

Microscopic Origin of the Entropy of Black Holes in General Relativity

Vijay Balasubramanian^{1,2,3,*} Albion Lawrence^{4,†} Javier M. Magán^{1,5,‡} and Martin Sasieta^{4,§}

¹David Rittenhouse Laboratory, University of Pennsylvania,
209 S. 33rd Street, Philadelphia, Pennsylvania 19104, USA

²Santa Fe Institute, 1399 Hyde Park Road, Santa Fe, New Mexico 87501, USA

³Theoretische Natuurkunde, Vrije Universiteit Brussel, Pleinlaan 2, B-1050 Brussels, Belgium

⁴Martin Fisher School of Physics, Brandeis University, Waltham, Massachusetts 02453, USA

⁵Instituto Balseiro, Centro Atómico Bariloche, 8400-S.C. de Bariloche, Río Negro, Argentina

 (Received 5 April 2023; revised 19 October 2023; accepted 18 December 2023; published 21 February 2024)

We construct an infinite family of microstates with geometric interiors for eternal black holes in general relativity with a negative cosmological constant in any dimension. Wormholes in the Euclidean path integral for gravity cause these states to have small, but nonzero, quantum mechanical overlaps that have a universal form. The overlaps have a dramatic consequence: The microstates span a Hilbert space of log dimension equal to the Bekenstein-Hawking entropy. The semiclassical microstates we construct contain Einstein-Rosen bridges of arbitrary size behind their horizons. Our results imply that all these bridges can be interpreted as quantum superpositions of wormholes of size at most exponential in the entropy.

DOI: [10.1103/PhysRevX.14.011024](https://doi.org/10.1103/PhysRevX.14.011024)

Subject Areas: Gravitation, String Theory

I. INTRODUCTION

In this article, we consider one old problem and one new problem concerning black holes. As we will show, these two problems turn out to be precisely related.

The old problem concerns the statistical origin and finiteness of the Bekenstein-Hawking black hole entropy [1,2]

$$S_{\text{BH}} = \frac{A}{4G}, \quad (1.1)$$

where A is the area of the black hole horizon, and we have chosen units to set $\hbar = 1$. One way to derive this equation is from an analysis of the partition function in Euclidean quantum gravity [3]. With different identifications of entropy S and area A , it also appears more generally in the context of quantum gravity with a negative cosmological constant [4–6] through the Ryu-Takayanagi conjecture [7,8]. Equation (1.1) suggests that the Hilbert space describing the quantum dynamics of black holes is finite dimensional, spanned by $e^{S_{\text{BH}}}$ orthogonal states, usually

dubbed the “black hole microstates.” The problem arises when one tries to find these microstates, thus providing a statistical account of black hole entropy.

Many routes towards solving this problem have been explored. On the one hand, we have approaches in string theory such as precise microstate counting [9] and the fuzzball approach [10]. The problem with these celebrated methods is that they apply to restricted scenarios and do not provide a satisfactory understanding of the universality of Eq. (1.1). On the other hand, we have approximate approaches, such as the interpretation of black hole entropy as entanglement entropy [11,12] or, relatedly, the association of black hole entropy with the entropy of thermally excited quantum fields in the vicinity of the horizon [13]. These approaches naively give an infinite answer, due to the infinity of quantum field theory modes in the ultraviolet. For sufficiently low-spin fields, at least, such divergences can be absorbed into a renormalization of Newton’s constant, so the denominator in Eq. (1.1) is understood as the observed low-energy Newton’s constant (see Refs. [14,15] for discussion). However, puzzles remain, such as the interpretation of the contribution of the bare Newton’s constant to Eq. (1.1), the validity of the approach in a full quantum theory of gravity, and the relationship of this interpretation to the state-counting interpretation when the latter is available [16].

The new problem concerns a conjecture about the behavior of the volume of the Einstein-Rosen bridge behind the horizon of an eternal black hole [18,19]. The conjecture suggests that the volume of such Einstein-Rosen bridges is related to the “quantum complexity” [20] of the underlying

*vijay@physics.upenn.edu

†albion@brandeis.edu

‡javier.magan@cab.cnea.gov.ar

§martinsasieta@brandeis.edu

Published by the American Physical Society under the terms of the [Creative Commons Attribution 4.0 International license](https://creativecommons.org/licenses/by/4.0/). Further distribution of this work must maintain attribution to the author(s) and the published article’s title, journal citation, and DOI.

state. In classical gravity, the volume of the Einstein-Rosen bridge grows linearly with respect to the asymptotic time. Thus, the conjecture predicts that the complexity, i.e., the number of gates of the minimal circuit simulating the time evolution of the quantum state, will increase linearly with time. On the other hand, it is known that the complexity of any circuit is bounded by the dimension of the Hilbert space on which the circuit acts (see Chapter 7 of Ref. [20] and the discussion in Ref. [21]). Thus, the conjecture also suggests that the volume of the Einstein-Rosen bridge must saturate at exponential time, and hence at a value exponentially large in the black hole entropy, namely, at $\mathcal{O}(e^{S_{\text{BH}}})$. However, in semiclassical gravity, the Einstein-Rosen bridge can grow indefinitely, and no saturation is observed [22].

We will show that there is a relation between these two problems: the expected finiteness of black hole entropy and the conjectured finiteness of Einstein-Rosen bridges. Indeed, we will provide a simple and universal solution to both problems in gravitational models in any dimension. To do so, we will only assume that general relativity, the well-established low-energy theory of gravity, has some ultraviolet completion with a well-defined semiclassical path integral approximation.

The first step in our analysis is to recognize that the problem at hand is not to find a *specific* family of $e^{S_{\text{BH}}}$ microstates. Indeed, below we will explicitly construct many different families of black hole microstates, each with infinitely many members [24]. These families will already, at the time-symmetric point, include geometries with Einstein-Rosen bridges of arbitrarily large volume, which connects our puzzle about time evolution of the interior Einstein-Rosen bridge to the puzzle regarding the apparently excessive number of microstates. However, we will show that the dimension of the Hilbert space spanned by any of these infinite families is actually finite and given by the exponential of the Bekenstein-Hawking entropy (1.1). This enormous reduction in the Hilbert space dimension as compared to the naive expectation originates from the existence of wormhole saddle points of the gravitational path integral, discussed recently in the context of heavy operator statistics in the AdS/CFT correspondence [25]. We demonstrate that these wormholes give rise to tiny, but universal, contributions to the quantum overlap of the candidate black hole microstates. Thus, states in the infinitely many families that we construct are not orthogonal, and a computation establishes that they span a Hilbert space of dimension that precisely equals the exponential of the Bekenstein-Hawking entropy [26]. The same overlaps will allow us to interpret Einstein-Rosen bridges of any volume as superpositions of wormholes of volume bounded by an exponential in the entropy.

Our results will generalize and clarify recent insights from low-dimensional gravity, in particular, concerning the spectral form factor in black hole physics [34–36], and the derivation of the Page curve via replica wormholes (see

Ref. [37] for a review). In particular, our construction will not use artifices like “end-of-the-world (EOW) branes” with unknown degrees of freedom, and it will be applicable to general relativity in general dimensions.

The rest of the paper is organized as follows. In Sec. II, we construct infinite families of geometric solutions to general relativity with a negative cosmological constant that have different interior geometries behind the horizons of a fixed-exterior eternal black hole. The two asymptotic regions of this black hole can have different masses. The interior regions include shells of dust moving within a long Einstein-Rosen bridge connecting the two asymptotic regions. Our states are semiclassically well defined and can be constructed via Lorentzian continuation of a Euclidean geometry. In the boundary dual conformal field theory, this construction has a well-defined interpretation in terms of the state-operator correspondence. Thus, these distinct interior geometries necessarily contribute to the dimension of the black hole Hilbert space. However, they naively lead to a black hole entropy far exceeding the Bekenstein-Hawking formula. The apparent overcounting resembles an old puzzle of Wheeler, concerning “bag of gold” geometries [38,39]. Thus, our construction sharpens a basic question: How do we count the independent degrees of freedom contributing to the entropy of a black hole.

In Sec. III, we show why the states we constructed in Sec. II are not orthogonal: They overlap quantum mechanically because of the effects of the wormholes found in Ref. [25]. These wormholes contribute to the Euclidean path integral, they exist for all dimensions and for all our states, and their action is straightforward to compute using general relativity. For most states in every family, the contribution from the wormhole is a universal functional of the partition function of the theory. We also compute higher moments of the overlaps, finding universal answers as well. Schematically, in the limit of heavy shells of dust with different masses, these wormholes contribute to the statistics of the overlaps so that, universally, the n th moment of the overlap is

$$\overline{\langle \Psi_{\mathbf{m}} | \Psi_{\mathbf{m}'} \rangle \dots \langle \Psi_{\mathbf{m}' \dots'} | \Psi_{\mathbf{m}} \rangle} \simeq \frac{Z(n\beta)^2}{Z(\beta)^{2n}}, \quad (1.2)$$

where the overline means we are performing the computation using the gravitational path integral, $Z(\beta)$ is the Euclidean partition function of the gravitational theory, and the square comes from the consideration of eternal black holes, which are naturally associated to twice the entropy.

In Sec. IV, we describe how such exponentially small, but nonvanishing overlaps determine a Hilbert space of finite dimension given by the exponential of black hole entropy. Physically, the excess microstates do not add to the entropy since they do not generate new orthogonal Hilbert space directions. We show this by demonstrating an explicit transition in the overlap matrix, the so-called Gram matrix

of the black hole microstates, as we increase the number of microstates under consideration. The transition happens when the Gram matrix goes from being positive definite to positive semidefinite, implying that the added vectors are not linearly independent. This transition does not depend on the specific set of microstates that we start with. Once the transition occurs, we do not generate new orthogonal Hilbert space directions. Our arguments provide a way of computing the entropy for general black holes, using the large, overcomplete basis of microstates that we construct.

In Sec. V, we describe the implications of our construction for the interior volumes of black holes. First, we compute the volumes of the Einstein-Rosen bridges associated with the families of black hole microstates constructed in Sec. II. We see that these volumes are simple functions of the proper masses (scaling dimensions in the AdS/CFT context) of the operators used to construct the states in the gravitational description. Since there is no limit on the energy or scaling dimension of such operators, there is no limit on the interior volumes even at a fixed time. This sharpens the problem regarding the conjecture that relates wormhole volume and quantum complexity since all these states are geometrical and semiclassical. We also compute the maximal transversal size of our Einstein-Rosen bridges, namely, the sizes of their “pythons” [40]. We then show that any state that has an Einstein-Rosen bridge whose size is superexponential in the entropy can be rewritten as a superposition of states with at most exponentially large wormholes. Thus, there is a transition beyond which the notion of “volume” requires refinement. If we define volume by a minimization over the possible linear definitions, this notion saturates at an exponentially large value. We use recent advances on spread complexity, the tridiagonalization of random matrices, and the associated Krylov basis [41,42] to construct a unitary toy model of these effects.

In Sec. VI, we end with conclusions and open problems. To understand the basic points of the paper, the reader can skip Secs. II C, III D, IV C, and V on a first read.

II. INFINITE GEOMETRIC FAMILIES OF BLACK HOLE MICROSTATES

The no-hair theorem states that all stationary black hole solutions of the Einstein equations are completely characterized by a few independent, externally observable, classical parameters, such as mass, electric charge, and angular momentum. This fact is sometimes incorrectly understood as an obstruction to finding black hole microstates. In fact, the same black hole can develop from many initial states that differ in their late-time interiors—we can regard these different interiors as examples of “microstates.” Likewise, an operation creating an excitation of a quantum field behind the horizon that does not modify the mass can also be regarded as making a microstate. In the holographic AdS/CFT context, such operations in the

interior of the black hole should be encoded in the exact dual CFT description [43]. This idea also appears in more general terms in the old notion of “black hole complementarity” [45].

Here, we will build infinite families of geometric microstates of a large eternal AdS black hole by exploring families of states with exteriors given by the black hole geometry but with distinct interiors arising from backreaction caused by shells of dust [46]. We will prepare these states via the Euclidean path integral, generalizing the construction in Refs. [50,51] for Sachdev-Ye-Kitaev model/Jackiw-Tietelboim (JT) gravity to a generic higher-dimensional AdS/CFT setup. Our states also generalize the infalling shell states considered in Ref. [52] to various numbers of shells behind the horizon of a spacetime with two asymptotic boundaries and interior backreaction. The dual CFT interpretation of our construction shows that these are pure states of a quantum theory of gravity with two asymptotically AdS boundaries.

We examine a small subset of the possible configurations that can be built using our construction. However, as we will argue later, the additional states would not increase the dimension of the spanned Hilbert space. In fact, our construction already provides an infinite number of candidate microstates, which naively overcount the Bekenstein-Hawking entropy. One of our main messages is that it does not really matter how we count states or which states we count. The only important thing is to count them carefully. Namely, we have to determine the actual dimension of the spanned Hilbert space. The main advantage of the families of states that we present below is that they can be described geometrically and permit explicit calculations.

A. State preparation and thin-shell operators

In detail, we consider states in a theory of gravity with two asymptotic AdS boundaries with topology $\mathbf{S}^{d-1} \times \mathbf{R}$, where \mathbf{R} represents time, and asymptotic AdS curvature radius ℓ . Quantum gravity with such boundary conditions is equivalent to the physics of two copies of a conformal field theory (CFT), dubbed CFT_L and CFT_R , living at the left and right boundaries and with Hamiltonians $H_L = H_R = H$. The spatial component of each of these boundaries is \mathbf{S}^{d-1} . The total Hilbert space is just the tensor product $\mathcal{H}_L^{\text{CFT}} \otimes \mathcal{H}_R^{\text{CFT}}$. The microcanonical Hilbert space dimension at a given energy in this theory is thus twice the microcanonical dimension of a single-sided black hole. We define the energy basis as

$$H_L|n, m\rangle = E_n|n, m\rangle, \quad H_R|n, m\rangle = E_m|n, m\rangle. \quad (2.1)$$

Quantum states in the doubled Hilbert space can be prepared by a Euclidean path integral for a single CFT on a finite cylinder, with the argument of the wave functional for each factor given by the value of the fields at the ends of the cylinder (Fig. 1). We insert an operator \mathcal{O}

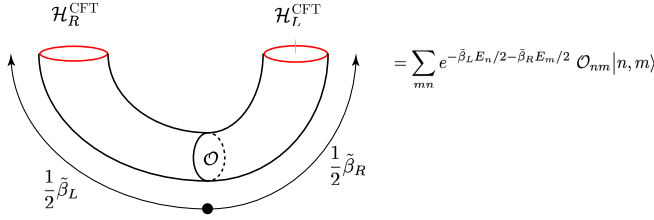


FIG. 1. Schematic form of the Euclidean path integral in the boundary field theory, which is two copies of a holographic CFT with Hilbert space \mathcal{H}^{CFT} . The cross section of the tube represents the spatial direction, and the length is Euclidean time. The operator \mathcal{O} is inserted in the circle at the bottom, and the tube corresponds to propagation in Euclidean time via the Hamiltonian of the underlying CFT, by $\tilde{\beta}_L$ ($\tilde{\beta}_R$) along the left (right) of the operator insertion.

in the single CFT at the spatial slice denoted by the bottom circle in Fig. 1. The cylinder extends by Euclidean time $\tilde{\beta}_L/2$ to the left and $\tilde{\beta}_R/2$ to the right. The resulting state can be described by starting with the operator

$$\rho_{\tilde{\beta}_L/2} \mathcal{O} \rho_{\tilde{\beta}_R/2}, \quad (2.2)$$

where $\rho_\beta = e^{-\beta H}$ is the unnormalized thermal density matrix, defined in a single copy of the CFT on $S^d \times \mathbf{R}$. It is identified with a state in a doubled Hilbert space via the usual isomorphism. We can write this schematically as

$$\begin{aligned} |\Psi\rangle &= |\rho_{\tilde{\beta}_L/2} \mathcal{O} \rho_{\tilde{\beta}_R/2}\rangle \\ &= \frac{1}{\sqrt{Z_1}} \sum_{n,m} e^{-\frac{1}{2}(\tilde{\beta}_L E_n + \tilde{\beta}_R E_m)} \mathcal{O}_{nm} |n, m\rangle, \end{aligned} \quad (2.3)$$

where $Z_1 = \text{Tr}(\mathcal{O}^\dagger e^{-\tilde{\beta}_L H} \mathcal{O} e^{-\tilde{\beta}_R H})$ normalizes these pure states. Note that the trace defining Z_1 is taken in a *single* copy of the CFT [53].

