Classification of Symmetry-Enriched Topological Quantum Spin Liquids

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We present a systematic framework to classify symmetry-enriched topological quantum spin liquids in two spatial dimensions. This framework can deal with all topological quantum spin liquids, which may be either Abelian or non-Abelian and chiral or nonchiral. It can systematically treat a general symmetry, which may include both lattice symmetry and internal symmetry, may contain antiunitary symmetry, and may permute anyons. The framework applies to all types of lattices and can systematically distinguish different lattice systems with the same symmetry group using their quantum anomalies, which are sometimes known as Lieb-Schultz-Mattis anomalies. We apply this framework to classify $\mathrm{U}(1)_{2N}$ chiral states and non-Abelian Ising$^{(\nu)}$ states enriched by a $p6\times \mathrm{SO}(3)$ or $p4\times \mathrm{SO}(3)$ symmetry and $\mathbb{Z}_N$ topological orders and $\mathrm{U}(1)_{2N}\times \mathrm{U}(1)_{-2N}$ topological orders enriched by a $p6m\times \mathrm{SO}(3)\times \mathbb{Z}_2^T$, $p4m\times \mathrm{SO}(3)\times \mathbb{Z}_2^T$, $p6m\times \mathbb{Z}_2^T$, or $p4m\times \mathbb{Z}_2^T$ symmetry, where $p6$, $p4$, $p6m$, and $p4m$ are lattice symmetries while SO(3) and $\mathbb{Z}_2^T$ are spin rotation and time-reversal symmetries, respectively. In particular, we identify symmetry-enriched topological quantum spin liquids that are not easily captured by the usual parton-mean-field approach, including examples with the familiar $\mathbb{Z}_2$ topological order.

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I. INTRODUCTION

Topological quantum spin liquids, which are more formally referred to as bosonic topological orders, are exotic gapped quantum phases of matter with long-range entanglement beyond the phenomenon of spontaneous symmetry breaking [1]. In two space dimensions, they can host anyons, i.e., quasiparticle excitations that are neither bosons nor fermions [2]. Besides being fundamentally interesting, they are also potential platforms for fault-tolerant quantum computation [3,4].

Roughly speaking, in the absence of symmetries, the universal properties of a topological quantum spin liquid are encoded in the properties of its anyons, such as their fusion rules and statistics. We refer to these properties as the topological properties of a topological quantum spin liquid. In the presence of symmetries, there can be interesting interplay between these topological properties and the symmetries. In particular, even with fixed topological properties, there can be sharply distinct topological quantum spin liquids that cannot evolve into each other without breaking the symmetries or encountering a quantum phase transition. These are known as different symmetry-enriched topological quantum spin liquids.

The goal of this paper is to classify symmetry-enriched topological quantum spin liquids in two space dimensions. That is, given the topological and symmetry properties, we would like to understand which symmetry-enriched topological quantum spin liquids are possible. This problem is first of fundamental conceptual interest, as understanding different types of quantum matter is one of the central goals of condensed matter physics. Moreover, although topological quantum spin liquids have been identified in certain lattice models and small-sized quantum simulators, its realization and detection in macroscopic quantum materials remains elusive [5]. The knowledge of which symmetry-enriched topological quantum spin liquids are possible is helpful for identifying the correct observable signatures of them, thus paving the way to the realization and detection of these interesting quantum phases in the future.

In the previous condensed matter literature, there are two widely used approaches to classify symmetry-enriched topological quantum spin liquids, and most of the other approaches can be viewed as variations or generalizations of them. The first approach is based on parton mean fields and projective symmetry groups, which starts by representing the microscopic degrees of freedom (such as spins) via certain fractional degrees of freedom and then studies the possible projective representations of the symmetries carried by these fractional degrees of freedom [6]. The advantages of this approach include its simplicity and its intimate connection to the microscopic degrees of freedom of the system.

However, this approach is not perfect. One of the main disadvantages of this approach is that it is not easily applicable to general topological quantum spin liquids. For example, it is inconvenient to apply this approach to classify symmetry-enriched $\mathbb{Z}_N$ topological orders with $N > 2$, because in this case the partons are interacting and a noninteracting parton mean field is inadequate. Also, it is often challenging to use this approach to study topological quantum spin liquids where anyons are permuted in a complicated manner by symmetries [7]. Moreover, even for quantum spin liquids that are often assumed to be captured by parton mean fields, some of their symmetry enrichment patterns may not be captured by any parton mean field. Reference [8] presented such a phenomenon for the gapless Dirac spin liquid, and in Sec. VIII we find that this phenomenon can occur even for the familiar $\mathbb{Z}_2$ topological quantum spin liquid. Another disadvantage of this approach is that the projective representations carried by the fractional degrees of freedom may not be the same as the symmetry properties of the physical anyons, which sometimes require one more nontrivial step to obtain. For examples, see Refs. [9,10].

The second approach is based on the categorical theoretic description of the topological quantum spin liquids and studies how the category theory (i.e., a powerful mathematical theory) corresponding to a topological quantum spin liquid can be consistently extended to include a symmetry group [7,11]. However, while this approach is fully general, it also has its disadvantages. Unlike the first approach, it has a relatively weaker connection with the microscopic properties of the physical system, and it is particularly inconvenient when applied to systems with lattice symmetries, which are often physically important. Specifically, within this approach all symmetry properties of the physical system are assumed to be captured by its symmetry group. However, such a description of the symmetries is inadequate for many purposes. For example, spin-$1/2$ systems defined on a triangular lattice, kagome lattice, and honeycomb lattice can all have the same symmetry group, say, the SO(3) spin rotational symmetry and the $p6m$ lattice symmetry. But they are physically distinct, and any symmetry-enriched topological quantum spin liquid that can emerge in one of these three systems cannot emerge in the other two [8,12]. A physically relevant classification of symmetry-enriched topological quantum spin liquids should take into account this distinction, which reflects some microscopic symmetry properties beyond the symmetry group. In the above example, these properties can be viewed as the lattice types.

In the present paper, we develop a framework to classify symmetry-enriched topological quantum spin liquids, which combines the advantages of the above two approaches while avoiding their disadvantages. Specifically, we use the language of category theory to directly describe the topological properties of a topological quantum spin liquid and the symmetry properties of the anyons, and we also keep track of the robust microscopic symmetry-related information of the physical system. Roughly speaking, this information includes

(i) the symmetry group, say, the SO(3) spin rotational symmetry and $p6m$ lattice symmetry;

(ii) how the microscopic degrees of freedom transform under the symmetry, say, whether they are spin-$1/2$'s or spin-1's;

(iii) the lattice type, say, whether the lattice is a triangular, kagome, or honeycomb lattice.

All three pieces of information above are taken into account in the aforementioned first approach, but the second approach usually considers only the first piece (see exceptions in Refs. [13–19] studying certain specific symmetry-enriched quantum spin liquids, but the methods therein have not been generalized to the generic case before). The reason why the other two pieces of information are physically relevant is because, as long as the symmetries are preserved, they are robust properties of the system that cannot change under perturbations [12]. Moreover, just like the symmetry group, these two pieces of information are often relatively easy to determine experimentally, and they are the basic aspects of any theoretical model. We make these concepts more precise in Sec. IV.

Ultimately, our framework takes as input (i) the topological properties of a topological quantum spin liquid and (ii) a collection of the above symmetry-related properties of a microscopic system and outputs which symmetry-enriched topological quantum spin liquids are possible. Each of these symmetry-enriched topological quantum spin liquids is characterized by the symmetry properties of its anyons. We apply this framework to classify various symmetry-enriched topological quantum spin liquids. We focus on examples with symmetry group $G = G_s \times G_{\mathrm{int}}$, where G_s is certain lattice symmetry and G_{int} is an on-site internal symmetry.

This framework relies on the state of the art in the studies of the quantum anomalies of lattice systems and topological quantum spin liquids. In particular, in Ref. [8], we and our coauthors obtained the precise characterizations of the full set of the quantum anomalies of a large class of lattice systems, which exactly encode the aforementioned robust microscopic symmetry-related information of the lattice system. These quantum anomalies are sometimes referred to as Lieb-Schultz-Mattis anomalies. Moreover, in Ref. [20], we developed a systematic framework to calculate the quantum anomaly of a generic symmetry-enriched topological quantum spin liquid.

We remark that our philosophy of encoding the robust microscopic symmetry-related information of a physical system into its quantum anomaly and using anomaly matching to classify quantum many-body states is very general. Besides classifying symmetry-enriched topological quantum spin liquids, it can also be used to classify other quantum states. In fact, we and our coauthors have applied it to classify certain symmetry-enriched gapless quantum spin liquids in Ref. [8]. Our anomaly-based philosophy can be viewed as a generalization of the theories of symmetry indicators or topological quantum chemistry, which classify band structures [21–25]. In fact, the robust microscopic symmetry-related information of a physical system we take is identical to what those theories take, but those theories apply only to weakly correlated systems that can be described by band theory, while ours applies to generic strongly correlated systems.

II. OUTLINE AND SUMMARY

The outline of the rest of the paper and summary of the main results are as follows.

(i) In Sec. III, we briefly review how to describe a symmetry-enriched topological quantum spin liquid and its quantum anomaly. The description is based on the category theory, but no previous knowledge of category theory is required.

(ii) In Sec. IV, we discuss the symmetry properties of a lattice system and how to encode them into the quantum anomaly of a lattice system.

(iii) In Sec. V, we present our general framework to classify symmetry-enriched topological quantum spin liquid, which may be Abelian or non-Abelian and chiral or nonchiral. This framework applies to topological quantum spin liquids with any symmetry, which may contain both lattice symmetry and internal symmetry, may contain antiunitary symmetry, and may permute anyons.

(iv) In Sec. VI, we apply the framework in Sec. V to classify the $\mathrm{U}(1)_{2N}$ topological quantum spin liquid enriched by a $p6\times \mathrm{SO}(3)$ or $p4\times \mathrm{SO}(3)$ symmetry, where $p6$ and $p4$ are lattice symmetries and SO(3) is the spin rotational symmetry. The results are summarized in Table V.

(v) In Sec. VII, we apply our framework in Sec. V to classify the non-Abelian $\mathrm{Ising}^{(\nu)}$ topological quantum spin liquid enriched by a $p6\times \mathrm{SO}(3)$ or $p4\times \mathrm{SO}(3)$ symmetry.

(vi) In Sec. VIII, we apply our framework in Sec. V to classify the $\mathbb{Z}_N$ topological quantum spin liquids enriched by one of these four symmetries: $p6m\times \mathrm{SO}(3)\times \mathbb{Z}_2^T$, $p4m\times \mathrm{SO}(3)\times \mathbb{Z}_2^T$, $p6m\times \mathbb{Z}_2^T$, and $p4m\times \mathbb{Z}_2^T$, where the $p6m$ and $p4m$ are lattice symmetries and SO(3) and $\mathbb{Z}_2^T$ are on-site spin rotational symmetry and time-reversal symmetry, respectively. The results are summarized in Table X. In particular, even for the simple case with $N=2$, we find many states beyond the description based on the usual parton mean field.

(vii) In Sec. IX, we apply our framework in Sec. V to classify the $\mathrm{U}(1)_{2N}\times \mathrm{U}(1)_{-2N}$ topological quantum spin liquids enriched by one of these four symmetries: $p6m\times \mathrm{SO}(3)\times \mathbb{Z}_2^T$, $p4m\times \mathrm{SO}(3)\times \mathbb{Z}_2^T$, $p6m\times \mathbb{Z}_2^T$, and $p4m\times \mathbb{Z}_2^T$. The results are summarized in Table X.

(viii) We close our paper in Sec. X.

(ix) Various appendixes contain further details. In particular, Appendix A presents a formal treatment of the connection between the data characterizing a topological order enriched by reflection symmetry and the data characterizing a topological order enriched by time-reversal symmetry. Appendix E presents the details of the symmetry fractionalization classes of the symmetry-enriched $\mathbb{Z}_2$ topological orders that are beyond the usual parton mean fields.

III. UNIVERSAL CHARACTERIZATION OF SYMMETRY-ENRICHED TOPOLOGICAL QUANTUM SPIN LIQUIDS

In this section, we review the universal characterization of symmetry-enriched topological quantum spin liquids. This characterization is divided into two parts. We first specify the topological order corresponding to the topological quantum spin liquid, which is reviewed in Sec. III A. After this, assuming the symmetry is an internal symmetry, we specify the global symmetry of the topological quantum spin liquid and how it acts on the topological order, which is reviewed in Sec. III B. We review the anomaly of a symmetry-enriched topological order in Sec. III C. Finally, we review the crystalline equivalence principle in Sec. III D, which allows us to connect a topological order with lattice symmetry to one with internal symmetry.

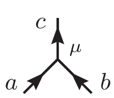
A. Topological order

The characteristic feature of a topological order is the presence of anyons, pointlike excitations that can have self-statistics other than bosonic or fermionic statistics, and nontrivial mutual braiding statistics. When multiple anyons are present, there may also be a protected degenerate state space. It is believed that the category theory can universally characterize all topological orders or the anyons therein. In this subsection, we briefly review the concepts relevant to this paper. Our review will be minimal and does not assume any knowledge of the category theory itself. For a more comprehensive review, see, e.g., Refs. [7,26,27] for a more physics-oriented introduction or Refs. [28–31] for a more mathematical treatment.

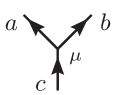
Anyons in a topological order are denoted by $a, b, c, \ldots$. A single anyon cannot be converted into a different single anyon via any local process. There is always a trivial anyon in all topological orders, obtained by performing some local operation on the ground state. Roughly speaking, in a many-body system where a topological order emerges at low energies, the quantum state is specified by two pieces of data: the global data characterizing the anyons, which cannot be changed by any local process, and the local data independent of the anyons, which can change by local operations.

For the purpose of this paper, two important properties of an anyon a are used: its quantum dimension d_a and topological spin θ_a. Suppose n anyons a are created, and the dimension of the degenerate state space scales as d_a^n when n is large. So the quantum dimension d_a effectively measures how rich the internal degree of freedom the anyon a carries. If $d_a = 1$, then a is said to be an Abelian anyon. Otherwise, it is a non-Abelian anyon. The topological spin θ_a measures the self-statistics of a. For bosons and fermions, the topological spin is ± 1. For a generic anyon, the topological spin can take other values.

Two anyons a and b can fuse into other anyons, expressed using the equation $a \times b \cong \sum_c N_{ab}^c c$, where N_{ab}^c are positive integers and c is said to be a fusion outcome of a and b. Generically, a and b may have multiple fusion outcomes, and there may also be multiple different ways to get each fusion outcome, which is why the right-hand side of the fusion equation involves a summation and N_{ab}^c can be larger than 1. Physically, fusion means that if we perform measurements only far away from the anyons a and b, we will think they look like the fusion outcome c. Diagrammatically, we can use

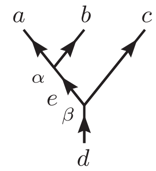


to represent a fusion process or

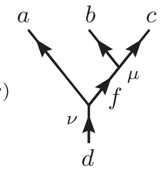


to represent a splitting process, which can be viewed as the reversed process of fusion. In the above, $\mu \in \{1, 2, \ldots, N_{ab}^c\}$, and the arrows can be viewed as the world lines of the anyons. If the trivial anyon is in the fusion product of two anyons, we say that these two anyons are antiparticles of each other, and we denote the antiparticle of a by $\bar{a}$.

When multiple anyons are present, we may imagine fusing them in different orders. Similarly, when a given anyon splits into multiple other anyons, it may also do so in different orders. For example, the two sides of the following equation represent two different orders of splitting an anyon d into three anyons: a, b, and c. Generically, the states obtained after these two processes are not identical, but they can be related by the so-called F symbol, which is a unitary matrix acting on the degenerate state space:

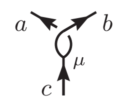


$$= \sum_{f,\mu,\nu} \left[F_d^{abc}\right]_{(e,\alpha,\beta),(f,\mu,\nu)}$$

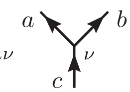


In this equation, e is a fusion outcome of a and b, where $\alpha \in \{1, \ldots N_{ab}^e\}$, d is a fusion outcome of c and e, where $\beta \in \{1, \ldots, N_{ec}^d\}$, f is a fusion outcome of b and c, where $\mu \in \{1, \ldots, N_{bc}^f\}$, and f and a also fuse into d, where $\nu \in \{1, \ldots, N_{af}^d\}$.

When anyons move around each other, the state may acquire some nontrivial braiding phase factor, and it may even be acted by a unitary matrix, in general. The statistics and braiding properties of the anyons are encoded in the R symbol, which is also a unitary matrix acting in the degenerate state space and is defined via the following diagram:



$$= \sum_{\nu} \left[R_c^{ab}\right]_{\mu\nu}$$



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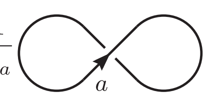
The topological spin can be expressed using the R symbol via

$$\theta_a = \theta_{\bar{a}} = \sum_{c,\mu} \frac{d_c}{d_a} \left[R_c^{aa}\right]_{\mu\mu} = \frac{1}{d_a}$$



The F and R symbols satisfy strong constraints and have some "gauge" freedom; i.e., two sets of $\{F, R\}$ data related by certain gauge transformations are physically equivalent. We remark that the F and R symbols can all be defined microscopically [32].

B. Global symmetry

In the presence of global symmetry, anyons can display interesting phenomena including (i) anyon permutation and (ii) symmetry fractionalization. We review the concepts relevant to this paper below, and more comprehensive reviews can be found in, e.g., Refs. [7,29,33]. In this subsection and the next, we assume that all symmetries are internal symmetries; i.e., they do not move the locations of the degrees of freedom. We comment on how to deal with the case with lattice symmetry in Sec. III D.

Before specifying any microscopic symmetry of interest, it is useful to first discuss the abstract topological symmetry group of a topological order, whose elements can be viewed as invertible maps that take one anyon into another, so that the fusion properties of the topological order are invariant. For unitary (antiunitary) topological symmetry action, the braiding properties, encoded in the F and R symbols, are preserved (conjugated). Later in the paper, we see many examples of topological symmetries of various topological orders.

Now we consider a microscopic symmetry described by a group G. Suppose $R_{\mathbf{g}}$ is a symmetry action, where $\mathbf{g}$ labels an element of G. This action can change anyons into other types; for example, it change an anyon a into another anyon ${}^{\mathbf{g}}a$. We use

$$\rho_{\mathbf{g}}\colon\ a \to {}^{\mathbf{g}}a \tag{1}$$

to represent the anyon permutation pattern. Mathematically, we may say that ρ_{g} describes a group homomorphism from G to the topological symmetry group of a topological order. From here, we see why the notion of topological symmetry is important: It encodes all possible anyon permutation patterns.

However, ρ_{g} by itself is insufficient to fully characterize a symmetry-enriched topological order. Consider creating three anyons a_1, a_2, and a_3 from the ground states, such that $a_1 \times a_2 \to \bar{a}_3$; i.e., these three anyons can fuse into the trivial anyon. After separating these anyons far away from each other, there are generically $N^{\bar{a}_3}_{a_1 a_2}$ degenerate such states. What effect does $R_{\mathbf{g}}$ have when it acts on a state in this degenerate space, denoted by $|\Psi_{a_1,a_2,a_3}\rangle$?

Since the state of a topological order is specified by the two pieces of information, the global one and the local one, the symmetry localization ansatz states that [34]

$$R_{\mathbf{g}}|\Psi_{a_1,a_2,a_3}\rangle = V^{(1)}_{\mathbf{g}} V^{(2)}_{\mathbf{g}} V^{(3)}_{\mathbf{g}} U_{\mathbf{g}}({}^{\mathbf{g}}a_1, {}^{\mathbf{g}}a_2; {}^{\mathbf{g}}\bar{a}_3)|\Psi_{{}^{\mathbf{g}}a_1,{}^{\mathbf{g}}a_2,{}^{\mathbf{g}}a_3}\rangle, \tag{2}$$

where $V^{(i)}_{\mathbf{g}}$ is a local unitary operation supported only around the anyon a_i, for $i = 1, 2, 3$, and $U_{\mathrm{g}}({}^{\mathbf{g}}a_1, {}^{\mathbf{g}}a_2; {}^{\mathbf{g}}\bar{a}_3)$ is a unitary matrix with rank $N^{\bar{a}_3}_{a_1 a_2}$, which acts on the degenerate state space and describes the symmetry action on the global part of the information contained in the state. Notice that the state appearing on the right-hand side is $|\Psi_{{}^{\mathbf{g}}a_1,{}^{\mathbf{g}}a_2,{}^{\mathbf{g}}a_3}\rangle$; i.e., generically, the anyons are permuted by the symmetry.

It can be shown that the local operations V satisfy

$$\eta_{a_i}(\mathbf{g},\mathbf{h}) V^{(i)}_{\mathbf{gh}}|\Psi_{a_1,a_2,a_3}\rangle = R_{\mathbf{g}} V^{(i)}_{\mathbf{h}} R^{-1}_{\mathbf{g}} V^{(i)}_{\mathbf{g}}|\Psi_{a_1,a_2,a_3}\rangle \tag{3}$$

for a pair of group elements $\mathbf{g}$ and $\mathbf{h}$. Here, $\eta_{a_i}(\mathbf{g},\mathbf{h})$ are generically nontrivial phase factors, implying that the anyons may carry fractional charge or projective quantum number under the symmetry; i.e., there can be symmetry fractionalization [35].

For a given topological order and a symmetry group G, it turns out that the data $\{\rho_{\mathbf{g}}; U_{\mathbf{g}}(a,b;c), \eta_a(\mathbf{g},\mathbf{h})\}$ completely characterize how this topological order is enriched by the symmetry G. We often call $U_{\mathbf{g}}(a,b;c)$ the U symbol and $\eta_a(\mathbf{g},\mathbf{h})$ the η symbol. These data $\{\rho_{\mathbf{g}}; U_{\mathbf{g}}(a,b;c), \eta_a(\mathbf{g},\mathbf{h})\}$ also satisfy strong constraints and have some "gauge" freedom; i.e., two sets of $\{\rho_{\mathbf{g}}; U_{\mathbf{g}}(a,b;c), \eta_a(\mathbf{g},\mathbf{h})\}$ data related by certain gauge transformations are physically equivalent. These gauge transformations are not explicitly used in this paper, but they are summarized in, e.g., Ref. [20]. Moreover, even if two sets of data $\{\rho_{\mathbf{g}}; U_{\mathbf{g}}(a,b;c), \eta_a(\mathbf{g},\mathbf{h})\}$ are not related by a gauge transformation, they still correspond to the physical state if they are related to each other by anyon relabeling that preserves the fusion and braiding properties [7,34]; i.e., such relabeling is precisely a unitary element in the topological symmetry group [36].

For a given anyon permutation pattern, one can show that distinct symmetry fractionalization classes form a torsor over $H^2_\rho(G,\mathcal{A})$. Namely, different possible symmetry fractionalization classes can be related to each other by elements of $H^2_\rho(G,\mathcal{A})$, where $\mathcal{A}$ is an Abelian group whose group elements correspond to the Abelian anyons in this topological order and the group multiplication corresponds to the fusion of these Abelian anyons. The subscript ρ represents the permutation action of G on these Abelian anyons. In particular, given an element $[t] \in H^2_\rho(G,\mathcal{A})$, we can go from one symmetry fractionalization class with data $\eta_a(\mathbf{g},\mathbf{h})$ to another with data $\tilde{\eta}_a(\mathbf{g},\mathbf{h})$ given by

$$\tilde{\eta}_a(\mathbf{g},\mathbf{h}) = \eta_a(\mathbf{g},\mathbf{h}) M_{a,t(\mathbf{g},\mathbf{h})}, \tag{4}$$

where $t(\mathbf{g},\mathbf{h}) \in \mathcal{A}$ is a representative 2-cocyle for the cohomology class $[t]$ and $M_{a,t(\mathbf{g},\mathbf{h})} = [\theta_{a\times t(\mathbf{g},\mathbf{h})}/\theta_a \theta_{t(\mathbf{g},\mathbf{h})}]$ is the braiding statistics between a and $t(\mathbf{g},\mathbf{h})$ [37].

In the case where no anyon is permuted by any symmetry, there is always a canonical notion of a trivial symmetry fractionalization class, where $\eta_a(\mathbf{g},\mathbf{h}) = 1$ for all anyon a and all $\mathbf{g},\mathbf{h} \in G$. In this case, an element of $H^2(G,\mathcal{A})$ is sufficient to completely characterize the symmetry fractionalization class.

Later in the paper, for a symmetry-enriched topological order, we often just specify $\rho_{\mathbf{g}}$, without explicitly specifying the U and η symbols. However, we specify the U and η symbols of the topological symmetry group of this topological order, which allows us to determine the U and η symbols of the microscopic symmetry as follows.

Denote the microscopic symmetry group by G and the topological symmetry group by G_0; then, $\rho_{\mathbf{g}}$ defines a group homomorphism $\varphi\colon G \to G_0$. Denote the U and a set of η symbols of G_0 by $U^{(0)}_{\mathbf{g_0}}(a,b;c)$ and $\eta^{(0)}_a(\mathbf{g}_0,\mathbf{h}_0)$, respectively, for any $\mathbf{g}_0,\mathbf{h}_0 \in G_0$. These U and η symbols are some data intrinsic to the topological order, independent of the microscopic symmetry G, just like the F and R symbols. The U and a set of η symbols of the microscopic symmetry G can be written, respectively, as

$$U_{\mathbf{g}}(a,b;c) = U_{\varphi(\mathbf{g})}(a,b;c),$$
$$\eta_a(\mathbf{g},\mathbf{h}) = \eta_a[\varphi(\mathbf{g}),\varphi(\mathbf{h})] \tag{5}$$

for any $\mathbf{g}, \mathbf{h} \in G$. Other symmetry fractionalization classes corresponding to other sets of η symbols can be related to this one via Eq. (4).

C. Anomalies of symmetry-enriched topological orders

A symmetry-enriched topological order can have a quantum anomaly. Roughly speaking, the anomaly characterizes the interplay between locality and the symmetry of the system. There are a few different definitions of quantum anomaly that are believed to be equivalent. In general, the anomaly can be viewed as an obstruction to gauging the symmetry, an obstruction to having a symmetric short-range entangled ground state, an obstruction to describing the system using a Hilbert space with a tensor product structure and on-site symmetry actions, or the boundary manifestation of a higher-dimensional bulk.

For a symmetry-enriched topological order, we can characterize its anomaly via anomaly indicators, a set of quantities expressed in terms of the data like $\{F_d^{abc}, R_c^{ab}, \rho_{\mathbf{g}}, U_{\mathbf{g}}(a,b;c), \eta_a(\mathbf{g},\mathbf{h})\}$. For example, consider a $\mathbb{Z}_2 \times \mathbb{Z}_2$ symmetry. The anomalies of $(2+1)$-dimensional bosonic systems with a $\mathbb{Z}_2 \times \mathbb{Z}_2$ symmetry are classified by $(\mathbb{Z}_2)^2$. Suppose the two generators of $\mathbb{Z}_2 \times \mathbb{Z}_2$ are C_1 and C_2. The two anomaly indicators can be given by $\mathcal{I}_3(C_1, C_2)$ and $\mathcal{I}_3(C_2, C_1)$, where

$$
\begin{aligned}
\mathcal{I}_3(C_1,C_2) = \frac{1}{D^2} \sum_{\substack{a,b,x,u \\ \mu\nu\tilde{\mu}\tilde{\nu}\rho\sigma\alpha \\ {}^{C_1}a=a \\ a\times b\times {}^{C_1}b \to {}^{C_2}a}} & d_b \frac{\theta_x}{\theta_a} \left(R_u^{b,{}^{C_1}b}\right)_{\rho\sigma} \\
& \times \left(F_{{}^{C_2}a}^{a,b,{}^{C_1}b}\right)^*_{(x,\tilde{\mu},\tilde{\nu})(u,\sigma,\alpha)} \left(F_{{}^{C_2}a}^{a,{}^{C_1}b,b}\right)_{({}^{C_1}x,\mu,\nu)(u,\rho,\alpha)} \\
& \times U_{C_1}^{-1}(a,b;x)_{\tilde{\mu}\mu} U_{C_1}^{-1}(x,{}^{C_1}b;{}^{C_2}a)_{\tilde{\nu}\nu} \\
& \times \frac{1}{\eta_b(C_1,C_1)} \frac{\eta_a(C_2,C_1)}{\eta_a(C_1,C_2)}
\end{aligned}
\tag{6}
$$

and $\mathcal{I}_3(C_2, C_1)$ is obtained from the above equation by swapping $C_1 \leftrightarrow C_2$ [20]. The reason for the subscript of $\mathcal{I}_3$ can be seen in Appendix D. The summation is over all anyon types a and b satisfying ${}^{C_1}a = a$ and that ${}^{C_2}a$ is a fusion outcome of $a \times b \times {}^{C_1} b$, x also denotes anyon types, and the Greek letters index different ways of getting a particular fusion outcome (e.g., $\mu = 1, 2, \ldots, N_{a^{C_1}b}^{C_2 a}$). Both $\mathcal{I}_3(C_1, C_2)$ and $\mathcal{I}_3(C_2, C_1)$ are actually partition functions of a $3+1$-dimensional $\mathbb{Z}_2 \times \mathbb{Z}_2$ symmetry-protected topological phase whose boundary is the relevant symmetry-enriched topological order, as discussed in detail in Ref. [20]. These two anomaly indicators take values in ± 1, and each set of values of these anomaly indicators specifies an element in the $(\mathbb{Z}_2)^2$ group, which classifies these anomalies.

Later in the paper, we discuss systems with symmetries different from $\mathbb{Z}_2 \times \mathbb{Z}_2$, but it turns out that many anomaly indicators for topological orders with those other symmetries can be expressed via $\mathcal{I}_{1,2}$.

D. Crystalline equivalence principle

In the above two subsections, our focus is topological orders with purely internal symmetries. But, as mentioned in the introduction, one of the main goals of this paper is to classify topological quantum spin liquids enriched by a general symmetry, which may contain both lattice symmetry and internal symmetry. The crystalline equivalence principle [38,39] provides a convenient way to describe a topological order with lattice symmetry (and possibly also internal symmetry) using a topological order with a purely internal symmetry.

More concretely, the crystalline equivalence principle asserts that for each topological phase with a symmetry group G, where G may contain both lattice symmetry and internal symmetry, there is a corresponding topological phase with only internal symmetries, where the symmetry group is still G, and all orientation reversing symmetries in the original topological phase should be viewed as antiunitary symmetries in the corresponding topological phase. For example, Appendix A explains how to translate the data characterizing a symmetry-enriched topological order with reflection symmetry into the data for a time-reversal symmetry-enriched topological order.

Strictly speaking, the above statement applies only to bosonic systems, which is the focus of the present paper. The fermionic version of this statement is still under development (see Refs. [40–43] for recent progress).

IV. SYMMETRY PROPERTIES AND QUANTUM ANOMALIES OF LATTICE SYSTEMS

In the introduction, we mentioned that the three pieces of robust symmetry-related information of a lattice system can be encoded in its quantum anomaly. In this section, we make this notion more precise. Although this idea is general, for concreteness, we focus on lattice spin systems in two spatial dimensions with one of these six symmetry groups: $p6 \times SO(3)$, $p4 \times SO(3)$, $p6m \times \mathbb{Z}_2^T$, $p4m \times \mathbb{Z}_2^T$, $p6m \times SO(3) \times \mathbb{Z}_2^T$, and $p4m \times SO(3) \times \mathbb{Z}_2^T$. Here, $p6$, $p4$, $p6m$, and $p4m$ are lattice symmetry groups, whose definitions are explained in Figs. 1 and 2. These lattice symmetries are assumed to only move the locations of the microscopic degrees of freedom, without acting on their internal states; i.e., there is no spin-orbit coupling. The SO(3) and $\mathbb{Z}_2^T$ are on-site spin rotational symmetry and time-reversal symmetry, respectively. These symmetry settings are relevant to many theoretical, experimental, and numerical studies, and the examples we consider in the later part of the paper are also based on these symmetry settings.

