Two-Dimensional Superfluidity of Exciton Polaritons Requires Strong Anisotropy

Ehud Altman
Department of Condensed Matter Physics, Weizmann Institute of Science, Rehovot 76100, Israel

Lukas M. Sieberer
Institute for Theoretical Physics, University of Innsbruck, A-6020 Innsbruck, Austria

Leiming Chen
College of Science, China University of Mining and Technology, Xuzhou, Jiangsu 221116, People's Republic of China

Sebastian Diehl
Institute of Theoretical Physics, TU Dresden, D-01062 Dresden, Germany, and Institute for Theoretical Physics, University of Innsbruck, A-6020 Innsbruck, Austria

John Toner
Department of Physics and Institute of Theoretical Science, University of Oregon, Eugene, Oregon 97403, USA
(Received 19 May 2014; published 19 February 2015)

Fluids of exciton polaritons, excitations of two-dimensional quantum wells in optical cavities, show collective phenomena akin to Bose condensation. However, a fundamental difference from standard condensates stems from the finite lifetime of these excitations, which necessitates continuous driving to maintain a steady state. A basic question is whether a two-dimensional condensate with long-range algebraic correlations can exist under these nonequilibrium conditions. Here, we show that such driven two-dimensional Bose systems cannot exhibit algebraic superfluid order except in low-symmetry, strongly anisotropic systems. Our result implies, in particular, that recent apparent evidence for Bose condensation of exciton polaritons must be an intermediate-scale crossover phenomenon, while the true long-distance correlations fall off exponentially. We obtain these results through a mapping of the long-wavelength condensate dynamics onto the anisotropic Kardar-Parisi-Zhang equation.

DOI: 10.1103/PhysRevX.5.011017 Subject Areas: Condensed Matter Physics, Optics, Statistical Physics

I. INTRODUCTION

One of the most striking discoveries to emerge from the study of nonequilibrium systems is that they sometimes exhibit ordered states that are impossible in their equilibrium counterparts. For example, it has been shown [1] that a two-dimensional "flock"—that is, a collection of moving, self-propelled entities—can develop long-ranged orientational order in the presence of finite noise (the nonequilibrium analog of temperature) and in the absence of both rotational symmetry-breaking fields and long-ranged interactions. In contrast, the Mermin-Wagner theorem [2] states that a two-dimensional equilibrium system with short-ranged interactions and rotation invariance (e.g., a two-dimensional ferromagnet with a two or more component magnetization) cannot order at finite temperature.

In this paper, we report an example of the opposite phenomenon: a driven, two-dimensional Bose system, such as a gas of polariton excitations in a two-dimensional isotropic quantum well [3], cannot exhibit off-diagonal algebraic correlations (i.e., two-dimensional superfluidity) [4]. In the polariton gas, the departure from thermal equilibrium is due to the incoherent pumping needed to counteract the intrinsic losses and maintain a constant excitation density.

The critical properties of related driven quantum systems have been the subject of numerous theoretical studies; in certain cases, it can be shown that the low-frequency correlation functions induced by driving are identical to those in equilibrium systems at an effective temperature set by the driving [5–10]. Such emergent equilibrium behavior occurs in three-dimensional bosonic systems, although nonequilibrium effects can change the dynamical critical behavior [9,10]. Here, we show that the nonequilibrium conditions imposed by the driving have a much more dramatic effect on two-dimensional Bose systems: effective equilibrium is never established in the generic isotropic case; instead, the nonequilibrium nature of the fluctuations inevitably destroys the condensate at long scales. We emphasize that dissipation alone, e.g., due to coupling to a dissipative equilibrium bath, would not have an adverse effect on the condensate. Rather, it is crucial to have both dissipation and driving, giving rise to a true nonequilibrium steady-state situation.

This conclusion follows from the known [11–14] connection between the complex Ginzburg-Landau equation (which describes the long-wavelength dynamics of a driven condensate) and the Kardar-Parisi-Zhang (KPZ) equation [15] or, in the anisotropic case, the anisotropic KPZ equation [16], which were originally formulated to describe randomly growing interfaces. The nonequilibrium fluctuations generated by the drive translate into the nonlinear terms of the KPZ equation.

Our results suggest that recent experiments [17–22] done with isotropic semiconductor quantum wells purporting to show evidence for the long-sought [23] Bose condensation of polariton excitations are, in fact, observing an intermediate-length-scale crossover phenomenon and not the true long-distance behavior of correlations. The condensate is destroyed at long distances despite the weak anisotropy, which is present in these experiments due to the splitting of transverse electric and transverse magnetic cavity modes [3,24], because that anisotropy proves to be far too weak to create sufficient anisotropy at reasonable laser-driving power. We remark that earlier work, which predicted long-range algebraic order in two-dimensional driven condensates [25], relied on a linear (Bogoliubov) theory. Although this theory may be valid on intermediate scales, our analysis shows that it fails at long distances due to the relevant nonlinearity.

On the other hand, performing the same mapping on the anisotropic complex Ginzburg-Landau equation leads, as noted by Grinstein *et al.* [13,14], to the anisotropic KPZ equation. These considerations suggest, as also noted by those authors, that algebraic order can prevail if the system is anisotropic, in the sense that the nonlinear coupling parameters are different in different directions. Then, the transition from this algebraically ordered phase to the disordered phase occurs by a standard equilibriumlike Kosterlitz-Thouless transition. Achieving such effective equilibrium requires very strong anisotropy, which may seem unnatural in the case of exciton polaritons in two-dimensional quantum wells. However, mapping a realistic model of such a system to the anisotropic KPZ equation shows that the anisotropy of the KPZ nonlinearities is a function of the driving laser power. Surprisingly, we find that even if the intrinsic anisotropy of the system is moderate, the effective anisotropy increases with pump power and eventually passes the threshold, allowing for an effective equilibrium description. Then, not only does an algebraically ordered phase occur, but it does so in a reentrant manner: The phase is entered and then left as the driving laser power is increased.

We emphasize that the results reported here are universal, in the sense that they apply to all driven open quantum systems with phase-rotation symmetry in two dimensions. They are thus of relevance to a variety of experiments in which competition between coherent and driven dissipative dynamics occur, such as microcavity arrays [26,27] or ultracold atoms [28]. However, at present, ensembles of exciton polaritons stand out as the most promising realization of this physics, as the ability to tune laser power provides a crucial “knob” that can be “turned” to make the system more nonequilibrium. (Note that this goal is opposite to that of some recent work, which has strived to reach the equilibrium limit [22].) In particular, below, we present parameter estimates that demonstrate that our predictions should be within reach of current technology.

II. MODEL

The dynamics of a driven dissipative system like a polariton condensate is determined both by coherent processes, such as the dispersion and scattering between polaritons, and independent dissipative processes induced by loss and the pumping field. A model of the condensate dynamics that incorporates these processes is

$$\partial_t\psi(\mathbf{x},t) = -\frac{\delta H_d}{\delta\psi^*} - i\frac{\delta H_c}{\delta\psi^*} + \zeta(\mathbf{x},t). \tag{1}$$

Here, ψ is the scalar complex order parameter, which describes the incipient condensate of linearly polarized polaritons [3]. The effective Hamiltonians H_ℓ ($\ell = c, d$) that generate the coherent and dissipative dynamics, respectively, read

$$H_\ell = \int_{x,y}\left[r_\ell|\psi|^2 + K_\ell^x|\partial_x\psi|^2 + K_\ell^y|\partial_y\psi|^2 + \frac{1}{2}u_\ell|\psi|^4\right]. \tag{2}$$

The last term $\zeta(\mathbf{x},t)$ in Eq. (1) is a zero-mean Gaussian white noise with short-ranged spatiotemporal correlations: $\langle\zeta^*(\mathbf{x},t)\zeta(\mathbf{x}',t')\rangle = 2\sigma\delta^d(\mathbf{x}-\mathbf{x}')\delta(t-t')$, $\langle\zeta(\mathbf{x},t)\times\zeta(\mathbf{x}',t')\rangle = 0$.