We show that these states correspond to two-sided black holes with independent masses $M_{+,-}$ and inverse temperatures $\beta_{R,L}$, respectively [54]. Note that the Euclidean times $\tilde{\beta}_{R,L}$ used to prepare the states through Euclidean evolution in Eq. (2.2) are not necessarily equal to the physical inverse temperatures of the black holes. We will derive the precise relationship for a certain class of operators \mathcal{O} below.

We can construct another family of states via the Euclidean path integral, with Euclidean time evolution by $\tilde{\beta}_R/2$ at the right boundary, followed by the insertion of an operator \mathcal{O}_1 , followed by evolution with $\tilde{\beta}_1$, then by another operator \mathcal{O}_2 , and finally by a further time evolution of $\tilde{\beta}_L/2$ up to the left boundary. This process prepares a pure state, which can be schematically written as

$$|\Psi\rangle = |\rho_{\tilde{\beta}_L/2} \mathcal{O}_2 \rho_{\tilde{\beta}_1} \mathcal{O}_1 \rho_{\tilde{\beta}_R/2}\rangle. \quad (2.4)$$

More generally, given operators $\mathcal{O}_1, \dots, \mathcal{O}_n$ and Euclidean lengths $\tilde{\beta}_1, \dots, \tilde{\beta}_{n-1}$, we can build

$$|\Psi\rangle = |\rho_{\tilde{\beta}_L/2} \mathcal{O}_n \rho_{\tilde{\beta}_{n-1}} \cdots \rho_{\tilde{\beta}_1} \mathcal{O}_1 \rho_{\tilde{\beta}_R/2}\rangle. \quad (2.5)$$

By construction, all these states belong to the Hilbert space $\mathcal{H}_L^{\text{CFT}} \otimes \mathcal{H}_R^{\text{CFT}}$ associated with the two asymptotic CFTs. We construct bulk duals for states of this form that contribute to the semiclassical Hilbert space of a black hole with fixed-exterior geometry, that is, with fixed ADM masses and exterior temperatures in each asymptotic exterior region. Thus, they are candidates for microstates of these black holes.

We are interested in states that have a simple semiclassical and geometrical description in the gravitational theory. To this end, we consider operators that create spherically symmetric “dust shells.” To do this, we start with any scalar operator \mathcal{O}_Δ with a scaling dimension Δ of $\mathcal{O}(1)$. Such operators are dual to scalar fields in the gravity theory with mass $m_\Delta \ell = \sqrt{\Delta(\Delta - d)}$. Standard constructions [55–58] show that the application of \mathcal{O}_Δ at a spherically symmetric distribution of points in the CFT is dual, after appropriate holographic renormalization [59–61], to applying the bulk scalar field to the same distribution of points at the AdS boundary. This process creates a shell of matter \mathcal{O} at the boundary that will propagate into the bulk. In our construction, any operator \mathcal{O}_Δ will suffice because, in an interacting theory, we expect that the algebra generated by \mathcal{O}_Δ will contain all operators of the theory, allowing us access to any state in the theory, although in a possibly complicated way.

We also require that the operator we are constructing is sufficiently heavy, i.e., that it creates states with masses of $\mathcal{O}(1/G)$, in units of the AdS radius ℓ , where G is Newton’s constant. With this choice, \mathcal{O} creates a spherically symmetric heavy shell of dust particles, which classically backreacts on the geometry at leading order in the $G \rightarrow 0$ expansion. To this end, we use a number of dust particles that scale parametrically as $n \sim \ell^{d-1}/G$, homogeneously distributed along the sphere [62]. The final dust-shell operator can be effectively described as a pressureless perfect fluid localized at the worldvolume \mathcal{W} of the shell. This fluid has the usual energy-momentum tensor

$$T_{\mu\nu}|_{\mathcal{W}} = \sigma u_\mu u_\nu, \quad (2.6)$$

where u^μ is the proper fluid velocity, tangent to \mathcal{W} , and σ is the surface density. The total rest mass of the dust shell is

$$m = \sigma V_\Omega r_\infty^{d-1} = n m_\Delta, \quad (2.7)$$

where $V_\Omega = \text{Vol}(\mathbf{S}^{d-1})$ is the volume of the sphere, and m_Δ is the mass of the individual operator insertions. The effective perfect fluid description of the operator works

for fluid densities σ that are sufficiently large, compared to one dust particle per unit volume, but also sufficiently small, when measured in Planck units; thus, we can trust the classical description.

B. Geometry of single-shell states

Our strategy for preparing bulk microstates is to find the bulk Euclidean geometry dual to the Euclidean calculation shown in Fig. 1. The full geometry is symmetric under Euclidean time reversal; the fields at the $\tau = 0$ spatial slice are fixed under this symmetry. Thus, we can continue to the Lorentzian signature at this slice.

1. Euclidean geometry

We start with one-shell microstates. The operator \mathcal{O} inserts a thin spherical domain wall of dust particles. This shell then propagates in Euclidean time. Since the mass is large in Planck units, we have to account for its back-reaction on the geometry.

More precisely, the worldvolume \mathcal{W} of the thin shell bisects the Euclidean manifold X and generates two connected components $X^\pm \subset X$, one on each side of \mathcal{W} . This process is depicted in Fig. 2. Given the spherical symmetry, the geometry of each component X^\pm is that of a Euclidean black hole

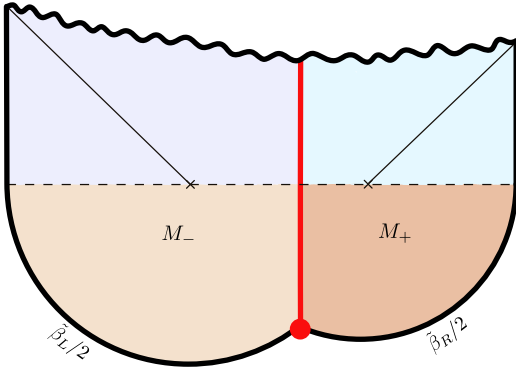


FIG. 2. Preparation of the state from the Euclidean path integral. The lower part of the figure (tan colors) shows time evolutions from a past Euclidean boundary with a shell operator (red dot) inserted. In the Euclidean section, the horizon is a point, here represented by crosses for the left and right black holes on either side of the shell. At $t = 0$, the Euclidean preparation geometry is matched onto a Lorentzian time evolution (blue and purple). The black hole horizons (thin black lines) become null surfaces, while the shell (red line) continues to move into the future behind both horizons. Spacetime is glued across the trajectory of the thin shell by Israel's junction conditions. Here, we have allowed for the possibility that the black holes on either side of the shell have different temperatures or masses. The solid black line is the spacetime boundary. For d -dimensional AdS geometries, this will be a cylinder $\mathbf{S}^{d-1} \times R$ on which the dual CFT lives.

$$ds_\pm^2 = f_\pm(r)d\tau_\pm + \frac{dr^2}{f_\pm(r)} + r^2 d\Omega_{d-1}^2, \quad (2.8)$$

where

$$f_\pm(r) = \frac{r^2}{\ell^2} + 1 - \frac{16\pi GM_\pm}{(d-1)V_\Omega r^{d-2}} \quad \text{for } d > 2, \quad (2.9)$$

$$f_\pm(r) = \frac{r^2}{\ell^2} - 8GM_\pm \quad \text{for } d = 2. \quad (2.10)$$

Here, r is a radial coordinate on the Euclidean disc, and $\tau_\pm \sim \tau_\pm + \beta_\pm$ are angular coordinates around periodic Euclidean time. In what follows, we will write all dimensional parameters in units of the AdS radius ℓ , effectively $\ell = 1$. Here, M_\pm is the ADM mass of the black hole in component X_\pm , with inverse temperature $\beta_\pm = 4\pi/f'(r_\pm)$, where f'_\pm is the derivative of f_\pm evaluated at the horizon radii r_\pm where $f_\pm(r) = 0$.

The shell follows a trajectory parametrized in terms of T , the synchronous proper time of the dust particles, via functions $r = R(T)$ and $\tau_\pm = \tau_\pm(T)$. The rest mass of the shell m is conserved along its trajectory. Its surface density at radius R is therefore defined by

$$\sigma = \frac{m}{V_\Omega R^{d-1}}, \quad (2.11)$$

where V_Ω is the volume of the unit transverse sphere. In the thin-shell formalism, the shell's dynamics, described just by its trajectory, is reduced to the motion of a nonrelativistic effective particle with zero total energy

$$\left(\frac{dR}{dT}\right)^2 + V_{\text{eff}}(R) = 0, \quad (2.12)$$

subject to the effective potential

$$V_{\text{eff}}(R) = -f_+(R) + \left(\frac{M_+ - M_-}{m} - \frac{4\pi Gm}{(d-1)V_\Omega R^{d-2}}\right)^2. \quad (2.13)$$

This potential arises from Israel's junction conditions [63] along the shell. Appendix A gives details about this gluing process and the derivation of the potential. Thus, each single-shell configuration for a black hole of fixed mass is characterized by the shell mass m .

Qualitatively, the thin shell starts at the boundary $R = r_\infty$ and falls inwards towards the tip of the geometry (the Euclidean horizon) at $R = r_\pm$. The trajectory is subject to a repulsive force in the Euclidean section. Thus, it bounces back at a minimum radius $R = R_* \geq r_\pm$ at the axis of time-reflection symmetry of the solution and then returns to $R = r_\infty$. In our calculations, we will send $r_\infty \rightarrow \infty$.

The Euclidean time elapsed by the shell $\Delta\tau_{\pm}$ as described in patches on either side of it, X^{\pm} , is

$$\Delta\tau_{\pm} = 2 \int_{R_*}^{r_{\infty}} \frac{dR}{f_{\pm}(R)} \sqrt{\frac{f_{\pm}(R) + V_{\text{eff}}(R)}{-V_{\text{eff}}(R)}}. \quad (2.14)$$

Therefore, the evolution of the one-shell states (2.3) is performed for the imaginary ‘‘preparation times,’’

$$\tilde{\beta}_L = \beta_L - \Delta\tau_-, \quad (2.15)$$

$$\tilde{\beta}_R = \beta_R - \Delta\tau_+, \quad (2.16)$$

which differ from the fixed inverse temperatures $\beta_{L,R}$ of the actual black holes (see Fig. 3), thus fixing their masses as well [64]. This condition ensures that the backreaction of the shell does not change the asymptotic mass of the geometry on either side. Notice that left and right asymptotic black hole regions can have different temperatures β_L and β_R because of the presence of the shell, which serves as a domain wall supporting this temperature difference. Here, the trajectory of the shell in Eq. (2.14) is fixed by the mass (temperature) of the background black hole and the mass of the shell, and the resulting elapsed Euclidean time in the trajectory fixes the required preparation time. If the mass of the shell is taken to zero, such solutions do not exist for different left or right black hole masses. Equation (2.14), together with Eqs. (2.15) and (2.16), ensures that the trajectory passes in between both horizons in the Euclidean geometry (Fig. 2) so that the Lorentzian continuation is an eternal black hole with a shell inside [65].

2. Lorentzian geometry

In the Lorentzian section (the blue and purple parts of Fig. 2), the shell follows the equation of motion

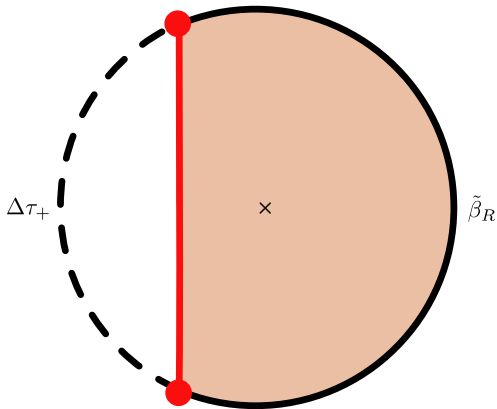


FIG. 3. We show here the Euclidean geometry of the right black hole, which is a disk of circumference β_R . The shell (red line) cuts the disk, with an associated Euclidean travel time $\Delta\tau_+$. Consistency fixes the state preparation temperature to satisfy $\beta_R = \tilde{\beta}_R + \Delta\tau_+$.

$\dot{R}^2 - V_{\text{eff}}(R) = 0$, arising from analytic continuation of the shell’s proper time $T \rightarrow iT$. The shell now emerges from the past singularity at finite proper time $T = -T_0$ in the past, reaches the radius $R = R_*$ inside the black hole at $T = 0$, and dives into the future singularity at $T = T_0$.

The geometry is that of two Schwarzschild-AdS black holes, with inverse temperatures $\beta_{L,R}$, glued along the trajectory of the shell’s worldvolume \mathcal{W} in the black hole interior (Fig. 2). The induced metric in the time-reflection symmetric slice is that of a ‘‘python’s lunch’’; that is, there is a bulge where the area of the transverse spatial spheres reaches a maximum, at $r = R_*$, between the two horizons [66].

3. Example: 2 + 1 dimensions

For $d = 2$, the shell’s trajectory can be explicitly evaluated. The BTZ horizon radii are given by $r_{\pm} = \sqrt{8GM_{\pm}}$, and the inverse temperatures are given by $\beta_{\pm} = 2\pi/r_{\pm}$. The effective potential (2.13) in this case is quadratic,

$$V_{\text{eff}}(R) = -(r^2 - R_*^2), \quad (2.17)$$

where the turning point R_* is

$$R_* = \sqrt{r_+^2 + \left(\frac{M_+ - M_-}{m} - 2Gm\right)^2}. \quad (2.18)$$

The solution for the shell’s trajectory (2.12) is

$$R(T) = R_* \cosh T, \quad (2.19)$$

where we have chosen the initial condition such that the shell passes through R_* at proper time $T = 0$. The Euclidean time elapsed by the shell (2.14) can also be computed analytically. We find

$$\Delta\tau_{\pm} = \beta_{\pm} \frac{\arcsin(r_{\pm}/R_*)}{\pi}. \quad (2.20)$$

C. Multishell states

We can also describe situations with multiple shells characterized by a mass vector $\mathbf{m} = \{m_1, m_2, \dots\}$. In the Euclidean picture, we can insert as many shells as we want behind the horizon, and the above analysis applies locally to each trajectory $R_i(T)$.

Such a multishell state will have the form (2.5). The region between any pair of shells is locally a black hole geometry and must have (see Fig. 4)

$$\beta_i = 2\tilde{\beta}_i + \Delta\tau_+^i + \Delta\tau_-^{i+1}, \quad (2.21)$$

where $\Delta\tau_{\pm}^i$ depends on m_i and on parameters M_{i-1} and M_i , which are ‘‘masses’’ characterizing the local geometries between the shells. From the consistency of the classical solution, the mass difference $M_i - M_{i-1}$ is constrained to

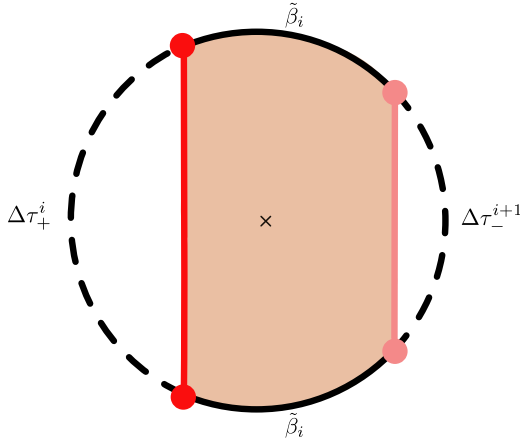


FIG. 4. Euclidean disk geometry (a Euclidean black hole) in between two shells (red lines). The shell travel times τ_+^i and τ_-^{i+1} plus twice the preparation temperature $\tilde{\beta}_i$ must be equal to the physical temperature β_i of this black hole geometry between the shells.

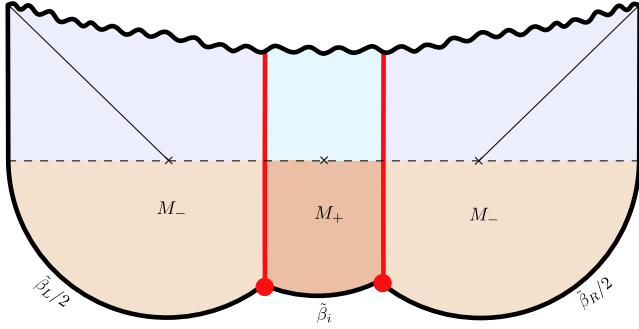


FIG. 5. Example of a multishell state of an eternal black hole with ADM masses M_- and interior geometry with associated mass parameter M_+ .

be upper bounded by the i th shell's gravitational self-energy at R_* .