Given such a symmetry group, different lattice systems can be organized into the so-called lattice homotopy classes [12,44]. Two lattice systems are in the same lattice

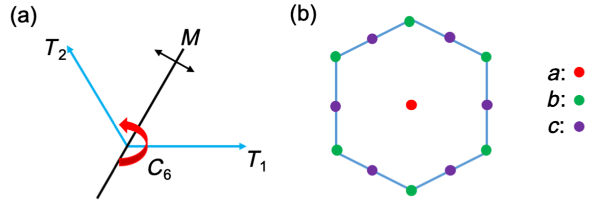


FIG. 1. (a) shows the generators of the $p6m$ group, including translations T_1 and T_2, a sixfold rotation C_6, and a mirror reflection M. The two translation vectors have the same length, and their angle is $2\pi/3$. The reflection axis of M bisects these two translation vectors. The $p6$ symmetry is generated by T_1, T_2, and C_6. Namely, $p6$ has no M compared to $p6m$. In (b), the hexagon is a translation unit cell of either $p6m$ or $p6$ lattice symmetry. There are three types of high-symmetry points, labeled by a, b, and c, and they form the sites of the triangular, honeycomb, and kagome lattices, respectively. The C_6 rotation center in (a) is at a type-a point.

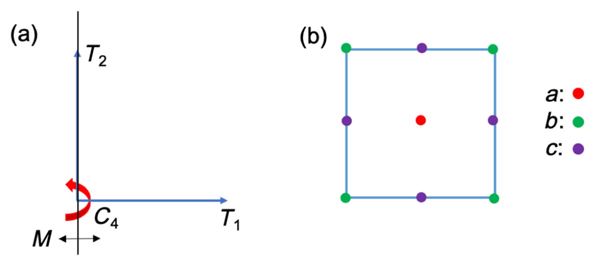


FIG. 2. (a) shows the generators of the $p4m$ group, including translations T_1 and T_2, a fourfold rotation C_4, and a mirror reflection M. The two translation vectors have the same length, and their angle is $\pi/2$. The reflection axis of M is parallel to the translation vector of T_2. The $p4$ symmetry is generated by T_1, T_2, and C_4. Namely, $p4$ has no M compared with $p4m$. In (b), the square is a translation unit cell of either $p4m$ or $p4$ lattice symmetry. There are three high-symmetry points, labeled by a, b, and c. Both type-a and type-b points form a square lattice, and type-c points form a checkerboard lattice. The C_4 rotation center in (a) is taken to be at a type-a point.

homotopy class if and only if they can be deformed into each other by these operations while preserving the lattice symmetry: (i) moving the microscopic degrees of freedom, (ii) identifying degrees of freedom with the same type of projective representation under the on-site symmetry, and (iii) adding or removing degrees of freedom with linear representation (i.e., trivial projective representation) under the on-site symmetry. Lattice systems within the same class share the same robust symmetry-related properties, while those in different classes have distinct symmetry properties and cannot be smoothly connected without breaking the symmetry. So the robust symmetry-related information of a lattice system is the lattice homotopy class it belongs to, while colloquially it is the three pieces of information mentioned in the introduction.

To make the above discussion less abstract, consider systems with $p6 \times \mathrm{SO}(3)$ symmetry. From Fig. 1, there are three types of high-symmetry points, forming a triangular, honeycomb, and kagome lattice, respectively. The on-site symmetry SO(3) has two types of projective representations: half-odd-integer spins and integer spins. According to the above discussion, although spin-$1/2$ systems defined on triangular, honeycomb, and kagome lattices have the same symmetry group, they are in different lattice homotopy classes and have sharply distinct symmetry properties, because they cannot be deformed into each other via the above operations.

Below, we enumerate all lattice homotopy classes in our symmetry settings. To this end, we first specify the types of projective representations under the internal symmetries we consider. As mentioned above, there are two types of projective representations for the SO(3) symmetry. For time-reversal symmetry $\mathbb{Z}_2^T$, there are also two types of projective representations: Kramers singlet and Kramers doublet. For symmetry $\mathrm{SO}(3) \times \mathbb{Z}_2^T$, there are actually four types of projective representations: integer spin under SO(3) while Kramers singlet under $\mathbb{Z}_2^T$, half-odd-integer spin under SO(3) while Kramers doublet under $\mathbb{Z}_2^T$, half-odd-integer spin under SO(3) while Kramers singlet under $\mathbb{Z}_2^T$, and integer spin under SO(3) while Kramers doublet under $\mathbb{Z}_2^T$. The first two types of projective representations are more common in physical systems and theoretical models than the last two, [45] so below we consider only the first two. Therefore, for all three types of internal symmetries we consider, i.e., SO(3), $\mathbb{Z}_2^T$, and $\mathrm{SO}(3) \times \mathbb{Z}_2^T$, there is a trivial projective representation and a nontrivial one under consideration.

Then, for lattice systems with the symmetry group being either $p6 \times \mathrm{SO}(3)$, $p6m \times \mathbb{Z}_2^T$, or $p6m \times \mathrm{SO}(3) \times \mathbb{Z}_2^T$, there are four different lattice homotopy classes [12].

(1) *Class "0"*.—A representative configuration: a system with degrees of freedom carrying only the trivial projective representation under the internal symmetry.
(2) *Class "a"*.—A representative configuration: a system with degrees of freedom carrying the nontrivial projective representation under the internal symmetry, which locate at the triangular lattice sites (type-a high-symmetry points in Fig. 1).
(3) *Class "c"*.—A representative configuration: a system with degrees of freedom carrying the nontrivial projective representation under the internal symmetry, which locate at the kagome lattice sites (type-c high-symmetry points in Fig. 1).
(4) *Class "$a + c$"*.—A representative configuration: a system with degrees of freedom carrying the nontrivial projective representation under the internal symmetry, which locate at both the triangular and kagome lattice sites (both type-a and type-c high-symmetry points in Fig. 1).

Note that a system with degrees of freedom carrying nontrivial projective representation that locate at the honeycomb lattice sites (type-b high-symmetry points) is in class 0 [12].

For lattice systems with the symmetry group being either $p4\times SO(3)$, $p4m\times \mathbb{Z}_2^T$, or $p4m\times SO(3)\times \mathbb{Z}_2^T$, there are eight different lattice homotopy classes. Using labels similar to the above, these are classes "0," "a," "b," "c," "$a+b$," "$a+c$," "$b+c$," and "$a+b+c$," respectively, where each label represents the type of the high-symmetry points at which the degrees of freedom carrying the nontrivial projective representation locate (0 means all degrees of freedom carry the regular representation, i.e., the trivial projective representation). Note that type-a and type-c high-symmetry points are physically distinct once the C_4 rotation center is specified, although they may look identical at the first glance.

To turn the above picture into useful mathematical formulations, Ref. [8] shows how to characterize each lattice homotopy class using its quantum anomaly, which in this context is also known as Lieb-Schultz-Mattis anomaly. Different lattice homotopy classes have different anomalies, and the lattice homotopy class 0 has a trivial anomaly. In the context of topological orders, the anomalies can be expressed via the anomaly indicators. For topological quantum spin liquids with $p6\times SO(3)$ symmetry, the anomaly indicators are given in Appendix D:

$$\begin{aligned} \mathsf{I}_1 &= \mathcal{I}_3(C_2U_\pi, C_2U'_\pi),\\ \mathsf{I}_2 &= \mathcal{I}_3(T_1C_2U_\pi, T_1C_2U'_\pi), \end{aligned} \tag{7}$$

where the expression of $\mathcal{I}_3$ is given by Eq. (6) and C_2 is a twofold rotation symmetry (i.e., $C_2\equiv C_6^3$), while U_π and U'_π are π spin rotations around two orthogonal axes. We can think of I_1 and I_2 as, respectively, detecting half-odd-integer spins at type-a and type-c high-symmetry points, which are, respectively, the twofold rotation centers of the C_2 and T_1C_2 symmetries. More generally, the values of these anomaly indicators for the four lattice homotopy classes enumerated above are shown in Table I.

For topological quantum spin liquids with $p4\times SO(3)$ symmetry, the anomaly indicators are

$$\begin{aligned} \mathsf{I}_1 &= \mathcal{I}_3(C_2U_\pi, C_2U'_\pi),\\ \mathsf{I}_2 &= \mathcal{I}_3(T_1T_2C_2U_\pi, T_1T_2C_2U'_\pi),\\ \mathsf{I}_3 &= \mathcal{I}_3(T_1C_2U_\pi, T_1C_2U'_\pi), \end{aligned} \tag{8}$$

TABLE I. Values of the anomaly indicators for the four lattice homotopy classes with symmetry group $p6\times SO(3)$.

	0	a	c	$a+c$
I_1	1	−1	1	−1
I_2	1	1	−1	−1

TABLE II. Values of the anomaly indicators for the eight lattice homotopy classes with symmetry group $p4\times SO(3)$.

	0	a	b	c	$a+b$	$a+c$	$b+c$	$a+b+c$
I_1	1	−1	1	1	−1	−1	1	−1
I_2	1	1	−1	1	−1	1	−1	−1
I_3	1	1	1	−1	1	−1	−1	−1

where C_2 is still a twofold rotational symmetry (but $C_2=C_4^2$ in this case), while U_π and U'_π are still π spin rotations around two orthogonal axes. We can think of I_1, I_2, and I_3 as, respectively, detecting half-odd-integer spins at type-a, type-b, and type-c high-symmetry point, which are, respectively, the twofold rotation centers of the C_2, $T_1T_2C_2$, and T_1C_2 symmetries. More generally, the values of these anomaly indicators for the eight lattice homotopy classes enumerated above are shown in Table II.

The anomaly indicators for the other symmetry groups we consider [i.e., $p6m\times SO(3)\times\mathbb{Z}_2^T$, $p4m\times SO(3)\times\mathbb{Z}_2^T$, $p6m\times\mathbb{Z}_2^T$, and $p4m\times\mathbb{Z}_2^T$] and their values in each lattice homotopy class are more complicated, and they are presented in Appendix D.

We remark that, strictly speaking, Eqs. (7) and (8) are anomaly indicators of topological quantum spin liquids with purely internal $p6\times SO(3)$ or $p4\times SO(3)$ symmetry, but the crystalline equivalence principle discussed in Sec. V still allows us to use them to classify symmetry-enriched topological quantum spin liquids.

Moreover, it is straightforward to generalize the above idea to other types of systems. For instance, to get the anomaly of a system with spin-orbit coupling, whose symmetry is just a subgroup of the symmetry of a system without spin-orbit coupling, one can simply start from the anomaly of a system without spin-orbit coupling and restrict the symmetry to that subgroup.

V. FRAMEWORK OF CLASSIFICATION

Now we are ready to present our framework to classify symmetry-enriched topological quantum spin liquids.

Our framework is based on the hypothesis of emergibility [8,46]. Namely, suppose the anomaly of the lattice system is ω; then, by tuning the parameters of this lattice system, a quantum many-body state (or its low-energy effective theory) with anomaly Ω can emerge at low energies if and only if the anomaly-matching condition holds: $\omega=\Omega$.

The "only if" part of this statement is established and well known [47]. The "if" part is hypothetical, but there is no known counterexample to it and it is supported by multiple nontrivial examples [48–51]. So we assume this hypothesis to be true and use it as our basis of analysis.

With this hypothesis, the framework to classify symmetry-enriched topological quantum spin liquids, or, equivalently, to obtain the possible symmetry-enriched topological quantum spin liquids that can emerge in the lattice system of interest, is as follows.

(1) Given the symmetry group, which may contain both lattice symmetry and internal symmetry, we first use the crystalline equivalence principle in Sec. III D to translate it into a purely internal symmetry.
(2) Based on the above internal symmetry and the topological quantum spin liquid, we use the method in Ref. [7] to obtain the classification of the internal symmetry-enriched topological quantum spin liquids.
(3) For each of the internal symmetry-enriched topological quantum spin liquids, we use the method in Ref. [20] to obtain its anomaly Ω.
(4) As discussed in Sec. IV, the original lattice system has its own quantum anomaly ω. We check if the anomaly-matching condition $\omega = \Omega$ holds. If it does (does not), then the corresponding symmetry-enriched topological quantum spin liquid can (cannot) emerge in this lattice system, according to the hypothesis of emergibility.

If there is only an internal symmetry but no lattice symmetry, then step 1 in the framework can be ignored. In this case, sometimes one is interested in only anomaly-free states; then, ω in step 4 should be the trivial anomaly. For steps 3 and 4, in our context Ω (ω) can be represented by the values of the anomaly indicators of the topological quantum spin liquid (lattice system), so checking whether $\omega = \Omega$ becomes checking whether the values of these two sets of anomaly indicators match. For instance, as discussed in Sec. IV, the anomaly ω for lattice systems with $p6 \times SO(3)$ symmetry can be specified by the values of $I_{1,2}$ defined in Eq. (7), and these values for different lattice homotopy classes are given in Table I. The anomaly Ω or the corresponding anomaly indicators $I_{1,2}$ for a $p6 \times SO(3)$ symmetric topological quantum spin liquid can be calculated using the results in Ref. [20], and we give detailed analysis below [see Eqs. (30) and (33) for some of the final results of the calculations].

We reiterate that the above framework can be straightforwardly generalized to classify quantum states other than symmetry-enriched topological quantum spin liquids. For example, it has been used to classify some gapless quantum spin liquids in Ref. [8].

In the following sections, we apply the above framework to obtain the classification of some representative two dimensional symmetry-enriched topological quantum spin liquids on various lattice systems.

VI. $U(1)_{2N}$ TOPOLOGICAL ORDERS: GENERALIZED ABELIAN CHIRAL SPIN LIQUIDS

Our first class of examples are topological quantum spin liquids with $U(1)_{2N}$ topological orders. These are Abelian chiral states, where the $N = 1$ case is the well-known Kalmeyer-Laughlin state [52,53], the $N = 2$ case is the $\nu = 2$ state in Kitaev's 16-fold way [26], and a general $U(1)_{2N}$ topological order can be obtained by putting bosons into an interacting bosonic integer quantum Hall state with Hall conductance $2N$ (in natural units) and coupling them to a dynamical U(1) gauge field [10,54–56]. We classify the $U(1)_{2N}$ topological quantum spin liquid enriched by $p6 \times SO(3)$ or $p4 \times SO(3)$ symmetry. As discussed in Sec. III D, these symmetries can be viewed as purely internal symmetries according to the crystalline equivalence principle. Our results are summarized in Table V.

The topological properties of the $U(1)_{2N}$ topological order can be described by either the Laughlin-$(1/2N)$ wave function or a Chern-Simons theory with Lagrangian $\mathcal{L} = -(2N/4\pi)\epsilon^{\mu\nu\lambda}A_\mu\partial_\nu A_\lambda$, with A a dynamical U(1) gauge field. These states also allow a description using noninteracting parton mean field. Specifically, one can consider $2N$ species of fermionic partons with an $SU(2N)$ gauge structure. When all species are in a Chern band with a unit Chern number, the resulting state is the $U(1)_{2N}$ topological order [57].

The above descriptions of these topological quantum spin liquids all suffer from some disadvantages. Concretely, the Laughlin wave function is a single specific state, and it cannot describe different symmetry-enriched states. To capture the symmetry actions in the Chern-Simons theory, one needs to invoke the concept of 2-group symmetries [58], which are not exact symmetries of the physical system. Also, in the $SU(2N)$ parton-mean-field description, the projective quantum number of the fermionic partons is not exactly the same as the symmetry fractionalization class of the anyons.

Below, we discuss the topological properties of these states in the language of Sec. III, which does not suffer from the above disadvantages, since it can describe general symmetry-enriched $U(1)_{2N}$ topological quantum spin liquids directly in terms of the symmetry properties of the anyons.

We label anyons in $U(1)_{2N}$ by (a), where a is an element in $\{0, \ldots, 2N-1\}$. These are all Abelian anyons with $d_{(a)} = 1$. The fusion rule is given by addition modulo $2N$, i.e.,

$$(a) \otimes (b) = ([a+b]_{2N}). \tag{9}$$

In this paper, we use the notation $[x]_y$ to denote x modulo y for any integer x and positive integer y, and $[x]_y$ takes values in $\{0, \ldots, y-1\}$. The F symbols can be written as

$$F^{(a)(b)(c)} = e^{(i\pi/2N)a(b+c-[b+c]_{2N})}, \tag{10}$$

and the R symbols are

$$R^{(a)(b)} = e^{(i\pi/2N)ab}, \tag{11}$$

which yield the topological spins

$$\theta_{(a)} = e^{(i\pi/2N)a^2}. \tag{12}$$

The topological symmetry of $\mathrm{U}(1)_{2N}$ is complicated for general N [59]. For $N=1$, there is no nontrivial topological symmetry. For $N\geq 2$, there is always a $\mathbb{Z}_2$ topological symmetry generated by the charge conjugation symmetry C, such that anyon $(a)\to([-a]_{2N})$ under C. For this topological symmetry, we can take the U symbols as

$$U_C\{(a),(b);([a+b]_{2N})\}=\begin{cases}(-1)^a, & b>0,\\ 1, & b=0\end{cases} \tag{13}$$

and a set of η symbols all equal to 1. From this set of η symbols, we can obtain all other possible η symbols via Eq. (4). When $2\leq N\leq 5$, this $\mathbb{Z}_2$ is the full topological symmetry group. When $N\geq 6$, there can be other topological symmetries [60]. In the later discussion, we consider general $N\geq 2$ but limit to the cases where the microscopic symmetry can permute anyons only as charge conjugation (i.e., we ignore anyon permutation patterns other than C, if any). To read off the results for $N=1$ from those for $N\geq 2$, we just need to ignore the cases where the microscopic symmetry permutes anyons.

A. Example: $\mathbb{Z}_2\times\mathrm{SO}(3)$

To illustrate our calculation of the anomaly of the $\mathrm{U}(1)_{2N}$ topological order with $p6\times\mathrm{SO}(3)$ or $p4\times\mathrm{SO}(3)$ symmetry, let us first discuss the example where the symmetry is $\mathbb{Z}_2\times\mathrm{SO}(3)$ in detail. It turns out that the calculation of the anomaly when the symmetry is $p6\times\mathrm{SO}(3)$ or $p4\times\mathrm{SO}(3)$ can be reduced to this example, by restricting $p6$ or $p4$ to its various $\mathbb{Z}_2$ subgroups.

The anomalies associated with the $\mathbb{Z}_2\times\mathrm{SO}(3)$ symmetry are classified by

$$H^4[\mathbb{Z}_2\times\mathrm{SO}(3),\mathrm{U}(1)]\cong\mathbb{Z}_2. \tag{14}$$

Hence, there is only one type of nontrivial anomaly, which can be detected by the anomaly indicator $\mathsf{I}=\mathcal{I}_3(C_2U_\pi,C_2U'_\pi)$, where C_2 is the generator of $\mathbb{Z}_2$ while U_π and U'_π are elements of SO(3), representing the π rotations about two orthogonal axes.

The SO(3) symmetry cannot permute anyons, because all elements of SO(3) are continuously connected to the identity element. Hence, only the generator of $\mathbb{Z}_2$, denoted by C_2 here, can permute anyons by charge conjugation, and there are two possibilities.

1. No anyon permutation

The first possibility is that the action of C_2 is trivial and there is no anyon permutation. For the case with $N=1$, this is the only possibility to be considered. Then, the symmetry fractionalization classes are classified by

$$H^2_{(1)}[\mathbb{Z}_2\times\mathrm{SO}(3),\mathbb{Z}_{2N}]=H^2(\mathbb{Z}_2,\mathbb{Z}_{2N})\oplus H^2[\mathrm{SO}(3),\mathbb{Z}_{2N}]=(\mathbb{Z}_2)^2. \tag{15}$$

Namely, there are two generators that generate four different symmetry fractionalization classes. To understand these symmetry fractionalization classes, we can directly write down representative cochains of them. A representative cochain of the first generator, which we denote by $\tilde{\beta}(x)$ and comes from $H^2(\mathbb{Z}_2,\mathbb{Z}_{2N})$, is

$$\tilde{\beta}(x)(C_2^i,C_2^j)=\frac{i+j-[i+j]_2}{2}=ij \mod 2N, \tag{16}$$

with $i,j\in\{0,1\}$. The reason for the name of this generator is explained in Appendix C. Physically, this generator detects whether the anyon (1) carries a fractional charge under the $\mathbb{Z}_2$ symmetry.

The second generator, which comes from $H^2[\mathrm{SO}(3),\mathbb{Z}_{2N}]$, detects whether the anyon (1) carries a half-odd-integer spin under the SO(3) symmetry, and we denote it by Nw_2, for reasons explained in Appendix C. To have a representative cochain of Nw_2, it is convenient to consider a $\mathbb{Z}_2\times\mathbb{Z}_2$ subgroup of SO(3) generated by U_π and U'_π, and an element in this subgroup can be written as $U_\pi^iU'^j_\pi$, with $i,j\in\{0,1\}$. Then, restricting SO(3) to this $\mathbb{Z}_2\times\mathbb{Z}_2$ subgroup, the representative cochain of Nw_2 is

$$(Nw_2)\left(U_\pi^{i_1}U'^{i_2}_\pi,U_\pi^{j_1}U'^{j_2}_\pi\right)=N(i_1j_1+i_2j_2+i_1j_2) \mod 2N. \tag{17}$$

So the symmetry fractionalization classes can be written as

$$w=n_1\cdot\tilde{\beta}(x)+n_2\cdot Nw_2 \tag{18}$$

and labeled as $\{n_1,n_2\}$ with $n_{1,2}\in\{0,1\}$. When the SO(3) symmetry is restricted to its $\mathbb{Z}_2\times\mathbb{Z}_2$ subgroup generated by U_π and U'_π, a representative cochain can be taken as

$$w\left(C_2^{i_1}U_\pi^{i_2}U'^{i_3}_\pi,C_2^{j_1}U_\pi^{j_2}U'^{j_3}_\pi\right)=n_1i_1j_1+n_2N(i_2j_2+i_3j_3+i_2j_3) \mod 2N. \tag{19}$$

Combining the above equation and Eq. (4), we get

$$\eta_{(a)}(C_2U_\pi,C_2U'_\pi)=\eta_{(a)}(C_2U_\pi,C_2U_\pi)=\exp\left(\frac{i\pi}{N}a(n_1+Nn_2)\right),$$
$$\eta_{(a)}(C_2U'_\pi,C_2U_\pi)=\exp\left(\frac{i\pi}{N}an_1\right). \tag{20}$$

Now we plug Eqs. (10), (11), (12), and (20) into Eq. (6) (the U symbols therein can all be taken as 1 since there is no anyon permutation) and get the anomaly indicator of the state in symmetry fractionalization class $\{n_1,n_2\}$:

$$\mathcal{I}_3(C_2U_\pi,C_2U'_\pi)=(-1)^{n_1n_2}. \tag{21}$$

2. C_2 acts as charge conjugation

The second possibility is that C_2 acts by charge conjugation. This possibility occurs only if $N \geq 2$. Then, the symmetry fractionalization classes are classified by

$$H^2_{(2)}[\mathbb{Z}_2 \times \mathrm{SO}(3), \mathbb{Z}_{2N}] = (\mathbb{Z}_2)^2. \quad (22)$$

There are also two generators that generate four different symmetry fractionalization classes, but these symmetry fractionalization classes are different from those in Sec. VI A 1. Explicitly, a representative cochain of the first generator, which we denote by Nx^2, is

$$(Nx^2)(C_2^i, C_2^j) = Nij \mod 2N, \quad (23)$$

with $i, j \in \{0, 1\}$. The second generator also detects whether the anyon (1) carries a half-odd-integer spin under the SO(3) symmetry, and we also denote it by Nw_2. The representative cochain restricted to the $\mathbb{Z}_2 \times \mathbb{Z}_2$ subgroup generated by U_π and U'_π is still given by Eq. (17).

So the symmetry fractionalization classes can be written as

$$w = n_1 \cdot Nx^2 + n_2 \cdot Nw_2 \quad (24)$$

and also labeled as $\{n_1, n_2\}$ with $n_{1,2} \in \{0, 1\}$. When the SO(3) symmetry is restricted to its $\mathbb{Z}_2 \times \mathbb{Z}_2$ subgroup generated by U_π and U'_π, a representative cochain can be taken as

$$w\left(C_2^{i_1} U_\pi^{i_2} U'^{i_3}_\pi, C_2^{j_1} U_\pi^{j_2} U'^{j_3}_\pi\right) = n_1 N i_1 j_1 + n_2 N(i_2 j_2 + i_3 j_3 + i_2 j_3) \mod 2N. \quad (25)$$

Combining the above equation and Eq. (4), we get

$$\eta_{(a)}(C_2 U_\pi, C_2 U'_\pi) = \eta_{(a)}(C_2 U_\pi, C_2 U_\pi) = \exp(i\pi a(n_1 + n_2)),$$
$$\eta_{(a)}(C_2 U'_\pi, C_2 U_\pi) = \exp(i\pi a n_1). \quad (26)$$

Now we plug Eqs. (10), (11), (12), (13), and (26) into Eq. (6) and get the anomaly indicator of the state in symmetry fractionalization class $\{n_1, n_2\}$:

$$\mathcal{I}_3(C_2 U_\pi, C_2 U'_\pi) = (-1)^{(n_1+1)n_2 N}. \quad (27)$$

Hence, when N is even, the anomaly is always absent; when N is odd, $n_1 = 0$, $n_2 = 1$ gives nonzero anomaly; otherwise, the anomaly is absent.

With this warm-up, we are ready to classify $\mathrm{U}(1)_{2N}$ topological quantum spin liquids enriched by $p6 \times \mathrm{SO}(3)$ or $p4 \times \mathrm{SO}(3)$ symmetry using the framework in Sec. V. The results are summarized in Table V.

B. $p6 \times \mathrm{SO}(3)$

The generator $T_{1,2}$ and SO(3) cannot permute anyons, [61] and only the generator C_6 can permute anyons by charge conjugation. Hence, there are two possibilities regarding how $p6 \times \mathrm{SO}(3)$ can permute anyons.

(1) *Trivial C_6 action: No anyon permutation.*—In this case, the possible symmetry fractionalization classes are classified by

$$H^2_{(1)}[p6 \times \mathrm{SO}(3), \mathbb{Z}_{2N}] = \mathbb{Z}_{2N} \oplus \mathbb{Z}_{(2N,6)} \oplus \mathbb{Z}_2, \quad (28)$$

whose elements can be written as

$$w = n_1 \cdot \tilde{\mathscr{B}}^{(1)}_{xy} + n_2 \cdot \tilde{\mathscr{B}}^{(1)}_{c^2} + n_3 \cdot Nw_2 \quad (29)$$

and labeled by $\{n_1, n_2, n_3\}$, with $n_1 \in \{0, \ldots, 2N-1\}$, $n_2 \in \{0, \ldots, (2N,6)-1\}$, and $n_3 \in \{0, 1\}$. Here, $\tilde{\mathscr{B}}^{(1)}_{xy}$, $\tilde{\mathscr{B}}^{(1)}_{c^2}$, and Nw_2 are generators of $\mathbb{Z}_{2N}$, $\mathbb{Z}_{(2N,6)}$, and $\mathbb{Z}_2$, respectively (the representative cochains and the reason for the names of these generators are given in Appendix C). Physically, we can think of $\tilde{\mathscr{B}}^{(1)}_{xy}$, $\tilde{\mathscr{B}}^{(1)}_{c^2}$, and Nw_2 as detecting whether the anyon (1) carries projective representation under translation symmetries C_6 and SO(3), respectively [62]. For each symmetry fractionalization class, the U and η symbols can be obtained via Eqs. (4), (5), and (13).

Without considering anomaly matching, $p6 \times \mathrm{SO}(3)$ symmetric $\mathrm{U}(1)_{2N}$ topological quantum spin liquids are classified by $\{n_1, n_2, n_3\}$, if no symmetry permutes anyons. Recall that two symmetry fractionalization classes related to each other by relabeling anyons are physically identical, so $\{n_1, n_2, n_3\}$ and $\{[-n_1]_{2N}, [-n_2]_{(2N,6)}, n_3\}$ are identified.

As argued in the introduction and Sec. IV, in systems with lattice symmetry, it is important to consider anomaly matching for the classification of symmetry-enriched topological quantum spin liquids. The values of the anomaly indicators for different lattice homotopy classes with $p6 \times \mathrm{SO}(3)$ symmetry are given in Table I. The anomaly indicators for the $\mathrm{U}(1)_{2N}$ state in the symmetry fractionalization class $\{n_1, n_2, n_3\}$ can be calculated in a way similar to Sec. VI A, which yields [63]

$$\mathsf{I}_1 = (-1)^{n_2 n_3}, \qquad \mathsf{I}_2 = (-1)^{(n_1+n_2)n_3}. \quad (30)$$

Therefore, by matching these anomaly indicators with Table I, we arrive at the classification in Table III.

(2) *Nontrivial C_6 action: C_6 as charge conjugation.*—In this case, the possible symmetry fractionalization classes are given by

$$H^2_{(2)}[p6 \times \mathrm{SO}(3), \mathbb{Z}_{2N}] = (\mathbb{Z}_2)^3, \quad (31)$$

TABLE III. Classification of symmetry-enriched $\mathrm{U}(1)_{2N}$ topological quantum spin liquids in lattice systems with $p6 \times \mathrm{SO}(3)$ symmetry, if no symmetry permutes anyons.

Lattice homotopy class	Symmetry fractionalization class
0	$\{n_1, n_2, 0\}$, or $\{n_1, n_2, 1\}$ with $n_{1,2}$ even
a	$\{n_1, n_2, 1\}$ with $n_{1,2}$ odd
c	$\{n_1, n_2, 1\}$ with n_1 odd, n_2 even
$a + c$	$\{n_1, n_2, 1\}$ with n_1 even, n_2 odd

TABLE IV. Classification of symmetry-enriched $\mathrm{U}(1)_{2N}$ topological quantum spin liquids in lattice systems with $p6 \times \mathrm{SO}(3)$ symmetry, if C_6 acts as charge conjugation.

Lattice homotopy class	Symmetry fractionalization class
0	$\{n_1, n_2, n_3\}$ for N even, $\{n_1, n_2, 0\}$ or $\{0, 1, 1\}$ for N odd
a	$\{1, 0, 1\}$ for N odd
c	$\{1, 1, 1\}$ for N odd
$a + c$	$\{0, 0, 1\}$ for N odd

whose elements can be written as

$$w = n_1 \cdot NB_{xy} + n_2 \cdot NA_c^2 + n_3 \cdot Nw_2 \quad (32)$$

and labeled by $\{n_1, n_2, n_3\}$ with $n_{1,2,3} \in \{0, 1\}$. Here, NB_{xy}, NA_c^2, and Nw_2 are generators of the three $\mathbb{Z}_2$ pieces, respectively (the representative cochains and the reason for the names of these generators are given in Appendix C). For each symmetry fractionalization class, the U and η symbols can be obtained via Eqs. (4), (5), and (13).

Similar to the previous case, without considering anomaly matching, $p6 \times \mathrm{SO}(3)$ symmetric $\mathrm{U}(1)_{2N}$ topological quantum spin liquids are classified by the above $\{n_1, n_2, n_3\}$ if C_6 acts as charge conjugation. Calculating the anomaly indicators for $\mathrm{U}(1)_{2N}$ with symmetry fractionalization class labeled by $\{n_1, n_2, n_3\}$ as before, we get

$$\mathsf{I}_1 = (-1)^{(n_2+1)n_3 N}, \quad \mathsf{I}_2 = (-1)^{(n_1+n_2+1)n_3 N}. \quad (33)$$

Therefore, by matching these anomaly indicators with Table I, we arrive at the classification in Table IV.

Summarizing all cases, the total number of different $p6 \times \mathrm{SO}(3)$ symmetry-enriched $\mathrm{U}(1)_{2N}$ topological quantum spin liquids is summarized in Table V. Note that this classification is complete for $N \leq 5$ but incomplete for $N \geq 6$, because we have assumed that the only way the symmetry can permute anyons is via charge conjugation,

TABLE V. Number of $p6 \times \mathrm{SO}(3)$ and $p4 \times \mathrm{SO}(3)$ symmetry-enriched $\mathrm{U}(1)_{2N}$ topological quantum spin liquids. For the case with a $p6 \times \mathrm{SO}(3)$ symmetry, each of the last two columns is written as a sum of two terms, representing the number of states where no anyon is permuted by symmetries and where C_6 acts as charge conjugation, respectively. For the case with $p4 \times \mathrm{SO}(3)$ symmetry, each of the last two columns is written as a sum of four terms, representing the number of states where no anyon is permuted by symmetries, where C_4 acts as charge conjugation while $T_{1,2}$ do not, where $T_{1,2}$ act as charge conjugation while C_4 does not, and where C_4 and $T_{1,2}$ all act as charge conjugation, respectively. The details of the symmetry fractionalization class of each state can be found in Tables III–IX.

