Erratum: Josephson-Anderson Relation and the Classical D'Alembert Paradox [Phys. Rev. X 11, 031054 (2021)]

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We report errors in the derivation in the paper. Although the main result, the detailed Josephson-Anderson relation in Eq. (3.50), remains correct, the derivation contains, in fact, two significant errors which exactly compensate each other. Both errors in the published derivation involve erroneous neglect of pressure forces at infinity. In addition to correcting the flawed derivations, we also take the opportunity in this Erratum to point out briefly the close connections of the Josephson-Anderson relation to prior results in the literature on classical incompressible fluids, which we discovered only subsequent to publication of the paper.

The main result of the paper and of this Erratum is a formula for the power injected into a viscous incompressible fluid at rest and filling all of space by a body moving through it at a prescribed velocity $-\mathbf{V}(t)$ or, equivalently, in the body frame, the power that is injected by a body at rest into a fluid moving past it with velocity $\mathbf{V}(t)$ at infinity. The paper considered for simplicity the case of constant velocity $\mathbf{V}(t) = \mathbf{V}$, except for a brief comment in Ref. [91] of the paper, so that the power injected was due solely to the drag force $\mathbf{F}_\omega$ arising from rotational fluid motions. In this Erratum, we consider the more general case, since it permits us to discuss more clearly the relations with prior results. Note that, in the case of general translation motion, the theorem of d'Alembert [1,2] does not apply and there is also an instantaneous force $\mathbf{F}_\phi$ exerted by the body even on the potential flow with velocity field $\mathbf{u}_\phi = \nabla\phi$. For example, see Ref. [3], Sec. VIII. 3, where it is shown that this latter force is given as the time derivative $\mathbf{F}_\phi = d\mathbf{I}_\phi/dt$ of an impulse

$$\mathbf{I}_\phi = -\int_{\partial B} \phi\hat{\mathbf{n}}dA \tag{E1}$$

with $\hat{\mathbf{n}}$ the unit normal vector on the body surface ∂B pointing into the fluid domain Ω. In that case, the total instantaneous force exerted by the body on the fluid is given exactly by $\mathbf{F} = \mathbf{F}_\omega + \mathbf{F}_\phi$, corresponding to the exact decomposition $\mathbf{u} = \mathbf{u}_\phi + \mathbf{u}_\omega$ and the separate momentum balances for each [3,4]. However, because $\mathbf{I}_\phi(t)$ remains bounded in time for a bounded velocity $\mathbf{V}(t)$, the time-averaged force from potential flow vanishes, $\langle \mathbf{F}_\phi \rangle = \mathbf{0}$, and only the rotational fluid motions provide a long-time effective source of drag.

The first erroneous result in the paper was Eq. (3.44), which claimed that

$$\mathbf{F}_\omega = \frac{d\mathbf{P}_\omega}{dt}, \tag{E2}$$

where $\mathbf{P}_\omega = \rho \int_\Omega \mathbf{u}_\omega dV$ is the total momentum in the rotational fluid motions with velocity $\mathbf{u}_\omega$. This result was derived from Eq. (3.16) in the paper, or

$$\partial_t \mathbf{u}_\omega = \mathbf{u} \times \boldsymbol{\omega} - \nu \nabla \times \boldsymbol{\omega} - \nabla h_\omega, \tag{E3}$$

where $h_\omega = p_\omega + \frac{1}{2}|\mathbf{u}_\omega|^2 + \mathbf{u}_\omega \cdot \mathbf{u}_\phi$ is the total pressure of the rotational motions. [Note that the original Eq. (3.16) in the paper also had a typo $+\nabla h_\omega$ for $-\nabla h_\omega$, although this did not affect the final result.] The result (3.44) in the paper was obtained by integrating Eq. (E3) over space to obtain

$$\frac{d\mathbf{P}_\omega}{dt} = \mathbf{F}_\omega - \rho \lim_{R\to\infty} \int_{S_R} \hat{\mathbf{x}} h_\omega dA, \tag{E4}$$

