-i_y, i_x) \in \mathfrak{B}$,
7. $W_r^{-1}(-i_y, i_x)W_{tx}(-i_y, i_x)W_r(-i_y - 1, i_x)W_{ty}(i_x, i_y + 1) \in \mathfrak{B}$,
8. $W_r^{-1}(-i_y, i_x)W_{ty}(-i_y, i_x)W_r(-i_y, i_x - 1)W_{tx}^{-1}(i_x, i_y) \in \mathfrak{B}$,
9. $W_t^{-1}(i_x, i_y)W_r^{-1}(-i_y, i_x)W_t(-i_y, i_x)W_r(-i_y, i_x) \in \mathfrak{B}$,
10. $W_t^{-1}(i_x, i_y)W_{py}^{-1}(i_x, -i_y)W_t(i_x, -i_y)W_{py}(i_x, -i_y) \in \mathfrak{B}$,
11. $W_t^{-1}(i_x, i_y)W_{tx}^{-1}(i_x + 1, i_y)W_t(i_x + 1, i_y)W_{tx}(i_x + 1, i_y) \in \mathfrak{B}$,
12. $W_t^{-1}(i_x, i_y)W_{ty}^{-1}(i_x, i_y + 1)W_t(i_x, i_y + 1)W_{ty}(i_x, i_y + 1) \in \mathfrak{B}$,
13. $W_t(i_x, i_y)W_t(i_x, i_y) \in \mathfrak{B}$.

2. $\text{SU}(2)$ spin-liquid classification

We present the mean-field *Ansatz* of the π -flux phase in Sec. II B. In Wen’s notation, it corresponds to the spin liquid $\text{SU}2Bn0$, and consequently has the following PSG:

$$\begin{aligned} W_{tx}(\mathbf{i}) &= (-)^{i_y} g_{tx}, & W_{px}(\mathbf{i}) &= (-)^{i_x} g_{px}, \\ W_{pxy}(\mathbf{i}) &= (-)^{i_x i_y} g_{pxy}, \\ W_{ty}(\mathbf{i}) &= g_{py}, & W_{py}(\mathbf{i}) &= (-)^{i_y} g_{py}, \\ W_t(\mathbf{i}) &= (-)^{i_x + i_y} g_t, \end{aligned} \quad (\text{B9})$$

where $g_\xi \in \text{SU}(2)$, $\xi = tx, ty, px, py, pxy, t$. All PSGs proximate to $\text{SU}2Bn0$ can be obtained by fixing the values of the g_ξ to a specific element in $\text{SU}(2)$ (the PSGs are only defined modulo the invariant gauge group). In Appendix B of Ref. [17], Wen enumerates which $\text{U}(1)$ and \mathbb{Z}_2 PSGs are proximate to $\text{SU}2Bn0$. All of the phases we consider must be identified with one of these options.

3. $\text{U}(1)$ spin-liquid classification

Wen [17] finds that the following $\text{U}(1)$ phases are proximate to $\text{SU}2Bn0$:

U1B000n	U1Bn10n	U1Cn0nn	U1C11nn
U1B0001	U1Bx10x	U1Cn0n1	U1C11nx
U1B001n	U1Bx11n	U1Cn0x1	U1C11xn
U1B0011	U1Bx11x	U1Cn01n	U1C11xx

In this section, we determine which of these lattice PSGs corresponds to the gapped and gapless U(1) spin liquids obtained by condensing Φ and Φ_1 , respectively.

Gapped U(1) spin liquid (D_f).—To compare with Wen's classification, we condense the Higgs fields in the z component. Therefore, for the gapped U(1) spin liquid D_f , only $\langle \Phi^z \rangle \neq 0$, and the PSG in Eq. (3.7) should be rewritten:

$$\begin{aligned} V_{tx} &= e^{i\theta_{tx}\sigma^z} i\sigma^x, & V_{px} &= e^{i\theta_{px}\sigma^z}, & V_r &= e^{i\theta_r\sigma^z}, \\ V_{ty} &= e^{i\theta_{ty}\sigma^z} i\sigma^x, & V_{py} &= e^{i\theta_{py}\sigma^z}, & V_t &= e^{i\theta_t\sigma^z} i\sigma^x. \end{aligned} \quad (\text{B10})$$

The resulting symmetry fractionalization is shown in Table I.

We identify this phase in several steps. We note that independent from θ_t , $(W_t U_t)^2 = -1$, and therefore, of the spin liquids proximate to SU2Bn0, only those with $W_t(\mathbf{i}) \propto \tau^0$ are possible candidates. Moreover, the U1B spin liquids all have $W_{tx} = (-)^{i_y} g_3(\theta_{tx})$, $W_{ty}(\mathbf{i}) = g_3(\theta_{ty})$, where $g_\ell(\theta) = e^{i\theta\tau^\ell}$. Inserting these into group relation no. 1 in Eq. (B8) returns -1 , again independent of the angles θ_{tx} and θ_{ty} , invalidating these options. This leaves four candidates: U1Cn0n1, U1Cn0x1, U1C11nx, and U1C11xx. We have computed the symmetry fractionalization of each of these phases and determined that D_f corresponds to U1Cn0n1 whose lattice PSG is

TABLE I. Symmetry fractionalization of U(1) spin liquids.

	Group relations	Gapped (D_f)	Gapless
1	$T_y^{-1} T_x T_y T_x^{-1}$	$-e^{-2i(\theta_{tx}-\theta_{ty})\sigma^z}$	-1
2	$P_y^{-1} T_x P_y T_x^{-1}$	$e^{2i\theta_{py}\sigma^z}$	$-e^{2i\theta_{tx}\sigma^z}$
3	$P_y^{-1} T_y P_y T_y$	$e^{2i\theta_{py}\sigma^z}$	-1
4	P_y^2	$e^{-2i\theta_{py}\sigma^z}$	-1
5	$P_y^{-1} R_{\pi/2} P_y R_{\pi/2}$	$e^{-2i\theta_t\sigma^z}$	1
6	$R_{\pi/2}^4$	$e^{-4i\theta_t\sigma^z}$	$e^{-4i\theta_t\sigma^z}$
7	$R_{\pi/2}^{-1} T_x R_{\pi/2} T_y$	$e^{2i\theta_r\sigma^z + i(\theta_{tx}-\theta_{ty})\sigma^z}$	$-e^{-i(\theta_{tx}+\theta_{ty})\sigma^z}$
8	$R_{\pi/2}^{-1} T_y R_{\pi/2} T_x^{-1}$	$e^{2i\theta_r\sigma^z - i(\theta_{tx}-\theta_{ty})\sigma^z}$	$e^{i(\theta_{tx}-\theta_{ty})\sigma^z}$
9	$R_{\pi/2}^{-1} T^{-1} R_{\pi/2} T$	$e^{2i\theta_t\sigma^z}$	1
10	$P_y^{-1} T^{-1} P_y T$	$e^{2i\theta_{py}\sigma^z}$	$e^{2i\theta_t\sigma^z}$
11	$T_x^{-1} T^{-1} T_x T$	$-e^{2i(\theta_t+\theta_{tx})\sigma^z}$	-1
12	$T_y^{-1} T^{-1} T_y T$	$-e^{2i(\theta_t+\theta_{ty})\sigma^z}$	-1
13	T^2	-1	$e^{2i\theta_t\sigma^z}$

U1Cn0n1:

$$\begin{aligned} W_{tx}(\mathbf{i}) &= (-)^{i_y} g_3(\theta_{tx}) i\tau^1, & W_{ty}(\mathbf{i}) &= g_3(\theta_{ty}) i\tau^1, \\ W_{px}(\mathbf{i}) &= (-)^{i_x} g_3(\theta_{px}), & W_{py}(\mathbf{i}) &= (-)^{i_y} g_3(\theta_{py}), \\ W_{pxy}(\mathbf{i}) &= (-)^{i_x i_y} g_3(\theta_{pxy}), & W_r(\mathbf{i}) &= (-)^{i_x i_y + i_x} g_3(\theta_r), \\ W_t(\mathbf{i}) &= (-)^{i_x + i_y} g_3(\theta_t) i\tau^1. \end{aligned} \quad (\text{B11})$$