Symmetry group	Lattice homotopy class	$N = 1$	Odd $N > 1$	Even N
$p6 \times \mathrm{SO}(3)$	0	5	$\left(\frac{5[N(N,3)+1]}{2}\right) + (5)$	$\left(\frac{5N(N,3)}{2} + 3\right) + (8)$
	a	1	$\left(\frac{N(N,3)+1}{2}\right) + (1)$	$\left(\frac{N(N,3)}{2}\right) + (0)$
	c	1	$\left(\frac{N(N,3)+1}{2}\right) + (1)$	$\left(\frac{N(N,3)}{2}\right) + (0)$
	$a + c$	1	$\left(\frac{N(N,3)+1}{2}\right) + (1)$	$\left(\frac{N(N,3)}{2} + 1\right) + (0)$
$p4 \times \mathrm{SO}(3)$	0	9	$\left(\frac{9(N+1)}{2}\right) + (9) + (9) + (9)$	$(9N + 6) + (12) + (20) + (20)$
	a	1	$\left(\frac{N+1}{2}\right) + (1) + (1) + (1)$	$(N) + (0) + (4) + (0)$
	b	1	$\left(\frac{N+1}{2}\right) + (1) + (1) + (1)$	$(N) + (0) + (0) + (4)$
	c	1	$\left(\frac{N+1}{2}\right) + (1) + (1) + (1)$	$(N + 2) + (4) + (0) + (0)$
	$a + b$	1	$\left(\frac{N+1}{2}\right) + (1) + (1) + (1)$	$(N) + (0) + (0) + (0)$
	$a + c$	1	$\left(\frac{N+1}{2}\right) + (1) + (1) + (1)$	$(N) + (0) + (0) + (0)$
	$b + c$	1	$\left(\frac{N+1}{2}\right) + (1) + (1) + (1)$	$(N) + (0) + (0) + (0)$
	$a + b + c$	1	$\left(\frac{N+1}{2}\right) + (1) + (1) + (1)$	$(N) + (0) + (0) + (0)$

while for $N \geq 6$ a symmetry can, in principle, permute anyons in other manners.

C. $p4 \times \mathrm{SO}(3)$

SO(3) cannot permute anyons, and both the generators $T_{1,2}$ and the generator C_4 can permute anyons by charge conjugation. Hence, there are four possibilities regarding how $p4 \times \mathrm{SO}(3)$ can permute anyons.

(1) *Trivial $T_{1,2}$ and C_4 action.*—In this case, the possible symmetry fractionalization classes are given by

$$H^2_{(1)}[p4 \times \mathrm{SO}(3), \mathbb{Z}_{2N}] = \mathbb{Z}_{2N} \oplus \mathbb{Z}_{(2N,4)} \oplus (\mathbb{Z}_2)^2, \tag{34}$$

whose elements can be labeled as

$$w = n_1 \cdot \tilde{\mathscr{B}}^{(1)}_{xy} + n_2 \cdot \tilde{\mathscr{B}}^{(1)}_{c^2} + n_3 \cdot \tilde{\beta}(A_{x+y}) + n_4 \cdot Nw_2, \tag{35}$$

with $n_1 \in \{0, \ldots, 2N-1\}$, $n_2 \in \{0, \ldots, (2N, 4) - 1\}$, and $n_{3,4} \in \{0, 1\}$. Here, $\tilde{\mathscr{B}}^{(1)}_{xy}$, $\tilde{\mathscr{B}}^{(1)}_{c^2}$, $\tilde{\beta}(A_{x+y})$, and Nw_2 are generators of $\mathbb{Z}_{2N}$, $\mathbb{Z}_{(2N,4)}$, and two $\mathbb{Z}_2$ pieces, respectively (the representative cochains and the reason for the names of these generators are given in Appendix C). For each symmetry fractionalization class, the U and η symbols can be obtained via Eqs. (4), (5), and (13).

Again, because symmetry fractionalization classes related by relabeling anyons are physically identical, different symmetry realizations on $\mathrm{U}(1)_{2N}$ in this case are specified by $\{n_1, n_2, n_3, n_4\}$, where $\{n_1, n_2, n_3, n_4\}$ is identified with $\{[-n_1]_{2N}, [-n_2]_{(2N,4)}, n_3, n_4\}$. Calculating the anomaly indicators for the $\mathrm{U}(1)_{2N}$ state with symmetry fractionalization class labeled by $\{n_1, n_2, n_3, n_4\}$, we get

$$\mathsf{I}_1 = (-1)^{n_2 n_4}, \quad \mathsf{I}_2 = (-1)^{(n_1+n_2)n_4}, \quad \mathsf{I}_3 = (-1)^{(n_2+n_3)n_4}. \tag{36}$$

Therefore, by matching these anomaly indicators with Table II, we arrive at the classification in Table VI.

(2) *Nontrivial C_4 action, trivial $T_{1,2}$ action.*—In this case, the possible symmetry fractionalization classes are given by

$$H^2_{(2)}[p4 \times \mathrm{SO}(3), \mathbb{Z}_{2N}] = (\mathbb{Z}_2)^4. \tag{37}$$

We can write these elements as

$$w = n_1 \cdot NB_{xy} + n_2 \cdot NB_{c^2} + n_3 \cdot \tilde{\beta}(A_{x+y}) + n_4 \cdot Nw_2, \tag{38}$$

with $n_{1,2,3,4} \in \{0, 1\}$. Here, NB_{xy}, NB_{c^2}, $\tilde{\beta}(A_{x+y})$, and Nw_2 are generators of the four $\mathbb{Z}_2$ pieces, respectively (the representative cochains and the reason for the names of these generators are given in Appendix C). For each symmetry fractionalization class, the U and η symbols can be obtained via Eqs. (4), (5), and (13).

Calculating the anomaly indicators for $\mathrm{U}(1)_{2N}$ with symmetry fractionalization class labeled by (n_1, n_2, n_3, n_4) as before, we get

$$\mathsf{I}_1 = (-1)^{n_2 n_4 N}, \qquad \mathsf{I}_2 = (-1)^{(n_1+n_2)n_4 N}, \qquad \mathsf{I}_3 = (-1)^{(n_2 N+n_3)n_4}. \tag{39}$$

Therefore, by matching these anomaly indicators with Table II, we arrive at the classification in Table VII.

(3) *Nontrivial $T_{1,2}$ action, trivial C_4 action.*—In this case, the possible symmetry fractionalization classes are given by

$$H^2_{(3)}[p4 \times \mathrm{SO}(3), \mathbb{Z}_{2N}] = \mathbb{Z}_{(2N,4)} \oplus (\mathbb{Z}_2)^3. \tag{40}$$

TABLE VI. Classification of symmetry-enriched $\mathrm{U}(1)_{2N}$ topological quantum spin liquids in lattice systems with $p4 \times \mathrm{SO}(3)$ symmetry, if no symmetry permutes anyons.

Lattice homotopy class	Symmetry fractionalization class
0	$\{n_1, n_2, n_3, 0\}$, or $\{n_1, n_2, 0, 1\}$ with $n_{1,2}$ even
a	$\{n_1, n_2, 1, 1\}$ with $n_{1,2}$ odd
b	$\{n_1, n_2, 0, 1\}$ with n_1 odd, n_2 even
c	$\{n_1, n_2, 1, 1\}$ with $n_{1,2}$ even
$a+b$	$\{n_1, n_2, 1, 1\}$ with n_1 even, n_2 odd
$a+c$	$\{n_1, n_2, 0, 1\}$ with $n_{1,2}$ odd
$b+c$	$\{n_1, n_2, 1, 1\}$ with n_1 odd, n_2 even
$a+b+c$	$\{n_1, n_2, 0, 1\}$ with n_1 even, n_2 odd

TABLE VII. Classification of symmetry-enriched $\mathrm{U}(1)_{2N}$ topological quantum spin liquids in lattice systems with $p4 \times \mathrm{SO}(3)$ symmetry, if C_4 acts as charge conjugation.

Lattice homotopy class	Symmetry fractionalization class
0	$\{n_1, n_2, n_3, 0\}$ or $\{n_1, n_2, 0, 1\}$ for N even, $\{n_1, n_2, n_3, 0\}$ or $\{0, 0, 0, 1\}$ for N odd
a	$\{1, 1, 1, 1\}$ for N odd
b	$\{1, 0, 0, 1\}$ for N odd
c	$\{n_1, n_2, 1, 1\}$ for N even, $\{0, 0, 1, 1\}$ for N odd
$a+b$	$\{0, 1, 1, 1\}$ for N odd
$a+c$	$\{1, 1, 0, 1\}$ for N odd
$b+c$	$\{1, 0, 1, 1\}$ for N odd
$a+b+c$	$\{0, 1, 0, 1\}$ for N odd

We can write these elements as

$$w = n_1 \cdot \tilde{\mathscr{B}}_{xy}^{(3)} + n_2 \cdot NB_{c^2} + n_3 \cdot NA_{x+y}^2 + n_4 \cdot Nw_2, \tag{41}$$

with $n_1 \in \{0, \ldots, (2N,4) - 1\}$ and $n_{2,3,4} \in \{0,1\}$. Here, $\tilde{\mathscr{B}}_{xy}^{(3)}$, NB_{c^2}, NA_{x+y}^2, and Nw_2 are generators of the $\mathbb{Z}_{(2N,4)}$ and the three $\mathbb{Z}_2$ pieces, respectively (the representative cochains and the reason for the names of these generators are given in Appendix C). For each symmetry fractionalization class, the U and η symbols can be obtained via Eqs. (4), (5), and (13).

Again, because symmetry fractionalization classes related by relabeling anyons are physically identical, different symmetry realizations on $\mathrm{U}(1)_{2N}$ in this case are specified by $\{n_1, n_2, n_3, n_4\}$, where $\{n_1, n_2, n_3, n_4\}$ is identified with $\{[-n_1]_{(2N,4)}, n_2, n_3, n_4\}$. Calculating the anomaly indicators for $\mathrm{U}(1)_{2N}$ with symmetry fractionalization class labeled by (n_1, n_2, n_3, n_4) as before, we get

$$\mathsf{I}_1 = (-1)^{(n_1+n_2N)n_4}, \qquad \mathsf{I}_2 = (-1)^{n_2 n_4 N}, \\ \mathsf{I}_3 = (-1)^{(n_2+n_3+1)n_4 N}. \tag{42}$$

Therefore, by matching these anomaly indicators with Table II, we arrive at the classification in Table VIII.

(4) *Nontrivial $T_{1,2}$ and C_4 action.*—In this case, the possible symmetry fractionalization classes are given by

$$H_{(4)}^2[p4 \times \mathrm{SO}(3), \mathbb{Z}_{2N}] = \mathbb{Z}_{(2N,4)} \oplus (\mathbb{Z}_2)^3. \tag{43}$$

We can label these elements as

$$w = n_1 \cdot \tilde{\mathscr{B}}_{xy}^{(4)} + n_2 \cdot NB_{c^2} + n_3 \cdot NA_{x+y}^2 + n_4 \cdot Nw_2, \tag{44}$$

with $n_1 \in \{0, \ldots, (2N,4) - 1\}$ and $n_{2,3,4} \in \{0,1\}$. Here, $\tilde{\mathscr{B}}_{xy}^{(4)}$, NB_{c^2}, $\widetilde{A_{x+y}^2}$, and Nw_2 are generators of the $\mathbb{Z}_{(2N,4)}$ and the three $\mathbb{Z}_2$ pieces, respectively (the representative cochains and the reason for the names of these generators are given in Appendix C). For each symmetry fractionalization class, the U and η symbols can be obtained via Eqs. (4), (5), and (13).

Again, because symmetry fractionalization classes related by relabeling anyons are physically identical, different symmetry realizations on $\mathrm{U}(1)_{2N}$ in this case are specified by $\{n_1, n_2, n_3, n_4\}$, where $\{n_1, n_2, n_3, n_4\}$ is identified with $\{[-n_1]_{(2N,4)}, n_2, n_3, n_4\}$. Calculating the anomaly indicators for $\mathrm{U}(1)_{2N}$ with symmetry fractionalization class labeled by (n_1, n_2, n_3, n_4) as before, we get

$$\mathsf{I}_1 = (-1)^{n_2 n_4 N}, \qquad \mathsf{I}_2 = (-1)^{(n_1+n_2N)n_4}, \\ \mathsf{I}_3 = (-1)^{(n_2+n_3+1)n_4 N}. \tag{45}$$

Therefore, by matching these anomaly indicators with Table II, we arrive at the classification in Table IX.

Summarizing all cases, the total number of different $p4 \times \mathrm{SO}(3)$ symmetry-enriched $\mathrm{U}(1)_{2N}$ topological quantum spin liquids is summarized in Table V. Note that this classification is complete for $N \leq 5$ but incomplete for $N \geq 6$, because we have assumed that the only way the symmetry can permute anyons is via charge conjugation, while for $N \geq 6$ a symmetry can, in principle, permute anyons in other manners.

VII. ISING$^{(\nu)}$ TOPOLOGICAL ORDERS: KITAEV'S NON-ABELIAN CHIRAL SPIN LIQUIDS

Our next class of examples are non-Abelian chiral spin liquid states, which we dub the "Ising$^{(\nu)}$ states," with ν an odd integer. Their topological properties are discussed in

TABLE VIII. Classification of symmetry-enriched $\mathrm{U}(1)_{2N}$ topological quantum spin liquids in lattice systems with $p4 \times \mathrm{SO}(3)$ symmetry, if translations act as charge conjugation.

Lattice homotopy class	Symmetry fractionalization class
0	$\{n_1, n_2, n_3, 0\}$, $\{0, n_2, n_3, 1\}$, or $\{2, n_2, n_3, 1\}$ for N even, $\{n_1, n_2, n_3, 0\}$ or $\{0,0,1,1\}$ for N odd
a	$\{1, n_2, n_3, 1\}$ for N even, $\{1,0,1,1\}$ for N odd
b	$\{1,1,0,1\}$ for N odd
c	$\{0,0,0,1\}$ for N odd
$a+b$	$\{0,1,0,1\}$ for N odd
$a+c$	$\{1,0,0,1\}$ for N odd
$b+c$	$\{1,1,1,1\}$ for N odd
$a+b+c$	$\{0,1,1,1\}$ for N odd

TABLE IX. Classification of symmetry-enriched $\mathrm{U}(1)_{2N}$ topological quantum spin liquids in lattice systems with $p4 \times \mathrm{SO}(3)$ symmetry, if translations and C_4 both act as charge conjugation.

Lattice homotopy class	Symmetry fractionalization class
0	$\{n_1, n_2, n_3, 0\}$, $\{0, n_2, n_3, 1\}$, or $\{2, n_2, n_3, 1\}$ for N even, $\{n_1, n_2, n_3, 0\}$ or $\{0,0,1,1\}$ for N odd
a	$\{1,1,0,1\}$ for N odd
b	$\{1, n_2, n_3, 1\}$ for N even, $\{1,0,1,1\}$ for N odd
c	$\{0,0,0,1\}$ for N odd
$a+b$	$\{0,1,0,1\}$ for N odd
$a+c$	$\{1,1,1,1\}$ for N odd
$b+c$	$\{1,0,0,1\}$ for N odd
$a+b+c$	$\{0,1,1,1\}$ for N odd

detail by Kitaev [26] and are reviewed below. The exactly solvable model in Ref. [26] has triggered enormous interest in realizing the $\mathrm{Ising}^{(1)}$ state in real materials [64–66]. We remark that usually these Kitaev quantum spin liquids are discussed in the context of spin-orbit coupled systems, but here we consider them in systems without spin-orbit coupling for simplicity. In particular, we classify $\mathrm{Ising}^{(\nu)}$ states in lattice systems with $p6\times\mathrm{SO}(3)$ or $p4\times\mathrm{SO}(3)$ symmetry.

The $\mathrm{Ising}^{(\nu)}$ state has three anyons $\{I,\sigma,\psi\}$, where the trivial anyon here is denoted I and the nontrivial fusion rules are given by

$$\psi\times\psi=I,\qquad \sigma\times\psi=\sigma,\qquad \sigma\times\sigma=I+\psi. \tag{46}$$

The nontrivial F symbols are

$$F_\sigma^{\psi\sigma\psi}=F_\psi^{\sigma\psi\sigma}=-1,$$
$$[F_\sigma^{\sigma\sigma\sigma}]_{ab}=\frac{\varkappa_\sigma}{\sqrt{2}}\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix}_{ab}. \tag{47}$$

Here, the column and row labels of the matrix take values I and ψ (in this order). All other F symbols are 1 if it is compatible with the fusion rule and 0 if it is not. $\varkappa_\sigma=(-1)^{(\nu^2-1)/8}$ is the Frobenius-Schur indicator of σ.

The nontrivial R symbols are

$$R^{\psi\psi}=-1,\qquad R_\sigma^{\psi\sigma}=R_\sigma^{\sigma\psi}=(-i)^\nu,$$
$$R_I^{\sigma\sigma}=\varkappa_\sigma e^{-i(\pi/8)\nu},\qquad R_\psi^{\sigma\sigma}=\varkappa_\sigma e^{i(3\pi/8)\nu}. \tag{48}$$

The topological spins are $\theta_\psi=-1$ and $\theta_\sigma=e^{i(\pi/8)\nu}$, and the chiral central charge $c_-=(\nu/2)$.

The topological symmetry of $\mathrm{Ising}^{(\nu)}$ is trivial, and no symmetry of $\mathrm{Ising}^{(\nu)}$ can permute anyons. The U symbol and a set of η symbols can all be chosen to be 1.

The symmetry fractionalization classes of $p6\times\mathrm{SO}(3)$ are classified by

$$H^2[p6\times\mathrm{SO}(3),\mathbb{Z}_2]\cong(\mathbb{Z}_2)^3. \tag{49}$$

We can label these elements as

$$w=n_1\cdot B_{xy}+n_2\cdot A_c^2+n_3\cdot w_2, \tag{50}$$

with $n_{1,2,3}\in\{0,1\}$. The symmetry fractionalization classes of $p4\times\mathrm{SO}(3)$ are given by

$$H^2[p4\times\mathrm{SO}(3),\mathbb{Z}_2]\cong(\mathbb{Z}_2)^4. \tag{51}$$

We can label these elements as

$$w=n_1\cdot B_{xy}+n_2\cdot B_{c^2}+n_3\cdot A_{x+y}^2+n_4\cdot w_2, \tag{52}$$

with $n_{1,2,3,4}\in\{0,1\}$. The representative cochains of these elements are presented in Appendix C. Physically, these generators can be viewed as detecting whether the non-Abelian anyon σ carries a projective quantum number under these global symmetries. For each symmetry fractionalization class, the U and η symbols can be obtained via Eqs. (4) and (5).

The above discussion implies that, without considering anomaly matching, there are in total $2^3=8$ different $p6\times\mathrm{SO}(3)$ symmetric $\mathrm{Ising}^{(\nu)}$ states and $2^4=16$ different $p4\times\mathrm{SO}(3)$ symmetric $\mathrm{Ising}^{(\nu)}$ states. Calculating the anomaly indicators for the $\mathrm{Ising}^{(\nu)}$ state in a way similar to the calculation for the $\mathrm{U}(1)_{2N}$ state, we find that, for any symmetry fractionalization class of either $p6\times\mathrm{SO}(3)$ or $p4\times\mathrm{SO}(3)$ symmetry, all anomaly indicators always evaluate to 1 and, hence, the anomaly is always absent. Therefore, all $p6\times\mathrm{SO}(3)$ or $p4\times\mathrm{SO}(3)$ symmetry-enriched $\mathrm{Ising}^{(\nu)}$ topological quantum spin liquids can emerge in lattice systems within lattice homotopy class 0 (including, for example, honeycomb lattice spin-1/2 system or spin-1 system on any lattice) but no other lattice homotopy class (including, for example, spin-1/2 system on triangular, kagome, square, and checkerboard lattices). We notice that, in most previous discussions of the $\mathrm{Ising}^{(1)}$ state in spin-orbit coupled systems, the underlying lattice systems indeed have a trivial anomaly, since they can be obtained from the lattice homotopy class 0 here by breaking certain symmetries.

VIII. $\mathbb{Z}_N$ TOPOLOGICAL ORDERS: GENERALIZED TORIC CODES

In this section, we consider the $\mathbb{Z}_N$ topological order, which is the $\mathbb{Z}_N$ generalization of the famous $\mathbb{Z}_2$ topological order [3,7,67–71]. The case with $N=2$ has been studied extensively in many different types of lattice systems. However, as mentioned in the introduction, when $N>2$, these states do not allow a description in terms of a simple parton mean field (instead, the partons have to be strongly interacting), and they are much less explored (see examples in Refs. [72–77]). Our framework in Sec. V allows us to classify a general $\mathbb{Z}_N$ topological quantum spin liquid enriched by a general symmetry. For concreteness, the symmetry we consider below is one of these four: $p6m\times\mathrm{SO}(3)\times\mathbb{Z}_2^T$, $p4m\times\mathrm{SO}(3)\times\mathbb{Z}_2^T$, $p6m\times\mathbb{Z}_2^T$, and $p4m\times\mathbb{Z}_2^T$, where $p6m$ and $p4m$ are lattice symmetries, while SO(3) and $\mathbb{Z}_2^T$ are on-site spin rotational symmetry and time-reversal symmetry, respectively.

In the $\mathbb{Z}_N$ topological order, there are N^2 anyons in total, which can be labeled by two integers as $a=(a_e,a_m)$, with $a_e,a_m\in\{0,\ldots,N-1\}$. Following the convention in the $\mathbb{Z}_2$ toric code, we call the anyon labeled by (1,0) as e and the anyon labeled by (0,1) as m. The fusion rules are elementwise addition modulo N, i.e.,

$$(a_e,a_m)\times(b_e,b_m)=([a_e+b_e]_N,[a_m+b_m]_N). \tag{53}$$

In a choice of gauge, the F symbols of this topological order are all 1, and the R symbols are given by

$$R^{ab} = e^{i(2\pi/N)a_m b_e}. \tag{54}$$

The topological symmetry group is complicated to determine for general N [59,78]. For $N=2$, the topological symmetry is $\mathbb{Z}_2 \times \mathbb{Z}_2^T$, generated by the unitary electric-magnetic duality symmetry S that exchanges e and m, i.e., $(a_e, a_m) \to (a_m, a_e)$, and an antiunitary symmetry T which exchanges e and m the same way as S. For this $\mathbb{Z}_2 \times \mathbb{Z}_2^T$ symmetry, we can choose the U symbols as

$$U_{\mathbf{g}}(a,b;c) = \begin{cases} (-1)^{a_m b_e}, & \mathbf{g} \text{ permutes anyons,} \\ 1, & \text{otherwise} \end{cases} \tag{55}$$

and a set of η symbols as

$$\eta_a(\mathbf{g},\mathbf{h}) = \begin{cases} (-1)^{a_e a_m}, & \mathbf{g},\mathbf{h} \text{ permute anyons,} \\ 1, & \text{otherwise,} \end{cases} \tag{56}$$

with (a_e, a_m) and (b_e, b_m) the anyon labels of anyons a and b, respectively.

For $N \geq 3$, there is always a $\mathbb{Z}_4^T \rtimes \mathbb{Z}_2$ topological symmetry. The antiunitary $\mathbb{Z}_4^T$ is generated by an action $T: (a_e, a_m) \to (a_m, [-a_e]_N)$, and the unitary $\mathbb{Z}_2$ is generated by an action $S: (a_e, a_m) \to (a_m, a_e)$. The two generators satisfy the relation

$$S^2 = \mathbf{1}, \qquad T^4 = \mathbf{1}, \qquad STS = T^{-1}. \tag{57}$$

For this $\mathbb{Z}_4^T \rtimes \mathbb{Z}_2$ symmetry, writing a group element as $\mathbf{g} = T^{g_1} S^{g_2}$, with $g_1 \in \{0,1,2,3\}$ and $g_2 \in \{0,1\}$, we can choose the U symbols as

$$U_{\mathbf{g}}(a,b;c) = \begin{cases} e^{(2\pi i/N)a_m b_e}, & g_1 + g_2 \text{is odd,} \\ 1, & \text{otherwise} \end{cases} \tag{58}$$

and a set of η symbols as

$$\eta_a(\mathbf{g},\mathbf{h}) = \begin{cases} e^{(2\pi i/N)a_e a_m}, & g_1+g_2, h_1+h_2 \text{ are odd,} \\ 1, & \text{otherwise.} \end{cases} \tag{59}$$

For certain $N \geq 3$, there can be other topological symmetries, in addition to the above $\mathbb{Z}_4 \rtimes \mathbb{Z}_2^T$ symmetry. For example, when $N=5$, the action $(a_e, a_m) \to ([3a_e]_5, [3a_m]_5)$ is an antiunitary topological symmetry. For simplicity, below we focus on the cases where $N = 2$, 3, 4.

The analysis of the classification is similar to the previous cases. In the present case, we need to understand the anomaly indicators of the $p6m \times \mathrm{SO}(3) \times \mathbb{Z}_2^T$, $p4m \times \mathrm{SO}(3) \times \mathbb{Z}_2^T$, $p6m \times \mathbb{Z}_2^T$, and $p4m \times \mathbb{Z}_2^T$ symmetries. These anomaly indicators and their values for different lattice homotopy classes can be found in Appendix D. Carrying out the procedure listed in Sec. V, we can obtain the classification. In Table X, we list the number of different symmetry-enriched $\mathbb{Z}_N$ topological quantum spin liquids in different lattice homotopy classes under these symmetries. The precise symmetry fractionalization classes in each case can be found in Appendix C. We also upload codes using which one can (i) see all symmetry fractionalization classes of the symmetry-enriched states within each lattice homotopy class and (ii) check which lattice homotopy class a given symmetry-enriched state belongs to [79]. Below, we comment on some of these results.

For the case with $N=2$ and the $p6m \times \mathrm{SO}(3) \times \mathbb{Z}_2^T$ symmetry, the classification was carried out for spin-1/2 systems on a triangular, kagome, and honeycomb lattice [14,15,80], which belongs to the lattice homotopy class a, c, and 0, respectively. For lattice homotopy classes a and c, our results agree with those in Refs. [14,15]. For the lattice homotopy class 0, using the parton-mean-field approach and assuming that one of e and m carries spin 1/2 under the SO(3) symmetry, Ref. [80] found 128 different states. We find 336 states in total, where 128 of them have one of e and m carrying half-odd-integer spin, and in the other 208 states both e and m carry integer spin, nine of which also have symmetries permuting e and m. For the case with $N=2$ and the $p4m \times \mathrm{SO}(3) \times \mathbb{Z}_2^T$ symmetry, Ref. [81] found 64 states on the square lattice spin-1/2 system, which belongs to our lattice homotopy class a, agreeing with our results.

For the case with $N=2$ and the $p4m \times \mathbb{Z}_2^T$ symmetry, using the parton-mean-field approach, Ref. [82] found 64 states on the square lattice system with Kramers doublet spins, which can all be obtained from the $p4m \times \mathrm{SO}(3) \times \mathbb{Z}_2^T$ symmetric $\mathbb{Z}_2$ topological quantum spin liquids by breaking the SO(3) symmetry. Suppose, in the $p4m \times \mathrm{SO}(3) \times \mathbb{Z}_2^T$ symmetric version of these states, the anyon e carries half-odd-integer spin under SO(3); then, projective quantum numbers of m are fixed for all these 64 states [14]. In particular, m experiences no nontrivial symmetry fractionalization pattern that simultaneously involves the time-reversal and lattice symmetries. The absence of such a symmetry fractionalization pattern still holds in the 64 "within-parton" $p4m \times \mathbb{Z}_2^T$ symmetric states obtained by breaking SO(3). However, in addition to these 64 states, we have found $117 - 64 = 53$ other states, with their symmetry fractionalization classes presented in Appendix E (anyons are not permuted by symmetries in all these 117 states). A common property of these 53 states is the presence of nontrivial symmetry fractionalization involving both the lattice symmetry and time-reversal symmetry for the anyon m; e.g., translation and time reversal may not commute for m. Furthermore, for all 117 states, the $C_2 \equiv C_4^2$ symmetry fractionalizes on the m anyon; i.e., effectively, $C_2^2 = -1$ for m. Usually, the interpretation of this phenomenon is that there is a background e anyon at

TABLE X. Number of various topological quantum spin liquids enriched by $p6m \times \mathrm{SO}(3) \times \mathbb{Z}_2^T$, $p6m \times \mathbb{Z}_2^T$, $p4m \times \mathrm{SO}(3) \times \mathbb{Z}_2^T$, or $p4m \times \mathbb{Z}_2^T$ symmetry, where the third, fourth, and fifth columns represent $\mathbb{Z}_2$, $\mathbb{Z}_3$, and $\mathbb{Z}_4$ topological orders, respectively, while the last two columns represent the $\mathrm{U}(1)_2 \times \mathrm{U}(1)_{-2}$ and $\mathrm{U}(1)_4 \times \mathrm{U}(1)_{-4}$ topological orders, respectively. The details of the symmetry fractionalization classes of each state can be found in Appendix C. For $\mathbb{Z}_2$ and $\mathbb{Z}_4$ topological orders, we also upload codes containing the symmetry fractionalization class for each state in each lattice homotopy class [79]. For $\mathbb{Z}_3$, $\mathrm{U}(1)_2 \times \mathrm{U}(1)_{-2}$, and $\mathrm{U}(1)_4 \times \mathrm{U}(1)_{-4}$ topological orders, all symmetry-enriched states are anomaly-free.

Symmetry group	Lattice homotopy class	$\mathbb{Z}_2$	$\mathbb{Z}_3$	$\mathbb{Z}_4$	$\mathrm{U}(1)_2 \times \mathrm{U}(1)_{-2}$	$\mathrm{U}(1)_4 \times \mathrm{U}(1)_{-4}$
$p6m \times \mathrm{SO}(3) \times \mathbb{Z}_2^T$	0	336	8	16 453	32	144
	a	8	0	70	0	0
	c	8	0	70	0	0
	$a+c$	4	0	82	0	0
$p6m \times \mathbb{Z}_2^T$	0	208	8	4725	16	72
	a	13	0	61	0	0
	c	13	0	61	0	0
	$a+c$	12	0	167	0	0
$p4m \times \mathrm{SO}(3) \times \mathbb{Z}_2^T$	0	3653	9	886 740	128	1344
	a	64	0	5008	0	0
	b	64	0	5008	0	0
	c	64	0	8872	0	0
	$a+b$	16	0	636	0	0
	$a+c$	16	0	656	0	0
	$b+c$	16	0	656	0	0
	$a+b+c$	8	0	318	0	0
$p4m \times \mathbb{Z}_2^T$	0	2629	9	280 852	64	672
	a	117	0	3491	0	0
	b	117	0	3491	0	0
	c	193	0	12 449	0	0
	$a+b$	33	0	513	0	0
	$a+c$	34	0	610	0	0
	$b+c$	34	0	610	0	0
	$a+b+c$	21	0	309	0	0

each square lattice site (the C_4 center), and the mutual braiding statistics between e and m yields $C_2^2 = -1$. However, for 16 of the 53 "beyond-parton" states, $(T_1C_2)^2 = (T_2C_2)^2 = -1$ for m, which seems to suggest that there are also background e anyons at the twofold rotation centers of T_1C_2 and T_2C_2, although microscopically there is no spin at those positions. So the analysis based on anomaly matching suggests that the simple picture where the fractionalization of rotational symmetries purely comes from background anyons is actually incomplete.

The above example shows that, even for simple states like the $\mathbb{Z}_2$ topological order, the parton-mean-field approach may miss some of their symmetry enrichment patterns, and our framework in Sec. V is more general. Note that here by "parton mean field" we are referring to the usual parton mean fields where the partons are noninteracting. If the partons are allowed to interact strongly, say, if they form nontrivial interacting symmetry-protected topological states under the projective symmetry group of the partons, symmetry-enriched states not captured by Ref. [82] may arise, but it is technically complicated to study them. Also, by using parton constructions other than the one in Ref. [82], one may also obtain states beyond those in Ref. [82], but it is challenging to make this approach systematic.