where S_R is the sphere of radius R centered at the body and $\hat{\mathbf{x}} = \mathbf{x}/r$ is the radially outward unit vector. It was then argued in the paper from the Poisson equation for h_ω that $h_\omega = O(r^{-3})$ because the source term is $O(r^{-5})$. This argument is, however, incorrect. A simple counterexample is the potential flow $\mathbf{u}_\phi$, whose total pressure obtained from the Bernoulli relation scales as $h_\phi \sim -\dot{\mathbf{I}}_\phi(t)\cdot\hat{\mathbf{x}}/4\pi\rho r^2$ for $r\to\infty$, even though the source term in its Poisson equation is $O(r^{-6})$. The correct result for rotational flow is similarly obtained from the Bernoulli relation $h_\omega = -\dot{\phi}_\omega + c(t)$ with

$$\phi_\omega \sim -\frac{\mathbf{I}_\omega(t)\cdot\hat{\mathbf{x}}}{4\pi\rho r^2} \tag{E5}$$

since the vorticity decays rapidly at infinity, $\omega = O(r^{-4})$, so that the asymptotic rotational flow is itself potential, $\mathbf{u}_\omega = \nabla\phi_\omega$. See Ref. [5], Eq. (19). Note that $\mathbf{I}_\omega$ is the impulse of rotational motions defined by Eq. (3.29) in the paper but now multiplied by mass density ρ so that it has units of momentum. Using these results for h_ω in Eq. (E4) gives

$$\frac{d\mathbf{P}_\omega}{dt} = \mathbf{F}_\omega - \frac{1}{3}\frac{d\mathbf{I}_\omega}{dt}. \tag{E6}$$

However, it was shown in Appendix B of the paper that $\mathbf{P}_\omega = \frac{2}{3}\mathbf{I}_\omega$ and the same result was obtained in Ref. [5], Eq. (27), using an identical argument. Substituting this result into Eq. (E6) yields

$$\mathbf{F}_\omega = \frac{d\mathbf{I}_\omega}{dt} \tag{E7}$$

rather than the erroneous result (E2) in Eq. (3.44) of the paper. Of course, Eq. (E7) is nothing other than the well-known relationship between force and fluid impulse [3,5].

The second erroneous result in the paper was Eq. (3.47), which claimed that

$$\frac{dE_{\rm int}}{dt} = +\rho\int_\Omega \mathbf{u}_\phi\cdot(\mathbf{u}\times\boldsymbol{\omega} - \nu\nabla\times\boldsymbol{\omega})dV, \tag{E8}$$

where $E_{\rm int} = \rho\int_\Omega \mathbf{u}_\phi\cdot\mathbf{u}_\omega dV$ is the interaction energy between potential and rotational flow. This result was derived from Eq. (3.8) in the paper, or

$$\partial_t(\mathbf{u}_\phi\cdot\mathbf{u}_\omega) + \nabla\cdot(h_\omega\mathbf{u}_\phi + h_\phi\mathbf{u}_\omega) = \mathbf{u}_\phi\cdot(\mathbf{u}\times\boldsymbol{\omega} - \nu\nabla\times\boldsymbol{\omega}) + \mathbf{A}(t)\cdot\mathbf{u}_\omega, \tag{E9}$$

where we have now included the term from the acceleration $\mathbf{A}(t) = \dot{\mathbf{V}}(t)$, which vanished for the constant-velocity problem discussed in the paper. Integrating Eq. (E8) over Ω and multiplying by ρ then yields

$$\frac{dE_{\rm int}}{dt} = +\rho\int_\Omega \mathbf{u}_\phi\cdot(\mathbf{u}\times\boldsymbol{\omega} - \nu\nabla\times\boldsymbol{\omega})dV + \mathbf{A}(t)\cdot\mathbf{P}_\omega(t) - \rho\mathbf{V}(t)\cdot\lim_{R\to\infty}\int_{S_R}\hat{\mathbf{x}}h_\omega dA, \tag{E10}$$