Gapless U(1) spin liquid.—The (continuum) PSG of the gapless spin liquid with $\langle \Phi_1^z \rangle \neq 0$ is

$$\begin{aligned} V_{tx} &= e^{i\theta_{tx}\sigma^z}, & V_{px} &= e^{i\theta_{px}\sigma^z}, & V_r &= e^{i\theta_r\sigma^z}, \\ V_{ty} &= e^{i\theta_{ty}\sigma^z} i\sigma^x, & V_{py} &= e^{i\theta_{py}\sigma^z} i\sigma^x, & V_t &= e^{i\theta_t\sigma^z}. \end{aligned} \quad (\text{B12})$$

From the symmetry fractionalization in Table I and the arguments in the previous section, we conclude that only U1B spin liquids with $W_t \propto \tau^0$ are possible candidates: U1B000n, U1Bn10n, U1B001n, U1Bx11n. Computing the symmetry fractionalization of these four spin liquids identifies U1Bx11n as the correct lattice analogue:

U1Bx11n:

$$\begin{aligned} W_{tx}(\mathbf{i}) &= (-)^{i_y} g_3(\theta_{tx}) \tau^0, & W_{ty}(\mathbf{i}) &= g_3(\theta_{ty}) \tau^0, \\ W_{px}(\mathbf{i}) &= (-)^{i_x} g_3(\theta_{px}) i\tau^1, & W_{py}(\mathbf{i}) &= (-)^{i_y} g_3(\theta_{py}) i\tau^1, \\ W_{pxy}(\mathbf{i}) &= (-)^{i_x i_y} g_3(\theta_{pxy}) i\tau^1, & W_r(\mathbf{i}) &= (-)^{i_x i_y + i_x} g_3(\theta_r), \\ W_t(\mathbf{i}) &= (-)^{i_x + i_y} g_3(\theta_t) \tau^0. \end{aligned} \quad (\text{B13})$$

In Appendix C 3, we show that this PSG has no lattice realization.

4. \mathbb{Z}_2 spin liquids

Wen divides the \mathbb{Z}_2 spin liquids into two classes. Their PSGs are

$$\begin{aligned} W_{tx}(\mathbf{i}) &= \tilde{\eta}^{i_y} \tau^0, & W_{px}(\mathbf{i}) &= \eta_{xpx}^{i_x} \eta_{xpy}^{i_y} g_{py}, \\ W_{pxy}(\mathbf{i}) &= (-)^{i_x i_y} g_{pxy}, \\ W_{ty}(\mathbf{i}) &= \tau^0, & W_{py}(\mathbf{i}) &= \eta_{xpy}^{i_x} \eta_{xpy}^{i_y} g_{px}, \\ W_t(\mathbf{i}) &= \eta_t^{i_x + i_y} g_t, \end{aligned} \quad (\text{B14})$$

where A spin liquids have $\tilde{\eta} = +1$ and B spin liquids have $\tilde{\eta} = -1$. Unlike for the SU(2) case, each of the group elements g_ξ takes only a single value. He labels these spin liquids by Z2A(g_{px}) $_{\eta_{xpx}}$ (g_{py}) $_{\eta_{xpy}}$ $g_{pxy}(g_t)_{\eta_t}$ and Z2B(g_{px}) $_{\eta_{xpx}}$ (g_{py}) $_{\eta_{xpy}}$ $g_{pxy}(g_t)_{\eta_t}$. An equivalent short-hand notation replaces $(\tau^0, \tau^1, \tau^2, \tau^3)$ and $(\tau_+^0, \tau_+^1, \tau_+^2, \tau_+^3)$ by $(0, 1, 2, 3)$ and $(\tau_-^0, \tau_-^1, \tau_-^2, \tau_-^3)$ by (n, x, y, z) (this is the notation used in the majority of the paper). There are 272 distinct such PSGs, however, though at least 72 of

these are anomalous and cannot be described with a mean-field Hamiltonian on the lattice.

We can determine the symmetry fractionalization of each of these PSGs using Eq. (B8), forming a table similar to Fig. 9, and this information is what leads to the identification in Fig. 10. It is clear that the symmetry fractionalization does not completely determine the PSG since both *s*PSG1 and *s*PSG5 have the same symmetry fractionalization as two different spin liquids. We will show that in both cases, a single lattice PSG can be associated with each of our continuum versions.

Our primary strategy will be to check which PSG's in Fig. 10 are proximate to U1Cn0n1. By studying Table I, we determine which values of θ_ξ give the \mathbb{Z}_2 symmetry fractionalization of the phases we are interested in. In both cases we find only a single possibility. We also verify that *s*PSG5 is proximate to U1Bx11n.

We note that the symmetry transformations in Table I depend on only five generators: $T_x, T_y, P_y, R_{\pi/2}, T$. To make contact with Wen's conventions, we also display the gauge transformations corresponding to $P_x = R_{\pi/2}P_yR_{\pi/2}^{-1}$ and $P_{xy} = R_{\pi/2}P_y^{-1}$; their forms are also determined by the angles $\theta_{tx}, \theta_{ty}, \theta_{py}, \theta_r$, and θ_t .

Lattice PSG of A_f phase (sPSG5).—We begin by determining which choice of angles of the gapped U(1) spin liquid returns the symmetry fractionalization of *s*PSG5. Setting $\theta_{tx} = 0$ fixes the remaining angles to be

$$\theta_{ty} = \pi, \quad \theta_{py} = \pm \frac{\pi}{2}, \quad \theta_r = 0, \pi, \quad \theta_t = 0, \pi. \quad (\text{B15})$$

The choices only result in gauge transformations differing by a minus sign and, except for W_{ty} , do not affect the symmetry fractionalization. In what follows we choose positive prefactors for all of the gauge transformations below. Modulo these considerations, this is the *only* PSG proximate to U1Cn0n1 with the same symmetry fractionalization as *s*PSG1. This gives

$$\begin{aligned} W_{tx} &= (-)^{i_y} i\tau^1, & W_{ty} &= -i\tau^1, \\ W_{px} &= (-)^{i_x} i\tau^3, & W_{py} &= (-)^{i_y} i\tau^3, \\ W_{pxy} &= (-)^{i_x i_y} i\tau^3, & W_r &= (-)^{i_x i_y + i_x} \tau^0, \\ W_t &= (-)^{i_x + i_y} i\tau^1. \end{aligned} \quad (\text{B16})$$

We can bring it into the form of Eq. (B14) by performing the gauge transformation

$$W(\mathbf{i}) = \begin{cases} (-)^{(i_x + i_y)/2} i\tau^2 & i_x + i_y = \text{even} \\ (-)^{(i_x + i_y - 1)/2} i\tau^3 & i_x + i_y = \text{odd.} \end{cases} \quad (\text{B17})$$

Under this transformation, the PSG in Eq. (B16) becomes

$$\begin{aligned} W_{tx} &= (-)^{i_y} \tau^0, & W_{ty} &= -\tau^0, \\ W_{px} &= (-)^{i_x + i_y} i\tau^3, & W_{py} &= (-)^{i_x + i_y} i\tau^3, \\ W_{pxy} &= (-)^{i_x(i_y + 1)} i\tau^3, & W_r &= (-)^{i_x(i_y + 1)} \tau^0, \\ W_t &= (-)^{i_x + i_y} i\tau^1. \end{aligned} \quad (\text{B18})$$

Upon shifting $i_y \rightarrow i_y + 1$, we recognize this PSG as Z2Bzz3x, and, rotating by 90° about the y axis, this becomes Z2Bxx1z. This identifies Z2Bxx1z as the unique lattice PSG capable of describing the phase A_f .