Equation (1) is widely known as the complex Ginzburg-Landau equation [29,30] or, in the context of polariton condensates, as the dissipative Gross-Pitaevskii equation [31,32], although usually only the isotropic (i.e., $K_\ell^x = K_\ell^y$), noise-free ($\zeta = 0$) case is considered (but, see Ref. [33]). Modifications of this equation, e.g., including higher powers of ψ and ζ, higher derivatives, or combinations of the two, can readily be shown to be irrelevant in the renormalization-group (RG) sense: they have no effect on the long-distance, long-time scaling properties of either the ordered phase or the transition into it [34].

Each of the parameters appearing in the model has a clear physical origin, as we now review. The coefficient r_d is the single-particle loss rate γ_l (spontaneous decay) offset by the pump rate γ_p; that is, $r_d = \gamma_l - \gamma_p$. We consider a situation in which both loss and pump processes are spatially homogeneous. The effective chemical potential r_c is completely arbitrary. Indeed, it can be adjusted by a temporally local gauge transformation $\psi(\mathbf{x},t) = \psi'(\mathbf{x},t)e^{i\omega t}$, such that $r_c{}' = r_c + \omega$. In the following, we choose r_c so that, in the absence of noise, the equation of motion has a stationary, spatially uniform solution.

The term proportional to u_c is the pseudopotential that describes the elastic scattering of two polaritons, whereas u_d is the nonlinear loss or, alternatively, a reduction of the pump rate with density that ensures saturation of particle number. The coefficients $K_c^{x,y} = 1/(2m_{x,y})$ (units are chosen such that $\hbar = 1$), where $m_{x,y}$ are the eigenvalues of the effective polariton mass tensor, with principal axes x, y. Under typical circumstances, the diffusionlike term K_d is expected to be small but is allowed by symmetry and so will always be generated [35,36]. Finally, the noise is given by the total rate of particles entering and leaving the system. In polariton condensates, where u_d reflects a nonlinear reduction of the pumping rate rather than an additional loss mechanism (see Appendix B), the noise strength at the steady state is simply set by the single-particle loss, i.e., $\sigma = \gamma_l$ [37].

Before proceeding, it is important to clarify under what conditions Eq. (1) describes an effective thermal equilibrium at all wavelengths. Imposing the additional condition that the field follows a thermal Gibbs distribution in the steady state translates to the simple requirement $H_d = RH_c$, where R is a multiplicative constant [33,38–40]. This condition can also be seen as a symmetry of the dynamics which ensures detailed balance [9,10] and is realized in dynamical systems that relax to thermal equilibrium [41,42]. In contrast, in a driven system, the relation $H_d = RH_c$ is not satisfied, in general, because the dissipative and coherent parts of the dynamics are generated by independent processes. This relation can, however, arise as an emergent symmetry at low frequencies and long wavelengths, as was shown to be the case for a three-dimensional driven condensate [9,10]. Below, we shall derive the hydrodynamic long-wavelength description of a two-dimensional driven condensate and determine if it flows to effective thermal equilibrium.

III. MAPPING TO A KPZ EQUATION

In the long-wavelength limit, Eq. (1) generically reduces to a KPZ equation [15] for the phase variable [13]. In the dissipationless case $H_d = 0$, the dynamics becomes totally different and exhibits the usual propagating long-wavelength Bogoliubov quasiparticles. We discuss this point further in Appendix A, after Eq. (A2). As in equilibrium, in a hydrodynamic description of the condensate, the order-parameter field is written in the amplitude-phase representation as $\psi(\mathbf{x},t) = [M_0 + \chi(\mathbf{x},t)]e^{i\theta(\mathbf{x},t)}$. Integrating out the gapped amplitude mode and keeping only terms that are not irrelevant in the sense of the renormalization group, we obtain a closed equation for θ (see Appendix A)

$$\partial_t\theta = D_x\partial_x^2\theta + D_y\partial_y^2\theta + \frac{\lambda_x}{2}(\partial_x\theta)^2 + \frac{\lambda_y}{2}(\partial_y\theta)^2 + \bar{\zeta}(\mathbf{x},t), \tag{3}$$

with ($\alpha = x, y$):

$$D_\alpha = K_d^\alpha\left[1 + \frac{K_c^\alpha}{K_d^\alpha}\frac{u_c}{u_d}\right],$$
$$\lambda_\alpha = 2K_c^\alpha\left[\frac{K_d^\alpha}{K_c^\alpha}\frac{u_c}{u_d} - 1\right] \tag{4}$$

and noise strength (replacing σ in the noise correlations above)

$$\Delta = \frac{(u_d^2 + u_c^2)\gamma_l}{2u_d(\gamma_p - \gamma_l)}. \tag{5}$$

Equation (3) is the anisotropic KPZ equation, originally formulated to describe the roughness of a growing surface due to random deposition of particles on it [15,16], in which case θ is the height of the interface. It reduces to the isotropic KPZ equation when $D_x = D_y$ and $\lambda_x = \lambda_y$. This reduction can also be achieved by a trivial rescaling of lengths if $\Gamma \equiv \lambda_y D_x/\lambda_x D_y = 1$. Thus, when $\Gamma \neq 1$, the system is anisotropic.

Crucially, the presence of the nonlinearity directly reflects nonequilibrium conditions [43]. Indeed, the coefficients λ_x, λ_y that measure the deviation from thermal equilibrium vanish identically when the conditions

$$K_c^x/K_d^x = K_c^y/K_d^y = u_c/u_d, \tag{6}$$

which follow from the equilibrium requirement that $H_d = RH_c$ are met.

It is furthermore important to note that our KPZ model differs from that formulated for a description of randomly growing interfaces [15] in that the analog of the interface-height variable in our model is actually a compact phase; hence, topological defects in this field are possible. This difference with the conventional KPZ equation also arises in "active smectics" [44].

Analysis of Eq. (3) in the absence of vortices is the analog of the low-temperature spin-wave (linear phase fluctuation) theory of the equilibrium XY model. Indeed, without the nonlinear terms, the KPZ equation reduces to linear diffusion, which would bring the field to an effective thermal equilibrium with power-law off-diagonal correlations (in $d = 2$). A transition to the disordered phase in this equilibrium situation can occur only as a Kosterlitz-Thouless (KT) transition through proliferation of topological defects in the phase field.

In a driven condensate, the nonlinear terms are, in general, present, and in two dimensions have the same canonical scaling dimension as the linear terms. A more careful RG analysis is therefore required to determine how the system behaves at long scales even without defect proliferation. Such an analysis has been done in Refs. [16,44] for the anisotropic KPZ equation. In this case, the flow is closed in the two-parameter space of scaled nonlinearity $g \equiv \lambda_x^2 \Delta / D_x^2 \sqrt{D_x D_y}$ and scaled anisotropy $\Gamma \equiv \lambda_y D_x / \lambda_x D_y$ and is given, to leading order in g, by

$$\frac{dg}{dl} = \frac{g^2}{32\pi}(\Gamma^2 + 4\Gamma - 1),$$
$$\frac{d\Gamma}{dl} = \frac{\Gamma g}{32\pi}(1 - \Gamma^2). \tag{7}$$

These flows are illustrated in Fig. 1. We see that in an isotropic system, (i.e., $\Gamma = 1$) the nonlinear coupling g, which embodies the nonequilibrium fluctuations, is relevant. Moreover, for a wide range of anisotropies (namely, all $\Gamma > 0$), the flow is attracted to the isotropic line: the system flows to strong coupling, with emergent rotational symmetry. On the other hand, if the anisotropy is sufficiently strong, so that $\Gamma < 0$, the nonlinearity becomes irrelevant and the system can flow to an effective equilibrium state at long scales. We emphasize that by "strong anisotropy," we mean only that the KPZ nonlinearities $\lambda_{x,y}$ have opposite signs; the diffusion constants $D_{x,y}$ must both be positive for reasons of stability. Furthermore, we do not assume strong anisotropy between D_x and D_y; our predictions apply even when these couplings are comparable in magnitude or, indeed, even if they are equal. The linear spatial extents of the system $L_{x,y}$ are also assumed to be comparable. That is, in no sense are we considering a "nearly one-dimensional" system.