We also notice that the number of $\mathbb{Z}_3$ topological quantum spin liquids is nonzero only in the lattice homotopy class 0. This phenomenon is actually true for general odd N. To see it, first notice all lattice homotopy classes except 0 have some mixed anomalies between the SO(3) symmetry and the lattice symmetry [8]. In order to match this anomaly, it is impossible for both e and m to carry integer spin. Suppose that e carries half-odd-integer spin, and consider threading an SO(3) monopole through the system. The monopole will be viewed as a π flux from the perspective of e. Then, the local nature of the monopole implies that it must trap an anyon that has π braiding statistics with e. For odd N, no such anyon exists, which leads to a contradiction. So $\mathbb{Z}_N$ topological quantum spin liquids with N odd cannot possibly arise in lattice homotopy class other than 0. Note that the above argument does not rely on the time-reversal symmetry, and it is valid no matter how the symmetries permute anyons.

For $\mathbb{Z}_N$ topological quantum spin liquids in systems belonging to a lattice homotopy class other than 0, which requires N to be even, anyons (1, 0) and $(0, N/2)$ cannot simultaneously carry half-odd-integer spin; otherwise, there would be a mixed anomaly between the SO(3) and time-reversal symmetries [83].

IX. $\mathrm{U}(1)_{2N} \times \mathrm{U}(1)_{-2N}$ TOPOLOGICAL ORDERS: GENERALIZATIONS OF THE DOUBLE-SEMION STATE

In this section, we consider the $\mathrm{U}(1)_{2N} \times \mathrm{U}(1)_{-2N}$ topological order, which is the generalization of the double-semion state, i.e., the case with $N = 1$. Effectively, this state can be obtained by stacking a $\mathrm{U}(1)_{2N}$ state, which is discussed in Sec. VI, on its time-reversal partner, the $\mathrm{U}(1)_{-2N}$ state. In addition, these states can also be constructed via the twisted quantum double models or the string-net models [84,85]. We would like to classify the $\mathrm{U}(1)_{2N} \times \mathrm{U}(1)_{-2N}$ topological order enriched by one of these four symmetries: $p6m \times \mathrm{SO}(3) \times \mathbb{Z}_2^T$, $p4m \times \mathrm{SO}(3) \times \mathbb{Z}_2^T$, $p6m \times \mathbb{Z}_2^T$, and $p4m \times \mathbb{Z}_2^T$.

In a $\mathrm{U}(1)_{2N} \times \mathrm{U}(1)_{-2N}$ topological quantum spin liquid, there are $4N^2$ anyons in total, which can be labeled by two integers as $a = (a_s, a_{\bar{s}})$, with $a_s, a_{\bar{s}} \in \{0, \ldots, 2N - 1\}$. Following the convention in the double-semion state, we call the anyon labeled by (1, 0) as s and the anyon labeled by (0, 1) as $\bar{s}$ (note in this convention that s and $\bar{s}$ are not antiparticles of each other). The fusion rules are element-wise addition modulo $2N$, i.e.,

$$(a_s, a_{\bar{s}}) \times (b_s, b_{\bar{s}}) = ([a_s + b_s]_{2N}, [a_{\bar{s}} + b_{\bar{s}}]_{2N}). \tag{60}$$

In a choice of gauge, the F symbols of the theory are

$$F^{abc} = \exp\bigg(i\frac{2\pi}{N}(a_s(b_s + c_s - [b_s + c_s]_{2N}) - a_{\bar{s}}(b_{\bar{s}} + c_{\bar{s}} - [b_{\bar{s}} + c_{\bar{s}}]_{2N}))\bigg) \tag{61}$$

and the R symbols are

$$R^{ab} = \exp\left(i\frac{2\pi}{N}(a_s b_s - a_{\bar{s}} b_{\bar{s}})\right). \tag{62}$$

The topological symmetry group is complicated to determine for general N, just like the $\mathrm{U}(1)_{2N}$ state [59,78]. Here, we list the topological symmetry group for $N = 1, 2$. For $N = 1$, the topological symmetry is $\mathbb{Z}_2^T$, generated by $\tilde{S}$ exchanging s and $\bar{s}$, i.e.,

$$(a_s, a_{\bar{s}}) \to (a_{\bar{s}}, a_s). \tag{63}$$

We can choose the U symbols and a set of η symbols all equal to 1.

For $N = 2$, the topological symmetry is $\mathbb{Z}_4^T \rtimes \mathbb{Z}_2^T$, generated by an order 2 antiunitary symmetry $\tilde{S}$ which exchanges s and $\bar{s}$, $\tilde{S}: (a_s, a_{\bar{s}}) \to (a_{\bar{s}}, a_s)$, and another order 4 antiunitary symmetry T, which permutes anyons in the following way $T: (a_s, a_{\bar{s}}) \to (a_{\bar{s}}, [-a_s]_{2N})$. The two generators satisfy the relation

$$\tilde{S}^2 = \mathbf{1}, \qquad T^4 = \mathbf{1}, \qquad \tilde{S}T\tilde{S} = T^{-1}. \tag{64}$$

An element in $\mathbb{Z}_4^T \rtimes \mathbb{Z}_2^T$ can be written as $T^{g_1}\tilde{S}^{g_2}$, with $g_1 \in \{0, \ldots, 3\}$ and $g_2 \in \{0, 1\}$. To define the U symbols, first we define the following function:

$$\tilde{U}(a_s, b_s) = \begin{cases} (-1)^{a_s} & b_s \neq 0, \\ 1 & b_s = 0. \end{cases} \tag{65}$$

Given an element $\mathbf{g} \in \mathbb{Z}_4^T \rtimes \mathbb{Z}_2^T$, the U symbols can be chosen such that

$$U_{\mathbf{g}}(a, b; c) = \begin{cases} 1 & g_1 = 0, \\ \tilde{U}(a_{\bar{s}}, b_{\bar{s}}) & g_1 = 1, \\ \tilde{U}(a_s, b_s)\tilde{U}(a_{\bar{s}}, b_{\bar{s}}) & g_1 = 2, \\ \tilde{U}(a_s, b_s) & g_1 = 3. \end{cases} \tag{66}$$

And a set of η symbols can be chosen to be all identity.

Carrying out the procedure in Sec. V in a manner similar to the previous examples, we can obtain the classification of $\mathrm{U}(1)_2 \times \mathrm{U}(1)_{-2}$ and $\mathrm{U}(1)_4 \times \mathrm{U}(1)_{-4}$ topological quantum spin liquids enriched by $p6m \times \mathrm{SO}(3) \times \mathbb{Z}_2^T$, $p4m \times \mathrm{SO}(3) \times \mathbb{Z}_2^T$, $p6m \times \mathbb{Z}_2^T$, or $p4m \times \mathbb{Z}_2^T$ symmetry. The results are summarized in Table X.

We notice that, in all symmetry groups considered here, $\mathrm{U}(1)_2 \times \mathrm{U}(1)_{-2}$ and $\mathrm{U}(1)_4 \times \mathrm{U}(1)_{-4}$ can arise only in the lattice homotopy class 0. Reference [86] presented a physical reason for this phenomenon. If we consider only the symmetry group $p6m \times \mathrm{SO}(3) \times \mathbb{Z}_2^T$ and $p4m \times \mathrm{SO}(3) \times \mathbb{Z}_2^T$, the following simpler argument can explain it. To be concrete, suppose the symmetry group is $p6m \times \mathrm{SO}(3) \times \mathbb{Z}_2^T$, and a similar argument can be made if the symmetry group is $p4m \times \mathrm{SO}(3) \times \mathbb{Z}_2^T$. Now suppose breaking the symmetry to $p6 \times \mathrm{SO}(3)$. Then, the system can be viewed as a $p6 \times \mathrm{SO}(3)$ symmetric $\mathrm{U}(1)_{2N}$ state on top of a $p6 \times \mathrm{SO}(3)$ symmetric $\mathrm{U}(1)_{-2N}$ state, and these two states must have opposite anomalies under the $p6 \times \mathrm{SO}(3)$ symmetry; otherwise, they cannot be connected by time reversal to form the original $p6m \times \mathrm{SO}(3) \times \mathbb{Z}_2^T$ symmetric state. Namely, after breaking $p6m \times \mathrm{SO}(3) \times \mathbb{Z}_2^T$ to $p6 \times \mathrm{SO}(3)$, there is no remaining anomaly, and the state is in lattice homotopy class 0. Now we ask which lattice homotopy class with a $p6m \times \mathrm{SO}(3) \times \mathbb{Z}_2^T$ symmetry becomes the lattice homotopy class 0 with a $p6 \times \mathrm{SO}(3)$ symmetry after this symmetry breaking. From the representative configurations of all lattice homotopy classes in Sec. IV, clearly only the lattice homotopy class 0 does.

X. DISCUSSION

In this paper, we have presented a general framework in Sec. V to classify symmetry-enriched topological quantum spin liquids in two spatial dimensions. This framework applies to all topological quantum spin liquids, which may be Abelian or non-Abelian and chiral or nonchiral. The symmetry we consider may include both lattice symmetry and internal symmetry, may contain antiunitary symmetry, and may permute anyons. We then apply this framework to various examples in Secs. VI–IX. As argued in the introduction, our framework combines the advantages of the previous approaches in the literature while avoiding their disadvantages. Indeed, we are able to identify symmetry-enriched topological quantum spin liquids that are not easily captured by the usual parton-mean-field approach (see examples in Sec. VIII), and we can systematically distinguish different lattice systems with the same symmetry group using their quantum anomalies.

We finish this paper by discussing some open questions.

(i) In this paper, we characterize a topological quantum spin liquid with a lattice symmetry by one with an internal symmetry via the crystalline equivalence principle in Sec. III D. However, it is more ideal to have a theory that directly describes topological quantum spin liquids with lattice symmetries.

Such a theory should be able to tell how an arbitrary symmetry acts on a state obtained by creating some anyons from the ground state and putting them at arbitrary positions. The symmetry action should be some analog of Eq. (2), but it is subtle to understand what constraints the analogs of $U_{\mathbf{g}}(a,b;c)$ and $\eta_a(\mathbf{g},\mathbf{h})$ should satisfy. So far, this question has been answered if the lattice symmetry contains only translation symmetry [13], but for cases with point group symmetries it is answered in a very specific case, where the lattice symmetry is reflection, and the state contains only two anyons that are (i) antiparticles of each other, (ii) transformed into each other under the reflection symmetry, and (iii) located at two reflection-related positions [87]. It is useful to have a complete theory that can answer this question in full generality. Such a theory is also helpful for the purpose of identifying observable signatures of different symmetry-enriched topological quantum spin liquids.

(ii) Strictly speaking, our classification is a classification of different patterns of how symmetries permute anyons and the symmetry fractionalization patterns. In principle, one should further consider how the classification is modified upon stacking an invertible state on the topological quantum spin liquid with the same symmetry. This question is subtle, because some nontrivial invertible states can be trivialized in the presence of a long-range entangled state [7,83,88]. We leave this problem for future study.

(iii) In this paper, we focus on how symmetry permutes anyons and the symmetry fractionalization classes, which can be viewed as the bulk properties of different symmetry-enriched topological quantum spin liquids. It is also interesting to explore their boundary properties in the future. In particular, sometimes the symmetry enrichment pattern may enforce the boundary of the topological quantum spin liquid to be gapless, even if it is nonchiral [89,90]. Similarly, it is intriguing to study the properties of defects in different symmetry-enriched topological quantum spin liquids and examine their potential to perform quantum computation [7,91].

(iv) It is useful to find numerical algorithms to identify the symmetry enrichment pattern of a topological quantum spin liquid that emerges in a lattice model that is not fine-tuned and find experimental methods to detect the symmetry enrichment pattern in experiments. Some previous proposals for various specific cases include Refs. [92–96], but it is useful to find algorithms and methods applicable to the general setting.

(v) After classifying different symmetry-enriched topological quantum spin liquids and finding methods to detect them numerically and experimentally, it is important to construct explicit models that realize these topological orders. For many topological orders enriched by internal symmetries, Refs. [97–100] construct their exactly solvable models with explicit Hamiltonians and ground-state wave functions. Moreover, there are many proposals for realizing symmetry-protected and symmetry-enriched topological states with lattice symmetries in the literature, including Refs. [42,44,101–106]. We anticipate that we can combine the above constructions to obtain exactly solvable models with concrete Hamiltonians that realize the symmetry-enriched topological quantum spin liquids discussed in this paper.

It will be also interesting to find quantum materials and develop quantum simulators to realize these different phases and explore interesting continuous quantum phase transitions out of them, which are beyond the conventional Landau-Ginzburg-Wilson-Fisher paradigm.

(vi) In this paper, our focus is topological quantum spin liquids in two spatial dimensions. It is interesting to generalize our work to other systems, such as fermionic systems, systems in higher dimensions, systems with spin-orbit coupling, gapless systems, and fractonic systems.

In particular, many interesting experimental systems feature spin-orbit couplings, and the general framework in the present paper can be straightforwardly extended to such systems. Because systems with spin-orbit coupling can often be obtained by breaking some symmetries in systems without spin-orbit coupling, there can be two main differences between these two types of systems. First, compared to systems without spin-orbit coupling, the distinction between some quantum phases may disappear in systems with spin-orbit coupling, since the latter has a smaller symmetry compared to the former and the relevant distinction may be well defined only in the presence of a larger symmetry. Second, there can be quantum phases that can be realized in systems with spin-orbit coupling but not in systems without spin-orbit coupling; i.e., they are incompatible with a larger symmetry. These are both intriguing phenomena that deserve future investigation.

Also, there are many experimental candidates of $(3+1)$-dimensional symmetry-enriched gapless U(1) quantum spin liquids in pyrochlores [107], and their classification has been discussed within the framework of projective symmetry groups [108–113]. As discussed in the present paper, the classification based on projective symmetry groups may be incomplete. Using more general approaches, U(1) quantum spin liquids with only internal symmetries have been classified [16,83,88,114], and some examples of their lattice symmetry-enriched versions have been constructed [115]. However, a systematic classification of $(3+1)$-dimensional U(1) quantum spin liquids enriched by both lattice and internal symmetries is lacking, and it is interesting to apply the idea in the present paper to those settings in the future.

Codes for checking anomaly matching and details of realizations for $\mathbb{Z}_2$ and $\mathbb{Z}_4$ topological order are available [116].

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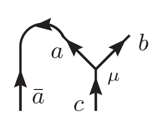
APPENDIX A: TRANSLATION BETWEEN THE CHARACTERIZATION OF REFLECTION SYMMETRY AND TIME-REVERSAL SYMMETRY

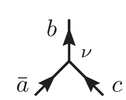
It is a common folklore that "reflection symmetry = time-reversal symmetry × charge conjugation." However, one precise formulation of the statement is based on the CPT theorem, which is formulated in relativistic quantum field theory and requires Lorentz symmetry as a premise [117]. In the context of topological order, even though Lorentz symmetry is not explicitly present, it is also widely believed that the statement also holds true. However, the precise correspondence between reflection symmetry and time-reversal symmetry, especially the matching between the data $\{\rho_{\mathbf{g}}; U_{\mathbf{g}}(a,b;c), \eta_a(\mathbf{g},\mathbf{h})\}$ for these two symmetries, is little known in the literature. We summarize this correspondence in this appendix.

For this purpose, we need more formal treatments of the topological order in terms of a unitary modular tensor category (UMTC), which go beyond what is reviewed in Sec. III, and we refer the interested readers to Sec. II in Ref. [20] for the basics. Following the convention of Refs. [7,87,118], we model the time-reversal symmetry action on the UMTC as a $\mathbb{C}$-antilinear functor and the (unitary) reflection symmetry action as an *antimonoidal* functor (also see Ref. [27] for slightly different treatments). Therefore, mathematically speaking, in this appendix we establish a precise correspondence between the data of a $\mathbb{C}$-antilinear functor and the data of an antimonoidal functor. We believe that such correspondence will imply the correspondence of the data $\{\rho_{\mathbf{g}}; U_{\mathbf{g}}(a,b;c), \eta_a(\mathbf{g},\mathbf{h})\}$ for these two symmetries on the explicit wave functions of the topological order, and we defer it to future study.

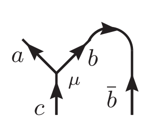
Throughout the appendix, we assume that the reflection symmetry is unitary. We can also consider antiunitary reflection symmetry, which in the crystalline equivalence principle should correspond to a unitary symmetry which does not reflect spacetime. Following the treatment in this appendix, we can similarly establish a correspondence between a $\mathbb{C}$-linear functor for the unitary symmetry and a $\mathbb{C}$-antilinear antimonoidal functor for the antiunitary reflection symmetry. The details can be worked out by following closely the treatment in this appendix, and we omit them.

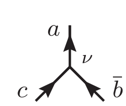
Recall that anyon lines may be "bent" using the A and B symbols, given diagrammatically by



$= \sum_{\nu} \left[A_c^{ab}\right]_{\mu\nu}$


, (A1)



$= \sum_{\nu} \left[B_c^{ab}\right]_{\mu\nu}$


. (A2)

They can be expressed in terms of F symbols by

$$[A_c^{ab}]_{\mu\nu} = \sqrt{\frac{d_a d_b}{d_c}} \varkappa_a^* \left[F_b^{\bar{a}ab}\right]^*_{1,(c,\mu,\nu)}, \tag{A3}$$

$$[B_c^{ab}]_{\mu\nu} = \sqrt{\frac{d_a d_b}{d_c}} \left[F_a^{ab\bar{b}}\right]_{(c,\mu,\nu),1}, \tag{A4}$$

where the phase $\varkappa_a$ is the Frobenius-Schur indicator

$$\varkappa_a = d_a F_{a11}^{a\bar{a}a}. \tag{A5}$$

Let us start with a time-reversal symmetry $\mathcal{T}$ and construct a unitary reflection symmetry $\mathcal{R}$ from $\mathcal{T}$ as follows. The action of $\mathcal{R}$ on anyon a is

$$\mathcal{R}: a \to {}^{\mathcal{T}}\bar{a}, \tag{A6}$$

and the action of $\mathcal{R}$ on the topological state $|a,b;c\rangle_\mu$ is

$$F_{\mathcal{R}}|a,b;c\rangle_\mu \equiv \sum_\nu U_{\mathcal{R}}({}^{\mathcal{R}}b, {}^{\mathcal{R}}a; {}^{\mathcal{R}}c)_{\mu\nu} |{}^{\mathcal{R}}b, {}^{\mathcal{R}}a; {}^{\mathcal{R}}c\rangle_\nu, \tag{A7}$$

where $U_{\mathcal{R}}(a,b;c)$ is an $N_{ab}^c \times N_{ab}^c$ matrix that is defined in terms of matrix multiplication as follows:

$$U_{\mathcal{R}}(a,b;c) \equiv U_{\mathcal{T}}(\bar{b},\bar{a};\bar{c})^* \left(B_{\bar{c}}^{\bar{b},\bar{a}}\right)^* \left(A_{\bar{b}}^{\bar{c},a}\right) \left(B_a^{c,\bar{b}}\right)^*, \tag{A8}$$

and we suppress indices when $N_{ab}^c > 1$. Note that the positions of ${}^{\mathcal{R}}a$ and ${}^{\mathcal{R}}b$ are flipped compared to a and b on the two sides of Eq. (A7). Therefore, we call $F_{\mathcal{R}}$ an antimonoidal functor instead of a monoidal functor. The extra factors take account of the "flipping" of anyons after charge conjugation.

Now we explicitly check that various consistency conditions for antimonoidal functors can indeed be satisfied. To preserve the structure of braiding and fusion, under the action of $\mathcal{R}$, the F and R symbols should transform according to

$$F_{\mathcal{R}}[F_{def}^{abc}] = U_{\mathcal{R}}({}^{\mathcal{R}}b, {}^{\mathcal{R}}a; {}^{\mathcal{R}}e) U_{\mathcal{R}}({}^{\mathcal{R}}c, {}^{\mathcal{R}}e; {}^{\mathcal{R}}d) \left(F_{{}^{\mathcal{R}}d}^{{}^{\mathcal{R}}c{}^{\mathcal{R}}b{}^{\mathcal{R}}a}\right)^{-1}_{{}^{\mathcal{R}}e{}^{\mathcal{R}}f} U_{\mathcal{R}}^{-1}({}^{\mathcal{R}}c, {}^{\mathcal{R}}b; {}^{\mathcal{R}}f) U_{\mathcal{R}}^{-1}({}^{\mathcal{R}}f, {}^{\mathcal{R}}a; {}^{\mathcal{R}}d) = F_{def}^{abc},$$

$$F_{\mathcal{R}}[R_c^{ab}] = U_{\mathcal{R}}({}^{\mathcal{R}}a, {}^{\mathcal{R}}b; {}^{\mathcal{R}}c) \left(R_{{}^{\mathcal{R}}c}^{{}^{\mathcal{R}}a{}^{\mathcal{R}}b}\right)^{-1} U_{\mathcal{R}}({}^{\mathcal{R}}b, {}^{\mathcal{R}}a; {}^{\mathcal{R}}c)^{-1} = R_c^{ab}. \tag{A9}$$

This is indeed satisfied if we let $U_{\mathcal{R}}$ be the expression in Eq. (A8), which can be proven by a straightforward diagrammatic manipulation.

Under a vertex basis gauge transformation, $\Gamma_c^{ab}: V_c^{ab} \to V_c^{ab}$, according to the right-hand side of Eq. (A8), $[U_{\mathcal{R}}(a,b;c)]_{\mu\nu}$ transforms in the following way:

$$\tilde{U}_{\mathcal{R}}(a,b,c)_{\mu\nu} = \sum_{\mu',\nu'} \left[\Gamma_{{}^{\bar{\mathcal{R}}}c}^{{}^{\bar{\mathcal{R}}}b{}^{\bar{\mathcal{R}}}a}\right]_{\mu,\mu'} U_{\mathbf{g}}(a,b,c)_{\mu'\nu'} [(\Gamma_c^{ab})^{-1}]_{\nu'\nu}, \tag{A10}$$

with the shorthand $\bar{\mathcal{R}} = \mathcal{R}^{-1}$. This is indeed what we expect from an antimonoidal functor. Here, we use the gauge-fixing condition $\Gamma_a^{a1} = \Gamma_a^{1a} = \Gamma_1^{\bar{a}a} = 1$ regarding the vertex basis gauge transformation. Under a symmetry action gauge transformation, $[U_{\mathcal{R}}(a,b;c)]_{\mu\nu}$ transforms in the following way:

$$\tilde{U}_{\mathcal{R}}(a,b,c)_{\mu\nu} = \frac{\gamma_{\mathcal{T}}(\bar{a})^* \gamma_{\mathcal{T}}(\bar{b})^*}{\gamma_{\mathcal{T}}(\bar{c})^*} U_{\mathcal{R}}(a,b,c)_{\mu\nu}, \tag{A11}$$

which is also what we expect.

Finally, we should write down how the η symbols match. This can be done by considering the consistency equation between U symbols and η symbols. For example, suppose $\mathbf{g}$ is some unitary symmetry that does not reverse orientation, and we have

$$\begin{aligned}\kappa_{\mathcal{R},\mathbf{g}}(a,b;c) &\equiv U_{\mathcal{R}}(a,b;c)^{-1} U_{\mathbf{g}}({}^{\bar{\mathcal{R}}}b, {}^{\bar{\mathcal{R}}}a; {}^{\bar{\mathcal{R}}}c)^{-1} U_{\mathcal{R}\mathbf{g}}(a,b;c) \\ &= \kappa_{\mathcal{T},\mathbf{g}}(\bar{b},\bar{a};\bar{c})^* = \frac{\eta_{\bar{a}}(\mathcal{T},\mathbf{g})^* \eta_{\bar{b}}(\mathcal{T},\mathbf{g})^*}{\eta_{\bar{c}}(\mathcal{T},\mathbf{g})^*}. \end{aligned} \tag{A12}$$

Hence, the correspondence between η symbols should be given by the following equation:

$$\eta_a(\mathcal{R},\mathbf{g}) = \eta_{\bar{a}}(\mathcal{T},\mathbf{g})^*. \tag{A13}$$

Following similar derivation, we have

$$\eta_a(\mathbf{g},\mathcal{R}) = \frac{\eta_{\bar{a}}(\mathbf{g},\mathcal{T})^*}{U_{\mathbf{g}}(\bar{a},a;1)}, \tag{A14}$$

$$\eta_a(\mathcal{R}_\mathbf{1},\mathcal{R}_\mathbf{2}) = \eta_a(\mathcal{T}_\mathbf{1},\mathcal{T}_\mathbf{2}) U_{\mathcal{T}_1}(a,\bar{a};1). \tag{A15}$$

It is straightforward to check that the desired consistency conditions for the η symbols of reflection symmetries are also satisfied.

APPENDIX B: WALLPAPER GROUP SYMMETRIES: GROUP STRUCTURE AND $\mathbb{Z}_2$ COHOMOLOGY

In this appendix, for the readers' convenience, we collect necessary information of the wallpaper group symmetries appearing in this paper, $p6$, $p6m$, $p4$, and $p4m$, together with its $\mathbb{Z}_2$ cohomology. This will be important in the identification of symmetry fractionalization classes and the calculation of anomaly matching. A complete list of the $\mathbb{Z}_2$ cohomology for all the 17 wallpaper group symmetries is collected in Ref. [8].

The $\mathbb{Z}_2$ cohomology of wallpaper group symmetries is presented in terms of its $\mathbb{Z}_2$ cohomology ring. The product in the $\mathbb{Z}_2$ cohomology ring is understood as the cup product. Namely, given $\omega\in H^k(G,\mathbb{Z}_2)$ and $\eta\in H^n(G,\mathbb{Z}_2)$, we can define $\omega\cup\eta\in H^{k+n}(G,\mathbb{Z}_2)$, which is abbreviated to $\omega\eta$ in this paper, such that

$$(\omega\cup\eta)(g_1,\ldots,g_{k+n})=\omega(g_1,\ldots,g_k)\cdot\eta(g_{k+1},\ldots,g_l). \tag{B1}$$

Here, $\cdot$ is simply the multiplication in $\mathbb{Z}_2$. By identifying a set of generators $A_\bullet$, $B_\bullet$, etc., we can identify all elements in the $\mathbb{Z}_2$ cohomology with the help of addition and cup product.

We define a set of functions that take integers as their arguments:

$$P(x)=\begin{cases}1, & x \text{ is odd},\\ 0, & x \text{ is even},\end{cases}\qquad P_c(x)=1-P(x),$$
$$[x]_a=\{y=x \ (\mathrm{mod}\ a)|0\le y<a\},$$
$$P_{ab}(x)=\begin{cases}1, & x=b \ (\mathrm{mod}\ a),\\ 0, & \text{otherwise}.\end{cases} \tag{B2}$$

When writing down the cohomology corresponding to the Lieb-Schultz-Mattis constraints, we also need the cohomology of SO(3) and $\mathrm{SO}(3)\times\mathbb{Z}_2^T$. We use $w_2\in H^2[\mathrm{SO}(3),\mathbb{Z}_2]$ to denote the second Stiefel-Whitney class of SO(3) and $t\in H^1(\mathbb{Z}_2,\mathbb{Z}_2)$ to denote the generator for the $\mathbb{Z}_2$ cohomology of the time-reversal $\mathbb{Z}_2^T$ symmetry.

1. $p6$

This group is generated by T_1, T_2, and C_6, two translations with translation vectors that have the same length and an angle of $2\pi/3$, and a sixfold rotational symmetry, such that

$$C_6^6=1,\qquad C_6T_1C_6^{-1}=T_1T_2,$$
$$C_6T_2C_6^{-1}=T_1^{-1},\qquad T_1T_2=T_2T_1. \tag{B3}$$

An arbitrary element in $p6$ can be written as $g=T_1^xT_2^yC_6^c$, with $x,y\in\mathbb{Z}$ and $c\in\{0,1,2,3,4,5\}$.

The $\mathbb{Z}_2$ cohomology ring of $p6$ is

$$\mathbb{Z}_2[A_c;B_{xy}]/(B_{xy}^2=B_{xy}A_c^2). \tag{B4}$$

Here, $H^1(p6,\mathbb{Z}_2)=\mathbb{Z}_2$, with generator $\xi_1=A_c$, which have a representative cochain:

$$\xi_1(g)=c. \tag{B5}$$

$H^2(p6,\mathbb{Z}_2)=\mathbb{Z}_2^2$, with generators $\lambda_1=B_{xy}$ and $\lambda_2=A_c^2$, and we can choose the representative cochains to be

$$\begin{aligned}B_{xy}(g_1,g_2)=&P_{60}(c_1)y_1x_2+P_{61}(c_1)\left(\frac{x_2(x_2-1)}{2}+y_1x_2-y_2(x_2+y_1)\right)\\&+P_{62}(c_1)\left(\frac{y_2(y_2+1)}{2}-x_2-y_2(x_2+y_1)\right)+P_{63}(c_1)(-x_2+y_2-y_1x_2)\\&+P_{64}(c_1)\left(\frac{x_2(x_2-1)}{2}+y_2-y_1x_2-y_2(x_2-y_1)\right)+P_{65}(c_1)\left(\frac{y_2(y_2+1)}{2}-y_2(x_2-y_1)\right),\end{aligned} \tag{B6}$$

$$A_c^2(g_1,g_2)=c_1c_2. \tag{B7}$$

According to Ref. [8], the anomalies of $p6\times\mathrm{SO}(3)$ symmetric lattice systems in lattice homotopy classes a, c, and $a+c$ can be, respectively, written as

$$\exp\left(\pi i(B_{xy}+A_c^2)w_2\right), \tag{B8}$$

$$\exp\left(\pi iB_{xy}w_2\right), \tag{B9}$$

$$\exp\left(\pi iA_c^2w_2\right). \tag{B10}$$

2. $p6m$

This group is generated by T_1, T_2, C_6, and M, where the first three generators have the same properties as those in $p6$ and the last one is a mirror symmetry whose mirror axis passes through the C_6 center and bisects T_1 and T_2, such that

$$M^2=1,\ \ MC_6M=C_6^{-1},\ \ MT_1M=T_2,\ \ MT_2M=T_1,$$
$$C_6^6=1,\quad C_6T_1C_6^{-1}=T_1T_2,$$
$$C_6T_2C_6^{-1}=T_1^{-1},\quad T_1T_2=T_2T_1. \tag{B11}$$

An arbitrary element in $p6m$ can be written as $g = T_1^x T_2^y C_6^c M^m$, with $x, y \in \mathbb{Z}$, $c \in \{0,1,2,3,4,5\}$, and $m \in \{0,1\}$.

The $\mathbb{Z}_2$ cohomology ring of $p6m$ is

$$\mathbb{Z}_2[A_c, A_m; B_{xy}]/(B_{xy}^2 = B_{xy}(A_c^2 + A_c A_m)). \tag{B12}$$

Here, $H^1(p6m, \mathbb{Z}_2) = \mathbb{Z}_2^2$, with generators $\xi_1 = A_c$ and $\xi_2 = A_m$, which have representative cochains

$$\xi_1(g) = c, \qquad \xi_2(g) = m. \tag{B13}$$

$H^2(p6m, \mathbb{Z}_2) = \mathbb{Z}_2^4$, with generators $\lambda_1 = B_{xy}$, $\lambda_2 = A_c^2$, $\lambda_3 = A_c A_m$, and $\lambda_4 = A_m^2$, and we can choose the representative cochains to be

$$\begin{aligned} B_{xy}(g_1,g_2) = {}& P_{60}(c_1)[P_c(m_1)y_1x_2 + m_1y_2(x_2+y_1)] \\ &+ P_{61}(c_1)\left[P_c(m_1)\left(\frac{x_2(x_2-1)}{2} + y_1x_2 - y_2(x_2+y_1)\right) + m_1\left(\frac{y_2(y_2-1)}{2} + y_1(-x_2+y_2)\right)\right] \\ &+ P_{62}(c_1)\left[P_c(m_1)\left(\frac{y_2(y_2+1)}{2} - x_2 - y_2(x_2+y_1)\right) + m_1\left(\frac{x_2(x_2+1)}{2} - y_2 - y_1x_2\right)\right] \\ &+ P_{63}(c_1)[P_c(m_1)(-x_2+y_2-y_1x_2) + m_1(x_2-y_2+y_2(x_2-y_1))] \\ &+ P_{64}(c_1)\left[P_c(m_1)\left(\frac{x_2(x_2-1)}{2} + y_2 - y_1x_2 - y_2(x_2-y_1)\right) + m_1\left(\frac{y_2(y_2-1)}{2} + x_2 + y_1(x_2-y_2)\right)\right] \\ &+ P_{65}(c_1)\left[P_c(m_1)\left(\frac{y_2(y_2+1)}{2} - y_2(x_2-y_1)\right) + m_1\left(\frac{x_2(x_2+1)}{2} + y_1x_2\right)\right], \end{aligned} \tag{B14}$$

$$A_c^2(g_1,g_2) = c_1c_2, \quad A_cA_m(g_1,g_2) = m_1c_2, \quad A_m^2(g_1,g_2) = m_1m_2. \tag{B15}$$

According to Ref. [8], the anomalies of $p6m \times \mathrm{SO}(3) \times \mathbb{Z}_2^T$ symmetric lattice systems in lattice homotopy classes a, c, and $a + c$ can be, respectively, written as

$$\exp\left(\pi i(B_{xy} + A_c^2 + A_cA_m)(w_2 + t^2)\right), \tag{B16}$$

$$\exp\left(\pi i B_{xy}(w_2 + t^2)\right), \tag{B17}$$

$$\exp\left(\pi i(A_c^2 + A_cA_m)(w_2 + t^2)\right). \tag{B18}$$

3. $p4$

This group is generated by T_1, T_2, and C_4, two translations with perpendicular translation vectors that have equal length, and a fourfold rotational symmetry, such that

$$C_4^4 = 1, \qquad C_4T_1C_4^{-1} = T_2, \qquad C_4T_2C_4^{-1} = T_1^{-1}, \qquad T_1T_2 = T_2T_1. \tag{B19}$$

An arbitrary element in $p4$ can be written as $g = T_1^x T_2^y C_4^c$, with $x, y \in \mathbb{Z}$ and $c \in \{0,1,2,3\}$.