Another way we could have reached this conclusion is by studying the mean-field *Ansatz* allowed by either of these PSGs. It turns out that the mean-field Hamiltonian corresponding to the other candidate PSG, Z2Bxx2z, cannot be gapped, whereas no such restrictions exist for Z2Bxx1z.

We also show that Z2Bxx1z is proximate to the gapless spin liquid U1Bx11n. In order to reproduce the symmetry fractionalization of *s*PSG5, the angles in Eq. (B13) must be

$$\theta_{tx} = \pm \frac{\pi}{2}, \quad \theta_{ty} = \mp \frac{\pi}{2}, \quad \theta_r = 0, \pi, \quad \theta_t = \pm \frac{\pi}{2}. \quad (\text{B19})$$

θ_{py} is undetermined, and therefore, unlike in the previous case, proximity to U1Bx11n does not fully determine the lattice PSG corresponding to *s*PSG5. The angles that are restricted indicate that

$$\begin{aligned} W_{tx} &= (-)^{i_y} i\tau^3, & W_{ty} &= -i\tau^3, \\ W_r &= (-)^{i_x(i_y + 1)} \tau^0, & W_t &= (-)^{i_x + i_y} i\tau^3. \end{aligned} \quad (\text{B20})$$

Rotating by 90° about the y axis takes $\tau^3 \rightarrow \tau^1$. We then observe that all of the gauge transformations shown above are equal to the corresponding gauge transformation in Eq. (B16). It can be shown that θ_{py} can be chosen to obtain Z2Bxx1z but not Z2Bxx2z. Therefore, only Z2Bxx1z is proximate to U1Bx11n.

*s*PSG1.—Performing the same analysis as above, we find that the only way for the symmetry fractionalization of U1Cn0n1 to return the symmetry fractionalization of *s*PSG1 is if the angles in Eq. (B11) are

$$\theta_{ty} = \pi, \quad \theta_{py} = \pm \frac{\pi}{2}, \quad \theta_r = 0, \pi, \quad \theta_t = \pm \frac{\pi}{2}, \quad (\text{B21})$$

where, again, we have set $\theta_{tx} = 0$. The gauge transformations associated with the symmetry generators are then

$$\begin{aligned} W_{tx} &= (-)^{i_y} i\tau^1, & W_{ty} &= -i\tau^1, \\ W_{px} &= (-)^{i_x} i\tau^3, & W_{py} &= (-)^{i_y} i\tau^3, \\ W_{pxy} &= (-)^{i_x i_y} i\tau^3, & W_r &= (-)^{i_x(i_y + 1)} \tau^0, \\ W_t &= (-)^{i_x + i_y} i\tau^2. \end{aligned} \quad (\text{B22})$$

Performing the gauge transformation in Eq. (B17), these become

$$\begin{aligned} W_{tx} &= (-)^{i_y} \tau^0, & W_{ty} &= -\tau^0, \\ W_{px} &= i(-)^{i_x+i_y} i\tau^3, & W_{py} &= i(-)^{i_x+i_y} i\tau^3, \\ W_{pxy} &= (-)^{i_x(i_y+1)} i\tau^3, & W_r &= (-)^{i_x(i_y+1)} \tau^0, \\ W_t &= i\tau^2. \end{aligned} \quad (\text{B23})$$

It is not difficult to see that this corresponds to Z2Bzz32, which is equivalent to Z2Bxx13.

APPENDIX C: LATTICE REALIZATIONS OF SPIN LIQUIDS

In this appendix, we use the lattice PSGs determined in Appendix B for the π -flux phase and A_f , B_f , and D_f to write down the corresponding lattice Hamiltonian. Doing so serves as further verification of the symmetry fractionalization used in the main text. Further, the calculation of the Berry phase in Sec. IV C 2 requires the lattice description of the gapped U(1) spin liquid corresponding to U1Cn0n1.

1. SUBn0 mean-field Hamiltonian

The *Ansatz* for the π -flux state is given in Eq. (B9). Gauge invariance and the form of the translational symmetry operations compels the mean-field parameters to take the following form:

$$u_{i,i+m} = (-)^{i_x m_y} i u_m^0. \quad (\text{C1})$$

In order for the mean-field Hamiltonian to be Hermitian, u_{ij}^\dagger must equal u_{ji} . This can be used to show that

$$(-)^{i_x m_y} i u_m^0 = -(-)^{i_x m_y} (-)^{m_x m_y} i u_m^0, \quad (\text{C2})$$

which indicates

$$u_{-m}^0 = -(-)^{m_x m_y} u_m^0. \quad (\text{C3})$$

Next, Eq. (B5) states that u_{ij} must be invariant under the action of all (projective) symmetry operations. In particular, acting with $P_x P_y$ and using Eq. (C3), we find

$$\begin{aligned} u_{i,i+m} &= W_{px} P_x W_{py} P_y [u_{i,i+m}] \\ &= -(-)^{i_x m_y} (-)^{m_x m_y} (-)^{m_x+m_y} i u_m^0 \tau^0. \end{aligned} \quad (\text{C4})$$

Similarly, the action of time reversal requires

$$u_{i,i+m} = W_t \mathcal{T} [u_{i,i+m}] = (-)^{i_x m_y} (-)^{m_x+m_y} i u_m^0 \tau^0. \quad (\text{C5})$$

Between these two equations, we conclude that $u_m^0 \neq 0$ only when $m_x + m_y = \text{odd}$. Finally, we relate mean-field

parameters for different m 's through the action of P_x , P_y , and P_{xy} :

$$\begin{aligned} u_{(-m_x, m_y)}^0 &= (-)^{m_x} u_{(m_x, m_y)}^0, & u_{(m_x, -m_y)}^0 &= (-)^{m_y} u_{(m_x, m_y)}^0, \\ u_{(m_y, m_x)}^0 &= (-)^{m_x m_y} u_{(m_x, m_y)}^0. \end{aligned} \quad (\text{C6})$$

The mean-field *Ansatz* we obtain is

$$u_{i,i+\hat{x}} = i\alpha\tau^0, \quad u_{i,i+\hat{y}} = (-)^{i_x} i\alpha\tau^0. \quad (\text{C7})$$

Inserting these hopping terms into Eq. (B3) (and dropping the constant), we obtain

$$H'_\pi = -i\alpha \sum_i [\psi_i^\dagger \psi_{i+\hat{x}} + (-)^{i_x} \psi_i^\dagger \psi_{i+\hat{y}} + \text{H.c.}]. \quad (\text{C8})$$

We now show that the low-energy theory is precisely the Dirac Hamiltonian. In momentum space, we find

$$\begin{aligned} H'_\pi &= 2\alpha \int_{-\pi/2}^{\pi/2} \frac{dk_x}{2\pi} \int_{-\pi/2}^{\pi/2} \frac{dk_y}{2\pi} \Psi_k^\dagger (\sin k_x \tilde{\tau}^3 \mu^3 \tau^0 \\ &\quad + \sin k_y \tilde{\tau}^1 \mu^3 \tau^0) \Psi_k, \end{aligned} \quad (\text{C9})$$

where $\Psi_k = (\psi_k, \psi_{k+\mathcal{Q}_x+\mathcal{Q}_y}, \psi_{k+\mathcal{Q}_x}, \psi_{k+\mathcal{Q}_y})^T$, with $\mathcal{Q}_x = (\pi, 0)$ and $\mathcal{Q}_y = (0, \pi)$, and

$$\begin{aligned} \tilde{\tau}^3 \mu^3 &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \\ \tilde{\tau}^1 \mu^3 &= \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, \\ \tilde{\tau}^0 \mu^1 &= \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}. \end{aligned} \quad (\text{C10})$$