The formulas (2.12) and (2.13) go through for the dynamics of each shell, with M_i and M_{i+1} replacing the black hole masses, since the thin-shell formalism is a local analysis in the vicinity of the shell. The preparation of such states is shown in Fig. 5.

III. OVERLAPS BETWEEN STATES FROM SEMICLASSICAL WORMHOLES

Consider a pair of black hole microstates $|\Psi_{\mathbf{m}}\rangle, |\Psi_{\mathbf{m}'}\rangle$, for generic shell configurations \mathbf{m}, \mathbf{m}' representing classically different interior geometries in the $G \rightarrow 0$ limit. For example, for one-shell states, this means configurations whose rest mass differences scale parametrically as $\Delta m = O(\ell^{d-2}/G)$. The objective of this section is to compute the overlaps and their products,

$$\begin{aligned} & \overline{\langle \Psi_{\mathbf{m}} | \Psi_{\mathbf{m}'} \rangle}, \quad \overline{\langle \Psi_{\mathbf{m}} | \Psi_{\mathbf{m}'} \rangle \langle \Psi_{\mathbf{m}'} | \Psi_{\mathbf{m}''} \rangle}, \\ & \overline{\langle \Psi_{\mathbf{m}} | \Psi_{\mathbf{m}'} \rangle \langle \Psi_{\mathbf{m}'} | \Psi_{\mathbf{m}''} \rangle \langle \Psi_{\mathbf{m}''} | \Psi_{\mathbf{m}'''} \rangle}, \dots, \end{aligned} \quad (3.1)$$

where the overline means that we are computing these quantities within the low-energy effective theory of general relativity with a negative cosmological constant, coupled to the shells [68]. We will typically restrict ourselves to $\beta_L = \beta_R = \beta$ so that the left and right black holes have the same mass, as these are the scenarios we need while counting black hole microstates.

At first glance, it would appear that if we compute the first inner product $\overline{\langle \Psi_{\mathbf{m}} | \Psi_{\mathbf{m}'} \rangle}$ in Eq. (3.1), then the remaining quantities will follow by multiplication. However, we will see that this is not the case because of nonperturbative effects in the gravitational path integral. To show this scenario, we start with the combined Euclidean action

$$\begin{aligned} I[X] = & -\frac{1}{16\pi G} \int_X (R - 2\Lambda) + \frac{1}{8\pi G} \int_{\partial X} K \\ & + \int_{\mathcal{W}} \sigma + I_{\text{ct}}. \end{aligned} \quad (3.2)$$

Here, Λ is the cosmological constant, K is the extrinsic curvature of the spacetime boundary, σ is the density of the shell, \mathcal{W} is the worldvolume of the shell, and I_{ct} are counterterms, localized at the asymptotic boundary ∂X , that remove divergences and renormalize the value of the on-shell action [59,61].

Recall that inner products are computed in terms of the gravitational path integral by summing over all Euclidean geometries that are consistent with the boundary conditions determined by the preparation of the bra and ket states. In the semiclassical approximation, the path integral then reduces to a sum over the exponential of the action of the classical saddle points X satisfying the boundary conditions. When $\mathbf{m} \neq \mathbf{m}'$, we need some way to join the different shells from the Euclidean boundary. Achieving this result requires a number of bulk interactions of the order of $\Delta \mathbf{m} = |\mathbf{m} - \mathbf{m}'|$ in Planck units, so we expect the result to be parametrically suppressed in the exponential of the mass difference $\Delta \mathbf{m}$. Thus, we neglect such contributions by taking the mass difference to be arbitrarily large.

In this limit, the inner product is

$$\overline{\langle \Psi_{\mathbf{m}} | \Psi_{\mathbf{m}'} \rangle} = \delta_{\mathbf{m}, \mathbf{m}'}. \quad (3.3)$$

Next, $\overline{\langle \Psi_{\mathbf{m}} | \Psi_{\mathbf{m}'} \rangle \langle \Psi_{\mathbf{m}'} | \Psi_{\mathbf{m}''} \rangle}$ is computed from the path integral with two disconnected asymptotic boundaries, which prepare the overlaps $\langle \Psi_{\mathbf{m}} | \Psi_{\mathbf{m}'} \rangle$ and $\langle \Psi_{\mathbf{m}'} | \Psi_{\mathbf{m}''} \rangle$, respectively. This quantity receives one contribution from a disconnected geometry—two copies of the saddle point contributing to $\overline{\langle \Psi_{\mathbf{m}} | \Psi_{\mathbf{m}'} \rangle}$, one associated with each of the two asymptotic boundaries. In our limit, in which the

difference in masses between different states is taken to be arbitrarily large, this quantity vanishes unless $\mathbf{m} = \mathbf{m}' = \mathbf{m}''$. If $\mathbf{m} = \mathbf{m}''$, i.e., the first bra and the final ket have the same shells (so that we are computing a “square” of the overlap), there is a second connected contribution—a semiclassical wormhole X_2 connecting the two boundaries, as described in Ref. [25] in the context of operator statistics. The full semiclassical expression then reads

$$\begin{aligned} \overline{\langle \Psi_{\mathbf{m}} | \Psi_{\mathbf{m}'} \rangle \langle \Psi_{\mathbf{m}'} | \Psi_{\mathbf{m}} \rangle} &\equiv \overline{|\langle \Psi_{\mathbf{m}} | \Psi_{\mathbf{m}'} \rangle|^2} \\ &= \delta_{\mathbf{m}, \mathbf{m}'} + \frac{Z_2}{Z_1 Z_1'}, \end{aligned} \quad (3.4)$$

where we have defined

$$Z_2 = e^{-I[X_2]}, \quad Z_1 = e^{-I[X]}, \quad Z_1' = e^{-I[X']} \quad (3.5)$$

as the renormalized action (3.2) of the wormhole X_2 . The normalization factors Z_1 and Z_1' associated with each shell [see Eq. (2.3)] are given by the classical saddles X and X' , respectively [69]. In fact, if $\mathbf{m} \neq \mathbf{m}''$, there will still be a connected contribution, but it will be exponentially suppressed if the mass difference of the shells $|\mathbf{m} - \mathbf{m}''|$ is taken to be large, by the same reasoning as above. We can continue similarly for the higher-order overlaps

$$\overline{\langle \Psi_{\mathbf{m}} | \Psi_{\mathbf{m}'} \rangle \langle \Psi_{\mathbf{m}'} | \Psi_{\mathbf{m}''} \rangle \dots \langle \Psi_{\mathbf{m}'} \dots | \Psi_{\mathbf{m}} \rangle}_c = \frac{Z_n}{Z_1 Z_1' \dots Z_1' \dots}, \quad (3.6)$$

which arise from the maximally connected n -boundary wormhole contributions $Z_n = e^{-I[X_n]}$. Below, we study $I[X]$ and $I[X_2]$ in detail, as well as the other $I[X_n]$, in certain limits, which we will need in later sections. Appendix B provides more details about wormholes with more than two boundaries.

A. Single-shell states

We start with single-shell states, $\mathbf{m} = m$ and $\mathbf{m}' = m'$. The normalization of these states is computed semiclassically by the space-time filling procedure described above. This process gives

$$Z_1 = e^{-I[X]}, \quad (3.7)$$

where X is the saddle-point manifold depicted in Fig. 6. From Eq. (2.14) and Fig. 3 with $\beta_L = \beta_R = \beta$, the effective temperature $\tilde{\beta}_m$ to prepare the state is determined by the real temperature β of the saddle-point black hole, and the Euclidean time $\Delta\tau$ elapsed by the shell, via the relation

$$\beta = \tilde{\beta}_m + \Delta\tau. \quad (3.8)$$

The preparation temperature is the same on both sides of the shell, so $\tilde{\beta}_L = \tilde{\beta}_R = \tilde{\beta}_m$. In order to evaluate the

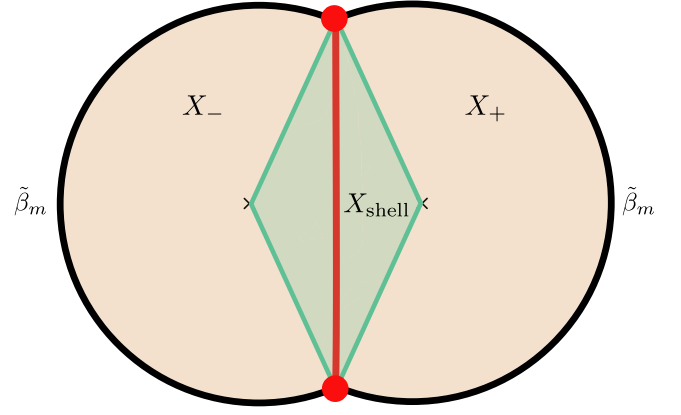


FIG. 6. Different regions in the decomposition of the Euclidean gravitational action $I[X]$ for the single-shell state. This figure shows the r, τ_{\pm} plane of the geometry. In this plane, τ_{\pm} are angular coordinates around the horizons r_{\pm} , represented by the crosses. The green region X_{shell} accounts for the intrinsic contribution from the shell. Its boundaries are constant Euclidean time hypersurfaces that join the horizons to the operator insertions at the boundary.

renormalized action $I[X]$ in Eq. (3.2), it is convenient to divide the manifold into $X = X_- \cup X_{\text{shell}} \cup X_+$ (Fig. 6). The region X_{shell} is bounded by the constant Euclidean time hypersurfaces that join the left or right horizon to the two operator insertions at r_{∞} . From additivity, the on-shell action decomposes as

$$I[X] = I[X_-] + I[X_+] + I[X_{\text{shell}}]. \quad (3.9)$$

The first two terms just depend on the exterior black hole geometries, and therefore

$$I[X_{\pm}] = \tilde{\beta}_m F(\beta) = \tilde{\beta}_m (M - S/\beta), \quad (3.10)$$

where $F(\beta) = -\beta^{-1} \log Z(\beta)$ is the renormalized free energy of the respective black hole and

$$S \equiv \frac{A}{4G} \quad (3.11)$$

is the Bekenstein-Hawking entropy. More explicitly, the renormalized action is given by [70]

$$I[X_{\pm}] = \tilde{\beta}_m F(\beta) = \frac{\tilde{\beta}_m V_{\Omega}}{16\pi G} (-r_+^d + r_+^{d-2} + c_d). \quad (3.12)$$

The constant c_d accounts for the Casimir energy of the CFT in even dimensions [59] ($c_d = -\frac{1}{2}, \frac{3}{8}, -\frac{5}{16}, \dots$ in $d = 2, 4, 6, \dots$). Equation (3.12) is computed by noticing that the solutions have angular symmetry around the thermal cycle. We compute the action of a fraction of the whole disk (i.e., regions X_+ and X_-), so the integral over Euclidean time then goes from zero to the preparation temperature.

The last term $I[X_{\text{shell}}]$ arises from the presence of the shell since its value vanishes as $m \rightarrow 0$ [71]. The Euclidean action associated with the region X_{shell} has the form

$$I[X_{\text{shell}}] = -\frac{1}{16\pi G} \int_{X_{\text{shell}}} (R - 2\Lambda) + \int_{\mathcal{W}} \sigma \quad (3.13)$$

before counterterms are added to remove long-distance divergences that develop in both terms as $r_\infty \rightarrow \infty$. Here, σ is the mass density of the shell, whose radius dependence is given by Eq. (2.11). The on-shell Einstein-Hilbert term gives two contributions:

$$R - 2\Lambda = -2d + \frac{16\pi G}{d-1} \delta(y), \quad (3.14)$$

where y is a normal coordinate to \mathcal{W} . The second term comes from the δ function contribution of the stress tensor of the shell, and the first term comes from the constant background curvature. Plugging this expression into Eq. (3.13) gives

$$I[X_{\text{shell}}] = \frac{d}{8\pi G} \text{Vol}(X_{\text{shell}}) + m \frac{d-2}{d-1} L[\gamma_{\mathcal{W}}], \quad (3.15)$$

where the $\text{Vol}(X_{\text{shell}})$ term arises from the bulk integral of the background curvature and the second term takes contributions from the δ function in Eq. (3.14) and from the integral over the shell in Eq. (3.13). Here, $L[\gamma_{\mathcal{W}}]$ is the proper length of the trajectory of the shell. Thus, the second term looks like the action of a heavy particle propagating in the (τ_\pm, r) plane. Explicitly, each term is computed from the integrals

$$L[\gamma_{\mathcal{W}}] = 2 \int_{R_*}^{r_\infty} \frac{dR}{\sqrt{-V_{\text{eff}}(R)}}, \quad (3.16)$$

$$\text{Vol}(X_{\text{shell}}) = \frac{4V_\Omega}{d} \int_{R_*}^{r_\infty} \frac{dR}{f_+(R)} \sqrt{\frac{f_+(R) + V_{\text{eff}}(R)}{-V_{\text{eff}}(R)}} (R^d - r_+^d). \quad (3.17)$$

This volume is the same as that seen from the left and right sides of the shell because we assume the two asymptotic geometries have the same mass, and hence the same horizon radius $r_+ = r_-$. Solving these integrals requires numerical treatment when $d > 2$. Summarizing, the partition function normalizing one-shell states is

$$-\log Z_1 = 2\tilde{\beta}_m F(\beta) + \frac{d}{8\pi G} \text{Vol}(X_{\text{shell}}) + m \frac{d-2}{d-1} L[\gamma_{\mathcal{W}}]. \quad (3.18)$$

1. Wormhole

We now calculate the semiclassical overlap squared, namely,

$$|\langle \Psi_{\mathbf{m}} | \Psi_{\mathbf{m}'} \rangle|^2 = \delta_{\mathbf{m}, \mathbf{m}'} + \frac{Z_2}{Z_1 Z_1'}, \quad (3.19)$$

where we recall that

$$Z_2 = e^{-I[X_2]} \quad (3.20)$$

is the exponential of the gravitational action of the wormhole X_2 , see Fig. 7. The normalizations Z_1 and Z_1' are the ones computed in the previous subsection. The wormhole

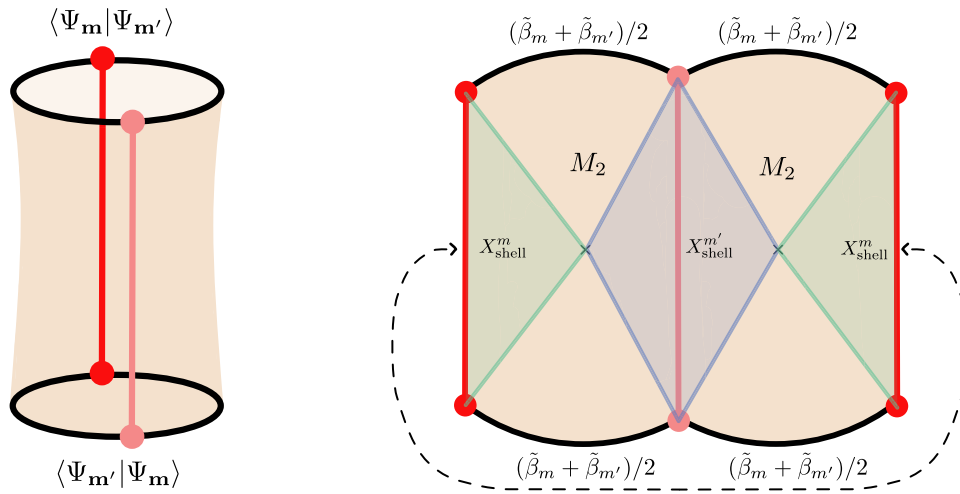


FIG. 7. Wormhole X_2 consisting of a pair of Euclidean black holes of the same mass, which are glued together along the trajectory of the two thin shells. The mass M_2 of the black hole that forms the wormhole is determined by the preparation temperatures and the saddle-point equations.