The $\mathbb{Z}_2$ cohomology ring of $p4$ is

$$\begin{aligned} \mathbb{Z}_2[A_c, A_{x+y}; B_{c^2}, B_{xy}]/(&A_c^2 = 0, A_{x+y}A_c = 0, \\ &B_{xy}A_{x+y} = B_{xy}A_c, B_{c^2}A_{x+y} = B_{xy}A_{x+y} + A_{x+y}^3, \\ &B_{xy}^2 = B_{xy}B_{c^2}). \end{aligned} \tag{B20}$$

Here, $H^1(p4, \mathbb{Z}_2) = \mathbb{Z}_2^2$, with generators $\xi_1 = A_{x+y}$ and $\xi_2 = A_c$, which have representative cochains

$$\xi_1(g) = x + y, \qquad \xi_2(g) = c. \tag{B21}$$

$H^2(p4, \mathbb{Z}_2) = \mathbb{Z}_2^3$, with generators $\lambda_1 = B_{xy}$, $\lambda_2 = B_{c^2}$, and $\lambda_3 = A_{x+y}^2$, and we can choose the representative cochains to be

$$B_{xy}(g_1,g_2) = P_c(c_1)y_1x_2 + P(c_1)y_2(y_1 + x_2), \tag{B22}$$

$$B_{c^2}(g_1,g_2) = \frac{[c_1]_4 + [c_2]_4 - [c_1+c_2]_4}{4}, \tag{B23}$$

$$A_{x+y}^2(g_1,g_2) = (x_1+y_1)(x_2+y_2). \tag{B24}$$

According to Ref. [8], the anomalies of $p4 \times \mathrm{SO}(3)$ symmetric lattice systems in lattice homotopy classes a, b,

c, $a+b$, $a+c$, $b+c$, and $a+b+c$ can be, respectively, written as

$$\exp\left(\pi i(B_{xy}+B_{c^2}+A_{x+y}^2)w_2\right), \tag{B25}$$

$$\exp\left(\pi i B_{xy} w_2\right), \tag{B26}$$

$$\exp\left(\pi i A_{x+y}^2 w_2\right), \tag{B27}$$

$$\exp\left(\pi i(B_{c^2}+A_{x+y}^2)w_2\right), \tag{B28}$$

$$\exp\left(\pi i(B_{xy}+B_{c^2})w_2\right), \tag{B29}$$

$$\exp\left(\pi i(B_{xy}+A_{x+y}^2)w_2\right), \tag{B30}$$

$$\exp\left(\pi i B_{c^2} w_2\right). \tag{B31}$$

4. *p*4*m*

This group is generated by T_1, T_2, C_4, and M, where the first three generators have the same properties as those in $p4$ and the last generator M is a mirror symmetry that flips the translation vector of T_1, such that

$$M^2=1,\quad MC_4M=C_4^{-1},\quad MT_1M=T_1^{-1},\quad MT_2M=T_2,$$
$$C_4^4=1,\quad C_4T_1C_4^{-1}=T_2,\quad C_4T_2C_4^{-1}=T_1^{-1},\quad T_1T_2=T_2T_1. \tag{B32}$$

An arbitrary element in $p4m$ can be written as $g=T_1^xT_2^yC_4^cM^m$, with $x,y\in\mathbb{Z}$, $c\in\{0,1,2,3\}$ and $m\in\{0,1\}$.

The $\mathbb{Z}_2$ cohomology ring of $p4m$ is

$$\begin{aligned}
&\mathbb{Z}_2[A_c,A_{x+y},A_m;B_{c^2},B_{xy}]/(A_c(A_c+A_m)=0,\\
&A_{x+y}A_c=0,\\
&B_{xy}A_{x+y}=B_{xy}(A_c+A_m),\\
&B_{c^2}A_{x+y}=B_{xy}A_{x+y}+A_{x+y}^3+A_{x+y}^2A_m,\\
&B_{xy}^2=B_{xy}B_{c^2}).
\end{aligned} \tag{B33}$$

Here, $H^1(p4m,\mathbb{Z}_2)=\mathbb{Z}_2^3$, with generators $\xi_1=A_{x+y}$, $\xi_2=A_c$, and $\xi_3=A_m$, which have representative cochains

$$\xi_1(g)=x+y,\qquad \xi_2(g)=c,\qquad \xi_3(g)=m. \tag{B34}$$

$H^2(p4m,\mathbb{Z}_2)=\mathbb{Z}_2^6$, with generators $\lambda_1=B_{xy}$, $\lambda_2=B_{c^2}$, $\lambda_3=A_{x+y}^2$, $\lambda_4=A_{x+y}A_m$, $\lambda_5=A_c^2$, and $\lambda_6=A_m^2$, and we can choose the representative cochains to be

$$B_{xy}(g_1,g_2)=P_c(c_1)y_1x_2+P(c_1)y_2(y_1+x_2), \tag{B35}$$

$$B_{c^2}(g_1,g_2)=\frac{[c_1]_4+(-1)^{m_1}[c_2]_4-[c_1+(-1)^{m_1}c_2]_4}{4}, \tag{B36}$$

$$\begin{aligned}
&A_{x+y}^2(g_1,g_2)=(x_1+y_1)(x_2+y_2),\\
&A_{x+y}A_m(g_1,g_2)=m_1(x_2+y_2),
\end{aligned} \tag{B37}$$

$$A_c^2(g_1,g_2)=c_1c_2,\qquad A_m^2(g_1,g_2)=m_1m_2. \tag{B38}$$

According to Ref. [8], the anomalies of $p4m\times SO(3)\times\mathbb{Z}_2^T$ symmetric lattice systems in lattice homotopy classes a, b, c, $a+b$, $a+c$, $b+c$, and $a+b+c$ can be, respectively, written as

$$\exp\left(\pi i(B_{xy}+B_{c^2}+A_{x+y}(A_{x+y}+A_m))(w_2+t^2)\right), \tag{B39}$$

$$\exp\left(\pi i B_{xy}(w_2+t^2)\right), \tag{B40}$$

$$\exp\left(\pi i A_{x+y}(A_{x+y}+A_m)(w_2+t^2)\right), \tag{B41}$$

$$\exp\left(\pi i(B_{c^2}+A_{x+y}(A_{x+y}+A_m))(w_2+t^2)\right), \tag{B42}$$

$$\exp\left(\pi i(B_{xy}+B_{c^2})(w_2+t^2)\right), \tag{B43}$$

$$\exp\left(\pi i(B_{xy}+A_{x+y}(A_{x+y}+A_m))(w_2+t^2)\right), \tag{B44}$$

$$\exp\left(\pi i B_{c^2}(w_2+t^2)\right). \tag{B45}$$

APPENDIX C: DETAILS OF REALIZATIONS: ANYON PERMUTATION PATTERNS AND SYMMETRY FRACTIONALIZATION CLASSES

In this appendix, for the topological orders appearing in this paper, we give the full details of all possible symmetry fractionalization classes given different anyon permutation patterns, including the explicit representative cochain for each generator of the symmetry fractionalization classes. For $\mathbb{Z}_2$ and $\mathbb{Z}_4$ topological orders, we also upload codes using which one can (i) see all symmetry fractionalization classes of the symmetry-enriched states within each lattice homotopy class and (ii) check which lattice homotopy class a given symmetry-enriched state belongs to [79]. As for $\mathbb{Z}_3$, $U(1)_2\times U(1)_{-2}$ and $U(1)_4\times U(1)_{-4}$ topological orders, all symmetry enrichment patterns lead to anomaly-free states.

As reviewed in Sec. III B, given a topological order and how the symmetry G permutes the anyons of this topological order, all possible symmetry fractionalization classes form a torsor over $H^2(G,\mathcal{A})$, where $\mathcal{A}$ is the group formed by Abelian anyons in this topological order [to simplify the notation, in this appendix we do not write down the subscript ρ of $H_\rho^2(G,\mathcal{A})$]. Given a reference set of η symbols for G, which can be chosen to come from the pullback of the η symbols of the topological symmetry using Eq. (5), we can identify all other symmetry fractionalization classes from Eq. (4). Hence, all we need to do is to identify all elements in $H^2(G,\mathcal{A})$. More precisely, we need to write down the representative cochains of all generators of $H^2(G, \mathcal{A})$. It turns out that elements in $H^2(G, \mathcal{A})$ can usually be determined by its relation to $H^2(G, \mathbb{Z}_2)$ and $H^2(G, \mathbb{Z})$. The $\mathbb{Z}_2$ cohomology of the wallpaper group symmetries involved in this paper has been collected in Appendix B, and the $\mathbb{Z}_2$ cohomology of all wallpaper group symmetries are worked out in Ref. [8]. Also, recall that we use t to denote the generator of $H^1(\mathbb{Z}_2^T, \mathbb{Z}_2)$, as in Appendix B.

Now we explain some technical tricks to identify the elements in $H^2(G, \mathcal{A})$. Let us specialize to the case where $\mathcal{A} = \mathbb{Z}_N$ (with a potential nontrivial G action on $\mathcal{A}$). Consider the projection map p from $\mathbb{Z}$ to $\mathbb{Z}_N$:

$$p\colon\ \mathbb{Z} \to \mathbb{Z}_N, \tag{C1}$$

which induces the map between $\mathbb{Z}$ cohomology and $\mathbb{Z}_N$ cohomology

$$p_*\colon\ H^k(G, \mathbb{Z}) \to H^k(G, \mathbb{Z}_N). \tag{C2}$$

Given an element $[\omega] \in H^k(G, \mathbb{Z})$ with some representative cochain ω, the representative cochain of $[p_*(\omega)]$ is identically ω with the outcome understood as an element in $\mathbb{Z}_N$ instead of $\mathbb{Z}$. We use $\tilde{\omega}$ to label the obtained element in $H^k(G, \mathbb{Z}_N)$.

To identify an element in $H^k(G, \mathbb{Z})$, usually it is helpful to consider the Bochstein homomorphism [119] associated to the short exact sequence $1 \to \mathbb{Z} \to \mathbb{Z} \to \mathbb{Z}_2 \to 1$:

$$\beta\colon\ H^{k-1}(G, \mathbb{Z}_2) \to H^k(G, \mathbb{Z}). \tag{C3}$$

In particular, for $k = 2$, consider an element $[x] \in H^1(G, \mathbb{Z}_2)$ with representative cochain x, and the representative cochain of $[\beta(x)]$ is given by

$$\beta(x)(\mathbf{g}, \mathbf{h}) = \frac{x(\mathbf{g}) + (-1)^{q(\mathbf{g})} x(\mathbf{h}) - x(\mathbf{gh})}{2}, \tag{C4}$$

where we demand that $x(\mathbf{g})$ takes values only in $\{0, 1\}$ and $q(\mathbf{g})$ denotes whether the $\mathbf{g}$ action on $\mathbb{Z}$ is trivial [$q(\mathbf{g}) = 0$ [or nontrivial [$q(\mathbf{g}) = 1$].

If $N = 2N'$ is even, we can also consider the map i from $\mathbb{Z}_2$ to $\mathbb{Z}_{2N'}$ defined by multiplication by N', i.e.,

$$i\colon\ \mathbb{Z}_2 \to \mathbb{Z}_{2N'}. \tag{C5}$$

It induces the map from $\mathbb{Z}_2$ cohomology to $\mathbb{Z}_{2N'}$ cohomology:

$$i_*\colon\ H^k(G, \mathbb{Z}_2) \to H^k(G, \mathbb{Z}_{2N'}). \tag{C6}$$

Utilizing the map i_*, we can use elements in $H^k(G, \mathbb{Z}_2)$ to identify elements in $H^k(G, \mathbb{Z}_{2N'})$. In particular, given an element $[\omega] \in H^k(G, \mathbb{Z}_2)$ with some representative cochain ω, the representative cochain of $[i_*(\omega)]$ is simply $N'\omega$. For clarity purposes, later we omit the bracket and use $N'\omega$ to label the obtained element in $H^k(G, \mathbb{Z}_{2N'})$.

The symmetry group G we consider usually takes the form $G_1 \times G_2$. In this situation, we can specify an element in the cohomology of G by specifying an element in the cohomology of G_1 or G_2. Namely, we can consider the projection

$$f\colon\ G \to G_1. \tag{C7}$$

It induces the map from the cohomology of G_1 to the cohomology of G:

$$f^*\colon\ H^k(G_1, \mathcal{A}) \to H^k(G, \mathcal{A}). \tag{C8}$$

Hence, given an element $[\omega] \in H^k(G_1, \mathcal{A})$ with some representative cochain ω, we can use it to specify $[f^*\omega]$. Writing an element in $g \in G$ as $g_1 g_2$ with $g_1 \in G_1$ and $g_2 \in G_2$, the representative cochain of $[f^*\omega]$ can be identified as

$$f^*\omega(g, h, \ldots) = \omega(g_1, h_1, \ldots). \tag{C9}$$

In this appendix, to simplify the notation, we do not explicitly write down the f^* symbol in the front and use the cohomology and cochain of a subgroup to implicitly refer to the cohomology and cochain of the total group. For example, we may specify an element $[\omega_1] \in H^k(G_1, \mathcal{A})$ and an element $[\omega_2] \in H^k(G_2, \mathcal{A})$; then, an element in $H^k(G, \mathcal{A})$ written as $\omega_1 + \omega_2$ really means $[f_1^*\omega_1] + [f_2^*\omega_2]$, where $f_{1,2}\colon G \to G_{1,2}$ is the projection from G to $G_{1,2}$.

It turns out that the above is enough to determine almost all symmetry fractionalization classes of our interest.

In this paper, for chiral topological orders, the symmetry groups we consider are $p6 \times \mathrm{SO}(3)$ and $p4 \times \mathrm{SO}(3)$. For nonchiral topological orders, we explicitly discuss the symmetry fractionalization classes for $p6m \times \mathrm{SO}(3) \times \mathbb{Z}_2^T$ and $p4m \times \mathrm{SO}(3) \times \mathbb{Z}_2^T$ in this appendix, and we can simply ignore all terms involving w_2 to get the corresponding symmetry fractionalization classes for $p6m \times \mathbb{Z}_2^T$ and $p4m \times \mathbb{Z}_2^T$, where w_2 is defined later and detects whether certain anyon carries an half-odd-integer spin under SO(3).

1. $\mathrm{U}(1)_{2N}$

In this case, we have $\mathcal{A} = \mathbb{Z}_{2N}$. The topological symmetry of $\mathrm{U}(1)_{2N}$ is complicated for general N. For $N = 1$, there is no nontrivial topological symmetry. For $N \geq 2$, there is always a $\mathbb{Z}_2$ subgroup of topological symmetry generated by the charge conjugation symmetry C such that anyon $(a) \to ([-a]_{2N})$ under C. For this topological symmetry, we can take

$$U_C\{(a), (b); ([a+b]_{2N})\} = \begin{cases} (-1)^a & b > 0 \\ 1 & b = 0 \end{cases} \tag{C10}$$

and a set of η symbols equal to 1. When $2 \leq N \leq 5$, this is the whole topological symmetry group. In the latter discussion, we consider general $N \geq 2$ but limit to the cases where G can act only as charge conjugation. For $N = 1$, we just need to ignore the cases where G can permute anyons nontrivially.

We start with explaining the symmetry fractionalization classes of the warm-up example in Sec. VI A, $\mathbb{Z}_2 \times \mathrm{SO}(3)$. Denote the generator of the $\mathbb{Z}_2$ group as C_2. Depending on whether C_2 permutes anyons or not, we have two possibilities.

(1) *Trivial C_2 action.*—The possible symmetry fractionalization classes are given by

$$H^2[\mathbb{Z}_2 \times \mathrm{SO}(3), \mathbb{Z}_{2N}] = \mathbb{Z}_2 \oplus \mathbb{Z}_2. \tag{C11}$$

We denote the generator of the first $\mathbb{Z}_2$ piece by $\tilde{\beta}(x)$, where $x \in H^1(\mathbb{Z}_2, \mathbb{Z}_2)$ is the nontrivial generator and the tilde is because it comes from the image of $p_*: H^2[\mathbb{Z}_2 \times \mathrm{SO}(3), \mathbb{Z}] \to H^2[\mathbb{Z}_2 \times \mathrm{SO}(3), \mathbb{Z}_{2N}]$, with trivial $\mathbb{Z}_2 \times \mathrm{SO}(3)$ action on $\mathbb{Z}$. We can explicitly write down the representative cochain of $\tilde{\beta}(x)$ according to Eq. (C4):

$$\tilde{\beta}(x)(C_2^i, C_2^j) = \frac{i + j - [i + j]_2}{2} = ij, \tag{C12}$$

where $i, j \in \{0, 1\}$.

We denote the generator of the second $\mathbb{Z}_2$ piece by Nw_2, where $w_2 \in H^2[\mathrm{SO}(3), \mathbb{Z}_2]$ is the second Stiefel-Whitney class and the N in the front is because it comes from the image of $i_*: H^2[\mathbb{Z}_2 \times \mathrm{SO}(3), \mathbb{Z}_2] \to H^2[\mathbb{Z}_2 \times \mathrm{SO}(3), \mathbb{Z}_{2N}]$. The explicit representative cochain of w_2 when restricted to the subgroup generated by the π rotations about two orthogonal axes is

$$w_2(U_\pi^{i_1} U_\pi'^{i_2}, U_\pi^{j_1} U_\pi'^{j_2}) = N(i_1 j_1 + i_2 j_2 + i_1 j_2) \mod 2. \tag{C13}$$

(2) *Nontrivial C_2 action.*—The possible symmetry fractionalization classes are given by

$$H^2[\mathbb{Z}_2 \times \mathrm{SO}(3), \mathbb{Z}_{2N}] = \mathbb{Z}_2 \oplus \mathbb{Z}_2. \tag{C14}$$

We denote the generator of the two $\mathbb{Z}_2$ pieces by Nx^2 and Nw_2, because both come from the image of $i_*: H^2[\mathbb{Z}_2 \times \mathrm{SO}(3), \mathbb{Z}_2] \to H^2[\mathbb{Z}_2 \times \mathrm{SO}(3), \mathbb{Z}_{2N}]$. In particular, the representative cochain of $x^2 \in H^2[\mathbb{Z}_2 \times \mathrm{SO}(3), \mathbb{Z}_2]$ is still

$$x^2(C_2^i, C_2^j) = \frac{i + j - [i + j]_2}{2} = ij, \tag{C15}$$

and the representative cochain of $Nx^2 \in H^2[\mathbb{Z}_2 \times \mathrm{SO}(3), \mathbb{Z}_{2N}]$ is simply multiplication of Eq. (C15) by N.

For $p6 \times \mathrm{SO}(3)$, there are two possible anyon permutation patterns, determined by whether C_6 permutes anyons or not. The classification of symmetry fractionalization classes and the generators for the two possibilities are listed in Table XI. The generators with tilde come from the image of $p_*: H^2[p6 \times \mathrm{SO}(3), \mathbb{Z}] \to H^2[p6 \times \mathrm{SO}(3), \mathbb{Z}_{2N}]$, with different actions of $p6 \times \mathrm{SO}(3)$ on $\mathbb{Z}$. Now we present the information about $H^2(p6, \mathbb{Z})$ and the generators for completeness.

(1) *Trivial action on $\mathbb{Z}$.*—

$$H^2(p6, \mathbb{Z}) = \mathbb{Z} \oplus \mathbb{Z}_6. \tag{C16}$$

TABLE XI. All possible symmetry fractionalization classes of $G = p6 \times \mathrm{SO}(3)$ and $G = p4 \times \mathrm{SO}(3)$ for $\mathrm{U}(1)_{2N}$, given all possible anyon permutation patterns. All generators with a tilde come from $H^2(G, \mathbb{Z})$ via Eq. (C2), and all generators with N in the front come from $H^2(G, \mathbb{Z}_2)$ via Eq. (C6). When counting the number of realizations in each case, overcounts due to the equivalence from relabeling anyons have been taken care of. To simplify the notation, in this table sometimes a single symbol can have different meanings. For example, $\tilde{\mathscr{B}}_{xy}^{(1)}$ for $p6 \times \mathrm{SO}(3)$ is different from $\tilde{\mathscr{B}}_{xy}^{(1)}$ for $p4 \times \mathrm{SO}(3)$, and their precise meanings and expressions can be found in the discussion regarding the $p6 \times \mathrm{SO}(3)$ and $p4 \times \mathrm{SO}(3)$ symmetries in this appendix.

Symmetry group	Action	$H^2(G, \mathcal{A})$	Realizations	Generators
$p6 \times \mathrm{SO}(3)$	Trivial	$\mathbb{Z}_{2N} \oplus \mathbb{Z}_{(2N,6)} \oplus (\mathbb{Z}_2)$	$4[N(N,3)+1]$	$\tilde{\mathscr{B}}_{xy}^{(1)}$, $\tilde{\mathscr{B}}_{c^2}^{(1)}$, Nw_2
	$C_6: (a) \to ([-a]_{2N})$	$(\mathbb{Z}_2)^3$	8	NB_{xy}, NA_c^2, Nw_2
$p4 \times \mathrm{SO}(3)$	Trivial	$\mathbb{Z}_{2N} \oplus \mathbb{Z}_{(2N,4)} \oplus (\mathbb{Z}_2)^2$	$8[N(N,2)+1]$	$\tilde{\mathscr{B}}_{xy}^{(1)}$, $\tilde{\mathscr{B}}_{c^2}^{(1)}$, $\tilde{\beta}(A_{x+y})$, Nw_2
	$C_4: (a) \to ([-a]_{2N})$	$(\mathbb{Z}_2)^4$	16	NB_{xy}, NB_{c^2}, $\tilde{\beta}(A_{x+y})$, Nw_2
	$T_{1,2}: (a) \to ([-a]_{2N})$	$\mathbb{Z}_{(2N,4)} \oplus (\mathbb{Z}_2)^3$	$8[(N,2)+1]$	$\tilde{\mathscr{B}}_{xy}^{(3)}$, NB_{c^2}, NA_{x+y}^2, Nw_2
	$T_{1,2}, C_4: (a) \to ([-a]_{2N})$	$\mathbb{Z}_{(2N,4)} \oplus (\mathbb{Z}_2)^3$	$8[(N,2)+1]$	$\tilde{\mathscr{B}}_{xy}^{(4)}$, NB_{c^2}, NA_{x+y}^2, Nw_2

We denote the generator of the $\mathbb{Z}$ piece and the $\mathbb{Z}_6$ piece by $\mathscr{B}_{xy}^{(1)}$ and $\mathscr{B}_{c^2}^{(1)}$, respectively, which have representative cochains

$$\begin{aligned}\mathscr{B}_{xy}^{(1)}(g_1,g_2) = {} & P_{60}(c_1)y_1x_2 + P_{61}(c_1)\left(\frac{x_2(x_2-1)}{2} + y_1x_2 - y_2(x_2+y_1)\right) \\ & + P_{62}(c_1)\left(\frac{y_2(y_2+1)}{2} - x_2 - y_2(x_2+y_1)\right) + P_{63}(c_1)(-x_2+y_2-y_1x_2) \\ & + P_{64}(c_1)\left(\frac{x_2(x_2-1)}{2} + y_2 - y_1x_2 - y_2(x_2-y_1)\right) + P_{65}(c_1)\left(\frac{y_2(y_2+1)}{2} - y_2(x_2-y_1)\right), \end{aligned} \quad \text{(C17)}$$

$$\mathscr{B}_{c^2}^{(1)}(g_1,g_2) = \frac{[c_1]_6 + [c_2]_6 - [c_1+c_2]_6}{6}. \quad \text{(C18)}$$

The representative cochain of $\mathscr{B}_{xy}^{(1)}$ has identically the same expression as Eq. (B6). Note that if we think of the expression as a representative cochain of $\mathbb{Z}_2$ cohomology, it does not matter whether we have a $+$ sign or $-$ sign in front of an integer, because we care about only the mod 2 value of the expression. However, we carefully choose the sign in Eq. (B6) or (C17) such that the expression is a $\mathbb{Z}$ cochain as well. Hence, we immediately see that the $\mathbb{Z}_2$ reduction of $\mathscr{B}_{xy}^{(1)}$ is B_{xy} in Eq. (B6). Likewise, the $\mathbb{Z}_2$ reduction of $\mathscr{B}_{c^2}^{(1)}$ is A_c^2 in Eq. (B7).

(2) *C_6 acts nontrivially on $\mathbb{Z}$.—*

$$H^2(p6,\mathbb{Z}) = 0. \quad \text{(C19)}$$

For $p4\times \mathrm{SO}(3)$, there are four possible anyon permutation patterns, determined by whether C_4 and $T_{1,2}$ permute anyons or not. The classification of symmetry fractionalization classes and the generators are listed in Table XI. Here, we present the information about $H^2(p4,\mathbb{Z})$ and the generators.

(1) *Trivial action on $\mathbb{Z}$.—*

$$H^2(p4,\mathbb{Z}) = \mathbb{Z}\oplus\mathbb{Z}_4\oplus\mathbb{Z}_2. \quad \text{(C20)}$$

We denote the generators of the $\mathbb{Z}$, $\mathbb{Z}_4$, and $\mathbb{Z}_2$ piece by $\mathscr{B}_{xy}^{(1)}$, $\mathscr{B}_{c^2}^{(1)}$, and $\beta^{(1)}(A_{x+y})$, respectively, which have representative cochains

$$\mathscr{B}_{xy}^{(1)}(g_1,g_2) = P_{40}(c_1)y_1x_2 - P_{41}(c_1)y_2(x_2+y_1) - P_{42}(c_1)y_1x_2 + P_{43}(c_1)y_2(y_1-x_2), \quad \text{(C21)}$$

$$\mathscr{B}_{c^2}^{(1)}(g_1,g_2) = \frac{[c_1]_4 + [c_2]_4 - [c_1+c_2]_4}{4}, \quad \text{(C22)}$$

$$\beta^{(1)}(A_{x+y})(g_1,g_2) = \frac{[x_1+y_1]_2 + [x_2+y_2]_2 - [x_1+x_2+y_1+y_2]_2}{2}. \quad \text{(C23)}$$

(2) *C_4 acts nontrivially on $\mathbb{Z}$.—*

$$H^2(p6,\mathbb{Z}) = \mathbb{Z}_2. \quad \text{(C24)}$$

We denote the generator by $\beta^{(2)}(A_{x+y})$, which has a representative cochain

$$\beta^{(2)}(A_{x+y})(g_1,g_2) = \frac{[x_1+y_1]_2 + (-1)^{c_1}[x_2+y_2]_2 - [x_1+x_2+y_1+y_2]_2}{2}. \quad \text{(C25)}$$

(3) *$T_{1,2}$ act nontrivially on $\mathbb{Z}$.—*

$$H^2(p6,\mathbb{Z}) = \mathbb{Z}_4. \quad \text{(C26)}$$

We denote the generator by $\mathscr{B}_{xy}^{(3)}$, which has a representative cochain

$$\mathscr{B}_{xy}^{(3)}(g_1,g_2) = (-1)^{\tilde{x}_1}[P_c(c_1)P(\tilde{y}_1)P(\tilde{x}_2) + P(c_1)P(\tilde{y}_1+\tilde{x}_2)P(\tilde{y}_2)]$$
$$\text{with} \quad \tilde{x} = x + P_{41}(c) + P_{42}(c), \quad \tilde{y} = y + P_{42}(c) + P_{43}(c). \tag{C27}$$

Note that the $\mathbb{Z}_2$ reduction of $\mathscr{B}_{xy}^{(3)}$ is actually $B_{xy} + B_{c^2} + A_{x+y}^2$.

(4) *Both $T_{1,2}$ and C_4 act nontrivially on $\mathbb{Z}$.*—

$$H^2(p6,\mathbb{Z}) = \mathbb{Z}_4. \tag{C28}$$

We denote the generator by $\mathscr{B}_{xy}^{(4)}$, which has a representative cochain

$$\mathscr{B}_{xy}^{(4)}(g_1,g_2) = (-1)^{x_1}[P_c(c_1)P(y_1)P(x_2) + P(c_1)P(y_1+x_2)P(y_2)]. \tag{C29}$$

2. Ising$^{(\nu)}$

In this case, we have $\mathcal{A} = \mathbb{Z}_2$, with no nontrivial topological symmetry. The classification of symmetry fractionalization classes and the generators are listed in Table XII.

3. $\mathbb{Z}_2$ topological order

In this case, we have $\mathcal{A} = (\mathbb{Z}_2)^2$. The topological symmetry is $\mathbb{Z}_2 \times \mathbb{Z}_2^T$. The unitary generator of the topological symmetry is the unitary electric-magnetic duality symmetry S that exchanges e and m, i.e.,

$$S: (a_e, a_m) \to (a_m, a_e), \tag{C30}$$

while the antiunitary generator is simply the antiunitary electric-magnetic duality symmetry that permutes anyons in the same way as S. We can choose the U symbol such that

TABLE XII. All possible symmetry fractionalization classes of $p6 \times \mathrm{SO}(3)$ and $p4 \times \mathrm{SO}(3)$ for Ising$^{(\nu)}$, where ν is an odd integer.

Symmetry group	Action	$H^2(G,\mathcal{A})$	Realizations	Generators
$p6 \times \mathrm{SO}(3)$	Trivial	$(\mathbb{Z}_2)^3$	8	B_{xy}, A_c^2, w_2
$p4 \times \mathrm{SO}(3)$	Trivial	$(\mathbb{Z}_2)^4$	16	B_{xy}, B_{c^2}, A_{x+y}^2, w_2

$$U_{\mathbf{g}}(a,b;c) = \begin{cases} (-1)^{a_m b_e} & \mathbf{g} \text{ permutes anyons,} \\ 1 & \text{otherwise.} \end{cases} \tag{C31}$$

And a set of η symbols can be chosen such that

$$\eta_a(\mathbf{g}_1,\mathbf{g}_2) = \begin{cases} (-1)^{a_e a_m} & \mathbf{g}_1,\mathbf{g}_2 \text{ permute anyons,} \\ 1 & \text{otherwise.} \end{cases} \tag{C32}$$

The classification of symmetry fractionalization classes and the generators are listed in Table XII. In the following, we explicitly comment on the cases involving symmetries that permute e and m.