Equivalently, writing $\Psi_k = (\psi_{1,1,k}, \psi_{1,2,k}, \psi_{2,1,k}, \psi_{2,2,k})^T$, we can identify the $\tilde{\tau}^\ell$'s with Pauli matrices acting on the first index of Ψ_k and the μ^ℓ 's with Pauli matrices acting on the second index. Finally, to make contact with the expression in Sec. II B, we express H'_π in terms of

$$\tilde{\Psi}_k = e^{i\pi\tilde{\tau}^2\mu^3/4} \Psi_k. \quad (\text{C11})$$

The resulting mean-field Hamiltonian is

$$H'_\pi = 2\alpha \int_{-\pi/2}^{\pi/2} \frac{dk_x}{2\pi} \int_{-\pi/2}^{\pi/2} \frac{dk_y}{2\pi} \tilde{\Psi}_k^\dagger (\sin k_x \tau^1 \mu^0 \sigma^0 - \sin k_y \tau^3 \mu^0 \sigma^0) \\ \cong -2\alpha \int \frac{d^2k}{(2\pi)^2} \tilde{\Psi}^\dagger (k_x \gamma^0 \gamma^x + k_y \gamma^0 \gamma^y) \tilde{\Psi}_k, \quad (\text{C12})$$

where we have rewritten the gauge-charged τ^ℓ 's as σ^ℓ 's (as done in the main body of the text) and used the fact that $\gamma^\mu = (\tilde{\tau}^y, i\tilde{\tau}^z, i\tilde{\tau}^x)$. It is clear that once dynamic gauge fields are included, this is equivalent to $\mathcal{L}_{\text{QCD}_3}$ in Eq. (2.20).

2. U1Cn0n1 mean-field Hamiltonian

We now use the *Ansatz* for Eq. (B11) to determine the lattice Hamiltonian corresponding to the gapped spin-liquid phase D_f . We show that it is precisely H'_π plus a term which breaks the SU(2) symmetry to U(1): $H_{D_f} = H'_\pi + H_1$.

Equation (B11) indicates that all bonds must be of the form

$$u_{i,i+m} = (-)^{i_x m_y} [i u_m^0 \tau^0 + (-)^{i_x+i_y} u_m^3 \tau^3]. \quad (\text{C13})$$

Further, Hermiticity of the Hamiltonian requires $u_{ij}^\dagger = u_{ji}$, and therefore,

$$(-)^{i_x m_y} [-i u_m^0 \tau^0 + (-)^{i_x+i_y} u_m^3 \tau^3] \\ = (-)^{i_x m_y} (-)^{m_x m_y} [i u_{-m}^0 \tau^0 + (-)^{i_x+i_y} (-)^{m_x+m_y} u_{-m}^3 \tau^3], \quad (\text{C14})$$

implying that

$$u_m^0 = -(-)^{m_x m_y} u_{-m}^0, \quad u_m^3 = (-1)^{m_x+m_y} (-)^{m_x m_y} u_{-m}^3. \quad (\text{C15})$$

Similarly, to satisfy Eq. (B5), u_{ij} must be invariant under 180° rotations:

$$u_{i,i+m} = W_{p_x} P_x W_{p_y} P_y [u_{i,i+m}] \\ = (-)^{i_x m_y} (-)^{m_x m_y} [-(-)^{m_x+m_y} i u_m^0 \tau^0 + (-)^{i_x+i_y} u_m^3 \tau^3], \quad (\text{C16})$$

where we have used the previous expression to relate u_m^ℓ and u_{-m}^ℓ . It follows that $u_m^0 = 0$ when $(m_x, m_y) = (\text{even}, \text{even})$ and that $u_m^3 = 0$ when $(m_x, m_y) = (\text{odd}, \text{odd})$. The *Ansatz* must also be invariant under T :

$$u_{i,i+m} = W_t \mathcal{T} [u_{i,i+m}] \\ = (-)^{i_x m_y} (-)^{m_x+m_y} [-i u_m^0 \tau^0 + (-)^{i_x+i_y} u_m^3 \tau^3], \quad (\text{C17})$$

showing that u_m^0 is nonzero only for $m_x + m_y = \text{odd}$ and that u_m^3 is only nonzero when $m_x + m_y = \text{even}$. Together, these give

$$u_{i,i+m} = \begin{cases} (-)^{i_x+i_y} u_m^3 & (m_x, m_y) = (\text{even}, \text{even}) \\ (-)^{i_x m_y} i u_m^0 \tau^0 & m_x + m_y = \text{odd}. \end{cases} \quad (\text{C18})$$

We can also show that the action of P_x , P_y , and P_{xy} implies the following relations:

$$u_{(m_x, m_y)}^\ell = (-)^{m_x} u_{(-m_x, m_y)}^\ell, \quad u_{(m_x, m_y)}^\ell = (-)^{m_y} u_{(m_x, -m_y)}^\ell, \\ u_{(m_x, m_y)}^\ell = (-)^{m_x m_y} u_{(m_y, m_x)}^\ell, \quad (\text{C19})$$

for $\ell = 0, 3$. Using these relations, we find

$$u_{i,i+\hat{x}} = i\alpha\tau^0, \quad u_{i,i+2\hat{x}} = (-)^{i_x+i_y} \beta\tau^3, \quad u_{i,i} = (-)^{i_x+i_y} a_0\tau^3, \\ u_{i,i+\hat{y}} = (-)^{i_x} i\alpha\tau^0, \quad u_{i,i+2\hat{y}} = (-)^{i_x+i_y} \beta\tau^3. \quad (\text{C20})$$

As expected, the nearest-neighbor bonds are identical to those we found for the π -flux phase in the previous section. The SU(2) symmetry is already broken to U(1) by the inclusion of the next-nearest-neighbor bonds and so this is all we consider.

As in the previous section, the mean-field Hamiltonian is obtained by inserting these hopping terms into Eq. (B3):

$$H_{D_f} = H'_\pi + H_1, \\ H_1 = \sum_i (-)^{i_x+i_y} [\beta(\psi_i^\dagger \tau^3 \psi_{i+2\hat{x}} + \psi_i^\dagger \tau^3 \psi_{i+2\hat{y}} + \text{H.c.}) \\ - a_0^3 \psi_i^\dagger \tau^3 \psi_i], \quad (\text{C21})$$

where H'_π is given above in Eq. (C9). In momentum space, this becomes

$$H_{D_f} = \int_{-\pi/2}^{\pi/2} \frac{dk_x}{2\pi} \int_{-\pi/2}^{\pi/2} \frac{dk_y}{2\pi} \tilde{\Psi}_k^\dagger \{ 2\alpha(\sin k_x \tilde{\tau}^3 \mu^3 \tau^0 + \sin k_y \tilde{\tau}^1 \mu^3 \tau^0) \\ - (2\beta[\cos 2k_x + \cos 2k_y] - a_0) \tilde{\tau}^0 \mu^1 \tau^3 \} \tilde{\Psi}_k, \quad (\text{C22})$$

where we have used the same notation as in the previous section: $\tilde{\Psi}_k = (\psi_k, \psi_{k+\mathcal{Q}_x+\mathcal{Q}_y}, \psi_{k+\mathcal{Q}_x}, \psi_{k+\mathcal{Q}_y})^T$, with $\mathcal{Q}_x = (\pi, 0)$ and $\mathcal{Q}_y = (0, \pi)$, and the matrices defined in Eq. (C10). In terms of $\tilde{\Psi}_k = e^{i\pi\tilde{\tau}^2 \mu^3/4} \Psi_k$:

$$H_{D_f} = \int_{-\pi/2}^{\pi/2} \frac{dk_x}{2\pi} \int_{-\pi/2}^{\pi/2} \frac{dk_y}{2\pi} \tilde{\Psi}_k^\dagger \{ 2\alpha(\sin k_x \tilde{\tau}^1 \mu^0 \sigma^0 - \sin k_y \tilde{\tau}^3 \mu^0 \sigma^0) \\ - (2\beta[\cos 2k_x + \cos 2k_y] - a_0) \tilde{\tau}^2 \mu^2 \sigma^3 \} \tilde{\Psi}_k, \quad (\text{C23})$$

where, again, we have rewritten the SU(2) matrices τ^ℓ as σ^ℓ in accord with the continuum notation. Expanding H_{D_f} about $\mathbf{k} = (0, 0)$, we obtain

$$H_{D_f} \cong \int \frac{d^2k}{(2\pi)^2} \tilde{\Psi}^\dagger [-2\alpha(k_x\gamma^0\gamma^x + k_y\gamma^0\gamma^y) + (4\beta - a_0)\gamma^0\mu^y\sigma^z] \tilde{\Psi}_k, \quad (\text{C24})$$

where $\gamma^\mu = (\tilde{\tau}^y, i\tilde{z}^z, i\tilde{\tau}^x)$. We conclude that the term which reduces the SU(2) symmetry down to U(1) is precisely equivalent to $\bar{\psi}\mu^y\sigma^z\psi \sim \text{tr}(\sigma^z\bar{X}\mu^y X)$.