X_2 consists of a pair of Euclidean black holes of the same mass. These are glued together along the trajectory of the two thin shells, as depicted in Fig. 6. The saddle-point equations were derived in Eq. (2.21) and follow from an Israel junction condition analysis. Here, they read

$$\beta_2 = \tilde{\beta}_m + \tilde{\beta}_{m'} + \Delta\tau_m + \Delta\tau_{m'}. \quad (3.21)$$

In this equation, the two preparation temperatures $\tilde{\beta}_m$ and $\tilde{\beta}_{m'}$ have already been fixed by Eq. (3.8). Then, in the wormhole solution, the equation of motion (2.14) fixes the elapsed Euclidean time for each shell as a function of the mass of the shell and the background black hole geometry parametrized by some β_2 . The constraint (3.21) then fixes β_2 . Having found this solution, the action of the wormhole after counterterm subtraction can be derived by the same logic as above, giving

$$I[X_2] = 2(\tilde{\beta}_m + \tilde{\beta}_{m'})F(\beta_2) + I[X_{\text{shell}}^m] + I[X_{\text{shell}}^{m'}]. \quad (3.22)$$

Including the normalizations Z_1 and Z_1' , we obtain the semiclassical overlap squared,

$$\begin{aligned} |\langle \Psi_m | \Psi_{m'} \rangle|^2 &= \delta_{m,m'} + \frac{Z_2}{Z_1 Z_1'} \\ &= \delta_{m,m'} + e^{-2(\tilde{\beta}_m + \tilde{\beta}_{m'})\Delta F - \Delta I[X_{\text{shell}}^m] - \Delta I[X_{\text{shell}}^{m'}]}, \end{aligned} \quad (3.23)$$

where $\Delta F = F(\beta_2) - F(\beta)$ and $\Delta I[X_{\text{shell}}^i] = I[X_{\text{shell}}^i]_{\beta_2} - I[X_{\text{shell}}^i]_{\beta}$ for $i = m, m'$.

2. 2+1 dimensions

For $d = 2$, we can work out the details analytically. First, the saddle-point equation (3.8) relating the state preparation and physical temperatures, combined with the expression for the Euclidean time elapsed across the whole shell trajectory (2.20), is solved by

$$\tilde{\beta}_m = \frac{\beta}{\pi} \arcsin \frac{r_+}{R_*}, \quad (3.24)$$

where $R_*^2 = r_+^2 + (2Gm)^2$. Thus, we see that $(\beta/2) \leq \tilde{\beta}_m \leq \beta$, where the lower bound is reached for small mass $m \ll M$, while the upper bound is reached for $m \gg M$.

In this case, the second term drops out of Eq. (3.15), and the volume of X_{shell} can be analytically computed, $\text{Vol}(X_{\text{shell}}) = 4\pi GmL[\gamma_{\mathcal{V}}]$. The result equals the standard propagator of a massive particle,

$$I[X_{\text{shell}}] = mL[\gamma_{\mathcal{V}}] = 2m \cosh^{-1} \left(\frac{r_\infty}{R_*} \right), \quad \text{for } d=2, \quad (3.25)$$

where we have evaluated the proper length for the explicit trajectory of the particle $R(T) = R_* \cosh T$, with $R_*^2 = r_+^2 + (2Gm)^2$.

To renormalize the logarithmic divergence of Eq. (3.25) as $r_\infty \rightarrow \infty$, we add the counterterm $I_{\text{ct}}[X_{\text{shell}}] = -m \log r_\infty$ and then take $r_\infty \rightarrow \infty$. The final renormalized action of the shell reads

$$I_{\text{ren}}[X_{\text{shell}}] = -2m \log R_* + 2m \log 2, \quad \text{for } d=2. \quad (3.26)$$

In addition, using Eq. (3.12), the total gravitational action of the saddle point is

$$-\log Z_1 = -\frac{\tilde{\beta}_m}{G} \left(\frac{\pi^2}{\beta^2} - \frac{1}{8} \right) - 2m \log R_* + 2m \log 2. \quad (3.27)$$

Next, given the action of the wormhole (3.22), we can derive an explicit analytical expression for the overlap squared:

$$\begin{aligned} |\langle \Psi_m | \Psi_{m'} \rangle|^2 &= \delta_{m,m'} + \frac{Z_2}{Z_1 Z_1'} \\ &= \delta_{m,m'} + e^{\frac{2(\tilde{\beta}_m + \tilde{\beta}_{m'})\pi^2}{G} \left(\frac{1}{\beta_2^2} - \frac{1}{\beta^2} \right) + 4m \log \frac{R_*(\beta_2)}{R_*(\beta)} + 4m' \log \frac{R'_*(\beta_2)}{R'_*(\beta)}}, \end{aligned} \quad (3.28)$$

where again $R_*(\beta) = \sqrt{(4\pi^2/\beta^2) + (2Gm)^2}$.

B. Higher moments

We can proceed similarly to compute higher moments of the overlap. As shown in Appendix B, the connected part acquires the form

$$\overline{\langle \Psi_{m_1} | \Psi_{m_2} \rangle \langle \Psi_{m_2} | \Psi_{m_3} \rangle \dots \langle \Psi_{m_n} | \Psi_{m_1} \rangle}_c = \frac{Z_n}{Z_1^{(m_1)} \dots Z_1^{(m_n)}}, \quad (3.29)$$

where $Z_n = e^{-I[X_n]}$ is the contribution from the n -boundary wormhole (Fig. 14) and $Z_1^{(m_i)}$ is the normalization of the state of a shell of mass m_i . The expressions for the n -boundary wormholes contributing to Z_n can be obtained in a straightforward manner using the building blocks we have derived already. For details, see Appendix B.

C. Limit of large mass and universality

To provide an account of black hole entropy in the next section, we need the contribution of n -boundary wormholes to the n th moment of the overlap. However, we only need the large mass $m_i \gg M$ limit, where we recall that the local mass of the shell inside the black hole can actually exceed the asymptotic mass of the black hole because of the effects of backreaction. In this limit, the Euclidean time elapsed by

each shell trajectory tends to zero, $\Delta\tau_i \rightarrow 0$ [see Eqs. (2.14) and (2.20)], and therefore $\tilde{\beta}_{m_i} \approx \beta$ [see Eq. (3.8)]. Also in this limit, $(d/8\pi G)\text{Vol}(X_{\text{shell}}) \approx (m/d-1)L[\gamma_{\mathcal{W}}]$ in Eq. (3.17) so that

$$I[X_{\text{shell}}] \approx mL[\gamma_{\mathcal{W}}] \approx 2m \log R_*, \quad (3.30)$$

after including the counterterms. The proper shell action therefore becomes constant, independent of the mass of the black hole, since $R_*^{d-1} \sim Gm\ell$ in this limit. The normalization of the one-shell states (3.18) is then given by

$$Z_1 \sim Z(\beta)^2 e^{-2m \log R_*}. \quad (3.31)$$

Similarly, the action of the wormhole X_2 becomes

$$Z_2 = e^{-I[X_2]} = Z(2\beta)^2 e^{-2m \log R_* - 2m' \log R'_*}. \quad (3.32)$$

This result can be derived by taking the large mass limit of our previous expressions, but it is also obvious from Fig. 8. Therefore, the overlap for $m \neq m'$ reads

$$\overline{|\langle \Psi_m | \Psi_{m'} \rangle|^2}_c = \frac{Z_2}{Z_1 Z'_1} \approx \frac{Z(2\beta)^2}{Z(\beta)^4}, \quad (3.33)$$

and it is universal in this limit, independently of the actual masses of the shells.

By similar reasoning, it is easy to show that the n -boundary wormhole introduced in Appendix B displays a universal form in this limit,

$$Z_n = e^{-I[X_n]} = Z(n\beta)^2 e^{-2 \sum_a m_a \log R_*^a}, \quad (3.34)$$

leading to the n th moment

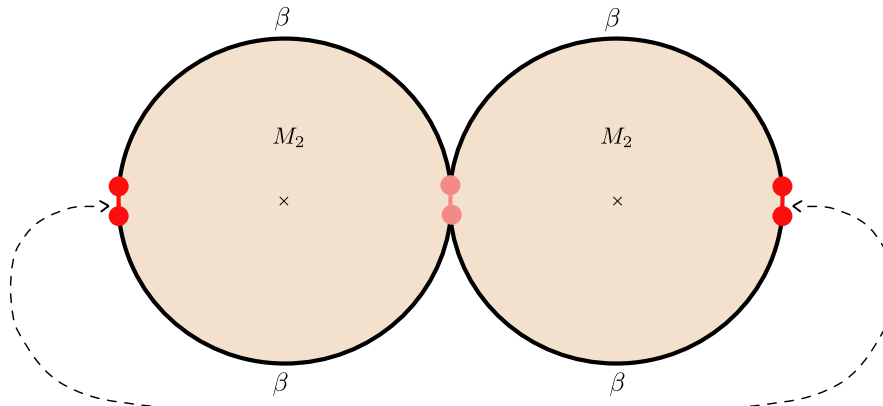


FIG. 8. Wormhole X_2 for large masses of the shells. The preparation of the state in this case is done with Euclidean time $\tilde{\beta}_m = \tilde{\beta}_{m'} \approx \beta$. The background black hole has inverse temperature 2β .

$$\begin{aligned} & \overline{|\langle \Psi_{m_1} | \Psi_{m_2} \rangle \langle \Psi_{m_2} | \Psi_{m_3} \rangle \dots \langle \Psi_{m_n} | \Psi_{m_1} \rangle|}_c \\ &= \frac{Z_n}{Z_1^{(m_1)} \dots Z_1^{(m_n)}} \approx \frac{Z(n\beta)^2}{Z(\beta)^{2n}}. \end{aligned} \quad (3.35)$$

A potential concern for this analysis is that the large shell mass limit pinches the wormhole geometry (Fig. 8). However, as we approach this limit, there is no singularity in the Euclidean geometry because the pinching points are at the boundary of space and remain infinitely far apart, and the Lorentzian continuation is also well defined. One might also worry that, in theory with extended states like string theory, we could have a condensation of light states in the pinching limit, requiring a modification of our analysis. However, no matter how large the mass is, the two-shell operator insertions are at infinite physical distance from each other, even though the Euclidean time elapsed goes to zero, because of a relative conformal factor. Thus, we do not expect extended states to give rise to a problem. This finding is consistent with the fact that we do not observe any singularities in the low-energy theory.

1. Multishell states

The overlaps between multishell states also have the same form in the limit where all of the masses $m_i \in \mathbf{m}$ are large, i.e., $m_i \gg M$. The Euclidean time elapsed by each shell vanishes in this regime, $\Delta\tau_i = 0$, and therefore $\tilde{\beta}_i \approx \beta_i$ as before. In this limit, the normalization of the state (3.7) becomes

$$Z_1 \approx Z(\beta)^2 e^{-2 \sum_i m_i \log R_*^i} \prod_{i=1}^{k-1} Z(2\beta_i), \quad (3.36)$$

where k is the number of shells. Each factor of $Z(2\beta_i)$ arises from the Euclidean black hole bounded by a pair of thin shells. The two factors of $Z(\beta)$ come from the exterior black holes. The intermediate factor is the action of each shell in the limit, which was derived above in Eq. (3.30).

Similarly, the wormhole action is now

$$\begin{aligned} Z_2 &= e^{-I[X_2]} \\ &\approx Z(2\beta)^2 e^{-2 \sum_i m_i \log R_i^* - 2 \sum_j m_j \log R_j^*} \\ &\quad \times \prod_{i=1}^{k-1} Z(2\beta_i) \prod_{j=1}^{k'-1} Z(2\beta_j), \end{aligned} \quad (3.37)$$

where k' is the number of shells in the second state. Therefore, the overlap reduces to the universal quantity

$$\overline{|\langle \Psi_{\mathbf{m}} | \Psi_{\mathbf{m}'} \rangle|^2} |_{\mathcal{C}} = \frac{Z_2}{Z_1 Z_1'} \approx \frac{Z(2\beta)^2}{Z(\beta)^4}, \quad (3.38)$$

independent of the number of shells, and of the interior geometries that are characterized by the parameters β_i and β'_i .

Finally, the n -boundary wormhole has the action

$$\begin{aligned} Z_n &= e^{-I[X_n]} \\ &\approx Z(n\beta)^2 e^{2 \sum_{\alpha} \sum_i m_{i\alpha} \log R_i^*} \prod_{\alpha=1}^n \prod_{i=1}^{k_{\alpha}-1} Z(2\beta_i), \end{aligned} \quad (3.39)$$

leading to the n th moment,

$$\begin{aligned} &\overline{\langle \Psi_{\mathbf{m}_1} | \Psi_{\mathbf{m}_2} \rangle \langle \Psi_{\mathbf{m}_2} | \Psi_{\mathbf{m}_3} \rangle \dots \langle \Psi_{\mathbf{m}_n} | \Psi_{\mathbf{m}_1} \rangle} |_{\mathcal{C}} \\ &= \frac{Z_n}{Z_1^{(\mathbf{m}_1)} \dots Z_1^{(\mathbf{m}_n)}} \approx \frac{Z(n\beta)^2}{Z(\beta)^{2n}}. \end{aligned} \quad (3.40)$$

This formula is a key result of our article. It states that wormholes in the quantum gravity path integral lead to universal, nonvanishing moments of the quantum overlaps of our black hole microstates. In Sec. IV, we will see how these overlaps lead to a bound on the dimension of the Hilbert space that equals the Bekenstein-Hawking entropy.

D. Microscopic interpretation of the overlaps

Inner products between quantum states in a Hilbert space are, by definition, complex numbers. As such, the product of a collection of them must factorize. However, this result is not what we found for the overlaps between shell microstates computed above via the rules prescribed by the semiclassical path integral of gravity. Indeed, the universal expression (3.40) manifestly displays nonfactorization, as a consequence of connected wormhole contributions to the products of overlaps [72]. In what follows, expanding on Ref. [25] (see also Ref. [74]), we will provide an interpretation of the semiclassical overlaps starting from a microscopic description of these states. Basically, we will view the semiclassical calculation as an approximation that is only sensitive to the magnitudes and not to the erratic phases of the real inner products between the corresponding quantum states.

We will argue that the results of the gravitational path integral are consistent with an assumption that the underlying theory, which in the present case is equivalent to the holographic dual CFT, satisfies something like the eigenstate thermalization hypothesis (ETH) [82,83]. ETH was originally formulated as a simple but powerful postulate that allows us to understand thermalization in isolated quantum systems. The postulate says that the matrix elements in the energy basis of a “simple” operator \mathcal{O} take the form

$$\langle E_n | \mathcal{O} | E_m \rangle = f(\bar{E}) \delta_{nm} + e^{-S(\bar{E}/2)} g(\bar{E}, \omega)^{1/2} R_{nm}, \quad (3.41)$$

where $|E_n\rangle$ are energy eigenstates. In this expression, we defined

$$\bar{E} \equiv \frac{E_n + E_m}{2}, \quad \omega = E_m - E_n. \quad (3.42)$$

The functions $f(\bar{E}), g(\bar{E}, \omega)$ are smooth functions of their arguments in the thermodynamic limit, and they encode information about the microcanonical one-point and two-point functions of the operator, respectively. The coefficients R_{nm} are erratic complex numbers of $\mathcal{O}(1)$ magnitude. The ETH then asserts that the R_{nm} entries can be viewed as independent random variables with zero mean and unit variance [84]. The domain of applicability of ETH is not fully understood, but it is generally expected to apply to quantum chaotic theories, such as the ones expected to describe black holes [34,87].

Following Ref. [25], the thin fine-grained operator should be regarded as simple: Applied near the spacetime boundary, it just creates a gas of particles in the low-energy effective theory [88]. As such, and in view of the chaotic Hamiltonian expected for the black hole dynamics, the shell operator should admit an ETH form (3.41). Under this assumption, Ref. [25] showed that the semiclassical calculation of the product of correlation functions of the thin-shell operator follows from the expected random character of the R_{nm} coefficients as follows. There are many shells of the same mass that differ in tiny, microscopic, perhaps Planck-scaled details, in, e.g., the precise positioning of the dust particles. These states should have an ETH description with coefficients of the same magnitude but randomly varying erratic phases. The semiclassical path integral should be interpreted as a coarse-grained quantity that does not have access to the precise microscopic phases and effectively averages over them, only preserving information about the magnitudes of amplitudes in which the phases cancel. In this identification, matching the semiclassical gravity results for the thin shells requires that $f(\bar{E})$ vanish, while the envelope function takes the form

$$g(\bar{E}, \omega) = e^{S(\bar{E}) - \alpha_- S(\bar{E} - \omega) - \alpha_+ S(\bar{E} + \omega) - I_{\text{shell}}(\bar{E}, \omega)}. \quad (3.43)$$

The coefficients $\alpha_{\pm} \equiv \Delta\tau_{\pm}/\beta_{\pm}$ are specified in terms of the shell's Euclidean travel times (2.14). The function $I_{\text{shell}}(\bar{E}, \omega)$ is the value of the shell's on-shell action (see Ref. [25]).