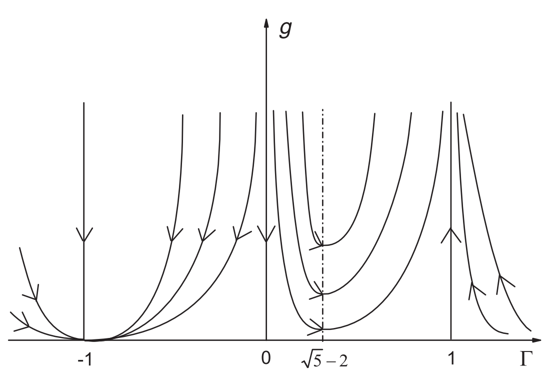

FIG. 1. The RG flow in the Γ-g parameter space for anisotropic driven Bose-Einstein condensation in $d = 2$. For $\Gamma < 0$ and $g > 0$, all flow lines go to a stable fixed point $(-1, 0)$; for $\Gamma > 0$ and $g > 0$, all flow lines go to infinity and approach the isotropic limit $\Gamma = 1$.

We will now discuss the physics of these two regimes, starting with the isotropic case, which is most relevant to current experiments with polariton condensates.

IV. ISOTROPIC SYSTEMS

As noted above, rotational symmetry is emergent at long scales if the anisotropy is not too strong at the outset. The limit of weak anisotropy is also the regime in which current experimental quantum-well polaritons lie. We therefore consider this case first.

On the line $\Gamma = 1$, the scaling of the nonlinear coupling $dg/dl = g^2/8\pi$ drives $g \to \infty$; in the growing surface problem, the system goes to the "rough" state, with height fluctuations scaling algebraically with length. The analogous behavior in the phase field θ would lead to stretched exponentially decaying order-parameter correlations. However, the fact that the phase field is compact implies that topological defects (vortices) in this field exist. Our expectation, based on analogy with equilibrium physics (which admittedly may be an untrustworthy analogy), is that vortices will unbind at the strong-coupling fixed point of the KPZ equation. Unbinding of vortices will lead to simple exponential correlations. Testing this expectation will be the object of future work.

We have thus established that the nonlinearity, no matter how weak, destroys the condensate at long distances, leading to either stretched or simple exponential decay of correlations throughout the isotropic regime. However, the effects of the nonlinearity only become apparent when g gets to be of order one. Solving the scaling equation, we see that $g(l_*) \sim 1$ at the characteristic "RG time" $l_* = 8\pi/g_0$; the corresponding length scale is

$$L_* = \xi_0 e^{l_*} = \xi_0 e^{8\pi/g_0}, \tag{8}$$

where ξ_0 is the mean-field healing length of the condensate. If the bare value of g_0 is small, then the scale L_* can be huge. On length scales smaller than L_*, the system is governed by the linearized isotropic KPZ equation, which, as noted earlier, is the same as an equilibrium XY model. Thus, all of the equilibrium physics associated with two-dimensional Bose-Einstein condensation, including power-law correlations and a Kosterlitz-Thouless defect-unbinding transition, can appear in a sufficiently small system.

As parameters, such as the pump power, are changed, the system can lose its apparent algebraic order in one of two ways: (i) The KPZ length L_* is gradually reduced below the system size or (ii) L_* remains large while the correlations within the system size L are destroyed by the unbinding of vortex-antivortex pairs at the scale L. The latter type of crossover would appear as a KT transition broadened by the finite size. Of course, for any given set of system parameters, a sufficiently large system ($L > L_*$) will always be disordered.

We shall now discuss how the system parameters determine what type of crossover, if any, will be seen in an experiment. We assume that the main tuning parameter is the pump power γ_p, and it will be convenient to track the behavior as a function of a dimensionless tuning parameter $x = \gamma_p/\gamma_l - 1$ and set $K_d = 0$ since this parameter is thought to be small in current experimental realizations. In Appendix B, we derive the parameters of the KPZ equation for a realistic model of a polariton condensate that is coupled to an excitonic reservoir. In particular, we obtain an expression for the bare dimensionless coupling constant g_0 in this model, which measures the bare deviation from equilibrium:

$$g_0 = \frac{\Delta\lambda^2}{D^3} = 2\bar{u}\bar{\gamma}^2\left(\frac{\bar{\gamma}^2 + (1+x)^2}{x(1+x)^3}\right). \tag{9}$$

Here, $\bar{u} \equiv u_c/K_c$ is the dimensionless interaction constant and $\bar{\gamma} \sim \gamma_l$ the dimensionless loss rate as defined in Appendix B. Note that g_0 diverges as we approach the mean-field transition at $x \to 0^+$ while it decays as $1/x^2$ as $x \to \infty$ at very high pump power.

Hence, in the latter regime, the KPZ length scale $L_* = \xi_0 \exp(8\pi/g_0)$ is certainly much larger than any reasonable system size. As the pump power is decreased and the system approaches the mean-field transition at $x = 0$, L_* drops sharply to a microscopic healing length $L_* \approx \xi_0$. L_* drops below the system size when $x \lesssim x_*$, where

$$\frac{x_*(1+x_*)^3}{\bar{\gamma}^2 + (1+x_*)^2} \approx \frac{\bar{u}\bar{\gamma}^2}{4\pi}\ln(L/\xi_0). \tag{10}$$

For pump powers corresponding to $x > x_*$, the system will appear to be at effective equilibrium and, hence, may sustain power-law order within its confines, whereas for pump power $x < x_*$, the nonequilibrium fluctuations become effective and destroy the algebraic correlations at the scale of the system size. However, it is possible that this crossover at x_* is preceded by the unbinding of vortices at values of $x = x_{\mathrm{KT}} > x_*$ while the finite system is still at effective equilibrium. This crossover behavior is depicted in Fig. 2.

To determine which crossover occurs in a particular system, let us estimate the value of the tuning parameter x_{KT} at which the putative Kosterlitz-Thouless transition would occur if the nonlinear term λ vanished or was negligible. Then, Eq. (3) obeys a fluctuation-dissipation relation with a temperature set by the noise $T = \Delta$. The KT transition would occur for an equilibrium XY model approximately at the point where $\Delta/D = \pi$. Expressing both Δ and D in terms of the tuning parameter x, we obtain the equation for the critical value x_{KT} at which the Kosterlitz-Thouless transition will appear to occur:

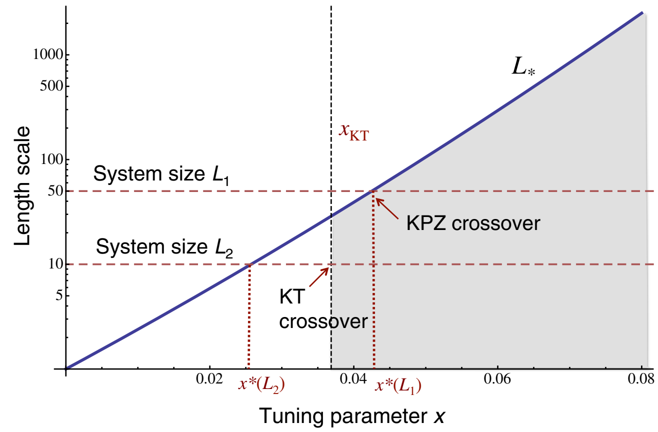

FIG. 2. Dependence of the emergent KPZ length scale L_* (in units of the microscopic healing length) on the tuning parameter $x = \gamma_p/\gamma_l - 1$. This curve was obtained by inserting the expression [Eq. (9)] for the bare coupling g_0 into Eq. (8) for L_*. While L_* is exponentially large when $x > \bar{u}\bar{\gamma}^2/2\pi$, it goes to a microscopic value ξ_0 at the mean-field transition $x = 0^+$. The shaded region marks the scales at which a system would exhibit algebraic correlations. Upon decreasing the tuning parameter x, a finite system will lose its algebraic order in one of two ways: (1) when L_* falls below the system size, as in the case of system L_1 shown, or (2) when a KT transitions occurs before L_* falls below the system size, as in the case of system L_2. Here, we have used $\bar{\gamma} = 0.8$ and $\bar{u} = 0.14$.

$$\frac{x_{\mathrm{KT}}(1+x_{\mathrm{KT}})}{\bar{\gamma}^2 + (1+x_{\mathrm{KT}})^2} \approx \frac{\bar{u}}{2\pi}. \tag{11}$$

Equations (10) and (11) for x_* and x_{KT} simplify when the interactions are weak $\bar{u} \ll 2\pi$ and the system is not too big $(\bar{u}/4\pi)\bar{\gamma}^2 \ln(L/\xi_0) \ll 1$. In this limit, which proves to hold in current experiments, x_* and x_{KT} are given approximately by $x_* \approx (\bar{u}/4\pi)\bar{\gamma}^2(1+\bar{\gamma}^2)\ln(L/\xi_0)$ and $x_{\mathrm{KT}} \approx (\bar{u}/4\pi)(1+\bar{\gamma}^2)$.

Under these conditions, we expect to see a crossover controlled by vortex unbinding through the KT mechanism, i.e., $x_{\mathrm{KT}} > x_*$, if the system size is $L < \xi_0 \exp(2/\bar{\gamma}^2)$. For larger system sizes, the crossover will be controlled by the nonlinearities of the KPZ equation.

In which regime do experiments on polariton condensates lie? In Ref. [45], parameter values for typical experimental systems have been deduced for the effective two-band polariton model described in Appendix B. Based on this analysis, we find $\bar{u}_c \approx 0.14$ and $\bar{\gamma} \approx 0.1$. With these parameters, the KT transition would be expected at $x_{\mathrm{KT}} \sim 0.02$ and the bare healing length at this point is $\xi_0 \sim 2\ \mu$m. Since the system size (spot size) is no more than approximately 100 μm in any current experiment, these values satisfy the conditions for the simplified expressions for x_* and x_{KT} discussed above. In addition, the small value of $\bar{\gamma}$ implies that the crossover will be dominated by KT physics; the nonlinear effects we have treated here will not be visible in these experiments. However, it should not be difficult to raise $\bar{\gamma}$ by reducing the cavity Q. (Note that the increased loss will require a concomitant increase of the laser power at threshold.) A moderate increase of $\bar{\gamma}$ to 0.75 will make it possible to enter the KPZ-dominated regime at attainable system sizes.

Several recent experiments in the field of exciton-polariton physics [46–48] have come nearly as close to equilibrium conditions as cold-atom experiments by realizing increasingly long polariton lifetimes (i.e., lowering $\bar{\gamma}$). From the perspective of the findings presented in this work, it may be viewed as an outstanding feature of exciton polaritons that both close-to-equilibrium conditions and the regime required to investigate the genuine nonequilibrium aspects predicted here are accessible in these systems.

V. STRONG ANISOTROPY

If the bare value of the anisotropy parameter is negative $\Gamma < 0$, then the RG equations (7) lead to a fixed point at $g = 0$. Because the nonlinear $\lambda_{x,y}$ terms in Eq. (3) are irrelevant in this region of parameter space, the linear (and, hence, equilibrium) version of the theory applies. Hence, it is possible, for $\Gamma < 0$, to obtain both a power-law phase and a KT defect-unbinding transition out of it.

To estimate the extent of this phase, we can utilize the RG flow of the anisotropic KPZ equation for $\Gamma < 0$ analyzed in Ref. [44]. In principle, we should add to these recursion relations terms coming from the vortices. Instead, we will follow Ref. [44] and assume that the vortex density is low enough that vortices only become important on length scales far longer than those at which the KPZ nonlinearities λ_α have become unimportant (i.e., those at which the scaled nonlinearity g has flowed to nearly 0). If vortices are so dilute, then we can use the recursion relations [Eq. (7)] for our problem, despite the fact that they were derived neglecting vortices.

Our strategy is then to use those recursion relations to flow to the linear regime, which, as noted earlier, is equivalent to an equilibrium XY model. In this regime, vortex unbinding is controlled by the (bare) parameters of Eq. (3) through the dimensionless noise strength $\kappa_0 \equiv \Delta/\sqrt{D_x D_y}$ and scaled anisotropy $\Gamma_0 = \lambda_y D_x/(\lambda_x D_y)$. Following Ref. [44], the critical point for vortex unbinding can be estimated by solving for the renormalized scaled noise $\kappa(l \to \infty) \equiv \kappa(\infty)$ as a function of the bare value using the RG equations of the noncompact KPZ equation; these equations include additional recursion relations for D_α and Δ as well as Eq. (7); for details, see Ref. [44]. The solution is $\kappa(\infty) = -\kappa_0(1-\Gamma_0)^2/(4\Gamma_0)$. The KT transition occurs at the point where this renormalized value $\kappa(\infty)$ of κ reaches π. Hence, the phase boundary in the κ_0-Γ_0 plane is a locus in the plane of bare scaled noise and anisotropy parameters given by [44]

$$\kappa_0 = -\frac{4\pi\Gamma_0}{(1-\Gamma_0)^2}. \tag{12}$$

The assumption in deriving this curve is that the dominant contribution to the stiffness renormalization comes from the nonlinear fluctuations rather than from bound vortex-antivortex pairs, which have been neglected.

There is a broad range of parameters for which a system enters a regime $\Gamma_0 = (D_x\lambda_y)/(D_y\lambda_x) < 0$, in which true power-law order and a KT transition exist. Using the effective (two-band) polariton model in Appendix B, we find the trajectory a realistic system would follow in the κ_0, Γ_0 plane with increasing pump power as a function of its microscopic parameters; see Fig. 3. This trajectory is given by

$$\Gamma_0 = \frac{[\nu_y(1+x)-\bar{\gamma}][\nu_x\bar{\gamma}+1+x]}{[\nu_x(1+x)-\bar{\gamma}][\nu_y\bar{\gamma}+1+x]}, \tag{13}$$

$$\kappa_0 = \frac{\bar{u}}{2x}\frac{[\bar{\gamma}^2+(1+x)^2]}{\sqrt{[\nu_y\bar{\gamma}+(1+x)][\nu_x\bar{\gamma}+(1+x)]}}, \tag{14}$$

with the ratios of the dissipative to coherent phase stiffnesses along the two directions $\nu_\alpha = K_d^\alpha/K_c^\alpha$. Now, consider gradually increasing the pump power, and hence x, from the mean-field threshold $x = 0$. For system parameters $\bar{\gamma} > \nu_y > \nu_x$, $\Gamma_0(x)$ starts out positive at $x = 0$, is reduced to negative values as x is increased past $x = \bar{\gamma}/\nu_y - 1$, and eventually runs off to $\Gamma_0 = -\infty$ at a finite value of x (namely, $x = \bar{\gamma}/\nu_x - 1$). If at the same time $\bar{u}$ is sufficiently small, then the experimental trajectory in the κ_0-Γ_0 plane is guaranteed to cross the dome marking the condensate (algebraic order) phase as determined in Eq. (12). The condition on $\bar{u}$ for this crossing to occur is