3. U1Bx11n mean-field Hamiltonian

In this section, we demonstrate that U1Bx11n has *no* lattice analogue. Referring to Eq. (B13), we see that gauge and translational symmetry requires

$$u_{i,i+m} = (-)^{i_x m_y} (iu_m^0 \tau^0 + u_m^3 \tau^3). \quad (\text{C25})$$

We relate $u_{-m}^{0,3} = u_m^{0,3}$ using the fact that $u_{i,i+m}^\dagger = u_{i+m,i}$:

$$u_{-m}^0 = -(-)^{m_x m_y} u_m^0, \quad u_{-m}^3 = (-)^{m_x m_y} u_m^3. \quad (\text{C26})$$

Then, acting on $u_{i,i+m}$ with $P_x P_y$ and \mathcal{T} gives

$$\begin{aligned} W_{p_x} P_x W_{p_y} P_y [u_{i,i+m}] &= (-)^{i_x m_y} (-)^{m_x m_y} (-)^{m_x + m_y} (-iu_m^0 \tau^0 + u_m^3 \tau^3), \\ \mathcal{T}[u_{i,i+m}] &= (-)^{i_x m_y} (-)^{m_x + m_y} (-iu_m^0 \tau^0 - u_m^3 \tau^3). \end{aligned} \quad (\text{C27})$$

Equating these expressions with $u_{i,i+m}$ implies that $u_m^0 \neq 0$ only for $m_x + m_y = \text{odd}$, as for SU2Bn0 and U1Cn0n1; it can be shown that they must satisfy identical constraints as the τ^0 bonds allowed by these PSGs. In particular, the nearest-neighbor values are identical to those in Eq. (C7). Conversely, there are no consistent solutions for u_m^3 : it

$$\begin{aligned} u_{(-m_x, m_y)}^0 &= (-)^{m_x} u_{(m_x, m_y)}^0, & u_{(-m_x, m_y)}^2 &= -u_{(m_x, m_y)}^2, & u_{(-m_x, m_y)}^3 &= u_{(m_x, m_y)}^3, \\ u_{(m_x, -m_y)}^0 &= (-)^{m_y} u_{(m_x, m_y)}^0, & u_{(m_x, -m_y)}^2 &= -u_{(m_x, m_y)}^2, & u_{(m_x, -m_y)}^3 &= u_{(m_x, m_y)}^3, \\ u_{(m_y, m_x)}^0 &= u_{(m_x, m_y)}^0, & u_{(m_y, m_x)}^2 &= -u_{(m_x, m_y)}^2, & u_{(m_y, m_x)}^3 &= u_{(m_x, m_y)}^3. \end{aligned} \quad (\text{C31})$$

These show that $u_m^{0,3}$ are restricted to take the same values as in Eq. (C20) for the gapped U(1) spin liquid, leaving the u_m^2 bonds to break the U(1) gauge symmetry down to \mathbb{Z}_2 . It turns out that its first nonzero value occurs at the sixth nearest neighbor:

$$\begin{aligned} u_{i,i+2\hat{x}+4\hat{y}} &= (-)^{i_x+i_y} \gamma \tau^2, & u_{i,i+2\hat{x}-4\hat{y}} &= -(-)^{i_x+i_y} \gamma \tau^2, \\ u_{i,i+4\hat{x}+2\hat{y}} &= -(-)^{i_x+i_y} \gamma \tau^2, & u_{i,i+4\hat{x}-2\hat{y}} &= (-)^{i_x+i_y} \gamma \tau^2. \end{aligned} \quad (\text{C32})$$

always vanishes and is therefore unable to break the SU(2) gauge symmetry to U(1).

4. Z2Bxx1z mean-field Hamiltonian

We choose a gauge such that Eq. (B16) describes the PSG of Z2Bxx1z. Translational symmetry and gauge invariance implies that

$$u_{i,i+m} = (-)^{i_x m_y} \{iu_m^0 \tau^0 + u_m^1 \tau^1 + (-)^{i_x+i_y} [u_m^2 \tau^2 + u_m^3 \tau^3]\}. \quad (\text{C28})$$

Hermiticity then requires

$$\begin{aligned} u_{-m}^0 &= -(-)^{m_x m_y} u_m^0, & u_{-m}^1 &= (-)^{m_x m_y} u_m^1, \\ u_{-m}^2 &= (-)^{m_x m_y} (-)^{m_x + m_y} u_m^3. \end{aligned} \quad (\text{C29})$$

Under the action of $P_x P_y$ and \mathcal{T} the *Ansatz* transforms as

$$\begin{aligned} W_{p_x} P_x W_{p_y} P_y [u_{i,i+m}] &= (-)^{i_x m_y} (-)^{m_x m_y} \{ -(-)^{m_x + m_y} iu_m^0 \tau^0 + (-)^{m_x + m_y} u_m^2 \tau^1 \\ &\quad + (-)^{i_x+i_y} [u_m^2 \tau^2 + u_m^3 \tau^3] \}, \\ \mathcal{T}[u_{i,i+m}] &= (-)^{i_x m_y} (-)^{m_x + m_y} \{ -iu_m^0 \tau^0 - u_m^1 \tau^1 \\ &\quad + (-)^{i_x+i_y} [u_m^2 \tau^2 + u_m^3 \tau^3] \}. \end{aligned} \quad (\text{C30})$$

These relations imply that $u_m^1 = 0$ for all \mathbf{m} , $u_m^0 \neq 0$ only for $m_x + m_y = 0$, and that $u_m^{2,3} \neq 0$ only for $(m_x, m_y) = (\text{even}, \text{even})$. By studying the action of P_x , P_y , and P_{xy} , we obtain the following relations:

The contribution of these bonds to the Hamiltonian is

$$\begin{aligned} H_2 &= \gamma \sum_i (-)^{i_x+i_y} [\psi_i^\dagger \tau^2 \psi_{i+2\hat{x}+4\hat{y}} - \psi_i^\dagger \tau^2 \psi_{i+2\hat{x}-4\hat{y}} \\ &\quad - \psi_i^\dagger \tau^2 \psi_{i+4\hat{x}+2\hat{y}} + \psi_i^\dagger \tau^2 \psi_{i+4\hat{x}-2\hat{y}} - \text{H.c.}], \end{aligned} \quad (\text{C33})$$

and the minimal Hamiltonian needed to describe Z2Bxx1z is $H_{A_f} = H_{D_f} + H_2$. In momentum space, we have

$$H_2 = 4\gamma \int_{-\pi/2}^{\pi/2} \frac{dk_x}{2\pi} \int_{-\pi/2}^{\pi/2} \frac{dk_y}{2\pi} (\sin 4k_x \sin 2k_y - \sin 2k_x \sin 4k_y) \Psi_k^\dagger \tilde{\tau}^0 \mu^2 \tau^2 \Psi_k, \quad (\text{C34})$$

where Ψ_k is defined in Eq. (C11), and the action of the Pauli matrices $\tilde{\tau}^\ell$ and μ^ℓ is given in Eq. (C10) and below. Once more, we change notation such that Pauli matrices acting on color space τ^ℓ become σ^ℓ 's and express H_2 in terms of the transformed fermion operator, $\tilde{\Psi}_k = e^{i\pi\tilde{\tau}^2\mu^3/4}\Psi_k$:

$$H_2 = 4\gamma \int_{-\pi/2}^{\pi/2} \frac{dk_x}{2\pi} \int_{-\pi/2}^{\pi/2} \frac{dk_y}{2\pi} (\sin 4k_x \sin 2k_y - \sin 2k_x \sin 4k_y) \tilde{\Psi}_k^\dagger \tau^y \mu^y \sigma^y \tilde{\Psi}_k \\ \cong -16\gamma \int \frac{d^2k}{(2\pi)^2} \tilde{\Psi}_k^\dagger [k_x k_y (k_x^2 - k_y^2) \gamma^0 \mu^y \sigma^y] \tilde{\Psi}_k. \quad (\text{C35})$$