The connection with the shell states presented here follows in a straightforward manner. Our states are defined as

$$|\Psi\rangle = |\rho_{\tilde{\beta}_L/2} \mathcal{O} \rho_{\tilde{\beta}_R/2}\rangle \\ = \frac{1}{\sqrt{Z_1}} \sum_{n,m} e^{-\frac{1}{2}(\tilde{\beta}_L E_n + \tilde{\beta}_R E_m)} \mathcal{O}_{nm} |n, m\rangle, \quad (3.44)$$

where \mathcal{O} is a shell operator and

$$Z_1 = \text{Tr}(\mathcal{O}^\dagger e^{-\tilde{\beta}_L H} \mathcal{O} e^{-\tilde{\beta}_R H}) \quad (3.45)$$

normalizes these pure states. Then, as described above, if we define the smooth function

$$f(E_n, E_m) \equiv S(\bar{E}) - \log g(\bar{E}, \omega) \\ = \alpha_- S(\bar{E} - \omega) + \alpha_+ S(\bar{E} + \omega) \\ + I_{\text{shell}}(\bar{E}, \omega), \quad (3.46)$$

we can reproduce the gravitational results by assuming an ETH form

$$|\Psi\rangle = \frac{1}{\sqrt{Z_1}} \sum_{n,m} e^{-\frac{1}{2}(\tilde{\beta}_L E_m + \tilde{\beta}_R E_n - f(E_n, E_m))} R_{mn} |m, n\rangle, \quad (3.47)$$

where the normalization becomes

$$Z_1 = \text{Tr}(\mathcal{O}^\dagger e^{-\tilde{\beta}_L H} \mathcal{O} e^{-\tilde{\beta}_R H}) \\ \approx \sum_{n,k} e^{-\tilde{\beta}_R E_n - \tilde{\beta}_L E_k - f(E_n, E_k)}. \quad (3.48)$$

These states look approximately random in the energy basis, in each microcanonical band. Indeed, projecting onto a microcanonical band of energies $[E, E + \Delta E]$ and normalizing the state, one obtains

$$|\Psi\rangle_E \simeq \frac{1}{\Omega_E} \sum_{n,m} R_{mn} |m, n\rangle. \quad (3.49)$$

Thus, we can regard the state preparation procedure, together with the projection on an energy window, as a procedure to generate an infinite number of random states in the band [89]. The infinite number of states arises by varying the mass of the shell operator \mathcal{O} used as the starting point or by using multiple shell states. In the holographic CFT, this result can be achieved by increasing the scaling dimension of the operator.

We now come back to the moments of inner products,

$$\overline{\langle \Psi_{m_1} | \Psi_{m_2} \rangle \langle \Psi_{m_2} | \Psi_{m_3} \rangle \cdots \langle \Psi_{m_k} | \Psi_{m_1} \rangle}. \quad (3.50)$$

In this expression, the overline means we are computing these products using the gravity path integral. By writing every state in terms of the shell operators used to prepare them, and assuming ETH for the thin shell, we can use the results in Ref. [25] to precisely interpret the overline in terms of an ETH average of the form

$$\overline{\mathcal{O}_{n_1 m_2} \mathcal{O}_{n_2 m_3} \cdots \mathcal{O}_{n_k m_1}}. \quad (3.51)$$

In this way, we arrive at a simple interpretation of the nonfactorization of the semiclassical inner products of our microstates: The semiclassical path integral only computes a coarse-grained average over the microstates that are consistent with the macroscopic semiclassical description appearing in the saddle point. The fine-grained phase of the overlap $\langle \Psi_{m_1} | \Psi_{m_2} \rangle$ depends erratically on the ETH coefficients of the operators \mathcal{O}_1 and \mathcal{O}_2 , and it averages out to zero semiclassically, $\overline{\langle \Psi_{m_1} | \Psi_{m_2} \rangle} = 0$ [92]. On the contrary, the magnitude of the overlap is a smooth quantity with some nonvanishing average value that is captured by the worm-hole contribution (3.50) for $k=2$. This finding follows and further supports the ideas of Refs. [36,75,80,85,93,94] connecting quantum chaos and semiclassical gravitational physics.

To end the discussion on the interpretation of the gravitational overlaps, we want to make it clear that, although ETH provides the most natural and physical interpretation of our results, it is important to notice that we do not need such an assumption in what follows and, in particular, in the derivation of the Bekenstein-Hawking degeneracy from the gravitational overlaps. The Bekenstein-Hawking entropy will follow from simple algebraic arguments, with the only input of the universal gravitational overlaps computed above.

1. Connection with the spectral form factor in chaotic theories

An interesting physical interpretation of our result (3.40) arises by considering the dynamics of chaotic quantum theories as follows. Given a quantum mechanical system with Hamiltonian H and a discrete spectrum, consider the so-called spectral form factor (SFF), defined by

$$\text{SFF}(t) = \frac{Z_{\beta-it} Z_{\beta+it}^*}{Z_\beta^2}, \quad (3.52)$$

where $Z_\beta = \sum_i e^{-\beta E_i}$ is the partition function of the Hamiltonian with eigenvalues E_i . The SFF is well known in the context of matrix models (see Ref. [95]), and it has recently been studied in relation to black hole dynamics

(see Ref. [34]). Consider also the TFD state, defined as follows:

$$|\psi_\beta\rangle \equiv \frac{1}{\sqrt{Z_\beta}} \sum_n e^{-\frac{\beta E_n}{2}} |n, n\rangle, \quad (3.53)$$

in the tensor product of the original Hilbert space with itself. Unitary evolution with a single, say, the left, Hamiltonian gives

$$|\psi_\beta(t)\rangle = e^{-iH_L t} |\psi_\beta\rangle = |\psi_{\beta+2it}\rangle. \quad (3.54)$$

The survival probability of this evolution, namely, the probability that the evolved state is found in the original thermofield double, equals the spectral form factor [41,96–99]

$$P(t) = |\langle \psi_{\beta+2it} | \psi_\beta \rangle|^2 = \text{SFF}(t). \quad (3.55)$$

The time average of the spectral form factor is

$$\begin{aligned} \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T dt \text{SFF}(t) \\ = \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T dt \frac{1}{Z(\beta)^2} \sum_{m,n} e^{-\beta(E_m + E_n) + i(E_m - E_n)t}. \end{aligned} \quad (3.56)$$

If the theory is chaotic, we expect level repulsion to remove all degeneracies in the spectrum. Hence, the time average gives

$$\lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T dt \text{SFF}(t) = \frac{Z(2\beta)}{Z(\beta)^2}. \quad (3.57)$$

Now, recall that the SFF is precisely the survival probability for the thermofield double state (3.55), i.e., the probability for the state to return to itself. In a chaotic theory, we expect an ergodic exploration of the Hilbert space. Thus, the time average in Eq. (3.57) computes the inner product between the initial state and a typical, essentially random, state in the Hilbert space. Notice that Eq. (3.40) for $n = 2$ is precisely the square of Eq. (3.57). The square appears because we are considering an eternal black hole, so the Hilbert space is doubled. This detail aside, we see that the gravitational path integral for the square of the overlap gives precisely the same result as the expected time-averaged survival amplitude in a chaotic theory, suggesting that the shell states we are computing are essentially random relative to each other.

The nonvanishing long time average of the spectral form factor is a proxy for the discreteness of the underlying Hamiltonian and its dimension. Finding this long time average by means of a gravitational computation has been a key goal in the context of quantum black holes (see

Ref. [34] and references therein). This goal was recently accomplished in the context of 2D JT gravity and its cousins [100,101], but here we have been able to find such typical inner products for gravity in general dimensions.

2. Reduced density matrices and quantum black hole hair

Consider the reduced density matrices $\rho_{R,L} = \text{Tr}_{L,R} |\Psi\rangle\langle\Psi|$ of each of the two boundary subsystems of our eternal black holes. In terms of the thin-shell operator making our microstates, their expression is

$$\rho_L = \frac{1}{Z_1} e^{-\frac{\beta_L H}{2}} \mathcal{O} e^{-\tilde{\beta}_R H} \mathcal{O}^\dagger e^{-\frac{\beta_L H}{2}}, \quad (3.58)$$

$$\rho_R = \frac{1}{Z_1} e^{-\frac{\beta_R H}{2}} \mathcal{O} e^{-\tilde{\beta}_L H} \mathcal{O}^\dagger e^{-\frac{\beta_R H}{2}}. \quad (3.59)$$

This finding can be verified by constructing the density matrix representation of the state in Eq. (2.2) and tracing over the left or right Hilbert spaces. These reduced density matrices are not exactly thermal. Since the classical geometries outside the horizon are equal to the black hole geometry, we say that these shell states display ‘‘quantum hair.’’ To analyze the effect of this quantum hair, we can use the gravitational path integral, interpreted in terms of the ETH language above, with the explicit representation of our states in the energy basis (3.47), to say that

$$\overline{R_{nk} R_{mk}^*} = \delta_{nm}. \quad (3.60)$$

It then follows that the semiclassical density matrices are diagonal in the energy basis (because they effectively average over the ETH phases). The reduced coarse-grained state at time $t = 0$ is then given by

$$\overline{\rho_{Rnm}} = p_n \delta_{nm}, \quad (3.61)$$

where the probabilities are

$$p_n = \frac{\sum_k e^{-\tilde{\beta}_R E_n - \tilde{\beta}_L E_k - f(E_n, E_k)}}{\sum_{n,k} e^{-\tilde{\beta}_R E_n - \tilde{\beta}_L E_k - f(E_n, E_k)}}. \quad (3.62)$$

These probabilities are close to the thermal values. To see this, first notice that the expectation value for the left or right energy is given in terms of the canonical ensemble at temperature $\beta_{L,R}$,

$$M_\pm = \langle \Psi | H_{R,L} | \Psi \rangle = \text{Tr}(\rho_{\beta_{R,L}} H_{R,L}). \quad (3.63)$$

The first equality simply expresses the fact that the ADM mass measured in the left or right external geometries must equal the expected value of the Hamiltonian in the state by the general reasoning of Ref. [59]. The second equality simply states that we could obtain the same result by taking the

expectation value of the left or right Hamiltonian in an exactly thermal density matrix. In other words, although our density matrix is not exactly thermal, it gives the same expectation value for the energy. In addition, we can consider the large mass limit $m \gg E$. In this case, we have $\tilde{\beta}_L = \beta_L$ and therefore

$$p_n = e^{-\beta_R(E_n - E_R) - S(E_R)} = \frac{e^{-\beta_R E_n}}{Z(\beta_R)}, \quad (3.64)$$

meaning that the reduced density matrix looks increasingly thermal. At small m , the expression is more complicated but not very illuminating for the present purposes. This structure of reduced density matrices suggests that there will be modes of the thermal radiation on both exterior geometries that will be sensitive to the fact that the reduced state is not thermal. Effectively, these modes detect the presence of the interior shell through its quantum hair. One might expect that the wavelength of these modes is determined by the distance of the shell to the horizon. When the shell is very far away (its mass is large), these modes become extremely low energy compared to the temperature and are suppressed by the corresponding Boltzmann factors, making them undetectable in practice. In a certain sense, the shell can be interpreted as a sort of soft firewall deep inside the black hole.

IV. COUNTING THE MICROSTATES OF BLACK HOLES

In Sec. II, we constructed several infinite families of black hole microstates with geometric descriptions whose existence sharpens the problem of black hole microstate counting. These states also challenge the holographic principle [102–104] since it appears that we can fit arbitrarily many inside a black hole of a given mass. However, as we showed in Sec. III, these states are not orthogonal because of nonperturbative effects in quantum gravity. In fact, they have universal overlaps with each other [see Eq. (3.40)]. Because of these overlaps, the dimension of the Hilbert space spanned by these states is smaller than the number of states. Below, we will calculate this dimension.

A. One-parameter family of states

First, we consider a convenient one-parameter family of states: the one-shell states with equal black hole temperatures on both sides. Equivalently, these states have the same ADM mass on both sides. Therefore, they are microstates of an eternal, finite-temperature black hole. For a given temperature β , our states are labeled by the rest mass m of the shell. In the context of AdS/CFT, this mass is related to the scaling dimension of the field by

$$m_n = n m_\Delta = n \ell^{-1} \sqrt{\Delta(\Delta - d)}, \quad (4.1)$$

where $n \sim \ell^{d-1}/G \gg 1$ is the number of operator insertions. Every shell backreacts on the interior geometry as described previously, leaving the outside geometry unchanged.

Given a field of mass m_Δ , we can thus create a discrete family of states labeled by n . In terms of the CFT, the family is defined as

$$|\Psi_n\rangle = \frac{1}{\sqrt{Z_1}} \sum_{i,j} e^{-\frac{1}{2}(\tilde{\beta}_L^{m_n} E_i + \tilde{\beta}_R^{m_n} E_j)} \mathcal{O}_{ij}^{m_n} |i, j\rangle, \quad (4.2)$$

where Z_1 normalizes the state and where we need the condition

$$\tilde{\beta}_L^{m_n} = \tilde{\beta}_R^{m_n} = \beta - \Delta\tau \quad (4.3)$$

on the Euclidean lengths used to prepare the state for the left or right physical black hole temperatures to be the same, i.e., $\beta_L = \beta_R = \beta$. Here, $\Delta\tau$ was defined in Eq. (2.14) as the Euclidean time required by the shell trajectory. Note that, as described in Sec. III, we can choose the initial size of the shell for any n so that all these states are within the regime of validity of our approximations. Specifically, in the Euclidean geometry, the minimum shell radius is R_* ; thus, in our limit, it can be verified that the shell density in Planck units never exceeds a maximum of $\mathcal{O}(\ell_p/\ell) \ll 1$.

Defined in this way, the difference in wormhole lengths between subsequent shell states is less than Planckian, and one cannot consider them to be geometrically different at a semiclassical level. This finding also implies that the inner product between subsequent states will not be exponentially suppressed; indeed, the universal result found earlier for geometrically different states does not apply [105]. For our purposes, it is convenient to choose an infinite family in which every member is geometrically different from the others at scales bigger than the Planck length. To achieve this goal, it is enough that subsequent states have wormholes with lengths differing by order $m = nm_\Delta$, where we take $n \sim \mathcal{O}(1/G_N)$. We thus consider microstates with masses

$$m_p = pm \quad p = 1, 2, \dots \quad (4.4)$$

This choice also ensures that as p grows, the overlap between different states is controlled by the universal answer derived before.

B. Microstate Gram matrix and the Hilbert space dimension

The infinite family of different, geometrical, and semiclassical states $|\Psi_p\rangle$ with masses $m_p = pm$ naively overcounts the Bekenstein-Hawking entropy. We now show that this is not actually the case [106]. The key question is not how many semiclassical or geometrical states we have at our disposal but rather what the dimension is of the Hilbert space they span. To find this dimension, we examine the

overlap matrix, i.e., the Gram matrix G of the microstates, whose entries are defined as

$$G_{pq} \equiv \langle \Psi_p | \Psi_q \rangle. \quad (4.5)$$

We consider this matrix G for $p, q = 1, \dots, \Omega$, with Ω finite. We then vary $\Omega = 1, \dots, \infty$. This process generates a sequence of $\Omega \times \Omega$ Gram matrices G of microstate overlaps.

Gram matrices are Hermitian and positive semidefinite by construction. They can always be written as

$$G = B^\dagger B, \quad (4.6)$$

with B being the matrix that has vectors $|\Psi_p\rangle$ as its columns. Here, B is a $\Omega \times \tilde{\Omega}$ matrix, where $\tilde{\Omega}$ is the dimension of the Hilbert space where the states $|\Psi_p\rangle$ live. Because it is positive semidefinite, its eigenvalues are positive or zero. In addition, from the definition of the Gram matrix, it is clear that the microstate vectors $|\Psi_p\rangle$ will be linearly independent if and only if the Gram matrix is positive *definite*, namely, if it has no zero eigenvalues. More generally, the rank of the Gram matrix will count the number of linearly independent vectors in the family $|\Psi_p\rangle$.

Therefore, for each Ω and associated Gram matrix G , we need to compute the number of zero eigenvalues of G . To this end, we first compute the resolvent of the matrix G , which is defined as

$$R_{ij}(\lambda) \equiv \left(\frac{1}{\lambda \mathbb{1} - G} \right)_{ij} = \frac{1}{\lambda} \delta_{ij} + \sum_{n=1}^{\infty} \frac{1}{\lambda^{n+1}} (G^n)_{ij}. \quad (4.7)$$

The density of eigenvalues of the Gram matrix then follows from the discontinuity across the real axis of the trace of the resolvent $R(\lambda) = \sum_{i=1}^{\Omega} R_{ii}(\lambda)$. To be precise, the density of eigenvalues is

$$D(\lambda) = \frac{1}{2\pi i} (R(\lambda - i\epsilon) - R(\lambda + i\epsilon)). \quad (4.8)$$

Reference [73] (see also Refs. [107–109]) showed how these types of resolvent matrices give rise to a Schwinger-Dyson equation. One starts by writing the definition of the resolvent in a diagrammatic expansion. In this version of

Eq. (4.7), the blobs are just the inner products $G_{ij} = \langle \Psi_i | \Psi_j \rangle$ that we have been discussing all along. They are just drawn horizontally instead of vertically as they were in our gravitational path integrals. The colored dots correspond to the shell insertions. To obtain Eq. (4.7) from this diagram, we note that the dashed lines are understood as “free propagators,” and they are assigned factors of $1/\lambda$.