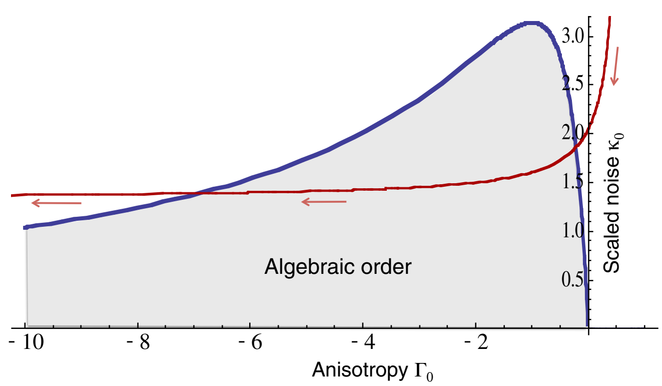

FIG. 3. Phase diagram of a generic anisotropic system exhibiting reentrance. The thin line marks the natural trajectory in an experiment in which only the pump power is varied. The arrows mark the direction of increasing pump power. Such an experiment will see a reentrant behavior, where the system starts in a disordered state, enters the power-law superfluid, and then goes back to a disordered state. Here, we have used the two-band model of Appendix B with $\bar{\gamma} = 1/2$, $\bar{u} = 2$, $\nu_x = 1/8$, and $\nu_y = 1/4$.

$$\bar{u} < 2\pi \frac{(\bar{\gamma} - \nu_y)}{\bar{\gamma}\nu_y(1 + \nu_y^2)}. \tag{15}$$

Thus, we not only naturally achieve the ordered phase in this anisotropic system by varying the driving but we do so in a reentrant manner: we enter the phase and then leave it again as the driving is increased. The analysis for $\gamma > \nu_y > \nu_x$ is the same if we take $\Gamma \to 1/\Gamma$.

We note that momentum-dependent splitting of transverse electric and transverse magnetic modes in typical exciton-polariton experiments induces a slight anisotropy in the coherent mass terms in these systems [3,24] while the dissipative mass terms K_d^α are typically very small because they are generated as second-order effects in the interactions [9,10]. Under these conditions, the regime of strong anisotropy cannot be reached with reasonable pump power.

VI. OUTLOOK

Our analysis can be extended to three dimensions. There, for weak deviations from equilibrium, i.e., a small bare value of the KPZ nonlinearity, it predicts a true Bose condensate which may be established through the dynamical phase transition described in Ref. [9], even in the isotropic case. However, beyond a critical strength of the deviation from equilibrium, there may be a different, nonequilibrium transition controlled by a strong-coupling fixed point of the three-dimensional KPZ equation [49]. The existence of such a transition would imply the possibility of a new nonequilibrium phase of matter with short-ranged order, distinct from the usual uncondensed state in that vortex loops do not proliferate. This possibility will be explored in future work. Other directions for future research are investigations of the interplay of the KPZ physics uncovered here with more realistic microscopic models of exciton-polariton condensates. These studies require taking into account both the polariton polarization [50,51] and the effects of disorder on phases and phase transitions [52] (see also Refs. [3,53] and references therein for further aspects of disorder in exciton polaritons), both of which are unavoidable in real experimental systems. It is worth noting that a recent analysis of the dissipative Gross-Pitaevskii equation in the presence of static disorder without noise found that the mean-field condensate is destroyed, through a completely different mechanism [54].

ACKNOWLEDGMENTS

Illuminating discussions with Daniel Podolsky and Anatoli Polkovnikov are gratefully acknowledged. We thank Austen Lamacraft for insightful comments on the manuscript. J. T. thanks Dung-hai Lee and the Physics Department of the University of California, Berkeley, CA, and the Max Planck Institute for the Physics of Complex Systems, Dresden, Germany, for their support (financial and otherwise) and hospitality while this work was being done. E. A. likewise thanks the Miller Institute at UC Berkeley and the Aspen Center for Physics under NSF Grant No. 1066293. L. C. also thanks the Max Planck Institute for the Physics of Complex Systems for support and hospitality and acknowledges support by the National Science Foundation of China (under Grant No. 11474354). We also thank the U.S. NSF for support by Awards No. EF-1137815 and No. 1006171 and the Simons Foundation for support by Award No. 225579 (J. T.), European Research Council Synergy Grant UQUAM, the Israel Science Foundation (E. A.), the Austrian Science Fund through the START Grant No. Y 581-N16 and the German Research Foundation through ZUK 64.

APPENDIX A: MAPPING TO THE KARDAR-PARISI-ZHANG EQUATION

Here, we review the mapping [13], in the long-wavelength limit, between the model (1) and an anisotropic KPZ equation [15,16]. We work in the amplitude-phase representation $\psi(\mathbf{x}, t) = [M_0 + \chi(\mathbf{x}, t)]e^{i\theta(\mathbf{x},t)}$, with M_0, χ, and θ all real. Here, M_0 is determined by requiring that $\chi = 0$, $\theta = 0$ is a static uniform solution of Eq. (1) in the absence of fluctuations $[\zeta(\mathbf{x}, t) = 0]$. The real and imaginary parts of Eq. (1) then give $M_0^2 = -r_d/u_d$ and $r_c = -u_c M_0^2$, respectively. We can satisfy the second condition by exploiting the freedom to choose r_c mentioned in the main text. As explained there, by varying the strength of the pump laser, one can experimentally control r_d, which determines the amplitude M_0. The mean-field transition occurs at the point $r_d = 0$ (i.e., when $\gamma_p = \gamma_l$), where the amplitude M_0 vanishes. For later convenience, we define the dimensionless tuning parameter $x \equiv \gamma_p/\gamma_l - 1$.

Plugging the amplitude-phase representation of ψ into Eq. (1) and linearizing in the amplitude fluctuations χ, we obtain the pair of equations

$$\partial_t\chi = -2u_d M_0^2\chi - K_c^x M_0\partial_x^2\theta - K_c^y M_0\partial_y^2\theta - K_d^x M_0(\partial_x\theta)^2 - K_d^y M_0(\partial_y\theta)^2 + \mathrm{Re}\zeta, \tag{A1}$$

$$M_0\partial_t\theta = -2u_c M_0^2\chi + K_d^x M_0\partial_x^2\theta + K_d^y M_0\partial_y^2\theta - K_c^x M_0(\partial_x\theta)^2 - K_c^y M_0(\partial_y\theta)^2 + \mathrm{Im}\zeta, \tag{A2}$$

where we have used the freedom discussed earlier to choose $r_c = -u_c M_0^2$ to simplify this expression.

Note that if we have no dissipation ($H_d = 0$), so that $u_d = 0$, both χ and θ are “slow” variables, in the sense of evolving at rates that vanish as the wave vector goes to 0. In this case, we can substitute Eq. (A1) into the time derivative of Eq. (A2) to obtain a wave equation for θ supplemented by irrelevant nonlinear corrections. This analysis gives the linear dispersion of the undamped Goldstone modes characteristic of a lossless condensate with exact particle number conservation. In contrast, without particle number conservation (i.e., in the presence of loss and drive), $u_d \neq 0$, and we can therefore neglect the $\partial_t \chi$ term (which vanishes as frequency $\omega \to 0$) on the left-hand side of Eq. (A1) relative to the $2u_d M_0^2 \chi$ on the right-hand side for any "hydrodynamic mode" (i.e., in the low-frequency limit). Doing so turns Eq. (A1) into a simple linear algebraic equation relating χ to spatial derivatives of θ. Substituting the solution for χ of this equation into Eq. (A2) gives Eq. (3), a closed equation for θ. The noise variable in that equation is related to the original noise through $\bar{\zeta} = (\mathrm{Im}\zeta - u_c \mathrm{Re}\zeta/u_d)/M_0$, and hence, $\langle \bar{\zeta}(\mathbf{x},t)\bar{\zeta}(\mathbf{x}',t')\rangle = 2\Delta\delta^d(\mathbf{x}-\mathbf{x}')\delta(t-t')$ with Δ given in Eq. (5). The stochastic equation for θ includes all terms that are marginal and relevant by canonical power counting while neglecting irrelevant terms like $\partial_t^2\theta$, $\partial_t \nabla^2\theta$, and $\partial_t(\nabla\theta)^2$ and terms with even more derivatives.