Notably, H_2 does *not* correspond to any of the continuum operators in the action we study in the main text, in particular, $\text{tr}(\sigma^a \tilde{X} \partial_0 X) \sim \tilde{\psi} \sigma^a \partial_0 \psi$. Instead, in the continuum language, H_2 is proportional to $\tilde{\psi} \mu^y \sigma^y \partial_x \partial_y (\partial_x^2 - \partial_y^2) \psi \sim \text{tr}(\sigma^y \tilde{X} \mu^y \partial_x \partial_y [\partial_x^2 - \partial_y^2] X)$. This is discussed in Sec. IV A 1.

APPENDIX D: SYMMETRY FRACTIONALIZATION OF CURRENT-LOOP ORDERED SPIN LIQUID

We can also use symmetry fractionalization to verify that the phase C_f corresponds to C_b . There are now only eight group relations, and these are listed in Table II. The PSG of the reduced symmetry group is defined by the gauge transformations

$$V_{ix} = i\sigma^y, \quad V_{py} = i\sigma^x, \\ V_{ix} = i\sigma^y, \quad V_{tpx} = i\sigma^z, \quad (\text{D1})$$

where the subscript tpx denotes the joint group action of TP_x . With these, we determine the fermionic symmetry

TABLE II. Symmetry fractionalization and twist factors for the fermionic and bosonic spinon and the vison in the phase \mathbb{Z}_2 spin liquid with current-loop order. By comparing with the result in Ref. [22], we are able to verify the equivalent of C_f and C_b .

Group relations	Fermionic	Vison	Twist	Bosonic
1 $T_y^{-1} T_x T_y T_x^{-1}$	-1	-1	1	1
2 $P_y^{-1} T_x P_y T_x^{-1}$	-1	-1	1	1
3 $P_y^{-1} T_y P_y T_y$	-1	1	1	-1
4 P_y^2	-1	1	-1	1
5 $T_x^{-1} (TP_x)^{-1} T_x (TP_x)$	-1	1	1	-1
6 $T_y^{-1} (TP_x)^{-1} T_y (TP_x)$	1	-1	1	-1
7 $P_y^{-1} (TP_x)^{-1} P_y (TP_x)$	-1	-1	1	1
8 $(TP_x)^2$	-1	1	1	-1

TABLE III. Symmetry action on the bosonic spinon and Higgs fields in the bosonic dual to the theories studied here, as presented in Eqs. (1.4) and (1.7) [22]. The spinon here is written as a two component spinor, $z = (z_\uparrow, z_\downarrow)^T$, with $i\sigma^y$ acting on the \uparrow, \downarrow indices. We note that $\mathcal{T}[z^*] = -i\sigma^y z^*$ and that $T_{x,y}[z^*] = i\sigma^y z$.

	\mathcal{T}	P_x	P_y	T_x	T_y
z_α	$i\sigma^y z$	z	z	$i\sigma^y z^*$	$i\sigma^y z^*$
Q_x	Q_x	$-Q_x$	Q_x	Q_x^*	Q_x^*
P	$-P$	P	P	P^*	P^*

fractionalization using the methods described in Sec. IV A. The results are shown in Table II under the column labeled ‘‘Fermionic’’.

Both the symmetry fractionalization of the vison and the twist factors for the reduced symmetry relations can be worked out from the ones already given; Table II lists these under the columns ‘‘Vison’’ and ‘‘Twist,’’ respectively.

In order to determine the bosonic symmetry fractionalization, we borrow notation from Ref. [22]. In Table III, the symmetry transformation properties of the bosonic spinon and Higgs fields in Eqs. (1.4) and (1.7) are reproduced. It will be convenient to express the bosonic spinon in terms of the four-component field $\mathcal{Z} = (z, z^*)^T = (z_\uparrow, z_\downarrow, z_\uparrow^*, z_\downarrow^*)^T$. We then let Pauli matrices τ^ℓ act on this new index, while σ matrices will act on the spin indices as before. The U(1) gauge transformations are expressed as $U(1)_g: \mathcal{Z} \rightarrow e^{i\theta\tau^z} \mathcal{Z}$. In this language, the symmetry transformations are expressed as

$$\mathcal{T}[\mathcal{Z}] = i\sigma^y \tau^z \mathcal{Z}, \quad P_{x,y}[\mathcal{Z}] = \mathcal{Z}, \quad T_{x,y}[\mathcal{Z}] = i\sigma^y \tau^x \mathcal{Z}. \quad (\text{D2})$$

Using these, we obtain the numbers in the column of Table II labeled ‘‘Boson’’.

Finally, we multiply the twist, vison, and boson columns and obtain the numbers in the fermion column, thereby verifying the equivalence of C_f and C_b .

APPENDIX E: LINEAR RESPONSE TO NONTRIVIAL FLUX

In this appendix, we calculate the relation in Eq. (4.10) in imaginary time. The residual U(1) gauge field a_μ couples to the current $J^\mu = \tilde{\psi} \gamma^\mu \sigma^x \psi$. The response function of an operator \mathcal{O} is $\chi_\mathcal{O}^\mu = \langle \mathcal{O}(x) J^\mu(x') \rangle$, and the linear response equation in momentum space is simply $\langle \mathcal{O}(q) \rangle = \chi_\mathcal{O}^\mu(p) A_\mu(q)$ (we specify to operators whose vacuum expectation values vanish in the absence of perturbations). Assuming $\mathcal{O} = \text{tr}(\tilde{X} M X)$, $\chi_\mathcal{O}^\mu(q)$ is represented by the Feynman diagram in Fig. 12 at leading order. We evaluate this as

$$\chi_\mathcal{O}^\mu(q) = - \int \frac{d^3p}{(2\pi)^3} \text{tr} \left[M \frac{\not{p} + i m \sigma^x \mu^y}{p^2 + m^2} \gamma^\mu \sigma^x \frac{\not{p} + \not{q} + i m \sigma^x \mu^y}{(p+q)^2 + m^2} \right]. \quad (\text{E1})$$

	Group relations	v	t	nPSG x									
				1	2	3	4	5	6	7	8	9	10
1	$T_y^{-1}T_xT_yT_x^{-1}$	-1	1	1	1	1	-1	-1	-1	-1	1	1	1
2	$P_x^{-1}T_xP_xT_x$	1	1	-1	1	-1	1	-1	1	1	1	1	-1
3	$P_y^{-1}T_xP_yT_x^{-1}$	-1	1	1	1	-1	-1	-1	1	-1	-1	1	-1
4	$P_x^{-1}T_yP_xT_y^{-1}$	-1	1	-1	1	-1	1	-1	1	1	1	1	-1
5	$P_y^{-1}T_yP_yT_y$	1	1	1	1	-1	-1	-1	1	-1	-1	1	-1
6	P_x^2	1	-1	-1	1	-1	1	-1	1	1	1	1	-1
7	P_y^2	1	-1	1	1	-1	-1	-1	1	-1	-1	1	-1
8	$P_x^{-1}P_y^{-1}P_xP_y$	-1	-1	1	1	1	1	1	1	1	1	1	1
9	$T_x^{-1}\mathcal{T}^{-1}T_x\mathcal{T}$	1	1	-1	-1	1	-1	1	1	1	1	1	-1
10	$T_y^{-1}\mathcal{T}^{-1}T_y\mathcal{T}$	1	1	1	1	-1	-1	1	1	1	-1	-1	1
11	$P_x^{-1}\mathcal{T}^{-1}P_x\mathcal{T}$	1	-1	-1	1	-1	1	-1	1	1	1	1	-1
12	$P_y^{-1}\mathcal{T}^{-1}P_y\mathcal{T}$	1	-1	1	1	-1	-1	-1	1	-1	-1	1	-1
13	\mathcal{T}^2	1	1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1

FIG. 11. Symmetry fractionalization of nematic PSGs (nPSGs) for spin liquids listed in Fig. 7. nPSG7 x (highlighted in blue) corresponds to the fermionic PSG determined in Ref. [21]. The columns labeled “V” and “T” list the vison fractionalization numbers and the twist factors, respectively.