where the last term used $\mathbf{u}_\phi \sim \mathbf{V}(t) + O(r^{-3})$. Exploiting again the Bernoulli relation for h_ω and the asymptotic formula (E5) yields finally

$$\frac{dE_{\rm int}}{dt} = +\rho\int_\Omega \mathbf{u}_\phi\cdot(\mathbf{u}\times\boldsymbol{\omega} - \nu\nabla\times\boldsymbol{\omega})dV + \mathbf{A}(t)\cdot\mathbf{P}_\omega(t) - \frac{1}{3}\dot{\mathbf{I}}_\omega(t)\cdot\mathbf{V}(t). \tag{E11}$$

This result differs from Eq. (3.47) in the paper because of the term proportional to $\mathbf{A}(t)$ but most essentially because of the term proportional to $\dot{\mathbf{I}}_\omega(t)$. The latter contribution arises from the rotational pressure force at infinity, which was erroneously assumed to vanish in the paper. The correction in Eq. (E11) invalidates also a statement made after Eq. (3.19) in the paper, that the sum of $E_{\rm int}(t)$ and $E_\omega(t) = \frac{1}{2}\rho\int_\Omega|\mathbf{u}_\omega|^2 dV$ "is conserved in the limit $\nu\to 0$, as long as solutions stay smooth in the limit." This statement is very generally untrue.

However, despite these two errors in the derivation of the paper, the main result in its Eq. (3.50) remains valid, by essentially the original argument. The derivation in the paper used a basic result, Eq. (3.42) or $E_{\rm int}(t) = \mathbf{P}_\omega(t)\cdot\mathbf{V}(t)$, which by time differentiation yields

$$\frac{dE_{\rm int}}{dt} = \mathbf{P}_\omega(t)\cdot\mathbf{A}(t) + \dot{\mathbf{P}}_\omega(t)\cdot\mathbf{V}(t). \tag{E12}$$

Substituting Eq. (E12) for $dE_{\rm int}(t)/dt$ into Eq. (E11) and using Eq. (E6) for $d\mathbf{P}_\omega(t)/dt$ then gives

$$\mathbf{A}(t)\cdot\mathbf{P}_\omega(t) + \left(\mathbf{F}_\omega - \frac{1}{3}\dot{\mathbf{I}}_\omega\right)\cdot\mathbf{V}(t) = \rho\int_\Omega \mathbf{u}_\phi\cdot(\mathbf{u}\times\boldsymbol{\omega} - \nu\boldsymbol{\nabla}\times\boldsymbol{\omega})dV + \mathbf{A}(t)\cdot\mathbf{P}_\omega(t) - \frac{1}{3}\dot{\mathbf{I}}_\omega(t)\cdot\mathbf{V}(t). \tag{E13}$$

None of the terms proportional to $\mathbf{A}(t)$ or to $\dot{\mathbf{I}}_\omega(t)$ were present in the paper, but, most crucially, they all cancel. Thus, the final result

$$-\mathbf{F}_\omega\cdot\mathbf{V}(t) = -\rho\int_\Omega \mathbf{u}_\phi\cdot(\mathbf{u}\times\boldsymbol{\omega} - \nu\boldsymbol{\nabla}\times\boldsymbol{\omega})dV \tag{E14}$$

is identical to Eq. (3.50), for the case with $\mathbf{V}(t) = \mathbf{V}$ and $\mathbf{F}_\omega = \mathbf{F}$ discussed in the paper.