Given this diagrammatic expansion, we can compute the resolvent matrix by using the gravitational path integral. As before, we denote a gravity computation in the leading approximation by an overline, and we arrive at

$$\overline{R_{ij}(\lambda)} = \frac{1}{\lambda} \delta_{ij} + \sum_{n=1}^{\infty} \frac{1}{\lambda^{n+1}} \overline{(G^n)_{ij}}, \quad (4.9)$$

This expansion can also be depicted graphically (Fig. 10). Interestingly, semiclassical gravity produces two terms when it is applied to the second term in the resolvent expansion depicted in Fig. 9; the first is disconnected while the second is a connected wormhole contribution. In the computation of this expansion in Fig. 10, we have only included diagrams that are “planar” in the sense explained in Ref. [73], since nonplanar diagrams will give subleading contributions in the limit of large entropy and large matrix G , which is the limit we are interested in.

The elements of $(G^n)_{ij}$ are products of the overlaps $\langle \Psi_i | \Psi_j \rangle$. Above, we showed, from the gravitational path integral, that these overlaps display a universal form for sufficiently high mass [110],

$$\overline{\langle \Psi_m | \Psi_{m'} \rangle \dots \langle \Psi_{m' \dots l} | \Psi_m \rangle} \simeq \frac{Z(n\beta)^2}{Z(\beta)^{2n}} \equiv \frac{Z_n}{Z_1^n}, \quad (4.10)$$

where n is the number of inner products on the left-hand side and where we recall that $Z(n\beta)^2 \neq Z_n$ because there are cancellations between the numerator and denominator. Notice that the right-hand side is universal, and it does not depend on the masses of the states.

Thus, we can perform the sum in Eq. (4.9) by observing that the expansion can be reorganized in a self-consistent way as depicted in Fig. 11. This case is analogous to the scenario developed in Ref. [73]. The upshot is that we obtain the Schwinger-Dyson equation

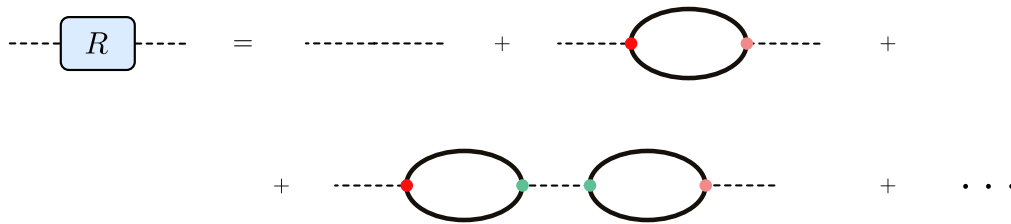


FIG. 9. Diagrammatic expansion of the resolvent matrix. The external dashed lines represent the (i, j) indices. The internal dashed lines represent summation over indices. The blobs represent the inner products $G_{ij} = \langle \Psi_i | \Psi_j \rangle$, and the color dots are the operator insertions.

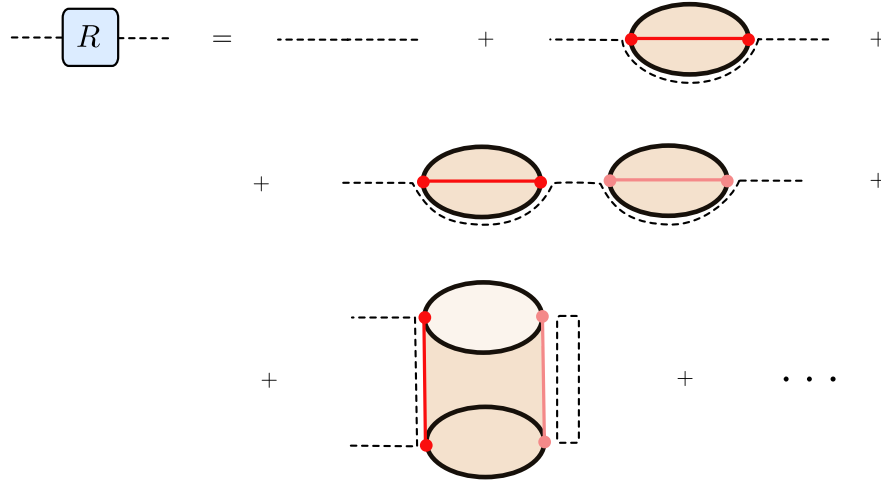


FIG. 10. Expansion of the resolvent as computed in the leading semiclassical gravity approximation. The colored blob in the second term represents the gravitational saddle point computing the first moment of the inner product. It corresponds to the second term in Fig. 9. The third and fourth terms are the two gravitational saddle-point contributions (disconnected and wormhole, respectively) to the computation of the third term in Fig. 9.

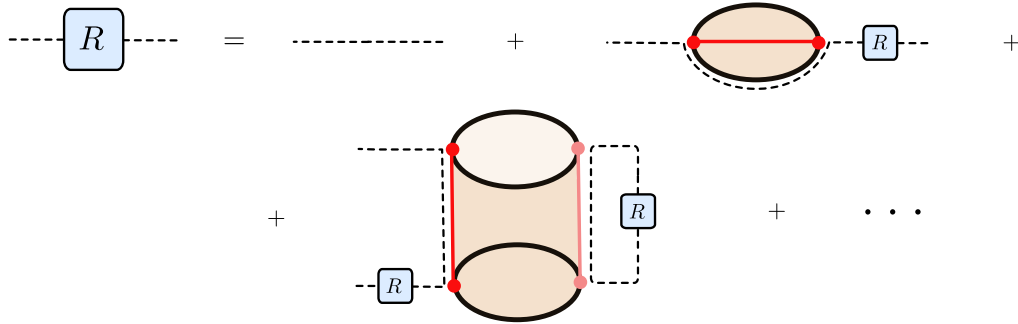


FIG. 11. Schwinger-Dyson equation for the resolvent, in the leading semiclassical approximation. This figure is a self-consistent reorganization of Fig. 10.

$$\overline{R_{ij}(\lambda)} = \frac{1}{\lambda} \delta_{ij} + \frac{1}{\lambda} \sum_{n=1}^{\infty} \frac{Z_n}{Z_1^n} \overline{R(\lambda)^{n-1} R_{ij}(\lambda)}, \quad (4.11)$$

$$Z(n\beta)^2 = \left(\int dE z(E) e^{-n\beta E} \right)^2, \quad (4.13)$$

where $R(\lambda)$ is the trace of the resolvent. Taking the trace of this equation, we arrive at

$$\lambda \overline{R(\lambda)} = \Omega + \sum_{n=1}^{\infty} \frac{Z_n}{Z_1^n} \overline{R(\lambda)^n}. \quad (4.12)$$

In order to count black hole microstates for a given energy from these overlaps, i.e., the microcanonical degeneracy, we have to first account for the fact that any entry in the Gram matrix is actually a sum over microcanonical windows in the energy basis [111]. This is because we constructed fixed-temperature, rather than fixed-mass, microstates. Thus, to proceed, we must project our states via an inverse Laplace transform into the microcanonical window around energy E ,

where $z(E)$ is the inverse Laplace transform of $Z(n\beta)$. The square appearing in the previous and subsequent formulas comes from the fact that we are considering eternal black holes, and the entropies or microcanonical degeneracies will be doubled since we have two copies of the same quantum gravity theory.

As in Ref. [73], we now define the functions

$$e^{\mathcal{S}} \equiv (z(E) \Delta E)^2, \quad \mathbf{Z}_n \equiv (z(E) e^{-n\beta E} \Delta E)^2. \quad (4.14)$$

Inserting these expressions after projecting Eq. (4.12) into the microcanonical band, we obtain the simple result

$$\lambda \overline{R(\lambda)} = \Omega + e^{\mathcal{S}} \sum_{n=1}^{\infty} \left(\frac{\overline{R(\lambda)}}{e^{\mathcal{S}}} \right)^n = \Omega + \frac{e^{\mathcal{S}} \overline{R(\lambda)}}{e^{\mathcal{S}} - \overline{R(\lambda)}}, \quad (4.15)$$

leading to a quadratic equation for the resolvent,

$$\overline{R(\lambda)}^2 + \left(\frac{e^{\mathbf{S}} - \Omega}{\lambda} - e^{\mathbf{S}} \right) \overline{R(\lambda)} + \frac{\Omega}{\lambda} e^{\mathbf{S}} = 0. \quad (4.16)$$

As we said above, the density of states follows from the discontinuity across the real axis of the trace of the resolvent of the Gram matrix [Eq. (4.8)] as computed in semiclassical gravity. This process finally gives

$$\overline{D(\lambda)} = \frac{e^{\mathbf{S}}}{2\pi\lambda} \sqrt{[\lambda - (1 - \Omega^{1/2} e^{-\mathbf{S}/2})^2][(1 + \Omega^{1/2} e^{-\mathbf{S}/2})^2 - \lambda]} + \delta(\lambda)(\Omega - e^{\mathbf{S}})\theta(\Omega - e^{\mathbf{S}}), \quad (4.17)$$

where $\mathbf{S} = A/4G$. Ultimately, the value $A/4G$ arises from evaluation of the gravitational action of the wormhole contribution to the overlap moments (4.13) and (4.14). This density of states has a continuous part and a singular part. The continuous part is supported on

$$(1 - \Omega^{1/2} e^{-\mathbf{S}/2})^2 < \lambda < (1 + \Omega^{1/2} e^{-\mathbf{S}/2})^2. \quad (4.18)$$

The eigenvalues accounted for by this part are all positive definite because this part is continuous and the measure at $\lambda = 0$ is zero [112]. The singular part counts the number of zero eigenvalues. It only appears when $\Omega > e^{\mathbf{S}}$ due to the Heaviside factor. Regardless, the number of positive eigenvalues, i.e., the rank of the Gram matrix, is accounted for by the continuous part of the distribution.

Thus, by separately integrating the continuous and singular parts of the eigenvalue distribution, we find the following:

- (i) For $\Omega < e^{\mathbf{S}}$, where \mathbf{S} is the Bekenstein-Hawking entropy, the Gram matrix G has no zero eigenvalues. The number of nonzero eigenvalues, i.e., the rank of G , equals the dimension spanned by the black hole microstates $|\Psi_p\rangle$ and is given by Ω . These statements are true, on average, in the effective random matrix ensemble for G that gravity provides.
- (ii) For $\Omega > e^{\mathbf{S}}$, the Gram matrix G has $\Omega - e^{\mathbf{S}}$ zero eigenvalues on average. The number of nonzero eigenvalues, i.e., the rank of G , equals the dimension spanned by the black hole microstates $|\Psi_p\rangle$ and is given by $e^{\mathbf{S}}$.
- (iii) The black hole microstate degeneracy, equal to the number of possible orthogonal states in a given energy band, is $e^{\mathbf{S}} = (z(E)\Delta E)^2$, equal to the exponential of the Bekenstein-Hawking entropy.

All of these statements are valid, on average, in the effective ensemble provided by semiclassical gravity. However, the variances associated with the densities of eigenvalues in random matrix theory [113] are suppressed by the dimension of the associated matrix. In this case, they will be suppressed by factors of $e^{-\mathbf{S}}$. It will be interesting to

compute the effects of these subleading corrections to the Bekenstein-Hawking entropy.

Intuitively, if we keep adding potential microstates to a system, there is a point at which these states cannot be orthogonal anymore. In our case, this point is controlled by the universal statistics of the inner product displayed in Sec. III, which is, in turn, controlled by the Bekenstein-Hawking entropy. Thus, the solution to the problem of understanding the entropy of general black holes is not to construct a specific set of $e^{\mathbf{S}}$ microstates. Indeed, there may be infinite numbers of such sets, even when they are constrained to be semiclassical and geometrical. The problem is really to show that any such choice gives rise to the same Hilbert space with the right Bekenstein-Hawking dimension.

We have proved the linear dependence of the semiclassical geometrical microstates that we examined. Still, we have not proved the completeness of the basis generated from these states. The basis we generate of course has the right dimension to explain the Bekenstein-Hawking entropy and is complete in its span. However, one could ask whether there are states in the black hole that cannot be expanded in this basis. If that occurs, the true entropy of the black hole is greater than the Bekenstein-Hawking entropy. We have derived the black hole degeneracy at leading order. However, there are subleading corrections to our results, and some of these could correspond to additional states that cannot be expanded in this way. These corrections will be nonperturbatively suppressed, and we expect that adding them could modify the Hilbert space dimension at subleading order in the $G \rightarrow 0$ expansion. That said, we indeed expect most states to be expandable in the basis generated from our shell states. For example, we could take the thermofield double (TFD) state without any shell, along with the small excitations around this background. These states again have nonzero, exponentially suppressed overlaps with all shell states that we have considered. This overlap can be computed by the same methods as in previous sections. The dominant contribution will come from a wormhole saddle with one shell behind the horizon. Thus, adding the TFD family of states to the Gram matrix will not increase its rank. Similarly, other semiclassical black hole microstates can be understood as superpositions of our dust-shell states, provided that the characteristic overlaps, given by wormhole configurations, have the correct magnitude.

We could have similarly considered microstates with inhomogeneities, microstates rotating with some angular momentum, or charged microstates. However, generically in statistical mechanics, such states make subleading contributions to the entropy. For example, very much like the shells of dust we are considering, most states of the particles of a gas in a room are approximately homogeneously spread over the volume. Indeed, configurations with inhomogeneities are suppressed in their phase-space

volume. The same is true if the system has, say, electric charge. For a given energy, there will be many more configurations of the system with equal numbers of positive and negative charges, and hence vanishing total charge. Similarly, at a given energy, there are also many more configurations with net vanishing rotation. These arguments can be made more precise in our context by considering an ensemble of shells in which the probability of having a dust particle in a certain position is constant along the sphere and in which the probability of having a dust particle with positive or negative electric charge is the same. This ensemble now contains microstates with inhomogeneities and charge, but average quantities such as the entropy will still be dominated by the spherically symmetric shells. Similar considerations also apply to dust-shell configurations with net angular momentum. Note also that if a shell has Planck-sized inhomogeneities, it will be semiclassically described by the same symmetric shells that we have studied.

Finally, we notice that subleading-in- G corrections to the black hole entropy can be easily incorporated using these methods. In particular, a class of corrections at next-to-leading order arises from the one-loop corrections to the overlaps of the thin-shell microstates in the large mass limit. Another class of corrections arises from the wormhole variance to the computation of the rank of the Gram matrix. The former will depend on the particular gravitational theory in question. For theories including charged black holes in the near-extremal limit, it has been recently understood that these corrections will universally become important at very low temperatures, and they will drive the density of states to zero for nonsupersymmetric black holes [114,115]. For BPS black holes, methods similar to the ones presented here have appeared in Ref. [116] to account for the extremal degeneracy. In Ref. [117], these quantum corrections to our construction for generic black holes will be described in detail.

C. Evaporating black holes and the Page curve

References [73,118] provide a proof of the semiclassical island formula [119–121] that reproduces the Page curve [122,123]. One of these scenarios, often called the “West Coast model” [73], allows detailed computation in the context of 2D JT gravity [124]. Meanwhile, the “East Coast model” [118] presents a general argument, based on gravitational path integrals in general dimensions. Some aspects of the West Coast model have been extended to three [126] and higher [52] dimensions as well.