If $M \propto \sigma^x$, it can be shown that the leading order term is quadratic in q . A q -linear piece is obtained by assuming that $\text{tr}(M\sigma^x) = 0$, in which case

$$\begin{aligned}
 \chi_{\mathcal{O}}^{\mu}(q) &= -m \int \frac{d^3p}{(2\pi)^3} \frac{1}{[p^2 + m^2][(p+q)^2 + m^2]} \\
 &\quad \times \{p_{\alpha} \text{tr}[M\gamma^{\alpha}\gamma^{\mu}\mu^{\nu}] + (p+q)_{\alpha} \text{tr}[M\mu^{\nu}\gamma^{\mu}\gamma^{\alpha}]\} \\
 &= \frac{m}{8\pi} \frac{iq_{\alpha}}{|q|} \arctan\left(\frac{|q|}{2m}\right) A_{\mu}(q) (\text{tr}[\mu^{\nu}M\gamma^{\alpha}\gamma^{\mu}] - \text{tr}[M\mu^{\nu}\gamma^{\mu}\gamma^{\alpha}]).
 \end{aligned} \tag{E2}$$

This is only nonzero for $M = \mu^{\nu}\gamma^{\nu}$. Expanding the inverse tangent in small q , we find

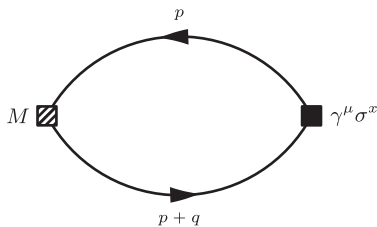


FIG. 12. Fermion bubble to calculate flux response.

$$\chi_{\nu}^{\mu}(q) \cong -\frac{1}{\pi} e^{\nu\alpha\mu} q_{\alpha} A_{\mu}(q) \cong \frac{i}{\pi} e^{\mu\nu\alpha} \partial_{\alpha} A_{\beta}(q). \tag{E3}$$

Returning to real time, we obtain the result in Eq. (4.10).

- [1] P. W. Anderson, *The Resonating Valence Bond State in La_2CuO_4 and Superconductivity*, *Science* **235**, 1196 (1987).
- [2] G. Baskaran and P. W. Anderson, *Gauge Theory of High-Temperature Superconductors and Strongly Correlated Fermi Systems*, *Phys. Rev. B* **37**, 580 (1988).
- [3] S. A. Kivelson, D. S. Rokhsar, and J. P. Sethna, *2e or Not 2e: Flux Quantization in the Resonating Valence Bond State*, *Europhys. Lett.* **6**, 353 (1988).
- [4] D. S. Rokhsar and S. A. Kivelson, *Superconductivity and the Quantum Hard-Core Dimer Gas*, *Phys. Rev. Lett.* **61**, 2376 (1988).
- [5] V. Kalmeyer and R. B. Laughlin, *Equivalence of the Resonating-Valence-Bond and Fractional Quantum Hall States*, *Phys. Rev. Lett.* **59**, 2095 (1987).
- [6] X. G. Wen, F. Wilczek, and A. Zee, *Chiral Spin States and Superconductivity*, *Phys. Rev. B* **39**, 11413 (1989).
- [7] N. Read and S. Sachdev, *Large N Expansion for Frustrated Quantum Antiferromagnets*, *Phys. Rev. Lett.* **66**, 1773 (1991).
- [8] S. Sachdev and N. Read, *Large N Expansion for Frustrated and Doped Quantum Antiferromagnets*, *Int. J. Mod. Phys. B* **05**, 219 (1991).