Given a group G with some element that permutes e and m, we have the following short exact sequence:

$$1 \to \tilde{G} \to G \to \mathbb{Z}_2 \to 1, \tag{C33}$$

where $\tilde{G}$ is the subgroup of G that does not permute anyons. From the Serre spectral sequence [119], we immediately see that

$$H^k(G, \mathbb{Z}_2 \oplus \mathbb{Z}_2) \cong H^k(\tilde{G}, \mathbb{Z}_2). \tag{C34}$$

Specifically, given an element $[\tilde{\omega}] \in H^2(\tilde{G},\mathbb{Z}_2)$ with a representative cochain ω, we can write down the representative cochain $\omega \in H^2(\tilde{G},\mathbb{Z}_2 \oplus \mathbb{Z}_2)$ as follows. First, choose an element not in $\tilde{G}$ such that $x^2 = \mathbf{1}$. Then, every $g \in G$ can be decomposed as $g = \tilde{g}x^i, i \in \{0,1\}$, where $i = 0$ if g is an element in G and $i = 1$ otherwise and $\tilde{g}$ is an element in $\tilde{G}$. Then we can write down the representative cochain $w(g_1,g_2)$ of G from the representative cochain $\tilde{w}$ of $\tilde{G}$, i.e.,

$$\omega(g_1,g_2) = (\tilde{w}(\tilde{g}_1, x^{i_1}\tilde{g}_2 x_1^{i_1}), \tilde{w}(x\tilde{g}_1 x, xx^{i_1}\tilde{g}_2 xx^{i_1})). \tag{C35}$$

We can think of the second term as the representative cochain obtained from the conjugation action of x on $\tilde{w}$. It is straightforward to check that w satisfies the cocycle equation and, thus, is the desired representative cochain. Therefore, for each case where some symmetry actions permute anyons, to identify the symmetry fractionalization classes, we need to identify $\tilde{G}$ that does not permute anyons. By simply calculating the cohomology of $\tilde{G}$, we can identify all the possible symmetry fractionalization classes.

Still, usually there can be some simplification, because sometimes we can identify ω and write down its representative cochain directly in terms of the $\mathbb{Z}_2$ cohomology of G. When this happens, to keep notations consistent, we still label ω using the $\mathbb{Z}_2$ cohomology of G. When we have to use the cohomology of $\tilde{G}$, we use $\mathtt{A}$ or $\mathtt{B}$ to emphasize that it refers to an element in the cohomology of the subgroup $\tilde{G}$.

When the time-reversal symmetry $\mathcal{T}$ permutes anyons, no matter how other symmetries act on anyons, $\tilde{G}$ will be isomorphic to $p6m \times \mathrm{SO}(3)$ or $p4m \times \mathrm{SO}(3)$ [120].

The symmetry fractionalization classes can be identified accordingly.

When the time-reversal symmetry does not permute anyons, $\tilde{G}$ will be the product of a subgroup $\tilde{G}_s$ of $p6m$ or $p4m$ and $\mathrm{SO}(3)\times\mathbb{Z}_2^T$. For the cases involving $p6m$, we have the following three possibilities:

(1) $M:(a_e,a_m)\to(a_m,a_e)$.—Then $\tilde{G}_s=p6$, generated by T_1, T_2, and C_6.
(2) $C_6:(a_e,a_m)\to(a_m,a_e)$.—Then $\tilde{G}_s=p31m$, generated by T_1, T_2, C_6^2, and M.
(3) $C_6,M:(a_e,a_m)\to(a_m,a_e)$.—Then $\tilde{G}_s=p3m1$, generated by T_1, T_2, C_6^2, and C_6^3M.

In these cases, it turns out that we can write down the representative cochains directly in terms of the representative cochains of the $\mathbb{Z}_2$ cohomology of $p6m$, and this is what we present in Table XIII.

For the cases involving $p4m$, we have the following seven possibilities. In these cases, the explicit representative cochains may not come from the representative cochains of the $\mathbb{Z}_2$ cohomology of $p4m$, and we really need the expressions of the representative cochains for each $\tilde{G}_s$. The $\mathbb{Z}_2$ cohomology of these $\tilde{G}_s$ and the representative cochains of their generators can all be found in Appendix E in Ref. [8], and we follow the notation there, except we change, e.g., A_x to $\mathbb{A}_x$ and use a wide tilde to emphasize that we are referring to the cochain and cohomology of a subgroup.

(1) $M:(a_e,a_m)\to(a_m,a_e)$.—Then $\tilde{G}_s=p4$, generated by T_1, T_2, and C_4.
(2) $C_4:(a_e,a_m)\to(a_m,a_e)$.—Then $\tilde{G}_s=pmm$, generated by T_1, T_2, C_4^2, and M. Now we have ten elements in the cohomology that are "asymmetric," and we can write down the representative cochains of them with the help of the representative cochains of pmm and Eq. (C35).
(3) $C_4,M:(a_e,a_m)\to(a_m,a_e)$.—Then $\tilde{G}_s=cmm$, generated by T_1, T_2, C_4^2, and C_4^3M. Now we have four elements in the cohomology that are asymmetric.
(4) $T_{1,2}:(a_e,a_m)\to(a_m,a_e)$.—Then $\tilde{G}_s=p4m$, generated by T_1T_2, $T_1^{-1}T_2$, C_4, and C_4M. Now we have six elements in the cohomology that are asymmetric.
(5) $T_{1,2},M:(a_e,a_m)\to(a_m,a_e)$.—Then $\tilde{G}_s=p4g$, generated by T_1T_2, $T_1^{-1}T_2$, C_4, and T_2M.
(6) $T_{1,2},C_4:(a_e,a_m)\to(a_m,a_e)$.—Then $\tilde{G}_s=p4g$, generated by T_1T_2, $T_1^{-1}T_2$, T_1C_4, and T_1T_2M.
(7) $T_{1,2},C_4,M:(a_e,a_m)\to(a_m,a_e)$.—Then $\tilde{G}_s=p4m$, generated by T_1T_2, $T_1^{-1}T_2$, T_1C_4, and $T_1T_2C_4M$. Now we have six elements in the cohomology that are asymmetric.

4. $\mathbb{Z}_N$ topological order ($N\geq 3$)

In this case, we have $\mathcal{A}=(\mathbb{Z}_N)^2$. The topological symmetry of $\mathbb{Z}_N$ is complicated for general N. For $N\geq 3$, there is always a subgroup $\mathbb{Z}_4^T\rtimes\mathbb{Z}_2$. The unitary $\mathbb{Z}_2$ is generated by the electric-magnetic duality symmetry S that exchanges e and m, i.e.,

$$S:(a_e,a_m)\to(a_m,a_e), \tag{C36}$$

and the antiunitary generator T of $\mathbb{Z}_4^T$ permutes anyons in the following way:

$$T:(a_e,a_m)\to(a_m,[-a_e]_N). \tag{C37}$$

The two generators satisfy the relation

$$S^2=\mathbf{1},\qquad T^4=\mathbf{1},\qquad STS=T^{-1}. \tag{C38}$$

An element in $\mathbf{g}\in\mathbb{Z}_4^T\rtimes\mathbb{Z}_2$ can be labeled by (g_1,g_2) with $g_1\in\{0,\ldots,3\}$, $g_2\in\{0,1\}$, which corresponds to the element $T^{g_1}S^{g_2}$. Given such element $\mathbf{g}$, the U symbols can be chosen such that

$$U_{\mathbf{g}}(a,b;c)=\begin{cases}e^{i(2\pi/N)a_mb_e} & g_1+g_2\equiv 1 \bmod 2,\\ 1 & g_1+g_2\equiv 0 \bmod 2.\end{cases} \tag{C39}$$

A specific choice of η symbols can be chosen such that

$$\eta_a(\mathbf{g},\mathbf{h})=\begin{cases}e^{i(2\pi/N)a_mb_e} & g_1+g_2\equiv h_1+h_2\equiv 1 \bmod 2,\\ 1 & \text{otherwise.}\end{cases} \tag{C40}$$

For $N=3$, 4, this is the full topological symmetry group.

To determine the anyon permutation patterns of $p6m\times\mathrm{SO}(3)\times\mathbb{Z}_2^T$ and $p4m\times\mathrm{SO}(3)\times\mathbb{Z}_2^T$, we just need to specify how the generators of the symmetry groups permute anyons. Because $\mathcal{T}^2=\mathbf{1}$, $\mathcal{T}$ can act on anyons in two ways: Either $\mathcal{T}:(a_e,a_m)\to([-a_e]_N,a_m)$ or $\mathcal{T}:(a_e,a_m)\to(a_e,[-a_m]_N)$. Because these two cases are related by relabeling anyons using the electric-magnetic duality S, we can specialize to the cases $\mathcal{T}:(a_e,a_m)\to(a_e,[-a_m]_N)$, and we need only to consider how $p6m$ or $p4m$ permutes anyons. For $p6m\times\mathrm{SO}(3)\times\mathbb{Z}_2^T$, there are four possible anyon permutation patterns, while for $p4m\times\mathrm{SO}(3)\times\mathbb{Z}_2^T$, there are eight possible anyon permutation patterns. The corresponding classification of symmetry fractionalization classes and the generators for $N=3$, 4 are listed in Tables XIV and XV, respectively. Specifically, since the symmetries cannot permute e and m, $H^2(G,\mathbb{Z}_N\times\mathbb{Z}_N)$ simply becomes the direct sum of two $H^2(G,\mathbb{Z}_N)$ pieces, with the actions on two $\mathbb{Z}_N$ pieces corresponding to symmetry actions on e or m, respectively. As discussed at the beginning of the appendix, $H^2(G,\mathbb{Z}_N)$ can all be obtained from the $\mathbb{Z}$ cohomology or $\mathbb{Z}_2$ cohomology of the symmetry groups. In particular, to obtain the full data of symmetry fractionalization classes,

TABLE XIII. All possible symmetry fractionalization classes of $p6m \times \mathrm{SO}(3) \times \mathbb{Z}_2^T$ and $p4m \times \mathrm{SO}(3) \times \mathbb{Z}_2^T$ for the $\mathbb{Z}_2$ topological order. When counting the number of realizations in each case, overcounts due to the equivalence from relabeling anyons have been taken care of. Removing all generators involving w_2, we obtain the symmetry fractionalization classes of $p6m \times \mathbb{Z}_2^T$ and $p4m \times \mathbb{Z}_2^T$ for the $\mathbb{Z}_2$ topological order. In the codes [79], the anyon permutation patterns are indexed according to the order of this table. For example, for symmetry group $p6m \times \mathrm{SO}(3) \times \mathbb{Z}_2^T$, anyon permutation pattern 2 represents $\mathcal{T}:(a_e, a_m) \to (a_m, a_e)$, and anyon permutation pattern 4 represents $C_6, \mathcal{T}:(a_e, a_m) \to (a_m, a_e)$.

Symmetry group	Action	$H^2(G,\mathcal{A})$	Realizations	Generators
$p6m \times \mathrm{SO}(3) \times \mathbb{Z}_2^T$	Trivial	$(\mathbb{Z}_2)^{16}$	32 896	$(B_{xy},0)$, $(A_c^2,0)$, $(A_cA_m,0)$, $(A_m^2,0)$, $(A_ct,0)$, $(A_mt,0)$, $(t^2,0)$, $(w_2,0)$, $(0,B_{xy})$, $(0,A_c^2)$, $(0,A_cA_m)$, $(0,A_m^2)$, $(0,A_ct)$, $(0,A_mt)$, $(0,t^2)$,$(0,w_2)$
	$\mathcal{T}:(a_e,a_m)\to(a_m,a_e)$ $M,\mathcal{T}:(a_e,a_m)\to(a_m,a_e)$ $C_6,\mathcal{T}:(a_e,a_m)\to(a_m,a_e)$ $C_6,M,\mathcal{T}:(a_e,a_m)\to(a_m,a_e)$	$(\mathbb{Z}_2)^5$	32	(B_{xy},B_{xy}), (A_c^2,A_c^2), (A_cA_m,A_cA_m), (A_m^2,A_m^2), (w_2,w_2)
	$M:(a_e,a_m)\to(a_m,a_e)$	$(\mathbb{Z}_2)^5$	32	(B_{xy},B_{xy}), (A_c^2,A_c^2), (A_ct,A_ct), (t^2,t^2), (w_2,w_2)
	$C_6:(a_e,a_m)\to(a_m,a_e)$ $C_6,M:(a_e,a_m)\to(a_m,a_e)$	$(\mathbb{Z}_2)^5$	32	(B_{xy},B_{xy}), (A_m^2,A_m^2), (A_mt,A_mt), (t^2,t^2), (w_2,w_2)
$p4m \times \mathrm{SO}(3) \times \mathbb{Z}_2^T$	Trivial	$(\mathbb{Z}_2)^{22}$	2 098 176	$(B_{xy},0)$, $(B_{c^2},0)$, $(A_{x+y}^2,0)$, $(A_{x+y}A_m,0)$, $(A_c^2,0)$, $(A_m^2,0)$, $(A_{x+y}t,0)$, $(A_ct,0)$, $(A_mt,0)$, $(t^2,0)$, $(w_2,0)$, $(0,B_{xy})$, $(0,B_{c^2})$, $(0,A_{x+y}^2)$, $(0,A_{x+y}A_m)$, $(0,A_c^2)$, $(0,A_m^2)$, $(0,A_{x+y}t)$, $(0,A_ct)$, $(0,A_mt)$, $(0,t^2)$, $(0,w_2)$
	$\mathcal{T}:(a_e,a_m)\to(a_m,a_e)$ $M,\mathcal{T}:(a_e,a_m)\to(a_m,a_e)$ $C_4,\mathcal{T}:(a_e,a_m)\to(a_m,a_e)$ $C_4,M,\mathcal{T}:(a_e,a_m)\to(a_m,a_e)$ $T_{1,2},\mathcal{T}:(a_e,a_m)\to(a_m,a_e)$ $T_{1,2},M,\mathcal{T}:(a_e,a_m)\to(a_m,a_e)$ $T_{1,2},C_4,\mathcal{T}:(a_e,a_m)\to(a_m,a_e)$ $T_{1,2},C_4,M,\mathcal{T}:(a_e,a_m)\to(a_m,a_e)$	$(\mathbb{Z}_2)^7$	128	(B_{xy},B_{xy}), (B_{c^2},B_{c^2}), (A_{x+y}^2,A_{x+y}^2), $(A_{x+y}A_m,A_{x+y}A_m)$, (A_c^2,A_c^2), (A_m^2,A_m^2), (w_2,w_2)
	$M:(a_e,a_m)\to(a_m,a_e)$	$(\mathbb{Z}_2)^7$	128	(B_{xy},B_{xy}), (B_{c^2},B_{c^2}), (A_{x+y}^2,A_{x+y}^2), $(A_{x+y}t,A_{x+y}t)$, (A_ct,A_ct), (t^2,t^2), (w_2,w_2)
	$C_4:(a_e,a_m)\to(a_m,a_e)$	$(\mathbb{Z}_2)^{14}$	8448	(B_{xy},B_{xy}), (B_{c^2},B_{c^2}), (t^2,t^2), (w_2,w_2), $(\mathtt{A}_x\mathtt{A}_m,\widetilde{\mathtt{A}_y\mathtt{A}_m}))$, $(\mathtt{A}_c\mathtt{A}_m,(\widetilde{\mathtt{A}_c+}\mathtt{A}_m)\mathtt{A}_m)$, $(\mathtt{A}_x\mathtt{A}_c,\mathtt{A}_y(\widetilde{\mathtt{A}_c}+\mathtt{A}_m))$, $(\mathtt{A}_x\widetilde{t,\mathtt{A}_y}t)$, $(\mathtt{A}_ct,(\widetilde{\mathtt{A}_c+}\mathtt{A}_m)t)$, $(\mathtt{A}_y\mathtt{A}_m,\widetilde{\mathtt{A}_x}\mathtt{A}_m)$, $((\mathtt{A}_c+\mathtt{A}_m)\widetilde{\mathtt{A}_m},\mathtt{A}_c\mathtt{A}_m)$, $(\mathtt{A}_y(\mathtt{A}_c+\widetilde{\mathtt{A}_m}),\mathtt{A}_x\mathtt{A}_c)$, $(\mathtt{A}_y\widetilde{t,\mathtt{A}_x}t)$, $((\mathtt{A}_c+\widetilde{\mathtt{A}_m})t,\mathtt{A}_ct)$
	$C_4,M:(a_e,a_m)\to(a_m,a_e)$	$(\mathbb{Z}_2)^{10}$	640	(B_{xy},B_{xy}), (B_{c^2},B_{c^2}), (A_{x+y}^2,A_{x+y}^2), $(A_{x+y}t,A_{x+y}t)$, (t^2,t^2), (w_2,w_2), $(\mathtt{A}_c^2,\widetilde{\mathtt{A}_c^2+}\mathtt{A}_m^2)$, $(\mathtt{A}_ct,(\widetilde{\mathtt{A}_c+}\mathtt{A}_m)t)$, $(\mathtt{A}_c^2+\widetilde{\mathtt{A}_m^2},\mathtt{A}_c^2)$, $((\mathtt{A}_c+\widetilde{\mathtt{A}_m})t,\mathtt{A}_ct)$
	$T_{1,2}:(a_e,a_m)\to(a_m,a_e)$	$(\mathbb{Z}_2)^{11}$	1152	(B_{c^2},B_{c^2}), (A_m^2,A_m^2), (A_mt,A_mt), (t^2,t^2), (w_2,w_2), $(\mathtt{B}_{xy},\mathtt{B}_{xy}+\mathtt{B}_{c^2}+\widetilde{\mathtt{A}_{x+y}}(\mathtt{A}_{x+y}+\mathtt{A}_m))$, $(\mathtt{A}_{x+y}^2,\widetilde{\mathtt{A}_{x+y}^2}+\mathtt{A}_c^2+\mathtt{A}_m^2)$,$(\mathtt{A}_{x+y}t,(\widetilde{\mathtt{A}_{x+y}}+\mathtt{A}_c+\mathtt{A}_m)t)$, $(\mathtt{B}_{xy}+\mathtt{B}_{c^2}+\widetilde{\mathtt{A}_{x+y}}(\mathtt{A}_{x+y}+\mathtt{A}_m),\mathtt{B}_{xy})$, $(\mathtt{A}_{x+y}^2+\widetilde{\mathtt{A}_c^2+}\mathtt{A}_m^2,\mathtt{A}_{x+y}^2)$, $((\mathtt{A}_{x+y}+\mathtt{A}_c\widetilde{+\mathtt{A}_m})t,\mathtt{A}_{x+y}t)$
	$T_{1,2},M:(a_e,a_m)\to(a_m,a_e)$	$(\mathbb{Z}_2)^7$	128	(B_{xy},B_{xy}), (B_{c^2},B_{c^2}), (A_m^2,A_m^2), (A_ct,A_ct), (A_mt,A_mt), (t^2,t^2), (w_2,w_2)
	$T_{1,2},C_4:(a_e,a_m)\to(a_m,a_e)$	$(\mathbb{Z}_2)^7$	128	(B_{xy},B_{xy}), (B_{c^2},B_{c^2}), (A_m^2,A_m^2) (A_ct,A_ct), (A_mt,A_mt), (t^2,t^2), (w_2,w_2)
	$T_{1,2},C_4,M:(a_e,a_m)\to(a_m,a_e)$	$(\mathbb{Z}_2)^{11}$	1152	(B_{c^2},B_{c^2}), (A_m^2,A_m^2), (A_mt,A_mt), (t^2,t^2), (w_2,w_2), $(\mathtt{B}_{xy},\mathtt{B}_{xy}+\mathtt{B}_{c^2}+\widetilde{\mathtt{A}_{x+y}}(\mathtt{A}_{x+y}+\mathtt{A}_m))$, $(\mathtt{A}_{x+y}^2,\widetilde{\mathtt{A}_{x+y}^2}+\mathtt{A}_c^2+\mathtt{A}_m^2)$,$(\mathtt{A}_{x+y}t,(\widetilde{\mathtt{A}_{x+y}}+\mathtt{A}_c+\mathtt{A}_m)t)$, $(\mathtt{B}_{xy}+\mathtt{B}_{c^2}+\mathtt{A}_{x+y}\widetilde{(\mathtt{A}_{x+y}}+\mathtt{A}_m),\mathtt{B}_{xy})$, $(\mathtt{A}_{x+y}^2+\widetilde{\mathtt{A}_c^2+}\mathtt{A}_m^2,\mathtt{A}_{x+y}^2)$, $((\mathtt{A}_{x+y}+\mathtt{A}_c\widetilde{+\mathtt{A}_m})t,\mathtt{A}_{x+y}t)$

TABLE XIV. All possible symmetry fractionalization classes of $p6m \times SO(3) \times \mathbb{Z}_2^T$ and $p4m \times SO(3) \times \mathbb{Z}_2^T$ for $\mathbb{Z}_3$ topological order, when the action of $\mathcal{T}$ is specified by $\mathcal{T}: (a_e, a_m) \to (a_e, [-a_m]_3)$. The cases where the action of $\mathcal{T}$ is $\mathcal{T}: (a_e, a_m) \to ([-a_e]_3, a_m)$ can easily obtained by duality. When counting the number of realizations in each case, overcounts due to the equivalence from relabeling anyons have been taken care of.

Symmetry group	Action	$H^2(G,\mathcal{A})$	Realizations	Generators
$p6m \times SO(3) \times \mathbb{Z}_2^T$	$M: (a_e, a_m) \to (a_e, [-a_m]_3)$	$\mathbb{Z}_1$	1	(0,0)
	$M: (a_e, a_m) \to ([-a_e]_3, a_m)$	$(\mathbb{Z}_3)^2$	5	$(\tilde{\mathscr{B}}_{xy}^{(2)}, 0)$, $(\tilde{\mathscr{B}}_{c^2}^{(2)}, 0)$
	$C_6: (a_e, a_m) \to ([-a_e]_N, [-a_m]_3)$ $M: (a_e, a_m) \to (a_e, [-a_m]_3)$	$\mathbb{Z}_1$	1	(0,0)
	$C_6: (a_e, a_m) \to ([-a_e]_3, [-a_m]_3)$ $M: (a_e, a_m) \to ([-a_e]_3, a_m)$	$\mathbb{Z}_1$	1	(0,0)
$p4m \times SO(3) \times \mathbb{Z}_2^T$	$M: (a_e, a_m) \to (a_e, [-a_m]_3)$	$\mathbb{Z}_1$	1	(0,0)
	$M: (a_e, a_m) \to ([-a_e]_3, a_m)$	$\mathbb{Z}_3$	2	$(\tilde{\mathscr{B}}_{xy}^{(2)}, 0)$
	$C_4: (a_e, a_m) \to ([-a_e]_3, [-a_m]_3)$ $M: (a_e, a_m) \to (a_e, [-a_m]_3)$	$\mathbb{Z}_1$	1	(0,0)
	$C_4: (a_e, a_m) \to ([-a_e]_3, [-a_m]_3)$ $M: (a_e, a_m) \to ([-a_e]_3, a_m)$	$\mathbb{Z}_1$	1	(0,0)
	$T_{1,2}: (a_e, a_m) \to ([-a_e]_3, [-a_m]_3)$ $M: (a_e, a_m) \to (a_e, [-a_m]_3)$	$\mathbb{Z}_1$	1	(0,0)
	$T_{1,2}: (a_e, a_m) \to ([-a_e]_3, [-a_m]_3)$ $M: (a_e, a_m) \to ([-a_e]_3, a_m)$	$\mathbb{Z}_1$	1	(0,0)
	$T_{1,2}, C_4: (a_e, a_m) \to ([-a_e]_3, [-a_m]_3)$ $M: (a_e, a_m) \to (a_e, [-a_m]_3)$	$\mathbb{Z}_1$	1	(0,0)
	$T_{1,2}, C_4: (a_e, a_m) \to ([-a_e]_3, [-a_m]_3)$ $M: (a_e, a_m) \to ([-a_e]_3, a_m)$	$\mathbb{Z}_1$	1	(0,0)

TABLE XV. All possible symmetry fractionalization classes of $p6m \times SO(3) \times \mathbb{Z}_2^T$ and $p4m \times SO(3) \times \mathbb{Z}_2^T$ for $\mathbb{Z}_4$ topological order, when the action of $\mathcal{T}$ is specified by $\mathcal{T}: (a_e, a_m) \to (a_e, [-a_m]_4)$. The cases where the action of $\mathcal{T}$ is $\mathcal{T}: (a_e, a_m) \to ([-a_e]_4, a_m)$ is related to these cases by relabeling, hence, are not distinct physical realizations. When counting the number of realizations in each case, overcounts due to the equivalence from relabeling anyons have been taken care of. Removing all generators involving w_2, we obtain the symmetry fractionalization classes of $p6m \times \mathbb{Z}_2^T$ and $p4m \times \mathbb{Z}_2^T$ for the $\mathbb{Z}_4$ topological order. In the codes [79], the anyon permutation patterns are indexed according to the order of this table. For example, for symmetry group $p6m \times SO(3) \times \mathbb{Z}_2^T$, anyon permutation pattern 1 represents $M: (a_e, a_m) \to (a_e, [-a_m]_4)$, and anyon permutation pattern 2 represents $M: (a_e, a_m) \to ([-a_e]_4, a_m)$.

Symmetry group	Action	$H^2(G,\mathcal{A})$	Realizations	Generators
$p6m \times SO(3) \times \mathbb{Z}_2^T$	$M: (a_e, a_m) \to (a_e, [-a_m]_4)$	$(\mathbb{Z}_2)^{16}$	2^{16}	$(2B_{xy}, 0)$, $(\tilde{\beta}(A_c), 0)$, $(2A_cA_m, 0)$, $(\tilde{\beta}(A_m), 0)$, $(2A_ct, 0)$, $(2A_mt, 0)$, $(\tilde{\beta}(t), 0)$, $(2w_2, 0)$, $(0, 2B_{xy})$, $(0, 2A_c^2)$, $(0, 2A_cA_m)$, $(0, 2A_m^2)$, $(0, \tilde{\beta}(A_c))$, $(0, \tilde{\beta}(A_m))$, $(0, 2t^2)$, $(0, 2w_2)$
	$M: (a_e, a_m) \to ([-a_e]_4, a_m)$	$\mathbb{Z}_4 \oplus (\mathbb{Z}_2)^{15}$	3×2^{15}	$(\tilde{\mathscr{B}}_{xy}^{(2)}, 0)$, $(\tilde{\mathscr{B}}_{c^2}^{(2)}, 0)$, $(2A_cA_m, 0)$, $(2A_m^2, 0)$, $(2A_ct, 0)$, $(2A_mt, 0)$, $(\tilde{\beta}(t), 0)$, $(2w_2, 0)$, $(0, 2B_{xy})$, $(0, 2A_c^2)$, $(0, 2A_cA_m)$, $(0, 2A_m^2)$, $(0, \tilde{\beta}(A_c))$, $(0, \tilde{\beta}(A_m))$, $(0, 2t^2)$, $(0, 2w_2)$
	$C_6: (a_e, a_m) \to ([-a_e]_4, [-a_m]_4)$ $M: (a_e, a_m) \to (a_e, [-a_m]_4)$	$(\mathbb{Z}_2)^{16}$	2^{16}	$(2B_{xy}, 0)$, $(2A_c^2, 0)$, $(2A_cA_m, 0)$, $(\tilde{\beta}(A_m), 0)$, $(2A_ct, 0)$, $(2A_mt, 0)$, $(\tilde{\beta}(t), 0)$, $(2w_2, 0)$, $(0, 2B_{xy})$, $(0, 2A_c^2)$, $(0, 2A_cA_m)$, $(0, 2A_m^2)$, $(0, \tilde{\beta}(A_c))$, $(0, \tilde{\beta}(A_m))$, $(0, 2t^2)$, $(0, 2w_2)$
	$C_6: (a_e, a_m) \to ([-a_e]_4, [-a_m]_4)$ $M: (a_e, a_m) \to ([-a_e]_4, a_m)$	$(\mathbb{Z}_2)^{16}$	2^{16}	$(2B_{xy}, 0)$, $(2A_c^2, 0)$, $(\tilde{\beta}(A_m), 0)$, $(2A_m^2, 0)$, $(2A_ct, 0)$, $(2A_mt, 0)$, $(\tilde{\beta}(t), 0)$, $(2w_2, 0)$, $(0, 2B_{xy})$, $(0, 2A_c^2)$, $(0, 2A_cA_m)$, $(0, 2A_m^2)$, $(0, \tilde{\beta}(A_c))$, $(0, \tilde{\beta}(A_m))$, $(0, 2t^2)$, $(0, 2w_2)$

(Table continued)

TABLE XV. *(Continued)*

Symmetry group	Action	$H^2(G,\mathcal{A})$	Realizations	Generators
$p4m\times \mathrm{SO}(3)\times \mathbb{Z}_2^T$	$M: (a_e,a_m)\to(a_e,[-a_m]_4)$	$(\mathbb{Z}_2)^{22}$	2^{22}	$(2B_{xy},0)$, $(2B_{c^2},0)$, $(\tilde{\beta}(A_{x+y}),0)$, $(2A_{x+y}A_m,0)$, $(\tilde{\beta}(A_c),0)$, $(\tilde{\beta}(A_m),0)$, $(2A_{x+y}t,0)$, $(2A_ct,0)$, $(2A_mt,0)$, $(\tilde{\beta}(t),0)$, $(2w_2,0)$, $(0,2B_{xy})$, $(0,2B_{c^2})$, $(0,2A_{x+y}^2)$, $(0,2A_{x+y}A_m)$, $(0,2A_c^2)$, $(0,2A_m^2)$, $(0,\tilde{\beta}(A_{x+y}))$, $(0,\tilde{\beta}(A_c))$, $(0,\tilde{\beta}(A_m))$, $(0,2t^2)$, $(0,2w_2)$
	$M: (a_e,a_m)\to([-a_e]_4,a_m)$	$(\mathbb{Z}_4)^2\oplus(\mathbb{Z}_2)^{20}$	10×2^{20}	$(\tilde{\mathscr{B}}_{xy}^{(2)},0)$, $(\tilde{\mathscr{B}}_{c^2}^{(2)},0)$, $(\tilde{\beta}(A_{x+y}),0)$, $(2A_{x+y}A_m,0)$, $(2A_c^2,0)$, $(2A_m^2,0)$, $(2A_{x+y}t,0)$, $(2A_ct,0)$, $(2A_mt,0)$, $(\tilde{\beta}(t),0)$, $(2w_2,0)$, $(0,2B_{xy})$, $(0,2B_{c^2})$, $(0,2A_{x+y}^2)$, $(0,2A_{x+y}A_m)$, $(0,2A_c^2)$, $(0,2A_m^2)$, $(0,\tilde{\beta}(A_{x+y}))$, $(0,\tilde{\beta}(A_c))$, $(0,\tilde{\beta}(A_m))$, $(0,2t^2)$, $(0,2w_2)$
	$C_4: (a_e,a_m)\to([-a_e]_4,[-a_m]_4)$ $M: (a_e,a_m)\to(a_e,[-a_m]_4)$	$(\mathbb{Z}_2)^{22}$	2^{22}	$(2B_{xy},0)$, $(2B_{c^2},0)$, $(\tilde{\beta}(A_{x+y}),0)$, $(2A_{x+y}A_m,0)$, $(\tilde{\beta}(A_m),0)$, $(2A_m^2,0)$, $(2A_{x+y}t,0)$, $(2A_ct,0)$, $(2A_mt,0)$, $(\tilde{\beta}(t),0)$, $(2w_2,0)$, $(0,2B_{xy})$, $(0,2B_{c^2})$, $(0,2A_{x+y}^2)$, $(0,2A_{x+y}A_m)$, $(0,2A_c^2)$, $(0,2A_m^2)$, $(0,\tilde{\beta}(A_{x+y}))$, $(0,\tilde{\beta}(A_c))$, $(0,\tilde{\beta}(A_m))$, $(0,2t^2)$, $(0,2w_2)$
	$C_4: (a_e,a_m)\to([-a_e]_4,[-a_m]_4)$ $M: (a_e,a_m)\to([-a_e]_4,a_m)$	$(\mathbb{Z}_2)^{22}$	2^{22}	$(2B_{xy},0)$, $(2B_{c^2},0)$, $(\tilde{\beta}(A_{x+y}),0)$, $(2A_{x+y}A_m,0)$, $(\tilde{\beta}(A_m),0)$, $(2A_m^2,0)$, $(2A_{x+y}t,0)$, $(2A_ct,0)$, $(2A_mt,0)$, $(\tilde{\beta}(t),0)$, $(2w_2,0)$, $(0,2B_{xy})$, $(0,2B_{c^2})$, $(0,2A_{x+y}^2)$, $(0,2A_{x+y}A_m)$, $(0,2A_c^2)$, $(0,2A_m^2)$, $(0,\tilde{\beta}(A_{x+y}))$, $(0,\tilde{\beta}(A_c))$, $(0,\tilde{\beta}(A_m))$, $(0,2t^2)$, $(0,2w_2)$
	$T_{1,2}: (a_e,a_m)\to([-a_e]_4,[-a_m]_4)$ $M: (a_e,a_m)\to(a_e,[-a_m]_4)$	$(\mathbb{Z}_2)^{22}$	2^{22}	$(2B_{xy},0)$, $(2B_{c^2},0)$, $(2A_{x+y}^2,0)$, $(2A_{x+y}A_m,0)$, $(\tilde{\beta}(A_c),0)$, $(\tilde{\beta}(A_m),0)$, $(2A_{x+y}t,0)$, $(2A_ct,0)$, $(2A_mt,0)$, $(\tilde{\beta}(t),0)$, $(2w_2,0)$, $(0,2B_{xy})$, $(0,2B_{c^2})$, $(0,2A_{x+y}^2)$, $(0,2A_{x+y}A_m)$, $(0,2A_c^2)$, $(0,2A_m^2)$, $(0,\tilde{\beta}(A_{x+y}))$, $(0,\tilde{\beta}(A_c))$, $(0,\tilde{\beta}(A_m))$, $(0,2t^2)$, $(0,2w_2)$
	$T_{1,2}: (a_e,a_m)\to([-a_e]_4,[-a_m]_4)$ $M: (a_e,a_m)\to([-a_e]_4,a_m)$	$\mathbb{Z}_4\oplus(\mathbb{Z}_2)^{21}$	3×2^{21}	$(\tilde{\mathscr{B}}_{xy}^{(6)},0)$, $(2B_{c^2},0)$, $(2A_{x+y}^2,0)$, $(\tilde{\beta}(A_m),0)$, $(2A_c^2,0)$, $(2A_m^2,0)$, $(2A_{x+y}t,0)$, $(2A_ct,0)$, $(2A_mt,0)$, $(\tilde{\beta}(t),0)$, $(2w_2,0)$, $(0,2B_{xy})$, $(0,2B_{c^2})$, $(0,2A_{x+y}^2)$, $(0,2A_{x+y}A_m)$, $(0,2A_c^2)$, $(0,2A_m^2)$, $(0,\tilde{\beta}(A_{x+y}))$, $(0,\tilde{\beta}(A_c))$, $(0,\tilde{\beta}(A_m))$, $(0,2t^2)$, $(0,2w_2)$
	$T_{1,2},C_4: (a_e,a_m)\to([-a_e]_4,[-a_m]_4)$ $M: (a_e,a_m)\to(a_e,[-a_m]_4)$	$\mathbb{Z}_4\oplus(\mathbb{Z}_2)^{21}$	3×2^{21}	$(\tilde{\mathscr{B}}_{xy}^{(7)},0)$, $(2B_{c^2},0)$, $(2A_{x+y}^2,0)$, $(2A_{x+y}A_m,0)$, $(2A_c^2,0)$, $(\tilde{\beta}(A_m),0)$, $(2A_{x+y}t,0)$, $(2A_ct,0)$, $(2A_mt,0)$, $(\tilde{\beta}(t),0)$, $(2w_2,0)$, $(0,2B_{xy})$, $(0,2B_{c^2})$, $(0,2A_{x+y}^2)$, $(0,2A_{x+y}A_m)$, $(0,2A_c^2)$, $(0,2A_m^2)$, $(0,\tilde{\beta}(A_{x+y}))$, $(0,\tilde{\beta}(A_c))$, $(0,\tilde{\beta}(A_m))$, $(0,2t^2)$, $(0,2w_2)$
	$T_{1,2},C_4: (a_e,a_m)\to([-a_e]_4,[-a_m]_4)$ $M: (a_e,a_m)\to([-a_e]_4,a_m)$	$(\mathbb{Z}_2)^{22}$	2^{22}	$(2B_{xy},0)$, $(2B_{c^2},0)$, $(2A_{x+y}^2,0)$, $(\tilde{\beta}(A_m),0)$, $(\tilde{\beta}(A_c),0)$, $(2A_m^2,0)$, $(2A_{x+y}t,0)$, $(2A_ct,0)$, $(2A_mt,0)$, $(\tilde{\beta}(t),0)$, $(2w_2,0)$, $(0,2B_{xy})$, $(0,2B_{c^2})$, $(0,2A_{x+y}^2)$, $(0,2A_{x+y}A_m)$, $(0,2A_c^2)$, $(0,2A_m^2)$, $(0,\tilde{\beta}(A_{x+y}))$, $(0,\tilde{\beta}(A_c))$, $(0,\tilde{\beta}(A_m))$, $(0,2t^2)$, $(0,2w_2)$

we need the cohomology and representative cochains of $H^2(p6m,\mathbb{Z})$ and $H^2(p4m,\mathbb{Z})$ with all possible actions on $\mathbb{Z}$, which we present here for completeness.