APPENDIX B: POLARITON-CONDENSATE MODEL WITH RESERVOIR

In the main text, we work with a complex Ginzburg-Landau equation describing the incipient condensate in the lower polariton band. Such a model clearly gives the correct universal physics. However, in order to find how the parameters of the anisotropic KPZ equation change as actual experimental parameters are varied, we must start from a more microscopic model of the polariton degrees of freedom.

The standard model for describing these systems is a two-fluid model which treats the excitonic range of the lower polariton band as a reservoir with local density n_R for the condensate which forms at zero momentum in the lower polariton band [3]. Here, we generalize the model slightly in order to include dissipative mass terms and anisotropy:

$$\partial_t\psi = \left[\sum_{\alpha=x,y}(iK_c^\alpha + K_d^\alpha)\partial_\alpha^2 - ir_c - \gamma_l - iu_c|\psi|^2 + Rn_R\right]\psi + \zeta,$$
$$\partial_t n_R = P - Rn_R|\psi|^2 - \gamma_R n_R, \tag{B1}$$

where $\langle \zeta^*(\mathbf{x},t)\zeta(\mathbf{x}',t')\rangle = 2\sigma\delta(\mathbf{x}-\mathbf{x}')\delta(t-t')$. It is usually assumed that the reservoir-relaxation rate γ_R is faster than all other scales. Hence, we may independently solve for the reservoir density, assuming it is time independent:

$$n_R = \frac{P}{\gamma_R + R|\psi|^2}. \tag{B2}$$

Substituting Eq. (B2) in Eq. (B1) for ψ, we obtain

$$\partial_t\psi = \left[\sum_\alpha (iK_c^\alpha + K_d^\alpha)\partial_\alpha^2 - ir_c - \gamma_l - iu_c|\psi|^2 + \frac{P}{\eta + |\psi|^2}\right] \times \psi + \zeta, \tag{B3}$$

where we have eliminated R and γ_R for the single parameter $\eta = \gamma_R/R$. We note that the amplitude of the white noise is given by the total loss rate (γ_l) and gain, and since in the steady state the loss and gain must be equal, we simply have $\sigma = \gamma_l$ in this case.

In the following, as in the main text, we work in the phase-amplitude representation $\psi(\mathbf{x},t) = [M_0 + \chi(\mathbf{x},t)]e^{i\theta(\mathbf{x},t)}$ and expand around the homogeneous mean-field solution. Let us therefore first solve for the mean-field steady state. The real part of the equation gives $\gamma_l = P/(\eta + M_0^2)$ from which we can deduce the condensate density $M_0^2 = P/\gamma_l - \eta$. The imaginary part of the equation is $r_c = -u_c M_0^2$. It is also worth noting that loss comes only from the term γ_l, since there is no two-particle loss term in this model. (Instead, saturation is reached due to the nonlinear reduction of the pump term.) Hence, in the steady state, when loss is equal to gain, the noise term is simply $\sigma = \gamma_l$.

We now proceed to write the equations of motion for χ and θ to linear order in χ

$$M_0^{-1}\partial_t\chi = -2\gamma_l^2 P^{-1} M_0\chi - K_c^\alpha\partial_\alpha^2\theta - K_d^\alpha(\partial_\alpha\theta)^2 + M_0^{-1}\mathrm{Re}\zeta,$$
$$\partial_t\theta = -2u_c\chi + K_d^\alpha\partial_\alpha^2\theta - K_c^\alpha(\partial_\alpha\theta)^2 - M_0^{-1}\mathrm{Im}\zeta. \tag{B4}$$

Now, as in the main text, we can eliminate χ to obtain the KPZ equation for θ, where $\alpha = x, y$ is summed over and

$$\partial_t\theta = D_\alpha\partial_\alpha^2\theta + \frac{1}{2}\lambda_\alpha(\partial_\alpha\theta)^2 + \bar{\zeta}, \tag{B5}$$

where

$$\bar{\zeta} = M_0^{-1}\left(\mathrm{Re}\zeta - \frac{u_c P}{\gamma_l^2}\mathrm{Im}\zeta\right). \tag{B6}$$

The noise parameter in $\langle\bar{\zeta}^*(\mathbf{x},t)\bar{\zeta}(\mathbf{x}',t')\rangle = 2\Delta\delta(\mathbf{x}-\mathbf{x}')\times\delta(t-t')$ is here given by

$$\Delta = \frac{\gamma_l^2/2}{P - \eta\gamma_l}\left(1 + \frac{u_c^2P^2}{\gamma_l^4}\right) = \left(1 + \frac{u_c^2\eta^2\gamma_p^2}{\gamma_l^4}\right)\frac{\gamma_l^2/2\eta}{\gamma_p - \gamma_l}$$
$$= \frac{u_c\bar{\gamma}}{2x}\left(1 + \frac{(1+x)^2}{\bar{\gamma}^2}\right), \tag{B7}$$

where we have defined $\gamma_p \equiv P/\eta$, the dimensionless tuning parameter $x = \gamma_p/\gamma_l - 1$, and the dimensionless loss parameter $\bar{\gamma} \equiv \gamma_l/(\eta u_c)$. Below, we will also need the dimensionless interaction strength $\bar{u} \equiv u_c/\sqrt{K_c^x K_c^y}$ and the ratios $\nu_\alpha = K_d^\alpha/K_c^\alpha$. The parameters of the anisotropic KPZ equation may now be written as

$$D_\alpha = K_c^\alpha\left(\frac{K_d^\alpha}{K_c^\alpha} + \frac{u_c P}{\gamma_l^2}\right) = K_c^\alpha\left(\nu^\alpha + \frac{1+x}{\bar{\gamma}}\right),$$
$$\lambda_\alpha = 2K_c^\alpha\left(\frac{K_d^\alpha u_c P}{K_c^\alpha \gamma_l^2} - 1\right) = 2K_c^\alpha\left(\nu^\alpha \frac{1+x}{\bar{\gamma}} - 1\right). \quad \text{(B8)}$$

In order to make contact with the main text, we note that the expressions for the diffusion constants D_α and nonlinear coefficients λ_α can be obtained from the predictions [Eq. (B8)] for the complex Ginzburg-Landau model [Eq. (1)] of the main text, if we make the replacement

$$u_d = \frac{\gamma_l^2}{P} = \frac{u_c\bar{\gamma}}{1+x}. \quad \text{(B9)}$$

The parameter K_d is thought to be small in isotropic two-dimensional quantum wells. If we take $K_d = 0$, then $D = K_c u_c/u_d = K_c(1+x)/\bar{\gamma}$ and $\lambda = -2K_c$. Equation (9) in the main text corresponds to this special case $K_d = 0$.