- [9] X. G. Wen, *Mean-Field Theory of Spin-Liquid States with Finite Energy Gap and Topological Orders*, *Phys. Rev. B* **44**, 2664 (1991).
- [10] F. A. Bais, P. van Driel, and M. de Wild Propitius, *Quantum Symmetries in Discrete Gauge Theories*, *Phys. Lett. B* **280**, 63 (1992).
- [11] T. Senthil and M. P. A. Fisher, *\mathbb{Z}_2 Gauge Theory of Electron Fractionalization in Strongly Correlated Systems*, *Phys. Rev. B* **62**, 7850 (2000).
- [12] M. Freedman, C. Nayak, K. Shtengel, K. Walker, and Z. Wang, *A Class of P, T-Invariant Topological Phases of Interacting Electrons*, *Ann. Phys. (Amsterdam)* **310**, 428 (2004).
- [13] T. H. Hansson, V. Oganesyan, and S. L. Sondhi, *Superconductors Are Topologically Ordered*, *Ann. Phys. (Amsterdam)* **313**, 497 (2004).
- [14] A. Y. Kitaev, *Fault-Tolerant Quantum Computation by Anyons*, *Ann. Phys. (Amsterdam)* **303**, 2 (2003).
- [15] X.-G. Wen, *Quantum Orders in an Exact Soluble Model*, *Phys. Rev. Lett.* **90**, 016803 (2003).
- [16] A. Y. Kitaev, *Anyons in an Exactly Solved Model and Beyond*, *Ann. Phys. (Amsterdam)* **321**, 2 (2006).
- [17] X.-G. Wen, *Quantum Orders and Symmetric Spin Liquids*, *Phys. Rev. B* **65**, 165113 (2002); arXiv:cond-mat/0107071.
- [18] X. Chen, Z.-C. Gu, Z.-X. Liu, and X.-G. Wen, *Symmetry Protected Topological Orders and the Group Cohomology of Their Symmetry Group*, *Phys. Rev. B* **87**, 155114 (2013).
- [19] A. M. Essin and M. Hermele, *Classifying Fractionalization: Symmetry Classification of Gapped \mathbb{Z}_2 Spin Liquids in Two Dimensions*, *Phys. Rev. B* **87**, 104406 (2013).
- [20] M. Barkeshli, H. Yao, and S. A. Kivelson, *Gapless Spin Liquids: Stability and Possible Experimental Relevance*, *Phys. Rev. B* **87**, 140402 (2013).
- [21] S. Chatterjee, Y. Qi, S. Sachdev, and J. Steinberg, *Superconductivity from a Confinement Transition Out of a Fractionalized Fermi Liquid with \mathbb{Z}_2 Topological and Ising-Nematic Orders*, *Phys. Rev. B* **94**, 024502 (2016).
- [22] S. Chatterjee, S. Sachdev, and M. Scheurer, *Intertwining Topological Order and Broken Symmetry in a Theory of Fluctuating Spin-Density Waves*, *Phys. Rev. Lett.* **119**, 227002 (2017).
- [23] X. Yang and F. Wang, *Schwinger Boson Spin-Liquid States on Square Lattice*, *Phys. Rev. B* **94**, 035160 (2016).
- [24] S. Chatterjee and S. Sachdev, *Insulators and Metals with Topological Order and Discrete Symmetry Breaking*, *Phys. Rev. B* **95**, 205133 (2017).
- [25] I. Affleck and J. B. Marston, *Large- n Limit of the Heisenberg-Hubbard Model: Implications for High- T_c Superconductors*, *Phys. Rev. B* **37**, 3774 (1988).
- [26] I. Affleck, Z. Zou, T. Hsu, and P. W. Anderson, *$SU(2)$ Gauge Symmetry of the Large- U Limit of the Hubbard Model*, *Phys. Rev. B* **38**, 745 (1988).
- [27] F. Yang and H. Yao, *Frustrated Resonating Valence Bond States in Two Dimensions: Classification and Short-Range Correlations*, *Phys. Rev. Lett.* **109**, 147209 (2012).
- [28] Y.-M. Lu, G. Y. Cho, and A. Vishwanath, *Unification of Bosonic and Fermionic Theories of Spin Liquids on the Kagome Lattice*, *Phys. Rev. B* **96**, 205150 (2017).
- [29] Y. Qi and L. Fu, *Detecting Crystal Symmetry Fractionalization from the Ground State: Application to \mathbb{Z}_2 Spin Liquids on the Kagome Lattice*, *Phys. Rev. B* **91**, 100401 (2015).
- [30] M. Zaletel, Y.-M. Lu, and A. Vishwanath, *Measuring Space-Group Symmetry Fractionalization in \mathbb{Z}_2 Spin Liquids*, *Phys. Rev. B* **96**, 195164 (2017).
- [31] W. Zheng, J.-W. Mei, and Y. Qi, *Classification and Monte Carlo Study of Symmetric \mathbb{Z}_2 Spin Liquids on the Triangular Lattice*, arXiv:1505.05351.
- [32] Y.-M. Lu, *Symmetric \mathbb{Z}_2 Spin Liquids and Their Neighboring Phases on Triangular Lattice*, *Phys. Rev. B* **93**, 165113 (2016).
- [33] C. Wang, A. Nahum, M. A. Metlitski, C. Xu, and T. Senthil, *Deconfined Quantum Critical Points: Symmetries and Dualities*, *Phys. Rev. X* **7**, 031051 (2017).
- [34] N. Read and S. Sachdev, *Valence-Bond and Spin-Peierls Ground States of Low-Dimensional Quantum Antiferromagnets*, *Phys. Rev. Lett.* **62**, 1694 (1989).
- [35] N. Read and S. Sachdev, *Spin-Peierls, Valence-Bond Solid, and Néel Ground States of Low-Dimensional Quantum Antiferromagnets*, *Phys. Rev. B* **42**, 4568 (1990).
- [36] T. Senthil, A. Vishwanath, L. Balents, S. Sachdev, and M. P. A. Fisher, *Deconfined Quantum Critical Points*, *Science* **303**, 1490 (2004).
- [37] T. Senthil, L. Balents, S. Sachdev, A. Vishwanath, and M. P. A. Fisher, *Quantum Criticality beyond the Landau-Ginzburg-Wilson Paradigm*, *Phys. Rev. B* **70**, 144407 (2004).
- [38] D. P. Arovas and A. Auerbach, *Functional Integral Theories of Low-Dimensional Quantum Heisenberg Models*, *Phys. Rev. B* **38**, 316 (1988).
- [39] S. Sachdev and R. Jalabert, *Effective Lattice Models for Two Dimensional Quantum Antiferromagnets*, *Mod. Phys. Lett. B* **04**, 1043 (1990).
- [40] F. D. M. Haldane, *$O(3)$ Nonlinear σ Model and the Topological Distinction between Integer- and Half-Integer-Spin Antiferromagnets in Two Dimensions*, *Phys. Rev. Lett.* **61**, 1029 (1988).
- [41] C.-M. Jian, A. Rasmussen, Y.-Z. You, and C. Xu, *Emergent Symmetry and Tricritical Points near the Deconfined Quantum Critical Point*, arXiv:1708.03050.
- [42] E. Lieb, T. Schultz, and D. Mattis, *Two Soluble Models of an Antiferromagnetic Chain*, *Ann. Phys. (N.Y.)* **16**, 407 (1961).
- [43] M. B. Hastings, *Lieb-Schultz-Mattis in Higher Dimensions*, *Phys. Rev. B* **69**, 104431 (2004).
- [44] G. Chen, A. Essin, and M. Hermele, *Majorana Spin Liquids and Projective Realization of $SU(2)$ Spin Symmetry*, *Phys. Rev. B* **85**, 094418 (2012).
- [45] A. Tanaka and X. Hu, *Many-Body Spin Berry Phases Emerging from the π -Flux State: Competition between Antiferromagnetism and the Valence-Bond-Solid State*, *Phys. Rev. Lett.* **95**, 036402 (2005).
- [46] T. Senthil and M. P. A. Fisher, *Competing Orders, Nonlinear Sigma Models, and Topological Terms in Quantum Magnets*, *Phys. Rev. B* **74**, 064405 (2006).
- [47] A. Abanov and P. Wiegmann, *Theta-Terms in Nonlinear Sigma-Models*, *Nucl. Phys.* **B570**, 685 (2000).
- [48] A. Nahum, P. Serna, J. T. Chalker, M. Ortuño, and A. M. Somoza, *Emergent $SO(5)$ Symmetry at the Néel to Valence-Bond-Solid Transition*, *Phys. Rev. Lett.* **115**, 267203 (2015).

- [49] M. Hermele, T. Senthil, M. P. A. Fisher, P. A. Lee, N. Nagaosa, and X.-G. Wen, *Stability of $U(1)$ Spin Liquids in Two Dimensions*, *Phys. Rev. B* **70**, 214437 (2004).
- [50] A. Polyakov, *Compact Gauge Fields and the Infrared Catastrophe*, *Phys. Lett. B* **59**, 82 (1975).
- [51] R. A. Jalabert and S. Sachdev, *Spontaneous Alignment of Frustrated Bonds in an Anisotropic, Three-Dimensional Ising Model*, *Phys. Rev. B* **44**, 686 (1991).
- [52] S. Sachdev and M. Vojta, *Translational Symmetry Breaking in Two-Dimensional Antiferromagnets and Superconductors*, [arXiv:cond-mat/9910231](https://arxiv.org/abs/cond-mat/9910231).
- [53] C. Xu and L. Balents, *Quantum Phase Transitions around the Staggered Valence-Bond Solid*, *Phys. Rev. B* **84**, 014402 (2011).
- [54] L. Fu, S. Sachdev, and C. Xu, *Geometric Phases and Competing Orders in Two Dimensions*, *Phys. Rev. B* **83**, 165123 (2011).
- [55] G. 't Hooft, *Magnetic Monopoles in Unified Gauge Theories*, *Nucl. Phys.* **B79**, 276 (1974).
- [56] A. M. Polyakov, *Particle Spectrum in Quantum Field Theory*, *JETP Lett.* **20**, 194 (1974).
- [57] N. Read and B. Chakraborty, *Statistics of the Excitations of the Resonating-Valence-Bond State*, *Phys. Rev. B* **40**, 7133 (1989).
- [58] N. Seiberg, T. Senthil, C. Wang, and E. Witten, *A Duality Web in $2 + 1$ Dimensions and Condensed Matter Physics*, *Ann. Phys. (Amsterdam)* **374**, 395 (2016).
- [59] A. Karch and D. Tong, *Particle-Vortex Duality from 3D Bosonization*, *Phys. Rev. X* **6**, 031043 (2016).
- [60] J. Murugan and H. Nastase, *Particle-Vortex Duality in Topological Insulators and Superconductors*, *J. High Energy Phys.* **05** (2017) 159.
- [61] M. A. Metlitski, A. Vishwanath, and C. Xu, *Duality and Bosonization of $(2 + 1)$ -Dimensional Majorana Fermions*, *Phys. Rev. B* **95**, 205137 (2017).
- [62] S. Kachru, M. Mulligan, G. Torroba, and H. Wang, *Non-supersymmetric Dualities from Mirror Symmetry*, *Phys. Rev. Lett.* **118**, 011602 (2017).
- [63] J.-Y. Chen, J. H. Son, C. Wang, and S. Raghu, *Exact Boson-Fermion Duality on a 3D Euclidean Lattice*, [arXiv: 1705.05841](https://arxiv.org/abs/1705.05841) [*Phys. Rev. Lett.* (to be published)].