Here, we have reproduced the advantages of the West Coast model within general relativity in general dimensions without including unknown degrees of freedom, thus giving a finer microscopic understanding of the East Coast model. In particular, the West Coast model assumes the existence of EOW branes equipped with somewhat mysterious color degrees of freedom. Microscopically, this assumes the

existence of k interior black hole microstates $|\psi_i\rangle_{\mathbf{B}}$ that are orthogonal to each other, or at least only have non-perturbatively small overlaps. At the same time, one assumes that the radiation of the black hole leaks into an external reservoir with the basis of states given by $\{|i\rangle_{\mathbf{R}}\}$. The full state of the system, composed of the black hole and the radiation reservoir, can be Schmidt decomposed into the form

$$|\Psi\rangle = \frac{1}{\sqrt{k}} \sum_{i=1}^k |\psi_i\rangle_{\mathbf{B}} |i\rangle_{\mathbf{R}}. \quad (4.19)$$

The entanglement entropy of the radiation is computed from the reduced density matrix

$$\rho_{\mathbf{R}} = \frac{1}{k} \sum_{i,j=1}^k |j\rangle\langle i|_{\mathbf{R}} \langle\psi_i|\psi_j\rangle = \frac{1}{k} \sum_{i,j=1}^k |j\rangle\langle i|_{\mathbf{R}} G_{ij} = \frac{G}{k}, \quad (4.20)$$

where G_{ij} is the Gram matrix of the microstates with an EOW brane inside the black hole. The connection between this model of black hole evaporation and our approach to black hole microstate counting is now clear. At the end of the day, both computations depend on the eigenvalue statistics of the Gram matrix of interior microstates. To relate our results to the analysis of the Page curve, we need to make the following associations:

$$k \rightarrow \Omega, \quad \rho_{\mathbf{R}} = \frac{G}{k} = \frac{G}{\Omega}, \quad \lambda_{\rho_{\mathbf{R}}} = \frac{\lambda_G}{\Omega}, \quad D(\lambda_{\rho_{\mathbf{R}}})_{\rho_{\mathbf{R}}} = \Omega D(\lambda_G), \quad (4.21)$$

where Ω is the dimension of the Gram matrix, $\lambda_{\rho_{\mathbf{R}}}$ and λ_G are eigenvalues of the density matrix $\rho_{\mathbf{R}}$ and the Gram matrix G , respectively, and D are the eigenvalue densities. These results lead to the following conclusions:

- (i) Our approach extends the West Coast model—in particular, the derivation of the density of states and Page curve—to general relativity in general dimensions. Everything follows from the universal overlaps between black hole microstates derived in Sec. III.
- (ii) Our approach clarifies the West Coast model by considering physical microstates, such as dust shells with different masses, which do not require the inclusion of mysterious EOW branes with novel degrees of freedom. In the West Coast model, the EOW brane states are taken to be naively orthogonal; then, wormhole contributions determine a nonperturbative overlap. In our case, the shells are classically orthogonal but also have a nonperturbative overlap.
- (iii) Our approach makes it clear that any basis, such as the EOW branes with color, cannot have infinitely many orthogonal members and that the true dimension of the Hilbert space is given by the Bekenstein-Hawking degeneracy.

These points have important implications. Concerning the extension of the West Coast model to a full theory of gravity, the authors of Ref. [127] argued that such resolutions of the Page curve puzzle might fail in theories with long-range gravity, such as general relativity itself [128]. Our results achieve such an extension to general relativity and help to demystify the nature of the degrees of freedom needed to achieve a resolution of the Page curve conundrum. The universality of gravitational dynamics implies that any complete, or indeed overcomplete, basis will work.

V. EINSTEIN-ROSEN VOLUME SATURATION AND COMPLEXITY

The authors of Refs. [18,19] conjectured that the volume of Einstein-Rosen (ER) bridges is related to the “quantum complexity” [20] of the underlying state. In classical gravity, these volumes grow linearly with asymptotic time, and the conjecture therefore predicts that “complexity” will increase linearly with time as well. However, the complexity of any circuit is bounded by the dimension of the Hilbert space on which the circuit acts (Chapter 7 of Ref. [20] and discussion in Ref. [21]). Thus, the conjecture makes a second prediction: The volume of an ER bridge must saturate in quantum gravity at a value exponential in the black hole entropy, i.e., at $\mathcal{O}(e^{S_{\text{BH}}})$. Clearly, no such saturation is observed semiclassically. One possibility is that we simply cannot make sense of semiclassical physics for times exponentially long in the entropy because the states cease to be geometrical by some still-undiscovered quantum effect [130]. Alternatively, the geometric volume may stop being related to complexity at an exponential time. As we will see below, our results suggest that the geometric volume, whatever its relation to complexity, cannot be measured as a linear operator in the theory since we will show that long wormholes can be written as a superposition of short wormholes.

A. Interior geometry of the microstates

We start by computing volumes of the Einstein-Rosen bridges for the families of shell states described in Sec. II. We also characterize the geometry of the python’s lunch within these bridges, i.e., regions of maximal transversal area. These quantities are conjectured to relate to the complexity of the underlying quantum state [19] and the complexity of interior reconstruction [40], respectively.

Using the expressions in Sec. II, the volume at time $t = 0$ in between the horizons for a geometry with a shell of mass m is

$$V(|\Psi_m\rangle) = 2V_\Omega \int_{r_+}^{R_*} \frac{r^{d-1} dr}{\sqrt{f_+(r)}}, \quad (5.1)$$

where R_* is the minimum radius to which the shell arrives. Here, V_Ω is the volume of the transverse sphere. According

to Ref. [19], the relative complexity between this state and the TFD is

$$\begin{aligned} \mathcal{C}(|\Psi_m\rangle) &= \frac{d-1}{8\pi G\ell} V(|\Psi_m\rangle) \\ &= \frac{d-1}{8\pi G\ell} (\text{Vol}(\Sigma) - \text{Vol}(\Sigma_0)), \end{aligned} \quad (5.2)$$

where Σ is the reflection-symmetric Cauchy slice in the geometry associated with $|\Psi_m\rangle$, and Σ_0 is the corresponding Cauchy slice of the eternal Schwarzschild black hole. Since the spacetime is the same outside the black hole, the relative complexity is associated with the stretching of the Einstein-Rosen bridge, given by Eq. (5.1).

For $d = 2$, we can compute the integral explicitly,

$$\mathcal{C}(|\Psi_m\rangle) = \frac{(R_*^2 - r_+^2)^{1/2}}{2G} = m\ell = nm_\Delta\ell = n\sqrt{\Delta(\Delta - d)}, \quad (5.3)$$

from the form of R_* in Eq. (2.18). This gives a strikingly simple relation between the Einstein-Rosen volume and the mass-scaling dimension for this family of states, supporting the proposal in Ref. [131], verified by different means in Refs. [132,133], relating quantum complexity and scaling dimensions in conformal field theories.

In higher dimensions, the relation between the volume and mass of the shell is more involved. However, for large enough shell masses, we obtain

$$R_*^{d-1} \approx \frac{4\pi Gm\ell}{(d-1)V_\Omega}. \quad (5.4)$$

Using this expression in Eq. (5.2), we obtain

$$\mathcal{C}(|\Psi_m\rangle) \approx \frac{V_\Omega R_*^{d-1}}{4\pi G} = \frac{m\ell}{d-1}, \quad (5.5)$$

again arriving at a simple relation with the scaling dimension.

For small masses, on the other hand, the solution to $V_{\text{eff}}(R_*) = 0$ in Eq. (2.13) satisfies

$$R_* \approx r_+ + \frac{\beta_+}{4\pi} \left(\frac{4\pi Gm}{(d-1)V_\Omega r_+^{d-2}} \right)^2. \quad (5.6)$$

In this regime, Eq. (5.2) gives

$$\mathcal{C}(|\Psi_m\rangle) \approx \frac{\beta_+ r_+ m}{4\pi\ell}. \quad (5.7)$$

Although still linear in the mass of the perturbation, the slope now depends on parameters of the original black hole.

1. Characterization of the python

The interior geometry of $|\Psi_m\rangle$ is a classical python's lunch, a geometry with a cross-sectional bulge at the position of the shell, $r = R_*$. For large $m \gg M$, this bulge has an area

$$S_b = \frac{\text{Area}_b}{4G} = \frac{V_\Omega R_*^{d-1}}{4G} \approx \frac{\pi}{d-1} m \ell. \quad (5.8)$$

Thus, most of the volume of the ER bridge comes from the region close to the bulge since the geometry is locally hyperbolic. To see this quantitatively, we can compute the proper length of the ER bridge,

$$L = 2 \int_{r_+}^{R_*} \frac{dr}{\sqrt{f_+(r)}} \approx 2\ell \log \frac{R_*}{r_+} \approx \frac{2\ell}{d-1} \log m \ell, \quad (5.9)$$

so that

$$S_b \approx \frac{\pi}{d-1} e^{\frac{d-1}{2}L} \quad \text{and} \quad \mathcal{C}(|\Psi_m\rangle) \approx \frac{1}{d-1} e^{\frac{d-1}{2}L}, \quad (5.10)$$

which is what we expect from the hyperbolic geometry of the black hole interior. Note that in order to have an exponentially large complexity in ℓ^{d-1}/G , the length of the wormhole need only scale linearly with ℓ^{d-1}/G . Thus, the single-shell pythons are relatively thick and short hyperbolic geometries. To build thin and long pythons, more similar to cylinders, we can concatenate multiple shells with reasonably low values of m . This model is depicted in Fig. 12.

2. Multiple shells

To construct long wormholes, we consider microstates with k shells $|\Psi_{\mathbf{m}}\rangle$ for $\mathbf{m} = (m_1, \dots, m_k)$. The interior volume is additive,

$$\begin{aligned} \mathcal{C}(|\Psi_{\mathbf{m}}\rangle) &= \sum_{i=1}^k \frac{(d-1)V_\Omega}{4\pi G \ell} \int_{r_+}^{R_*^i} \frac{r^{d-1} dr}{\sqrt{f_+(r)}} \\ &\approx \sum_i \frac{m_i \ell}{d-1}. \end{aligned} \quad (5.11)$$

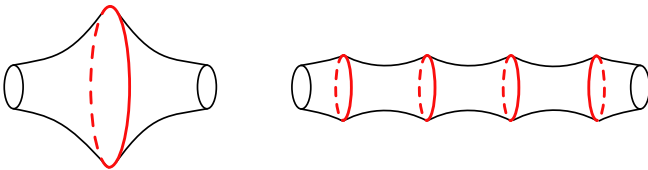


FIG. 12. Geometry of the Einstein-Rosen bridge for the microstates. The single-shell states with large m contain large portions of a hyperbolic geometry in the interior, while the multishell states can have a large volume while keeping the spherical section relatively small.

For $d = 2$, we again have

$$\mathcal{C}(|\Psi_{\mathbf{m}}\rangle) = \sum_{i=1}^k m_i \ell. \quad (5.12)$$

The python now contains multiple lunches at R_*^i (see Ref. [45]), with associated entropy

$$S_{b_i} = \frac{\text{Area}_{b_i}}{4G} = \frac{V_\Omega (R_*^i)^{d-1}}{4G} \approx \frac{\pi}{d-1} m_i \ell \quad (5.13)$$

and total length

$$L = \sum_i 2 \int_{r_+}^{R_*^i} \frac{dr}{\sqrt{f_+(r)}} \approx \sum_i \frac{2\ell}{d-1} \log m_i \ell. \quad (5.14)$$

Thus, we can build wormholes with high volumes and small bulges by considering many shells, each with $m_i \ell \sim \ell^{d-1}/G_N$.

3. Summary

Since the mass (or, equivalently, the dimension) of the shell is unbounded from above, we conclude that we can construct well-controlled geometrical states with Einstein-Rosen bridges of any size. In particular, there is no evident saturation of either the wormhole volume or the size of the python's lunch, both of which can be superexponential in the black hole entropy already at $t = 0$. Thus, superexponential wormholes need not be obstructed by quantum effects, as they can be classically well defined.

B. Saturation of Einstein-Rosen bridges

We now restrict ourselves, without loss of generality, to the one-shell states $|\Psi_p^\beta\rangle$ with masses $m_p = pm$, where $p = 1, 2, \dots$ and $m \sim \mathcal{O}(\ell^d/G)$. We are now explicitly labeling the states with the inverse temperature β of the black hole, with associated ADM energy E .

In Sec. IV, we have shown that, when projected onto the microcanonical window $[E, E + \Delta E]$ associated with the black hole, using the orthogonal projector Π_E , the unnormalized states

$$|\Psi_p^E\rangle = \Pi_E |\Psi_p^\beta\rangle, \quad (5.15)$$

for $p = 1, \dots, e^{S(E)}$, likely generate a basis of the microcanonical black hole Hilbert space

$$\mathcal{H}_E = \text{Span}\{|\Psi_p^E\rangle : p = 1, \dots, e^{S(E)}\}. \quad (5.16)$$

All of these basis elements have shell masses (and hence wormhole sizes) at most exponential in entropy and are thus “short” wormholes. The microcanonical Hilbert space dimension is set by the Bekenstein-Hawking entropy

$$e^{S(E)} \equiv (z(E)\Delta E)^2. \quad (5.17)$$

Now, consider a one-shell state $|\Psi_q^\beta\rangle$, which has a super-exponential Einstein-Rosen bridge, $q \gg e^{S(E)}$. Evidently, its projection into the microcanonical window can be expanded in the previous basis built out of shorter wormholes,

$$q > e^{S(E_i)} \rightarrow |\Psi_q^E\rangle = \sum_{p=1}^{e^{S(E)}} c_p |\Psi_p^E\rangle. \quad (5.18)$$

The wave function of the full state $|\Psi_q^\beta\rangle$ that we prepared in the previous section will, however, spread into different microcanonical windows, and if we want to generate it with short wormholes, we will need to consider states $|\Psi_p^{E_i}\rangle$ associated with different energy windows E_i . These states are constructed from projections of short wormhole states with different temperatures $|\Psi_p^{\beta_i}\rangle$ into microcanonical windows, $\Pi_{E_i}|\Psi_p^{\beta_i}\rangle$, specified by the ADM masses E_i of the corresponding large black holes. Therefore, we can expand the state as

$$|\Psi_q^\beta\rangle = \sum_i \sum_{p=1}^{e^{S(E_i)}} c_p^i |\Psi_p^{E_i}\rangle. \quad (5.19)$$

Equivalently, we can also write the state in a basis of fixed-temperature states. These states are linearly independent when projected to the corresponding microcanonical windows; thus, they are also linearly independent in the infinite-dimensional Hilbert space. Making the change of basis, we obtain

$$|\Psi_q^\beta\rangle = \sum_i \sum_p^{e^{S(E_i)}} \alpha_p^i |\Psi_p^{\beta_i}\rangle \quad (5.20)$$

in terms of some new coefficients α_p^i . In this last expression, we have explicitly written the large wormhole as a linear superposition of *geometric* short wormhole states $|\Psi_p^{\beta_i}\rangle$ for different temperatures of the black hole. The wave function of the long wormhole in the short wormhole basis, α_p^i , will follow an approximately thermal distribution, so the sum (5.20) is dominated by short wormholes with the temperature β . To see this, one can explicitly make the computation $\alpha_p^i \approx \langle \Psi_q^\beta | \Psi_p^{\beta_i} \rangle$. The overlap is given in terms of a generalization of the two-boundary Euclidean wormhole of Sec. III, for states with different temperatures. In the large mass limit, repeating the calculation that led to Eq. (3.33) shows that the overlap is independent of p , and it is given by the wormhole contribution

$$|\alpha_p^i| \sim \frac{Z(\beta + \beta_i)}{Z(\beta)Z(\beta_i)}. \quad (5.21)$$

For large temperatures $\beta_i \ll \beta$, the coefficients α_p^i are suppressed by the free energy. For small temperatures $\beta_i \gg \beta$, the amplitude becomes constant, of value $Z(\beta)^{-1}$.

Thus, the wormhole state $|\Psi_q^\beta\rangle$ has most of its support around the microcanonical energy window, defined by the equation of state that relates the exterior temperature β and the mass of the black hole M_{BH} , up to exponentially decaying Gibbs tails in the distribution [134]. We conclude that, when $q > e^{S(M_{\text{BH}})}$, i.e., for superexponential values, the shell states can be written as complicated linear superpositions of, at most, exponentially large wormholes. This finding demonstrates a sense in which the volume of Einstein-Rosen bridges is indeed bounded, and it relates this bounding quantum mechanically to the finiteness of black hole entropy.