Finally, to facilitate estimating the scales on which the phenomena discussed here can be observed in current experiments, we summarize the relations between the dimensionless quantities used in the main text and the parameters of the commonly used theoretical model of isotropic exciton-polariton systems. There, the diffusion constant K_d is taken to be 0 [although, in some cases, such a term is included effectively in a complex prefactor $(1+i\Omega)\partial_t\psi = \cdots$ on the right-hand side of Eq. (B1), which models frequency-dependent pumping [36] or energy relaxation [35]]; the coefficient $K_c = \hbar^2/(2m_{\rm LP})$ is related to the effective mass of the lower polariton (LP) branch; then, the mean-field healing length can be expressed in terms of the experimental parameters in Eq. (B1) as

$$\xi_0 = \frac{\hbar}{\sqrt{2m_{\rm LP}u_c(\frac{P}{\gamma_l} - \frac{\gamma_R}{R})}}. \quad \text{(B10)}$$

The healing length sets a natural scale for the KPZ crossover length L_* [Eq. (8)]. In order to compare the latter to the spatial extent L of the condensate (as explained in the main text, only for $L > L_*$ can the KPZ crossover be observed), we have to express the bare dimensionless nonlinearity g_0 [Eq. (9)] in terms of the parameters of the microscopic model. This expression can be obtained by inserting the relations

$$\bar{u} = \frac{2m_{\rm LP}u_c}{\hbar^2},$$
$$\bar{\gamma} = \frac{R\gamma_l}{\gamma_R u_c},$$
$$x = \frac{PR}{\gamma_R\gamma_l} - 1, \quad \text{(B11)}$$

in Eq. (9).

[1] J. Toner and Y. Tu, *Long-Range Order in a Two-Dimensional Dynamical XY Model: How Birds Fly Together*, Phys. Rev. Lett. **75**, 4326 (1995).

[2] N. D. Mermin and H. Wagner, *Absence of Ferromagnetism or Antiferromagnetism in One or Two Dimensional Isotropic Heisenberg Models*, Phys. Rev. Lett. **17**, 1133 (1966).

[3] I. Carusotto and C. Ciuti, *Quantum Fluids of Light*, Rev. Mod. Phys. **85**, 299 (2013).

[4] We will throughout this paper follow the literature on the equilibrium Kosterlitz-Thouless transition [55,56] and refer to an algebraically ordered phase as a "superfluid phase," even though the nature of its response (i.e., whether or not it has a well-defined superfluid density) remains an open question for future investigation. We note, however, that it has been shown by Keeling [57] that, within the Gaussian approximation, superfluid response persists in the presence of drive and dissipation in two dimensions, concomitant with algebraically decaying correlations in the same approximation. Since the anisotropic KPZ equation approaches a Gaussian fixed point in the algebraically ordered phase, it seems reasonable to expect superfluid response in this phase.

[5] A. Mitra, S. Takei, Y. Kim, and A. Millis, *Nonequilibrium Quantum Criticality in Open Electronic Systems*, Phys. Rev. Lett. **97**, 236808 (2006).

[6] S. Gopalakrishnan, B. L. Lev, and P. M. Goldbart, *Atom-Light Crystallization of Bose-Einstein Condensates in Multimode Cavities: Nonequilibrium Classical and Quantum Phase Transitions, Emergent Lattices, Supersolidity, and Frustration*, Phys. Rev. A **82**, 043612 (2010).

[7] E. D. Torre, E. Demler, T. Giamarchi, and E. Altman, *Dynamics and Universality in Noise-Driven Dissipative Systems*, Phys. Rev. B **85**, 184302 (2012).

[8] E. G. D. Torre, S. Diehl, M. D. Lukin, S. Sachdev, and P. Strack, *Keldysh Approach for Nonequilibrium Phase Transitions in Quantum Optics: Beyond the Dicke Model in Optical Cavities*, Phys. Rev. A **87**, 023831 (2013).

[9] L. M. Sieberer, S. D. Huber, E. Altman, and S. Diehl, *Dynamical Critical Phenomena in Driven-Dissipative Systems*, Phys. Rev. Lett. **110**, 195301 (2013).

[10] L. M. Sieberer, S. D. Huber, E. Altman, and S. Diehl, *Nonequilibrium Functional Renormalization for Driven-Dissipative Bose-Einstein Condensation*, Phys. Rev. B **89**, 134310 (2014).

[11] G. I. Sivashinsky, *Nonlinear Analysis of Hydrodynamic Instability in Laminar Flames—I. Derivation of Basic Equations*, Acta Astronaut. **4**, 1177 (1977).

[12] Y. Kuramoto, *Chemical Oscillations, Waves, and Turbulence*, Springer Series in Synergetics (Springer, Berlin, 1984).

[13] G. Grinstein, D. Mukamel, R. Seidin, and C. H. Bennett, *Temporally Periodic Phases and Kinetic Roughening*, Phys. Rev. Lett. **70**, 3607 (1993).

[14] G. Grinstein, C. Jayaprakash, and R. Pandit, *Conjectures about Phase Turbulence in the Complex Ginzburg-Landau Equation*, Physica (Amsterdam) **90D**, 96 (1996).

[15] M. Kardar, G. Parisi, and Y.-C. Zhang, *Dynamic Scaling of Growing Interfaces*, Phys. Rev. Lett. **56**, 889 (1986).

[16] D. E. Wolf, *Kinetic Roughening of Vicinal Surfaces*, Phys. Rev. Lett. **67**, 1783 (1991).

[17] J. Kasprzak, M. Richard, S. Kundermann, A. Baas, P. Jeambrun, J. M. J. Keeling, F. M. Marchetti, M. H. Szymanska, R. Andre, J. L. Staehli, V. Savona, P. B. Littlewood, B. Deveaud, and L. S. Dang, *Bose-Einstein Condensation of Exciton Polaritons*, Nature (London) **443**, 409 (2006).

[18] R. Balili, V. Hartwell, D. Snoke, L. Pfeiffer, and K. West, *Bose-Einstein Condensation of Microcavity Polaritons in a Trap*, Science **316**, 1007 (2007).

[19] H. Deng, G. S. Solomon, R. Hey, K. H. Ploog, and Y. Yamamoto, *Spatial Coherence of a Polariton Condensate*, Phys. Rev. Lett. **99**, 126403 (2007).

[20] G. Roumpos, M. Lohse, W. H. Nitsche, J. Keeling, M. H. Szymanska, P. B. Littlewood, A. Löffler, S. Höfling, L. Worschech, A. Forchel, and Y. Yamamoto, *Power-Law Decay of the Spatial Correlation Function in Exciton-Polariton Condensates*, Proc. Natl. Acad. Sci. U.S.A. **109**, 6467 (2012).

[21] V. V. Belykh, N. N. Sibeldin, V. D. Kulakovskii, M. M. Glazov, M. A. Semina, C. Schneider, S. Höfling, M. Kamp, and A. Forchel, *Coherence Expansion and Polariton Condensate Formation in a Semiconductor Microcavity*, Phys. Rev. Lett. **110**, 137402 (2013).

[22] W. H. Nitsche, N. Y. Kim, G. Roumpos, C. Schneider, M. Kamp, S. Hoefling, A. Forchel, and Y. Yamamoto, *Algebraic Order and the Berezinskii-Kosterlitz-Thouless Transition in an Exciton-Polariton Gas*, arXiv:1401.0756.

[23] A. Imamoglu, R. J. Ram, S. Pau, and Y. Yamamoto, *Nonequilibrium Condensates and Lasers without Inversion: Exciton-Polariton Lasers*, Phys. Rev. A **53**, 4250 (1996).

[24] I. A. Shelykh, A. V. Kavokin, Y. G. Rubo, T. C. H. Liew, and G. Malpuech, *Polariton Polarization-Sensitive Phenomena in Planar Semiconductor Microcavities*, Semicond. Sci. Technol. **25**, 013001 (2010).

[25] A. Chiocchetta and I. Carusotto, *Non-equilibrium Quasi-condensates in Reduced Dimensions*, Europhys. Lett. **102**, 67007 (2013).