Since the original publication, we have become aware of a substantial body of closely related work. An especially valuable review is Ref. [6], which discusses prior work of Burgers, Lighthill, Kambe, Howe, Wu, Quartapelle, Napolitano, and others. We can thus refer the reader to Ref. [6] for a nearly exhaustive discussion of this prior literature. We note one paper of Chang [7], which was apparently overlooked in Ref. [6], whose Eq. (E11) gives a formula for the pressure force only. However, this formula is of mixed type, with fluid velocity $\mathbf{u}$ in the body frame of reference and with pressure P and potential ϕ in the fluid rest frame. If this formula is transformed consistently to the body frame, it yields readily our detailed Josephson-Anderson relation, Eq. (3.50). Perhaps the most comprehensive results are those of Howe in Ref. [8], who considers a rigid body in arbitrary translational and rotational motion and who derives formulas for full vector forces and moments, generalizing ours for the drag force along the direction of motion. It is worth quoting the main force formula (2.11) derived in Ref. [8], which represents F_i, the ith component of the force on a body moving rigidly with velocity $\bar{\mathbf{U}} = \mathbf{U} + \Omega\times(\mathbf{x}-\mathbf{x}_0)$, as

$$F_i = \frac{\partial}{\partial t}\left(\rho\int_{\partial B(t)}\phi_i(\hat{\mathbf{n}}\cdot\bar{\mathbf{U}})dA\right) + \rho\int_{\partial B(t)}\left(\frac{\partial X_i}{\partial t} + \mathbf{u}\cdot\boldsymbol{\nabla}X_i\right)(\hat{\mathbf{n}}\cdot\bar{\mathbf{U}})dA - \rho\int_{\Omega(t)}\boldsymbol{\nabla}X_i\cdot(\mathbf{u}\times\boldsymbol{\omega} - \nu\boldsymbol{\nabla}\times\boldsymbol{\omega})dV. \tag{E15}$$

Unlike our result in the body frame, Howe's expression is written in the fluid rest frame, so that the integration regions $\partial B(t)$ and $\Omega(t)$ are explicitly time dependent. However, quoting from Ref. [8], one may regard $X_i = x_i - x_{0i} - \phi_i$ "as the velocity potential of irrotational flow past the body (imagined to be temporarily at rest at $\mathbf{x}_0$) which has unit speed in the i direction at large distances from the body." Thus, V_iX_i is the velocity potential considered in our paper, and from this fact one may readily derive our Eq. (3.50) as a special case of Howe's more general result. It should be emphasized, however, that our proof is considerably simpler and yields also Howe's more general formulas. In fact, our key intermediate result $E_{\rm int}(t) = \mathbf{P}_\omega(t)\cdot\mathbf{V}(t)$ can be generalized to Howe's setting as

$$P_i = \rho\int_{\Omega(t)}\boldsymbol{\nabla}X_i\cdot\mathbf{u}dV + \rho\int_{\partial B(t)}X_i(\hat{\mathbf{n}}\cdot\bar{\mathbf{U}})dA, \tag{E16}$$

where now $P_i = \rho\int_{\Omega(t)}u_i dV$ is the i component of the total fluid momentum in the reference frame with the fluid at rest at infinity. In that case, time differentiation of Eq. (E16) readily yields Howe's general result (E15) with the observation that, in the fluid frame of reference, the impulse $\mathbf{I}$ in the far-field formula for h analogous to Eq. (E5) must be replaced by $\mathbf{J} = \mathbf{I} + \rho|B|\mathbf{V}(t)$, where $|B|$ is the volume of the body [3,5]. In a following work, we shall present this derivation in detail and discuss more fully the connections of the Josephson-Anderson relation with prior work on classical fluids.

It is curious that all of the work on classical fluids cited in Ref. [6] made no reference to the corresponding work on quantum superfluids discussed in our paper, and vice versa. The two literatures seem to have developed entirely in parallel, until their close relations were pointed out in our work. In particular, the detailed Josephson-Anderson relation derived by Huggins [9] for internal flows seems to have been overlooked in the classical fluids literature, despite being a close analog of the results reviewed in Ref. [6] for external flows. Huggins' relation for internal flows has recently been generalized to streamwise periodic geometries convenient for numerical simulations [10] and will be investigated in future computational studies. Likewise, our paper seems to have been the first to point out that validity of the Josephson-Anderson relation persists in the infinite Reynolds limit and, thus, is connected with Onsager's ideal turbulence theory. This infinite Reynolds number limit has now been established in Ref. [11] with mathematical rigor.

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