For $p6m$, we have

(1) *Trivial action on* $\mathbb{Z}$.—

$$H^2(p6m,\mathbb{Z})=(\mathbb{Z}_2)^2. \tag{C41}$$

We denote the generators of the two $\mathbb{Z}_2$ pieces by $\beta^{(1)}(A_c)$ and $\beta^{(1)}(A_m)$, respectively, which have representative cochains

$$\beta^{(1)}(A_c)(g_1,g_2)=\frac{[c_1]_2+[c_2]_2-[c_1+c_2]_2}{2}, \tag{C42}$$

$$\beta^{(1)}(A_m)(g_1,g_2)=\frac{[m_1]_2+[m_2]_2-[m_1+m_2]_2}{2}. \tag{C43}$$

(2) *M acts nontrivially on* $\mathbb{Z}$.—

$$H^2(p6m,\mathbb{Z})=\mathbb{Z}\oplus\mathbb{Z}_6. \tag{C44}$$

We denote the generators of the $\mathbb{Z}$ piece and the $\mathbb{Z}_6$ piece by $\mathscr{B}_{xy}^{(2)}$ and $\mathscr{B}_{c^2}^{(2)}$, respectively, which have representative cochains

$$\begin{aligned}\mathscr{B}_{xy}^{(2)}(g_1,g_2)=&P_{60}(c_1)[P_c(m_1)y_1x_2+m_1y_2(x_2+y_1)]\\&+P_{61}(c_1)\left[P_c(m_1)\left(\frac{x_2(x_2-1)}{2}+y_1x_2-y_2(x_2+y_1)\right)+m_1\left(\frac{y_2(y_2-1)}{2}+y_1(-x_2+y_2)\right)\right]\\&+P_{62}(c_1)\left[P_c(m_1)\left(\frac{y_2(y_2+1)}{2}-x_2-y_2(x_2+y_1)\right)+m_1\left(\frac{x_2(x_2+1)}{2}-y_2-y_1x_2\right)\right]\\&+P_{63}(c_1)[P_c(m_1)(-x_2+y_2-y_1x_2)+m_1(x_2-y_2+y_2(x_2-y_1)]\\&+P_{64}(c_1)\left[P_c(m_1)\left(\frac{x_2(x_2-1)}{2}+y_2-y_1x_2-y_2(x_2-y_1)\right)+m_1\left(\frac{y_2(y_2-1)}{2}+x_2+y_1(x_2-y_2)\right)\right]\\&+P_{65}(c_1)\left[P_c(m_1)\left(\frac{y_2(y_2+1)}{2}-y_2(x_2-y_1)\right)+m_1\left(\frac{x_2(x_2+1)}{2}+y_1x_2\right)\right],\end{aligned} \tag{C45}$$

$$\mathscr{B}_{c^2}^{(2)}(g_1,g_2)=\frac{[c_1]_6+(-1)^{m_1}[c_2]_6-[c_1+(-1)^{m_1}c_2]_6}{6}. \tag{C46}$$

Note that the $\mathbb{Z}_2$ reduction of $\mathscr{B}_{c^2}^{(2)}$ is actually $A_c^2+A_cA_m$.

(3) C_6 *acts nontrivially on* $\mathbb{Z}$.—

$$H^2(p6m,\mathbb{Z})=\mathbb{Z}_2. \tag{C47}$$

We denote the generator by $\beta^{(3)}(A_m)$, which has a representative cochain

$$\beta^{(3)}(A_m)(g_1,g_2)=\frac{[m_1]_2+(-1)^{c_1}[m_2]_2-[m_1+m_2]_2}{2}. \tag{C48}$$

(4) *Both* C_6 *and M act nontrivially on* $\mathbb{Z}$.—

$$H^2(p6m,\mathbb{Z})=\mathbb{Z}_2. \tag{C49}$$

We denote the generator by $\beta^{(4)}(A_m)$, which has a representative cochain

$$\beta^{(4)}(A_m)(g_1, g_2) = \frac{[m_1]_2 + (-1)^{c_1+m_1}[m_2]_2 - [m_1 + m_2]_2}{2}. \tag{C50}$$

For $p4m$, we have

(1) *Trivial action on* $\mathbb{Z}$.—

$$H^2(p4m, \mathbb{Z}) = (\mathbb{Z}_2)^3. \tag{C51}$$

We denote the generators of the three $\mathbb{Z}_2$ pieces by $\beta^{(1)}(A_{x+y})$, $\beta^{(1)}(A_c)$, and $\beta^{(1)}(A_m)$, respectively, which have representative cochains

$$\beta^{(1)}(A_{x+y})(g_1, g_2) = \frac{[x_1 + y_1]_2 + [x_2 + y_2]_2 - [x_1 + x_2 + y_1 + y_2]_2}{2}, \tag{C52}$$

$$\beta^{(1)}(A_c)(g_1, g_2) = \frac{[c_1]_2 + [c_2]_2 - [c_1 + (-1)^{m_1}c_2]_2}{2}, \tag{C53}$$

$$\beta^{(1)}(A_m)(g_1, g_2) = \frac{[m_1]_2 + [m_2]_2 - [m_1 + m_2]_2}{2}. \tag{C54}$$

(2) *M acts nontrivially on* $\mathbb{Z}$.—

$$H^2(p4m, \mathbb{Z}) = \mathbb{Z} \oplus \mathbb{Z}_4 \oplus \mathbb{Z}_2. \tag{C55}$$

We denote the generators of the $\mathbb{Z}$, $\mathbb{Z}_4$, and $\mathbb{Z}_2$ piece by $\mathscr{B}^{(2)}_{xy}$, $\mathscr{B}^{(2)}_{c^2}$, and $\beta^{(2)}(A_{x+y})$, respectively, which have representative cochains

$$\mathscr{B}^{(2)}_{xy}(g_1, g_2) = [P_{40}(c_1) - P_{42}(c_1)](-1)^{m_1}y_1x_2 - P_{41}(c_1)y_2[y_1 + (-1)^{m_1}x_2] + P_{43}(c_1)y_2[y_1 - (-1)^{m_1}x_2], \tag{C56}$$

$$\mathscr{B}^{(2)}_{c^2}(g_1, g_2) = \frac{[c_1]_4 + (-1)^{m_1}[c_2]_4 - [c_1 + (-1)^{m_1}c_2]_4}{4}, \tag{C57}$$

$$\beta^{(2)}(A_{x+y})(g_1, g_2) = \frac{[x_1 + y_1]_2 + (-1)^{m_1}[x_2 + y_2]_2 - [x_1 + x_2 + y_1 + y_2]_2}{2}. \tag{C58}$$

(3) C_4 *acts nontrivially on* $\mathbb{Z}$.—

$$H^2(p4m, \mathbb{Z}) = (\mathbb{Z}_2)^2. \tag{C59}$$

We denote the generators of the two $\mathbb{Z}_2$ pieces by $\beta^{(3)}(A_{x+y})$ and $\beta^{(3)}(A_m)$, respectively, which have representative cochains

$$\beta^{(3)}(A_{x+y})(g_1, g_2) = \frac{[x_1 + y_1]_2 + (-1)^{c_1}[x_2 + y_2]_2 - [x_1 + x_2 + y_1 + y_2]_2}{2}, \tag{C60}$$

$$\beta^{(3)}(A_m)(g_1, g_2) = \frac{[m_1]_2 + (-1)^{c_1}[m_2]_2 - [m_1 + m_2]_2}{2}. \tag{C61}$$

(4) *Both* C_4 *and M act nontrivially on* $\mathbb{Z}$.—

$$H^2(p4m, \mathbb{Z}) = (\mathbb{Z}_2)^2. \tag{C62}$$

We denote the generators of the two $\mathbb{Z}_2$ pieces by $\beta^{(4)}(A_{x+y})$ and $\beta^{(4)}(A_m)$, respectively, which have representative cochains

$$\beta^{(4)}(A_{x+y})(g_1,g_2) = \frac{[x_1+y_1]_2 + (-1)^{c_1+m_1}[x_2+y_2]_2 - [x_1+x_2+y_1+y_2]_2}{2}, \tag{C63}$$

$$\beta^{(4)}(A_m)(g_1,g_2) = \frac{[m_1]_2 + (-1)^{c_1+m_1}[m_2]_2 - [m_1+m_2]_2}{2}. \tag{C64}$$

(5) *$T_{1,2}$ acts nontrivially on $\mathbb{Z}$.—*

$$H^2(p4m,\mathbb{Z}) = (\mathbb{Z}_2)^2. \tag{C65}$$

We denote the generator of the two $\mathbb{Z}_2$ pieces by $\beta^{(5)}(A_c)$ and $\beta^{(5)}(A_m)$, respectively, which have representative cochains

$$\beta^{(5)}(A_c)(g_1,g_2) = \frac{[c_1]_2 + (-1)^{x_1+y_1}[c_2]_2 - [c_1+(-1)^{m_1}c_2]_2}{2}, \tag{C66}$$

$$\beta^{(5)}(A_m)(g_1,g_2) = \frac{[m_1]_2 + (-1)^{x_1+y_1}[m_2]_2 - [m_1+m_2]_2}{2}. \tag{C67}$$

(6) *Both $T_{1,2}$ and M act nontrivially on $\mathbb{Z}$.—*

$$H^2(p4m,\mathbb{Z}) = \mathbb{Z}_4 \oplus \mathbb{Z}_2. \tag{C68}$$

We denote the generators of the $\mathbb{Z}_4$ and $\mathbb{Z}_2$ piece by $\mathscr{B}_{xy}^{(6)}$ and $\beta^{(6)}(A_c)$, respectively, which have representative cochains

$$\begin{aligned}&\mathscr{B}_{xy}^{(6)}(g_1,g_2) = (-1)^{\tilde{x}_1}[P_c(c_1)P(\tilde{y}_1)P(\tilde{x}_2) + P(c_1)P(\tilde{y}_1+\tilde{x}_2)P(\tilde{y}_2)]\\ &\quad\text{with}\;\; \tilde{x} = x + P_{41}(c) + P_c(m)P_{42}(c) + P(m)P_{40}(c), \quad \tilde{y} = y + P_{42}(c) + P_c(m)P_{43}(c) + P(m)P_{41}(c),\end{aligned} \tag{C69}$$

$$\beta^{(6)}(A_m)(g_1,g_2) = \frac{[m_1]_2 + (-1)^{x_1+y_1+m_1}[m_2]_2 - [m_1+m_2]_2}{2}. \tag{C70}$$

Note that the $\mathbb{Z}_2$ reduction of $\mathscr{B}_{xy}^{(7)}$ is actually $B_{xy} + B_{c^2} + A_{x+y}(A_{x+y} + A_m)$.

(7) *Both $T_{1,2}$ and C_4 act nontrivially on $\mathbb{Z}$.—*

$$H^2(p4m,\mathbb{Z}) = \mathbb{Z}_4 \oplus \mathbb{Z}_2. \tag{C71}$$

We denote the generators of the $\mathbb{Z}_4$ and $\mathbb{Z}_2$ pieces by $\beta^{(7)}(A_{x+y})$ and $\beta^{(7)}(A_m)$, respectively, which have representative cochains

$$\mathscr{B}_{xy}^{(7)}(g_1,g_2) = (-1)^{x_1}[P_c(c_1)P(y_1)P(x_2) + P(c_1)P(y_1+x_2)P(y_2)], \tag{C72}$$

$$\beta^{(7)}(A_m)(g_1,g_2) = \frac{[m_1]_2 + (-1)^{x_1+y_1+c_1}[m_2]_2 - [m_1+m_2]_2}{2}. \tag{C73}$$

(8) *All of $T_{1,2}$, C_4, and M act nontrivially on $\mathbb{Z}$.—*

$$H^2(p4m,\mathbb{Z}) = (\mathbb{Z}_2)^2. \tag{C74}$$

We denote the generators of the two $\mathbb{Z}_2$ pieces by $\beta^{(8)}(A_c)$ and $\beta^{(8)}(A_m)$, respectively, which have representative cochains

$$\beta^{(8)}(A_c)(g_1,g_2)=\frac{[c_1]_2+(-1)^{x_1+y_1+c_1+m_1}[c_2]_2-[c_1+(-1)^{m_1}c_2]_2}{2}, \tag{C75}$$

$$\beta^{(8)}(A_m)(g_1,g_2)=\frac{[m_1]_2+(-1)^{x_1+y_1+c_1+m_1}[m_2]_2-[m_1+m_2]_2}{2}. \tag{C76}$$

5. $\mathrm{U}(1)_2\times\mathrm{U}(1)_{-2}$ (double semion)

In this case, we have $\mathcal{A}=(\mathbb{Z}_2)^2$. The topological symmetry group is $\mathbb{Z}_2^T$, generated by $\tilde{S}$ exchanging s and $\bar{s}$, i.e.,

$$\tilde{S}\colon\ (a_s,a_{\bar{s}})\to(a_{\bar{s}},a_s). \tag{C77}$$

We can choose the U symbols and a set of η symbols all equal to 1. Therefore, the anyon permutation patterns are completely fixed. The classification of symmetry fractionalization patterns and the generators are listed in Table XVI. It turns out that all symmetry fractionalization classes lead to anomaly-free states.

6. $\mathrm{U}(1)_4\times\mathrm{U}(1)_{-4}$

In this case, we have $\mathcal{A}=(\mathbb{Z}_4)^2$. For $N=2$, the topological symmetry is $\mathbb{Z}_4^T\rtimes\mathbb{Z}_2^T$, generated by an order 2 antiunitary symmetry $\tilde{S}$ which exchanges s and $\bar{s}$, i.e.,

$$\tilde{S}\colon\ (a_s,a_{\bar{s}})\to(a_{\bar{s}},a_s), \tag{C78}$$

and another order 4 antiunitary symmetry T, which permutes anyons in the following way:

$$T\colon\ (a_s,a_{\bar{s}})\to(a_{\bar{s}},[-a_s]_{2N}). \tag{C79}$$

The two generators satisfy the relation

$$\tilde{S}^2=\mathbf{1},\qquad T^4=\mathbf{1},\qquad \tilde{S}T\tilde{S}=T^{-1}. \tag{C80}$$

An element in $\mathbb{Z}_4^T\rtimes\mathbb{Z}_2^T$ can be written as $T^{g_1}\tilde{S}^{g_2}$, with $g_1\in\{0,\ldots,3\}$ and $g_2\in\{0,1\}$. To define the U symbols, first we define the following function:

$$\tilde{U}(a_s,b_s)=\begin{cases}(-1)^{a_s} & b_s\neq 0,\\ 1 & b_s=0.\end{cases} \tag{C81}$$

Given an element $\mathbf{g}\in\mathbb{Z}_4^T\rtimes\mathbb{Z}_2^T$, the U symbols can be chosen such that

$$U_{\mathbf{g}}(a,b;c)=\begin{cases}1 & g_1=0,\\ \tilde{U}(a_{\bar{s}},b_{\bar{s}}) & g_1=1,\\ \tilde{U}(a_s,b_s)\tilde{U}(a_{\bar{s}},b_{\bar{s}}) & g_1=2,\\ \tilde{U}(a_s,b_s) & g_1=3.\end{cases} \tag{C82}$$

And a set of η symbols can be chosen to be all identity.

Because $\mathcal{T}^2=\mathbf{1}$, $\mathcal{T}$ can act on anyons in two ways: Either $\mathcal{T}:(a_s,a_{\bar{s}})\to(a_{\bar{s}},a_s)$ or $\mathcal{T}:(a_s,a_{\bar{s}})\to([-a_{\bar{s}}]_4,[-a_s]_4)$. Because these two cases are related by relabeling anyons using $T\tilde{S}$, we can specialize to the cases $\mathcal{T}:(a_s,a_{\bar{s}})\to(a_{\bar{s}},a_s)$, and we need only to consider how $p6m$ or $p4m$ permutes anyons. For the $p6m\times\mathrm{SO}(3)\times\mathbb{Z}_2^T$ case, there are four possible anyon permutation patterns, while for the $p4m\times\mathrm{SO}(3)\times\mathbb{Z}_2^T$ case, there are eight possible anyon permutation patterns.

The corresponding classification of symmetry fractionalization classes and the generators are listed in Table XVII. Specifically, since $\mathcal{T}$ permutes s and $\bar{s}$, $H^2[p6m\times\mathrm{SO}(3)\times\mathbb{Z}_2^T,\mathbb{Z}_4\oplus\mathbb{Z}_4]$ or $H^2[p6m\times\mathrm{SO}(3)\times\mathbb{Z}_2^T,\mathbb{Z}_4\oplus\mathbb{Z}_4]$ is isomorphic to $H^2[p6m\times\mathrm{SO}(3),\mathbb{Z}_4]$ or $H^2[p4m\times\mathrm{SO}(3),\mathbb{Z}_4]$, respectively, with the $\mathbb{Z}_4$ corresponding to the diagonal (0,0), (1,1), (2,2), (3,3) anyons. As discussed at the beginning of the appendix, $H^2(G,\mathbb{Z}_4)$ can all be obtained from the $\mathbb{Z}$ cohomology or $\mathbb{Z}_2$ cohomology of the symmetry groups, and we list the $\mathbb{Z}$ cohomology of $p6m$ or

TABLE XVI. All possible symmetry fractionalization classes of $p6m\times\mathrm{SO}(3)\times\mathbb{Z}_2^T$ and $p4m\times\mathrm{SO}(3)\times\mathbb{Z}_2^T$ for $\mathrm{U}(1)_2\times\mathrm{U}(1)_{-2}$ (double semion theory).

Symmetry group	Action	$H^2(G,\mathcal{A})$	Realizations	Generators
$p6m\times\mathrm{SO}(3)\times\mathbb{Z}_2^T$	$M,\mathcal{T}:(a_s,a_{\bar{s}})\to(a_{\bar{s}},a_s)$	$(\mathbb{Z}_2)^5$	32	(B_{xy},B_{xy}), (A_c^2,A_c^2), (A_cA_m,A_cA_m), (A_m^2,A_m^2), (w_2,w_2)
$p4m\times\mathrm{SO}(3)\times\mathbb{Z}_2^T$	$M,\mathcal{T}:(a_s,a_{\bar{s}})\to(a_{\bar{s}},a_s)$	$(\mathbb{Z}_2)^7$	128	(B_{xy},B_{xy}), (B_{c^2},B_{c^2}), (A_{x+y}^2,A_{x+y}^2), $(A_{x+y}A_m,A_{x+y}A_m)$, (A_c^2,A_c^2), (A_m^2,A_m^2), (w_2,w_2)

TABLE XVII. All possible symmetry fractionalization classes of $p6m \times SO(3) \times \mathbb{Z}_2^T$ and $p4m \times SO(3) \times \mathbb{Z}_2^T$ for $U(1)_4 \times U(1)_{-4}$, when the action of $\mathcal{T}$ is specified by $\mathcal{T}: (a_s, a_{\bar{s}}) \to (a_{\bar{s}}, a_s)$. The cases where the action of $\mathcal{T}$ is $\mathcal{T}: (a_s, a_{\bar{s}}) \to ([-a_{\bar{s}}]_4, [-a_s]_4)$ is related to these cases by relabeling, hence, are not distinct physical realizations. When counting the number of realizations in each case, overcounts due to the equivalence from relabeling anyons have been taken care of.

Symmetry group	Action	$H^2(G,\mathcal{A})$	Realizations	Generators
$p6m \times SO(3) \times \mathbb{Z}_2^T$	$M: (a_s, a_{\bar{s}}) \to (a_{\bar{s}}, a_s)$	$(\mathbb{Z}_2)^5$	32	$(2B_{xy}, 2B_{xy})$, $(\tilde{\beta}(A_c), \tilde{\beta}(A_c))$, $(2A_cA_m, 2A_cA_m)$, $(\tilde{\beta}(A_m), \tilde{\beta}(A_m))$, $(2w_2, 2w_2)$
	$M: (a_s, a_{\bar{s}}) \to ([-a_{\bar{s}}]_4, [-a_s]_4)$	$\mathbb{Z}_4 \oplus (\mathbb{Z}_2)^4$	48	$(\tilde{\mathscr{B}}_{xy}^{(2)}, \tilde{\mathscr{B}}_{xy}^{(2)})$, $(\tilde{\mathscr{B}}_{c^2}^{(2)}, \tilde{\mathscr{B}}_{c^2}^{(2)})$, $(2A_cA_m, 2A_cA_m)$, $(2A_m^2, 2A_m^2)$, $(2w_2, 2w_2)$
	$C_6: (a_s, a_{\bar{s}}) \to ([-a_s]_4, [-a_{\bar{s}}]_4)$ $M: (a_s, a_{\bar{s}}) \to (a_{\bar{s}}, a_s)$	$(\mathbb{Z}_2)^5$	32	$(2B_{xy}, 2B_{xy})$, $(2A_c^2, 2A_c^2)$, $(2A_cA_m, 2A_cA_m)$, $(\tilde{\beta}(A_m), \tilde{\beta}(A_m))$, $(2w_2, 2w_2)$
	$C_6: (a_s, a_{\bar{s}}) \to ([-a_s]_4, [-a_{\bar{s}}]_4)$ $M: (a_s, a_{\bar{s}}) \to ([-a_{\bar{s}}]_4, [-a_s]_4)$	$(\mathbb{Z}_2)^5$	32	$(2B_{xy}, 2B_{xy})$, $(2A_c^2, 2A_c^2)$, $(\tilde{\beta}(A_m), \tilde{\beta}(A_m))$, $(2A_m^2, 2A_m^2)$, $(2w_2, 2w_2)$
$p4m \times SO(3) \times \mathbb{Z}_2^T$	$M: (a_s, a_{\bar{s}}) \to (a_{\bar{s}}, a_s)$	$(\mathbb{Z}_2)^7$	128	$(2B_{xy}, 2B_{xy})$, $(2B_{c^2}, 2B_{c^2})$, $(\tilde{\beta}(A_{x+y}), \tilde{\beta}(A_{x+y}))$, $(2A_{x+y}A_m, 2A_{x+y}A_m)$, $(\tilde{\beta}(A_c), \tilde{\beta}(A_c))$, $(\tilde{\beta}(A_m), \tilde{\beta}(A_m))$, $(2w_2, 2w_2)$
	$M: (a_s, a_{\bar{s}}) \to ([-a_{\bar{s}}]_4, [-a_s]_4)$	$(\mathbb{Z}_4)^2 \oplus (\mathbb{Z}_2)^5$	320	$(\tilde{\mathscr{B}}_{xy}^{(2)}, \tilde{\mathscr{B}}_{xy}^{(2)})$, $(\tilde{\mathscr{B}}_{c^2}^{(2)}, \tilde{\mathscr{B}}_{c^2}^{(2)})$, $(\tilde{\beta}(A_{x+y}), \tilde{\beta}(A_{x+y}))$, $(2A_{x+y}A_m, 2A_{x+y}A_m)$, $(2A_c^2, 2A_c^2)$, $(2A_m^2, 2A_m^2)$, $(2w_2, 2w_2)$
	$C_4: (a_s, a_{\bar{s}}) \to ([-a_s]_4, [-a_{\bar{s}}]_4)$ $M: (a_s, a_{\bar{s}}) \to (a_{\bar{s}}, a_s)$	$(\mathbb{Z}_2)^7$	128	$(2B_{xy}, 2B_{xy})$, $(2B_{c^2}, 2B_{c^2})$, $(\tilde{\beta}(A_{x+y}), \tilde{\beta}(A_{x+y}))$, $(2A_{x+y}A_m, 2A_{x+y}A_m)$, $(\tilde{\beta}(A_m), \tilde{\beta}(A_m))$, $(2A_m^2, 2A_m^2)$, $(2w_2, 2w_2)$
	$C_4: (a_s, a_{\bar{s}}) \to ([-a_s]_4, [-a_{\bar{s}}]_4)$ $M: (a_s, a_{\bar{s}}) \to ([-a_{\bar{s}}]_4, [-a_s]_4)$	$(\mathbb{Z}_2)^7$	128	$(2B_{xy}, 2B_{xy})$, $(2B_{c^2}, 2B_{c^2})$, $(\tilde{\beta}(A_{x+y}), \tilde{\beta}(A_{x+y}))$, $(2A_{x+y}A_m, 2A_{x+y}A_m)$, $(\tilde{\beta}(A_m), \tilde{\beta}(A_m))$, $(2A_m^2, 2A_m^2)$, $(2w_2, 2w_2)$
	$T_{1,2}: (a_s, a_{\bar{s}}) \to ([-a_s]_4, [-a_{\bar{s}}]_4)$ $M: (a_s, a_{\bar{s}}) \to (a_{\bar{s}}, a_s)$	$(\mathbb{Z}_2)^7$	128	$(2B_{xy}, 2B_{xy})$, $(2B_{c^2}, 2B_{c^2})$, $(2A_{x+y}^2, 2A_{x+y}^2)$, $(2A_{x+y}A_m, 2A_{x+y}A_m)$, $(\tilde{\beta}(A_c), \tilde{\beta}(A_c))$, $(\tilde{\beta}(A_m), \tilde{\beta}(A_m))$, $(2w_2, 2w_2)$
	$T_{1,2}: (a_s, a_{\bar{s}}) \to ([-a_s]_4, [-a_{\bar{s}}]_4)$ $M: (a_s, a_{\bar{s}}) \to ([-a_{\bar{s}}]_4, [-a_s]_4)$	$\mathbb{Z}_4 \oplus (\mathbb{Z}_2)^6$	192	$(\tilde{\mathscr{B}}_{xy}^{(6)}, \tilde{\mathscr{B}}_{xy}^{(6)})$, $(2B_{c^2}, 2B_{c^2})$, $(2A_{x+y}^2, 2A_{x+y}^2)$, $(\tilde{\beta}(A_m), \tilde{\beta}(A_m))$, $(2A_c^2, 2A_c^2)$, $(2A_m^2, 2A_m^2)$, $(2w_2, 2w_2)$
	$T_{1,2}, C_4: (a_s, a_{\bar{s}}) \to ([-a_s]_4, [-a_{\bar{s}}]_4)$ $M: (a_s, a_{\bar{s}}) \to (a_{\bar{s}}, a_s)$	$\mathbb{Z}_4 \oplus (\mathbb{Z}_2)^6$	192	$(\tilde{\mathscr{B}}_{xy}^{(7)}, \tilde{\mathscr{B}}_{xy}^{(7)})$, $(2B_{c^2}, 2B_{c^2})$, $(2A_{x+y}^2, 2A_{x+y}^2)$, $(2A_{x+y}A_m, 2A_{x+y}A_m)$, $(2A_c^2, 2A_c^2)$, $(\tilde{\beta}(A_m), \tilde{\beta}(A_m))$, $(2w_2, 2w_2)$
	$T_{1,2}, C_4: (a_s, a_{\bar{s}}) \to ([-a_s]_4, [-a_{\bar{s}}]_4)$ $M: (a_s, a_{\bar{s}}) \to ([-a_{\bar{s}}]_4, [-a_s]_4)$	$(\mathbb{Z}_2)^7$	128	$(2B_{xy}, 2B_{xy})$, $(2B_{c^2}, 2B_{c^2})$, $(2A_{x+y}^2, 2A_{x+y}^2)$, $(\tilde{\beta}(A_m), \tilde{\beta}(A_m))$, $(\tilde{\beta}(A_c), \tilde{\beta}(A_c))$, $(2A_m^2, 2A_m^2)$, $(2w_2, 2w_2)$

$p4m$ in Appendix C 4. It turns out that all symmetry fractionalization classes lead to anomaly-free states.