[26] M. J. Hartmann, F. G. S. L. Brandao, and M. B. Plenio, *Quantum Many-Body Phenomena in Coupled Cavity Arrays*, Laser Photonics Rev. **2**, 527 (2008).

[27] A. A. Houck, H. E. Tureci, and J. Koch, *On-Chip Quantum Simulation with Superconducting Circuits*, Nat. Phys. **8**, 292 (2012).

[28] I. Bloch, J. Dalibard, and W. Zwerger, *Many-Body Physics with Ultracold Gases*, Rev. Mod. Phys. **80**, 885 (2008).

[29] M. Cross and P. Hohenberg, *Pattern Formation outside of Equilibrium*, Rev. Mod. Phys. **65**, 851 (1993).

[30] I. S. Aranson and L. Kramer, *The World of the Complex Ginzburg-Landau Equation*, Rev. Mod. Phys. **74**, 99 (2002).

[31] M. Wouters and I. Carusotto, *Excitations in a Nonequilibrium Bose-Einstein Condensate of Exciton Polaritons*, Phys. Rev. Lett. **99**, 140402 (2007).

[32] J. Keeling and N. G. Berloff, *Spontaneous Rotating Vortex Lattices in a Pumped Decaying Condensate*, Phys. Rev. Lett. **100**, 250401 (2008).

[33] R. Graham and T. Tel, *Steady-State Ensemble for the Complex Ginzburg-Landau Equation with Weak Noise*, Phys. Rev. A **42**, 4661 (1990).

[34] The gradient terms shown are the only ones at second order in gradients allowed in systems that have inversion symmetry about either the $x$or the y axis; we will restrict our discussion here to such systems.

[35] M. Wouters, T. C. H. Liew, and V. Savona, *Energy Relaxation in One-Dimensional Polariton Condensates*, Phys. Rev. B **82**, 245315 (2010).

[36] M. Wouters and I. Carusotto, *Superfluidity and Critical Velocities in Nonequilibrium Bose-Einstein Condensates*, Phys. Rev. Lett. **105**, 020602 (2010).

[37] In reality, the system might also be coupled to a particle-conserving bath, such as phonons in the solid, which we have not included. While such a coupling is irrelevant in the RG sense, if it is strong and induces fast relaxation toward the bath equilibrium, it may renormalize the parameters of the stochastic equation of motion (1). However, such a bath will not restore detailed balance, and none of the universal results presented here will change.

[38] R. Graham, in *Quantum Statistics in Optics and Solid-State Physics*, Springer Tracts in Modern Physics Vol. 66 (Springer-Verlag, Berlin, 1973), p. 1–97.

[39] U. Deker and F. Haake, *Fluctuation-Dissipation Theorems for Classical Processes*, Phys. Rev. A **11**, 2043 (1975).

[40] U. C. Täuber, in *Ageing and the Glass Transition, edited by Malte Henkel, Michel Pleimling, and Roland Sanctuary*, Lecture Notes in Physics Vol. 716 (Springer-Verlag, Berlin, 2007), p. 295–348.

[41] P. Hohenberg and B. Halperin, *Theory of Dynamic Critical Phenomena*, Rev. Mod. Phys. **49**, 435 (1977).

[42] Y. M. Kagan, B. V. Svistunov, and G. V. Shlyapnikov, *Erratum: Kinetics of Bose Condensation in an Interacting Bose Gas [Sov. Physics - JETP 74, 279-285 (February 1992)]*, J. Exp. Theor. Phys. **75**, 387 (1992).

[43] More precisely, the KPZ equation describes nonequilibrium conditions in $d > 1$. In fact, in one dimension, it can be mapped "accidentally" onto the noisy Burgers equation and thus may also occur in an equilibrium context. An explicit example has been identified in Ref. [58].

[44] L. Chen and J. Toner, *Universality for Moving Stripes: A Hydrodynamic Theory of Polar Active Smectics*, Phys. Rev. Lett. **111**, 088701 (2013).

[45] K. G. Lagoudakis, M. Wouters, M. Richard, A. Baas, I. Carusotto, R. André, L. S. Dang, and B. Deveaud-Plédran, *Quantized Vortices in an Exciton-Polariton Condensate*, Nat. Phys. **4**, 706 (2008).

[46] B. Nelsen, G. Liu, M. Steger, D. W. Snoke, R. Balili, K. West, and L. Pfeiffer, *Dissipationless Flow and Sharp Threshold of a Polariton Condensate with Long Lifetime*, Phys. Rev. X **3**, 041015 (2013).

[47] T. Jacqmin, I. Carusotto, I. Sagnes, M. Abbarchi, D. D. Solnyshkov, G. Malpuech, E. Galopin, A. Lemaître, J. Bloch, and A. Amo, *Direct Observation of Dirac Cones and a Flatband in a Honeycomb Lattice for Polaritons*, Phys. Rev. Lett. **112**, 116402 (2014).

[48] A. Dreismann, P. Cristofolini, R. Balili, G. Christmann, F. Pinsker, N. G. Berloff, Z. Hatzopoulos, P. G. Savvidis, and J. J. Baumberg, *Coupled Counterrotating Polariton Condensates in Optically Defined Annular Potentials*, Proc. Natl. Acad. Sci. U.S.A. **111**, 8770 (2014).

[49] M. P. A. Fisher and G. Grinstein, *Nonlinear Transport and $1/f^\alpha$ Noise in Insulators*, Phys. Rev. Lett. **69**, 2322 (1992).

[50] D. N. Krizhanovskii, D. Sanvitto, I. A. Shelykh, M. M. Glazov, G. Malpuech, D. D. Solnyshkov, A. Kavokin, S. Ceccarelli, M. S. Skolnick, and J. S. Roberts, *Rotation of the Plane of Polarization of Light in a Semiconductor Microcavity*, Phys. Rev. B **73**, 073303 (2006).

[51] L. Klopotowski, M. D. Martín, A. Amo, L. Viña, I. A. Shelykh, M. M. Glazov, G. Malpuech, A. V. Kavokin, and R. André, *Optical Anisotropy and Pinning of the Linear Polarization of Light in Semiconductor Microcavities*, Solid State Commun. **139**, 511 (2006).

[52] G. Malpuech, D. D. Solnyshkov, H. Ouerdane, M. M. Glazov, and I. Shelykh, *Bose Glass and Superfluid Phases of Cavity Polaritons*, Phys. Rev. Lett. **98**, 206402 (2007).

[53] V. Savona, *Effect of Interface Disorder on Quantum Well Excitons and Microcavity Polaritons*, J. Phys. Condens. Matter **19**, 295208 (2007).

[54] A. Janot, T. Hyart, P. R. Eastham, and B. Rosenow, *Superfluid Stiffness of a Driven Dissipative Condensate with Disorder*, Phys. Rev. Lett. **111**, 230403 (2013).

[55] V. L. Berezinskii, *Destruction of Long-Range Order in One-Dimensional and Two-Dimensional Systems Possessing a Continuous Symmetry Group. II. Quantum Systems*, Sov. Phys. JETP **34**, 610 (1972).

[56] J. M. Kosterlitz and D. J. Thouless, *Ordering, Metastability and Phase Transitions in Two-Dimensional Systems*, J. Phys. C **6**, 1181 (1973).

[57] J. Keeling, *Superfluid Density of an Open Dissipative Condensate*, Phys. Rev. Lett. **107**, 080402 (2011).

[58] M. Kulkarni and A. Lamacraft, *Finite-Temperature Dynamical Structure Factor of the One-Dimensional Bose Gas: From the Gross-Pitaevskii Equation to the Kardar-Parisi-Zhang Universality Class of Dynamical Critical Phenomena*, Phys. Rev. A **88**, 021603 (2013).