APPENDIX D: ANOMALY INDICATORS

In this appendix, we first write down the anomaly indicators for $\mathbb{Z}_2^T$, $\mathbb{Z}_2^T \times \mathbb{Z}_2^T$, $\mathbb{Z}_2 \times \mathbb{Z}_2$, and $SO(3) \times \mathbb{Z}_2^T$ symmetries, where $\mathbb{Z}_2^T$ denotes an antiunitary order 2 symmetry group. These anomaly indicators are all derived in Ref. [20] (also see Refs. [87,121] for the $\mathbb{Z}_2^T$ symmetry). As explained in Ref. [20] in detail (see Sec. VI therein), the anomaly indicators of many other groups, including $p6 \times SO(3)$, $p4 \times SO(3)$, $p6m \times SO(3) \times \mathbb{Z}_2^T$, $p4m \times SO(3) \times \mathbb{Z}_2^T$, $p6m \times \mathbb{Z}_2^T$, and $p4m \times \mathbb{Z}_2^T$, can be obtained by restricting these groups to some of their $\mathbb{Z}_2^T$, $\mathbb{Z}_2^T \times \mathbb{Z}_2^T$, and $\mathbb{Z}_2 \times \mathbb{Z}_2$ subgroups. So we can use the known anomaly indicators in Ref. [20] to write down all anomaly indicators for all symmetry groups appearing in this paper and identify the anomaly accordingly. These anomaly indicators are also recorded in this appendix. Using the expressions of the anomaly of each lattice homotopy class for these symmetries in Ref. [8], which are written in terms of group cohomology, we can further obtain the values of the anomaly indicators for each lattice homotopy class of these symmetries.

(i) $\mathbb{Z}_2^T$.—The anomalies for the group $\mathbb{Z}_2^T$ in $(2+1)$D are classified by $\mathbb{Z}_2 \oplus \mathbb{Z}_2$, and the two $\mathbb{Z}_2$ factors correspond to the "in-cohomology" and "beyond-cohomology" piece of the anomaly, respectively. The anomaly indicator for the beyond-cohomology piece is given by

$$\mathcal{I}_0 = \frac{1}{D}\sum_a d_a^2 \theta_a. \tag{D1}$$

This is also related to the chiral central charge c_- by the formula $\mathcal{I}_0 = \exp[2\pi i(c_-/8)]$. The anomaly indicator for the in-cohomology piece is given by

$$\mathcal{I}_1(\mathcal{T}) = \frac{1}{D}\sum_{{}^{\mathcal{T}}a=a} d_a \theta_a \times \eta_a(\mathcal{T},\mathcal{T}), \tag{D2}$$

where $\mathcal{T}$ is the generator of the $\mathbb{Z}_2^T$ symmetry.

(ii) $\mathbb{Z}_2^T \times \mathbb{Z}_2^T$.—The anomalies for the group $\mathbb{Z}_2^T \times \mathbb{Z}_2^T$ in $(2+1)$D are classified by $(\mathbb{Z}_2)^4$. Suppose the two antiunitary generators of $\mathbb{Z}_2^T \times \mathbb{Z}_2^T$ are $\mathcal{T}_1$ and $\mathcal{T}_2$. The four anomaly indicators can be given by $\mathcal{I}_0$, $\mathcal{I}_1(\mathcal{T}_1)$, $\mathcal{I}_1(\mathcal{T}_2)$, and

$$\begin{aligned}\mathcal{I}_2(\mathcal{T}_1,\mathcal{T}_2) = \frac{1}{D^3} \sum_{\substack{a,b,c,x,y,u,v \\ \mu_x\nu_x\mu_y\nu_y\tilde{\mu}_x\tilde{\nu}_x\tilde{\mu}_y\tilde{\nu}_y\rho\sigma\tau\alpha\beta\gamma\delta \\ {}^{\mathcal{T}_1}a\times{}^{\mathcal{T}_2}c\times c\to a \\ {}^{\mathcal{T}_1}c\times c\times b\to{}^{\mathcal{T}_2}b}} & d_c d_v \frac{\theta_v}{\theta_a\theta_b}\left(R_u^{{}^{\mathcal{T}_1}c,{}^{\mathcal{T}_2}c}\right)_{\rho\sigma}\left(F_v^{a,{}^{\mathcal{T}_1\mathcal{T}_2}c,{}^{\mathcal{T}_2}y}\right)^*_{({}^{\mathcal{T}_1}x,\tilde{\mu}_x,\alpha)(b,\tilde{\mu}_y,\tau)}\left(F_{{}^{\mathcal{T}_2}y}^{{}^{\mathcal{T}_2}c,{}^{\mathcal{T}_1}c,y}\right)^*_{(u,\rho,\beta)({}^{\mathcal{T}_2}b,\mu_y,\tilde{\nu}_y)} \\ & \times\left(F_x^{{}^{\mathcal{T}_1}x,{}^{\mathcal{T}_1}c,{}^{\mathcal{T}_2}c}\right)^*_{({}^{\mathcal{T}_1}a,\tilde{\nu}_x,\mu_x)(u,\sigma,\gamma)}\left(F_v^{{}^{\mathcal{T}_1}x,u,y}\right)^*_{(x,\gamma,\delta)({}^{\mathcal{T}_2}y,\beta,\alpha)}\left(F_v^{x,c,b}\right)^*_{(a,\nu_x,\tau)(y,\nu_y,\delta)} \\ & \times U^{-1}_{\mathcal{T}_1}({}^{\mathcal{T}_1}a,{}^{\mathcal{T}_2}c;x)_{\mu_x\tilde{\mu}_x}U^{-1}_{\mathcal{T}_1}(x,c;a)_{\nu_x\tilde{\nu}_x}U^{-1}_{\mathcal{T}_2}({}^{\mathcal{T}_1}c,y;{}^{\mathcal{T}_2}b)^*_{\mu_y\tilde{\mu}_y}U^{-1}_{\mathcal{T}_2}(c,b;y)^*_{\nu_y\tilde{\nu}_y} \\ & \times\eta_a(\mathcal{T}_1,\mathcal{T}_1)\eta_b(\mathcal{T}_2,\mathcal{T}_2)\frac{\eta_c(\mathcal{T}_2,\mathcal{T}_1)}{\eta_c(\mathcal{T}_1,\mathcal{T}_2)}.\end{aligned} \tag{D3}$$

(iii) $\mathbb{Z}_2 \times \mathbb{Z}_2$.—The anomalies for the group $\mathbb{Z}_2 \times \mathbb{Z}_2$ in $(2+1)$D are classified by $(\mathbb{Z}_2)^2$. Suppose the two generators of $\mathbb{Z}_2 \times \mathbb{Z}_2$ are C_1 and C_2. The two anomaly indicators can be given by $\mathcal{I}_3(C_1,C_2)$ and $\mathcal{I}_3(C_2,C_1)$, where

$$\begin{aligned}\mathcal{I}_3(C_1,C_2) = \frac{1}{D^2}\sum_{\substack{a,b,x,u \\ \mu\nu\tilde{\mu}\tilde{\nu}\rho\sigma\alpha \\ {}^{C_1}a=a \\ a\times b\times{}^{C_1}b\to{}^{C_2}a}} & d_b\frac{\theta_x}{\theta_a}\left(R_u^{b,{}^{C_1}b}\right)_{\rho\sigma}\left(F_{{}^{C_2}a}^{a,b,{}^{C_1}b}\right)^*_{(x,\tilde{\mu},\tilde{\nu})(u,\sigma,\alpha)}\left(F_{{}^{C_2}a}^{a,{}^{C_1}b,b}\right)_{({}^{C_1}x,\mu,\nu)(u,\rho,\alpha)} \\ & \times U^{-1}_{C_1}(a,b;x)_{\tilde{\mu}\mu}U^{-1}_{C_1}(x,{}^{C_1}b;{}^{C_2}a)_{\tilde{\nu}\nu}\times\frac{1}{\eta_b(C_1,C_1)}\frac{\eta_a(C_2,C_1)}{\eta_a(C_1,C_2)}.\end{aligned} \tag{D4}$$

(iv) $\mathrm{SO}(3)\times\mathbb{Z}_2^T$.—The anomalies for the group $\mathrm{SO}(3)\times\mathbb{Z}_2^T \equiv \mathrm{SO}(3)\times\mathbb{Z}_2^T$ in $(2+1)$D are classified by $(\mathbb{Z}_2)^4$. Suppose that the generator of $\mathbb{Z}_2^T$ is $\mathcal{T}$, and let U_π be a π rotation in SO(3). The four anomaly indicators can be given by $\mathcal{I}_0$, $\mathcal{I}(\mathcal{T})$, $\mathcal{I}(\mathcal{T}U_\pi)$, and

$$\mathcal{I}_4 = \frac{1}{D}\sum_a d_a^2\theta_a e^{i2\pi q_a}, \tag{D5}$$

where $q_a \in \{0, \frac{1}{2}\}$ denotes whether anyon a carries linear representation ($q_a = 0$) or spinor representation ($q_a = \frac{1}{2}$) under SO(3) symmetry.

From these known anomaly indicators, we can construct the anomaly indicators of the symmetry groups appearing in this paper by restricting to subgroups. We need to find a complete list of subgroups such that all nontrivial elements are nonzero after restricting to at least one such subgroup. If the condition is satisfied, we indeed find a complete set of anomaly indicators.

(v) $p6\times\mathrm{SO}(3)$.—The anomalies for the group $p6\times\mathrm{SO}(3)$ in $(2+1)$D are classified by $(\mathbb{Z}_2)^2$. The two anomaly indicators can be given by

$$\begin{aligned}\mathsf{I}_1 &= \mathcal{I}_3(C_2U_\pi, C_2U'_\pi), \\ \mathsf{I}_2 &= \mathcal{I}_3(T_1C_2U_\pi, T_1C_2U'_\pi).\end{aligned} \tag{D6}$$

The values of these anomaly indicators in each lattice homotopy class with $p6\times \mathrm{SO}(3)$ symmetry are given in Table I.

(vi) $p4\times \mathrm{SO}(3)$.—The anomalies for the group $p4\times \mathrm{SO}(3)$ in $(2+1)$D are classified by $(\mathbb{Z}_2)^3$. The three anomaly indicators can be given by

$$\mathsf{I}_1=\mathcal{I}_3(C_2U_\pi, C_2U'_\pi),\qquad \mathsf{I}_2=\mathcal{I}_3(T_1T_2C_2U_\pi, T_1T_2C_2U'_\pi),\qquad \mathsf{I}_3=\mathcal{I}_3(T_1C_2U_\pi, T_1C_2U'_\pi). \tag{D7}$$

The values of these anomaly indicators in each lattice homotopy class with $p6\times \mathrm{SO}(3)$ symmetry are given in Table II.

(vii) $p6m\times \mathrm{SO}(3)\times \mathbb{Z}_2^T$ and $p6m\times \mathbb{Z}_2^T$.—The anomalies for the group $p6m\times \mathrm{SO}(3)\times \mathbb{Z}_2^T$ in $(2+1)$D are classified by $(\mathbb{Z}_2)^{22}$. The complete list of anomaly indicators can be given by

$$\begin{aligned}
&\mathsf{I}_0=\mathcal{I}_0,\\
&\mathsf{I}_1=\mathcal{I}_1(\mathcal{T}),\qquad \mathsf{I}_2=\mathcal{I}_1(M),\qquad \mathsf{I}_3=\mathcal{I}_1(C_2\mathcal{T}),\qquad \mathsf{I}_4=\mathcal{I}_1(C_2M),\\
&\mathsf{I}_5=\mathcal{I}_2(\mathcal{T},C_2\mathcal{T}),\qquad \mathsf{I}_6=\mathcal{I}_2(\mathcal{T},M),\qquad \mathsf{I}_7=\mathcal{I}_2(C_2\mathcal{T},M),\qquad \mathsf{I}_8=\mathcal{I}_2(C_2\mathcal{T},C_2M),\qquad \mathsf{I}_9=\mathcal{I}_2(M,C_2M),\\
&\mathsf{I}_{10}=\mathcal{I}_1(T_1T_2C_2\mathcal{T}),\qquad \mathsf{I}_{11}=\mathcal{I}_2(M,T_1T_2C_2\mathcal{T}),\qquad \mathsf{I}_{12}=\mathcal{I}_1(\mathcal{T},T_1T_2C_2\mathcal{T}),\qquad \mathsf{I}_{13}=\mathcal{I}_1(M,T_1T_2C_2M),\\
&\mathsf{I}_{14}=\mathcal{I}_1(\mathcal{T}U_\pi),\qquad \mathsf{I}_{15}=\mathcal{I}_1(MU_\pi),\qquad \mathsf{I}_{16}=\mathcal{I}_1(C_2\mathcal{T}U_\pi),\qquad \mathsf{I}_{17}=\mathcal{I}_1(C_2MU_\pi),\\
&\mathsf{I}_{18}=\mathcal{I}_1(T_1T_2C_2\mathcal{T}U_\pi),\qquad \mathsf{I}_{19}=\mathcal{I}_4,\qquad \mathsf{I}_{20}=\mathcal{I}_3(C_2U_\pi,U'_\pi),\qquad \mathsf{I}_{21}=\mathcal{I}_3(M\mathcal{T}U_\pi,U'_\pi).
\end{aligned}\tag{D8}$$

The values of these anomaly indicators in each lattice homotopy class with $p6m\times \mathrm{SO}(3)\times \mathbb{Z}_2^T$ symmetry are given in Table XVIII.

The anomalies for the group $p6m\times \mathbb{Z}_2^T$ in $(2+1)$D are classified by $(\mathbb{Z}_2)^{14}$. The complete set of anomaly indicators for $p6m\times \mathbb{Z}_2^T$ can be obtained by simply ignoring all anomaly indicators involving U_π; i.e., this set consists of I_0–I_{13}. The values of these anomaly indicators in each lattice homotopy class with $p6m\times \mathbb{Z}_2^T$ symmetry are also given by Table XVIII (after removing I_{14}–I_{21}).

(viii) $p4m\times \mathrm{SO}(3)\times \mathbb{Z}_2^T$ and $p4m\times \mathbb{Z}_2^T$.—The anomalies for the group $p4m\times \mathrm{SO}(3)\times \mathbb{Z}_2^T$ in $(2+1)$D are classified by $(\mathbb{Z}_2)^{31}$. The complete list of anomaly indicators can be given by

$$\begin{aligned}
&\mathsf{I}_0=\mathcal{I}_0,\\
&\mathsf{I}_1=\mathcal{I}_1(\mathcal{T}),\qquad \mathsf{I}_2=\mathcal{I}_1(M),\qquad \mathsf{I}_3=\mathcal{I}_1(C_2\mathcal{T})\mathsf{I}_4=\mathcal{I}_1(C_4M),\\
&\mathsf{I}_5=\mathcal{I}_2(\mathcal{T},C_2\mathcal{T}),\qquad \mathsf{I}_6=\mathcal{I}_2(\mathcal{T},M),\qquad \mathsf{I}_7=\mathcal{I}_2(\mathcal{T},C_4M),\qquad \mathsf{I}_8=\mathcal{I}_2(C_2\mathcal{T},M),\qquad \mathsf{I}_9=\mathcal{I}_2(C_2\mathcal{T},C_4M),\\
&\mathsf{I}_{10}=\mathcal{I}_1(T_1M),\qquad \mathsf{I}_{11}=\mathcal{I}_1(T_1C_2\mathcal{T}),\qquad \mathsf{I}_{12}=\mathcal{I}_1(T_1T_2C_2\mathcal{T}),\\
&\mathsf{I}_{13}=\mathcal{I}_2(\mathcal{T},T_1M),\qquad \mathsf{I}_{14}=\mathcal{I}_2(\mathcal{T},T_1C_2\mathcal{T}),\qquad \mathsf{I}_{15}=\mathcal{I}_2(\mathcal{T},T_1T_2C_2\mathcal{T}),\\
&\mathsf{I}_{16}=\mathcal{I}_2(T_2C_2\mathcal{T},M),\qquad \mathsf{I}_{17}=\mathcal{I}_2(M,T_2C_2M),\qquad \mathsf{I}_{18}=\mathcal{I}_2(T_1T_2^{-1}C_2\mathcal{T},C_4M),\qquad \mathsf{I}_{19}=\mathcal{I}_2(T_1T_2C_2\mathcal{T},T_1M),\\
&\mathsf{I}_{20}=\mathcal{I}_4,\qquad \mathsf{I}_{21}=\mathcal{I}_1(\mathcal{T}U_\pi),\qquad \mathsf{I}_{22}=\mathcal{I}_1(MU_\pi),\qquad \mathsf{I}_{23}=\mathcal{I}_1(C_2\mathcal{T}U_\pi),\qquad \mathsf{I}_{24}=\mathcal{I}_1(C_4MU_\pi),\\
&\mathsf{I}_{25}=\mathcal{I}_1(T_1MU_\pi),\qquad \mathsf{I}_{26}=\mathcal{I}_1(T_1C_2\mathcal{T}U_\pi),\qquad \mathsf{I}_{27}=\mathcal{I}_1(T_1T_2C_2\mathcal{T}U_\pi),\\
&\mathsf{I}_{28}=\mathcal{I}_3(C_4M\mathcal{T}U_\pi,U'_\pi),\qquad \mathsf{I}_{29}=\mathcal{I}_3(M\mathcal{T}U_\pi,U'_\pi),\qquad \mathsf{I}_{30}=\mathcal{I}_3(T_1M\mathcal{T}U_\pi,U'_\pi).
\end{aligned}\tag{D9}$$

The values of these anomaly indicators in each lattice homotopy class with $p4m\times \mathrm{SO}(3)\times \mathbb{Z}_2^T$ symmetry are given in Table XIX.

The anomalies for the group $p4m\times \mathbb{Z}_2^T$ in $(2+1)$D are classified by $(\mathbb{Z}_2)^{20}$. The complete set of anomaly indicators for $p4m\times \mathbb{Z}_2^T$ can be obtained by simply ignoring all anomaly indicators involving U_π; i.e., this set consists of I_0–I_{19}. The values of these anomaly indicators in each lattice homotopy class with $p4m\times \mathbb{Z}_2^T$ symmetry can be obtained from Table XIX (after removing I_{20}–I_{30}).

TABLE XVIII. Values of the anomaly indicators for the four lattice homotopy classes with symmetry group $p6m \times \mathrm{SO}(3) \times \mathbb{Z}_2^T$. The anomaly indicators not listed in the table are all 1 for all lattice homotopy classes.

	0	a	c	$a+c$
I_3	1	−1	1	−1
I_5	1	−1	1	−1
I_{10}	1	1	−1	−1
I_{12}	1	1	−1	−1
I_{20}	1	−1	1	−1

TABLE XIX. Values of the anomaly indicators for the eight lattice homotopy classes with symmetry group $p4m \times \mathrm{SO}(3) \times \mathbb{Z}_2^T$. The anomaly indicators not listed in the table are all 1 for all lattice homotopy classes.

	0	a	b	c	$a+b$	$a+c$	$b+c$	$a+b+c$
I_3	1	−1	1	1	−1	−1	1	−1
I_5	1	−1	1	1	−1	−1	1	−1
I_{11}	1	1	1	−1	1	−1	−1	−1
I_{12}	1	1	−1	1	−1	1	−1	−1
I_{14}	1	1	1	−1	1	−1	−1	−1
I_{15}	1	1	−1	1	−1	1	−1	−1

APPENDIX E: SYMMETRY FRACTIONALIZATION CLASSES OF THE "BEYOND-PARTON" $\mathbb{Z}_2$ TOPOLOGICAL QUANTUM SPIN LIQUIDS

In Sec. VIII, we have found 117 different $p4m \times \mathbb{Z}_2^T$ symmetry-enriched $\mathbb{Z}_2$ topological quantum spin liquids in lattice homotopy class a, where 64 of them were identified using the parton-mean-field approach [82], while the other 53 are beyond the usual parton mean field. It turns out that there is no anyon permutation for any of these 117 states. In this appendix, we present the details of the symmetry fractionalization class of each of these 53 "beyond-parton" states, summarized in Table XX.

According to Table XIII, without anyon permutation the symmetry fractionalization classes are classified by $(\mathbb{Z}_2)^{20}$, which can be viewed as ten different quantum numbers for each of e and m. These quantum numbers are recorded in Table XX for each of the 53 states. Their physical meanings are clear. For example, the column for $(C_2)^2$ being 1 (−1) for an anyon means this anyon carries trivial (nontrivial) projective quantum number under C_2, which roughly speaking says that $C_2^2 = -1$ for this anyon. Similarly, $T_1 \mathcal{T} T_1^{-1} \mathcal{T}^{-1}$ being 1 (−1) for an anyon means that the translation T_1 and time reversal $\mathcal{T}$ commute (anticommute) for this anyon.

From Table XX we can see that, in all these 53 states, the anyon e is a Kramers doublet under the time-reversal symmetry, while the anyon m is a Kramers singlet.

TABLE XX. The symmetry fractionalization class of the 53 beyond-parton $p4m \times \mathbb{Z}_2^T$ symmetry-enriched $\mathbb{Z}_2$ topological quantum spin liquids in lattice homotopy class a.

e										m									
$(C_2)^2$	$(T_1T_2C_2)^2$	$(T_1C_2)^2$	M^2	$(T_1M)^2$	$(C_4M)^2$	$T_1\mathcal{T}T_1^{-1}\mathcal{T}^{-1}$	$C_4\mathcal{T}C_4^{-1}\mathcal{T}^{-1}$	$M\mathcal{T}M^{-1}\mathcal{T}^{-1}$	$\mathcal{T}^2$	$(C_2)^2$	$(T_1T_2C_2)^2$	$(T_1C_2)^2$	M^2	$(T_1M)^2$	$(C_4M)^2$	$T_1\mathcal{T}T_1^{-1}\mathcal{T}^{-1}$	$C_4\mathcal{T}C_4^{-1}\mathcal{T}^{-1}$	$M\mathcal{T}M^{-1}\mathcal{T}^{-1}$	$\mathcal{T}^2$
1	1	1	1	1	1	1	1	1	−1	−1	1	−1	1	1	1	−1	1	1	1
1	1	1	1	1	−1	1	−1	1	−1	−1	1	−1	1	1	1	−1	1	1	1
1	1	−1	1	−1	1	1	1	1	−1	−1	1	−1	1	1	1	−1	1	1	1
1	1	−1	1	−1	−1	1	−1	1	−1	−1	1	−1	1	1	1	−1	1	1	1
1	1	1	1	1	1	1	1	1	−1	−1	1	−1	1	1	1	−1	1	−1	1
1	1	−1	1	−1	1	1	1	1	−1	−1	1	−1	1	1	1	−1	1	−1	1
1	1	1	1	1	1	1	1	1	−1	−1	1	−1	1	1	1	−1	−1	1	1
1	1	−1	1	−1	1	1	1	1	−1	−1	1	−1	1	1	1	−1	−1	1	1
1	1	1	1	1	1	1	1	1	−1	−1	1	−1	1	1	1	−1	−1	−1	1
1	1	1	1	1	−1	1	−1	1	−1	−1	1	−1	1	1	1	−1	−1	−1	1
1	1	−1	1	−1	1	1	1	1	−1	−1	1	−1	1	1	1	−1	−1	−1	1
1	1	−1	1	−1	−1	1	−1	1	−1	−1	1	−1	1	1	1	−1	−1	−1	1
1	−1	1	1	1	1	1	1	1	−1	−1	1	−1	1	1	1	−1	−1	−1	1
1	−1	1	1	1	−1	1	−1	1	−1	−1	1	−1	1	1	1	−1	−1	−1	1
1	−1	−1	1	−1	1	1	1	1	−1	−1	1	−1	1	1	1	−1	−1	−1	1

(Table continued)

TABLE XX. *(Continued)*

e										m									
$(C_2)^2$	$(T_1T_2C_2)^2$	$(T_1C_2)^2$	M^2	$(T_1M)^2$	$(C_4M)^2$	$T_1\mathcal{T}T_1^{-1}\mathcal{T}^{-1}$	$C_4\mathcal{T}C_4^{-1}\mathcal{T}^{-1}$	$M\mathcal{T}M^{-1}\mathcal{T}^{-1}$	$\mathcal{T}^2$	$(C_2)^2$	$(T_1T_2C_2)^2$	$(T_1C_2)^2$	M^2	$(T_1M)^2$	$(C_4M)^2$	$T_1\mathcal{T}T_1^{-1}\mathcal{T}^{-1}$	$C_4\mathcal{T}C_4^{-1}\mathcal{T}^{-1}$	$M\mathcal{T}M^{-1}\mathcal{T}^{-1}$	$\mathcal{T}^2$
1	−1	−1	1	−1	−1	1	−1	1	−1	−1	1	−1	1	1	1	−1	−1	−1	1
1	1	1	1	1	1	−1	1	1	−1	−1	1	1	1	−1	1	1	1	−1	1
1	1	1	1	1	1	−1	1	1	−1	−1	1	1	1	−1	1	1	−1	−1	1
1	1	1	1	1	−1	−1	−1	1	−1	−1	1	1	1	−1	1	1	−1	−1	1
1	−1	1	1	1	1	−1	1	1	−1	−1	1	1	1	−1	1	1	−1	−1	1
1	−1	1	1	1	−1	−1	−1	1	−1	−1	1	1	1	−1	1	1	−1	−1	1
1	1	1	1	1	1	−1	1	1	−1	−1	1	1	1	−1	1	−1	1	1	1
1	1	1	1	1	−1	−1	−1	1	−1	−1	1	1	1	−1	1	−1	1	1	1
1	−1	1	1	1	1	−1	1	1	−1	−1	1	1	1	−1	1	−1	1	1	1
1	−1	1	1	1	−1	−1	−1	1	−1	−1	1	1	1	−1	1	−1	1	1	1
1	1	1	1	1	1	−1	1	1	−1	−1	1	1	1	−1	1	−1	−1	1	1
1	1	1	1	1	1	1	1	1	−1	−1	1	1	1	1	1	1	1	−1	1
1	1	−1	1	−1	1	1	1	1	−1	−1	1	1	1	1	1	1	1	−1	1
1	1	1	1	1	1	1	1	1	−1	−1	1	1	1	1	1	1	−1	1	1
1	1	1	1	1	1	−1	1	1	−1	−1	1	1	1	1	1	1	−1	1	1
1	1	1	−1	−1	1	1	−1	−1	−1	−1	1	1	1	1	1	1	−1	1	1
1	1	1	−1	−1	1	−1	−1	−1	−1	−1	1	1	1	1	1	1	−1	1	1
1	1	1	1	−1	1	1	1	1	−1	−1	1	1	1	1	1	1	−1	1	1
1	1	1	1	−1	1	−1	1	1	−1	−1	1	1	1	1	1	1	−1	1	1
1	1	1	−1	1	1	1	−1	−1	−1	−1	1	1	1	1	1	1	−1	1	1
1	1	1	−1	1	1	−1	−1	−1	−1	−1	1	1	1	1	1	1	−1	1	1
1	1	−1	1	−1	1	1	1	1	−1	−1	1	1	1	1	1	1	−1	1	1
1	1	−1	1	−1	1	−1	1	1	−1	−1	1	1	1	1	1	1	−1	1	1
1	1	−1	−1	1	1	1	−1	−1	−1	−1	1	1	1	1	1	1	−1	1	1
1	1	−1	−1	1	1	−1	−1	−1	−1	−1	1	1	1	1	1	1	−1	1	1
1	1	−1	1	1	1	1	1	1	−1	−1	1	1	1	1	1	1	−1	1	1
1	1	−1	1	1	1	−1	1	1	−1	−1	1	1	1	1	1	1	−1	1	1
1	1	−1	−1	−1	1	1	−1	−1	−1	−1	1	1	1	1	1	1	−1	1	1
1	1	−1	−1	−1	1	−1	−1	−1	−1	−1	1	1	1	1	1	1	−1	1	1
1	1	1	1	1	1	1	1	1	−1	−1	1	1	1	1	1	1	−1	−1	1
1	1	1	1	1	−1	1	−1	1	−1	−1	1	1	1	1	1	1	−1	−1	1
1	1	−1	1	−1	1	1	1	1	−1	−1	1	1	1	1	1	1	−1	−1	1
1	1	−1	1	−1	−1	1	−1	1	−1	−1	1	1	1	1	1	1	−1	−1	1
1	1	1	1	1	1	−1	1	1	−1	−1	1	1	1	1	1	−1	1	−1	1
1	1	1	1	1	1	−1	1	1	−1	−1	1	1	1	1	1	−1	−1	−1	1
1	1	1	1	1	−1	−1	−1	1	−1	−1	1	1	1	1	1	−1	−1	−1	1
1	−1	1	1	1	1	−1	1	1	−1	−1	1	1	1	1	1	−1	−1	−1	1
1	−1	1	1	1	−1	−1	−1	1	−1	−1	1	1	1	1	1	−1	−1	−1	1

Furthermore, for the anyon m there is always some nontrivial symmetry fractionalization simultaneously involving the lattice and time-reversal symmetries. For example, translation and time reversal may not commute on m. In contrast, in all 64 states identified using the parton-mean-field approach in Ref. [82], m experiences no symmetry fractionalization that involves both lattice and time-reversal symmetries. Moreover, we remark that, for all 117 states, the $C_2 \equiv C_4^2$ symmetry fractionalizes on the m anyon, i.e., effectively $C_2^2 = -1$ for m. Usually, the interpretation of this phenomenon is that there is a background e anyon at each square lattice site (the C_4 center), and the mutual braiding statistics between e and m yields $C_2^2 = -1$. However, for 16 of the 53 beyond-parton states, $(T_1C_2)^2 = (T_2C_2)^2 = -1$ for m, which seems to suggest that there are also background e anyons at the twofold rotation centers of T_1C_2 and T_2C_2, although microscopically there is no spin at those positions. So the analysis based on anomaly matching suggests that the simple picture where the fractionalization of rotational symmetries purely comes from background anyons is actually incomplete.

[1] All symmetries discussed in this paper are 0-form invertible symmetries, unless otherwise stated.

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[36] Namely, for any unitary element $\mathbf{k}_0$ in the topological symmetry group (which does not have to be a microscopic symmetry), we can relabel each anyon a by ${}^{\mathbf{k}_0}a$ (under this relabeling, the data $\{F^{abc}_d, R^{ab}_c\}$ are the same as $\{F^{{}^{\mathbf{k}_0}a\,{}^{\mathbf{k}_0}b\,{}^{\mathbf{k}_0}c}_{{}^{\mathbf{k}_0}d}$, $R^{{}^{\mathbf{k}_0}a\,{}^{\mathbf{k}_0}b}_{{}^{\mathbf{k}_0}c}\}$ up to gauge transformations). Then, the symmetry-enriched topological order with data $\{\rho_{\mathbf{g}}\colon a \to {}^{\mathbf{g}}a; U_{\mathbf{g}}(a, b; c), \eta_a(\mathbf{g}, \mathbf{h})\}$ is the same as the one with data $\{\rho'_{\mathbf{g}}\colon a \to \mathbf{k}_0[{}^{\mathbf{g}}({}^{\overline{\mathbf{k}_0}}a)]; U'_{\mathbf{g}}(a, b; c) = U_{\mathbf{g}}({}^{\overline{\mathbf{k}_0}}a, {}^{\overline{\mathbf{k}_0}}b; {}^{\overline{\mathbf{k}_0}}c)$, $\eta'_a(\mathbf{g}, \mathbf{h}) = \eta_{\overline{\mathbf{k}_0}a}(\mathbf{g}, \mathbf{h})\}$.

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[61] The generator $T_{1,2}$ cannot permute anyons because $C_6 T_2 C_6^{-1} = T_1^{-1}$, implying that T_1 and T_2 should permute anyons in opposite ways, which, combined with $C_6 T_1 C_6^{-1} = T_1 T_2$, implies that neither T_1 nor T_2 can permute anyons.

[62] In fact, the generator A_c^2 actually detects whether the anyon (1) simultaneously carries projective representation under four different symmetries, generated by C_6, $T_1 C_6^3$, $T_2 C_6^3$, and $T_1 T_2 C_6^3$, respectively.

[63] Another way to calculate these anomaly indicators is as follows. We can restrict the $p6 \times SO(3)$ symmetry into two of its $\mathbb{Z}_2 \times SO(3)$ subgroups, where the $\mathbb{Z}_2$ in the first subgroup is generated by $C_2 \equiv C_6^3$ and in the second it is generated by $T_1 C_2$. By comparing the representative cochains of B_{xy}, A_c^2, and Nw_2 in Appendix C with Eq. (19), we see that, after the restriction to the first $\mathbb{Z}_2 \times SO(3)$ subgroup, the $p6 \times SO(3)$ symmetry fractionalization class labeled by $\{n_1, n_2, n_3\}$ becomes the class labeled by $\{n_2, n_3\}$ in Sec. VI A 1, while after the restriction to the second $\mathbb{Z}_2 \times SO(3)$ subgroup, the class labeled by $\{n_1, n_2, n_3\}$ becomes the class labeled by $\{n_1 + n_2, n_3\}$ in Sec. VI A 1. According to Eq. (21), the anomaly indicators in Eq. (7) for $U(1)_{2N}$ with the symmetry fractionalization class labeled by $\{n_1, n_2, n_3\}$ is Eq. (30).

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