C. Concerning volume operators

These considerations make it clear that the volume of the interior cannot be measured by a linear operator in quantum mechanics. To sharpen this point, we can try to define a “volume operator” in the naive Hilbert space of infinite dimension spanned by the family of thin-shell states by assigning a geometric volume to each of them [135]:

$$\hat{V}_{\text{naive}}|\Psi_p\rangle = V_p|\Psi_p\rangle, \quad p \in \{1, \dots, \infty\}, \quad (5.22)$$

where the V_p on the right side is the classically computed volume. However, we have shown that the fundamental Hilbert space of the black hole is spanned by a finite number of these states, in fact, by precisely e^S of them. Therefore, it is clear that the observable \hat{V}_{naive} in Eq. (5.22) cannot exist in the fundamental description of the black hole since there cannot be so many linearly independent eigenstates. We could try to remedy this by defining a linear operator restricted to subexponential states

$$\hat{V}|\Psi_p\rangle = V_p|\Psi_p\rangle, \quad p \in \{1, \dots, e^S\}. \quad (5.23)$$

However, if $q \geq e^S$, this linear operator obviously fails to provide the geometric volume of the state,

$$\langle \Psi_q | \hat{V} | \Psi_q \rangle \approx \sum_{p=1}^{e^S} |\alpha_p|^2 V_p \neq V_q. \quad (5.24)$$

This finding suggests that the geometric volume of the interior states of the black hole must be encoded in a nonlinear or state-dependent way in the fundamental description. Such a semiclassical volume function could still exist in the Hilbert space of the black hole. In the next section, we describe linear volume operators related to sensible notions of state complexity that are correspondingly bounded above.

D. Toy model of interior growth and saturation of complexity

We have shown that the superexponential wormhole states can be written as superpositions of exponential ones. With this in mind, in this section, we will construct a toy model describing a time evolution between wormholes of different length. We will then define a notion of volume in these universes, which is related to the ‘‘spread complexity’’ of the underlying state in a basis of volume eigenstates, recently introduced in Ref. [41] (reviewed in Appendix C). This notion of volume and spread complexity will saturate at values exponential in the entropy [136].

We start with the family of one-shell states used above, which are

$$|\Psi_p\rangle, \quad m_p = pm, \quad m \sim \mathcal{O}(\ell^{d-2}/G). \quad (5.25)$$

As proved previously, projecting these states to energy windows E_i of size ΔE gives rise to finite-dimensional Hilbert spaces,

$$\mathcal{H}_i = \text{Span}\{|\Psi_p^i\rangle : p = 1, \dots, e^{S(E_i)}\}. \quad (5.26)$$

In this scenario, we can create a toy model of black hole interior growth by defining, for each window of energy E , a ‘‘simple’’ hopping Hamiltonian that produces transitions between subsequent shell states of growing volume. This method is shown schematically in Fig. 13. At late times, the wave function will be completely spread out over the subexponential wormhole microstates, but the amplitudes will keep evolving with time. We showed that all the classically superexponential wormholes are superpositions of these subexponential ones, which means that detailed differences between amplitudes in the superposition, including the phases, must be involved in determining the classical length. The details of this will depend on the choice of the hopping parameters, a question that we will leave for future work.

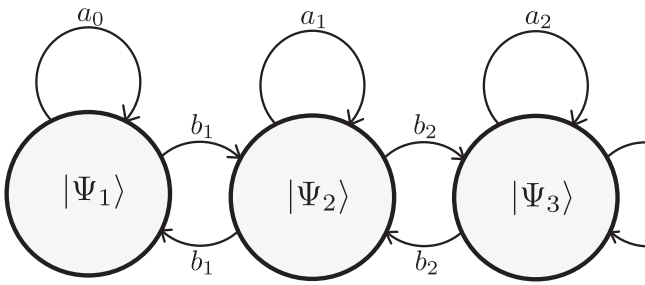


FIG. 13. Simple Hamiltonian that effects the transition between subsequent shell states. This Hamiltonian is simple since it acts on $\mathcal{O}(S)$ shell particles at a given time.

The Hamiltonian takes the form [137]

$$H = \begin{pmatrix} a_0 & b_1 & & & & & & & \\ b_1 & a_1 & b_2 & & & & & & \\ & b_2 & a_2 & b_3 & & & & & \\ & & \ddots & \ddots & \ddots & & & & \\ & & & \ddots & \ddots & \ddots & & & \\ & & & & b_{e^{S(E)}-2} & a_{e^{S(E)}-2} & b_{e^{S(E)}-1} & & \\ & & & & & b_{e^{S(E)}-1} & a_{e^{S(E)}-1} & & \end{pmatrix}, \quad (5.27)$$

for some choice of a and b . This Hamiltonian is ‘‘simple’’ because it acts on small groups of the dust particles making up the shells at any given time, thereby locally changing the dust density and the wormhole volume. Starting from the state with the shortest wormhole, namely, $|\Psi_1^E\rangle$, time evolution with this Hamiltonian will make the expected wormhole size grow since the wave function will move along the ‘‘1D chain.’’

We study the spread complexity of this evolution. As explained in detail in Appendix C, this is the minimum spread of the wave function over all choices of basis of the Hilbert space. The solution to this minimization is the Krylov basis. For a given initial state $|\Psi\rangle$ and a Hamiltonian H , the Krylov basis is the one that arises from the Gram-Schmidt procedure applied to $H^n|\Psi\rangle$. The Krylov basis is simple to compute in our scenario, in which we start with $|\Psi_1^E\rangle$ and evolve with the tridiagonal Hamiltonian H because the Krylov basis is just the basis of the shell states themselves, $|\Psi_p^E\rangle$. The wave function can be expanded in this basis,

$$|\Psi_1(t)\rangle = \sum_p \psi_p(t) |\Psi_p^E\rangle. \quad (5.28)$$

In terms of the probabilities of each of the shell states as time evolves, spread complexity can be computed as

$$C_{\text{spread}}(t) = C_{\mathcal{K}}(t) = \sum_p p |\psi_p(t)|^2. \quad (5.29)$$

This notion of complexity is just the expectation value of the position operator along the 1D chain. Since the position in the chain is the mass of the shell in appropriate units and the classical volume is proportional to the mass as well, the spread complexity and the expected Einstein-Rosen volume in the shell basis match. For example, we can use the precise analytical formulas in $d = 2$. When acting on shell states, we thus identify the position operator [138] in the chain with a linear volume operator as

$$\hat{p}|\Psi_p\rangle = p|\Psi_p\rangle, \quad \hat{C} \equiv \frac{d-1}{8\pi G\ell} \hat{V}|\Psi_p\rangle = pm\ell|\Psi_p\rangle. \quad (5.30)$$

Now, we equate the spread complexity position operator to a “holographic complexity operator” as

$$\hat{C} = m\ell\hat{p}. \quad (5.31)$$

Tridiagonal Hamiltonians such as Eq. (5.27) were studied extensively in quantum chaotic models in Refs. [41,42]. Following that work, the spread complexity will increase for an exponentially long time and saturate at an exponentially large value. By construction in our toy model, the average wormhole volume as measured by the linear volume operator defined above will also show this behavior. Meanwhile, the state itself can continue to evolve with time, and the classical volume, no longer simply related to the volume operator we constructed, or to the complexity of time evolution, could continue to increase.

VI. SUMMARY AND DISCUSSION

The key insight in our approach is that, to understand the microscopic origin of black hole entropy, it is enough to construct any well-controlled and sufficiently large space of states. The rank of the matrix of overlaps between these states then reveals the Hilbert space dimension. We have shown that there are many families of such microstates that are geometrical and under semiclassical control. These states are highly atypical in the Hilbert space and do not have the standard Schwarzschild interior. Nevertheless, they are sufficient to demonstrate that the Bekenstein-Hawking entropy can be explained as the dimension of an underlying quantum Hilbert space of the black hole.

Our results followed from the appearance of quantum wormholes that contribute to the overlaps between apparently distinct classical states. These are nonperturbative effects in quantum gravity. We also showed that all our results could be understood phenomenologically by assuming that the geometric configurations we construct are random phase superpositions of energy eigenstates.

An outstanding question is as follows: Why is the semiclassical gravitational path integral so useful? It can compute the black hole entropy [139], and moreover, it appears to resolve some aspects of the information paradox, such as the expected decay of the Page curve at late times in black hole evaporation. At first sight, this seems very unlikely because both questions seem to require access to the complete space of microstates. Here, we have shown that there are enough states under semiclassical control to give access to the dimension of the full Hilbert space. This finding makes it possible for the semiclassical path integral to correctly answer questions that depend on the Hilbert space dimension, like the entropy of black holes or the decay of the Page curve.

Finally, our results do not explicitly use any details of string theory, AdS/CFT, or any other formulation of quantum gravity. Indeed, the only assumptions we have

used are as follows: (a) There is some ultraviolet completion, and (b) the semiclassical Euclidean path integral provides sensible information about the ultraviolet completion. With these assumptions, our results provide an explanation for the entropy of black holes in any theory that has general relativity coupled to massive matter as a low-energy limit. In particular, this construction works in specific top-down models of quantum gravity, such as those appearing in string theory and AdS/CFT since these theories contain the necessary ingredients to construct our microstates. Perhaps this explains the universality of the Bekenstein-Hawking entropy formula. We have worked in universes with a negative cosmological constant because there are additional tools in this case that allow us to put our state-construction methods on an entirely firm footing. However, there are no obstacles to repeating the analysis in, for example, asymptotically flat spacetimes [141].

ACKNOWLEDGMENTS

We would like to thank José Barbón, Horacio Casini, and Roberto Emparan for useful discussions. V. B. and J. M. are supported in part by the Department of Energy through DE-SC0013528 and QuantISED DE-SC0020360, as well as by the Simons Foundation through the It From Qubit Collaboration (Grant No. 38559). A. L. and M. S. are supported in part by the Department of Energy through DE-SC0009986 and QuantISED DE-SC0020360.

APPENDIX A: THIN-SHELL FORMALISM

In this appendix, we provide a review of the thin-shell formalism. The objective is to glue two Euclidean Schwarzschild-AdS regions X^\pm with local geometry

$$ds_\pm^2 = f_\pm(r)d\tau_\pm + \frac{dr^2}{f_\pm(r)} + r^2 d\Omega_{d-1}^2. \quad (A1)$$

The gluing of the geometries along the trajectory \mathcal{W} of a codimension-one domain is performed using Israel’s junction conditions. We denote by (h_{ab}^\pm, K_{ab}^\pm) the induced metrics h_{ab}^\pm and extrinsic curvatures K_{ab}^\pm in terms of the metric on each side X^\pm . These quantities are evaluated at \mathcal{W} . Denoting $\Delta h_{ab} = h_{ab}^+ - h_{ab}^-$ and $\Delta K_{ab} = K_{ab}^+ - K_{ab}^-$, the junction conditions are simply

$$\Delta h_{ab} = 0, \quad (A2)$$

$$\Delta K_{ab} - h_{ab}\Delta K = -8\pi G S_{ab}, \quad (A3)$$

where $\Delta K = h^{ab}\Delta K_{ab}$ [142], and S_{ab} is the energy-momentum of the thin domain wall. For a dust shell, we have $S_{ab} = -\sigma u_a u_b$, where the minus sign in S_{ab} comes from the analytic continuation $\sigma \rightarrow -\sigma$ in the Lorentzian fluid.

The junction conditions determine the motion for the shell, namely, $R = R(T)$, with T the synchronous proper time of the shell. First, the angular parts of Eq. (A3) impose the conservation of

$$m = \sigma V_{\Omega} R^{d-1} \quad (\text{A4})$$

along \mathcal{W} , which is the rest mass of the shell. Second, from the continuity of the metric (A2), we arrive at

$$f_{\pm} \dot{\tau}_{\pm} = \sqrt{-\dot{R}^2 + f_{\pm}}, \quad (\text{A5})$$

where we use the notation $\dot{x} = dx/dT$ and the square root can have either sign, giving different particular trajectories.

The remaining component of Eq. (A3) finally gives

$$\kappa_{+} \sqrt{-\dot{R}^2 + f_{+}(R)} - \kappa_{-} \sqrt{-\dot{R}^2 + f_{-}(R)} = \frac{8\pi Gm}{(d-1)V_{\Omega} R^{d-2}}, \quad (\text{A6})$$

where $\kappa_{\pm} = \text{sign}(\dot{\tau}_{\pm})$ is the sign of the extrinsic curvature. We now square this expression, getting an effective equation of motion for a nonrelativistic particle of zero total energy,

$$\dot{R}^2 + V_{\text{eff}}(R) = 0, \quad (\text{A7})$$

where the effective potential reads

$$V_{\text{eff}}(R) = -f_{+}(R) + \left(\frac{M_{+} - M_{-}}{m} - \frac{4\pi Gm}{(d-1)V_{\Omega} R^{d-2}} \right)^2. \quad (\text{A8})$$

In this nonrelativistic dynamics describing the trajectory of the shell in the Euclidean geometry, the shell starts at the boundary $R = r_{\infty}$ and bounces at $R = R_{*}$ with $V(R_{*}) = 0$. Notice that $R_{*} \geq r_{\pm}$ for the respective horizon radii. The time the shell takes to complete this motion can be computed from Eq. (A5). It is given by the integral

$$\Delta\tau_{\pm} = 2 \int_{R_{*}}^{\infty} \frac{dR}{f_{\pm}} \sqrt{\frac{f_{\pm} + V_{\text{eff}}}{-V_{\text{eff}}}}. \quad (\text{A9})$$

1. 2+1 dimensions

The $d = 2$ case can be worked out more explicitly. For the BTZ black hole, the event horizon radii are $r_{\pm} = \sqrt{8GM_{\pm}}$, the inverse temperatures are $\beta_{\pm} = 2\pi/r_{\pm}$, and the effective potential (A8) reads

$$V_{\text{eff}}(R) = -(r^2 - R_{*}^2), \quad (\text{A10})$$

where the turning point takes the explicit form

$$R_{*} = \sqrt{r_{+}^2 + \left(\frac{M_{+} - M_{-}}{m} - 2Gm \right)^2}. \quad (\text{A11})$$

The exact solution for the shell's motion in Eq. (A7) can be explicitly found,

$$R(T) = R_{*} \cosh T, \quad (\text{A12})$$

where we have chosen the initial conditions forcing the shell to pass through R_{*} at proper time $T = 0$. Finally, the Euclidean time elapsed by the shell (A9) during its motion can also be computed to give

$$\Delta\tau_{\pm} = \beta_{\pm} \frac{\arcsin(r_{\pm}/R_{*})}{\pi}. \quad (\text{A13})$$

APPENDIX B: HIGHER MOMENTS AND MULTIBOUNDARY WORMHOLES

In this appendix, we provide details of the semiclassical computation of higher moments of the overlaps for one-shell states. The connected part of the n th moment is

$$\overline{\langle \Psi_{m_1} | | \Psi_{m_2} \rangle \langle \Psi_{m_2} | | \Psi_{m_3} \rangle \dots \langle \Psi_{m_n} | | \Psi_{m_1} \rangle} |_{\text{c}} = \frac{Z_n}{Z_1^{m_1} \dots Z_1^{m_n}}, \quad (\text{B1})$$

where

$$Z_n = e^{-I[X_n]} \quad (\text{B2})$$

is the contribution from the gravitational action of the n -boundary wormhole X_n that we now construct. The normalizations $Z_1^{m_i}$ are the ones computed in Sec. III.

The wormhole X_n with n boundaries consists of a pair of Euclidean black holes of the same temperature β_n [143], which are glued together along the trajectory of the n thin shells. This case is shown in Fig. 14 for the sixth moment. The saddle-point equation reads

$$\beta_n = \sum_{i=1}^n (\tilde{\beta}_{m_i} + \Delta\tau_{m_i}). \quad (\text{B3})$$

In this equation, the n preparation temperatures $\tilde{\beta}_{m_i}$ have already been fixed by Eq. (3.8). Then, in the wormhole solution, the equation of motion (2.14) fixes the elapsed Euclidean time for each shell as a function of the mass of the shell and the background black hole geometry parametrized by β_n . The constraint (B3) then fixes β_n .

Having found this solution, the action of the wormhole after counterterm subtraction can be derived by the same logic as in Sec. III, giving

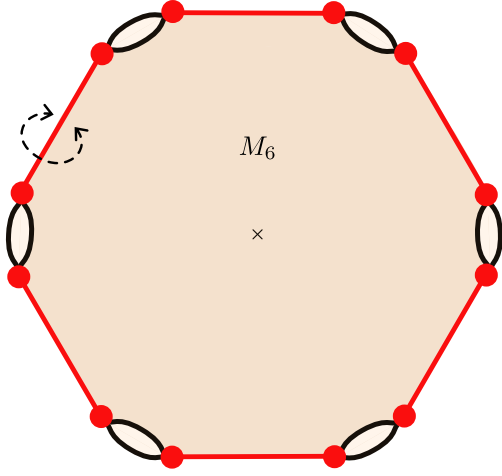


FIG. 14. Wormhole X_6 , consisting of a pair of Euclidean black holes of the same mass, M_6 , which are glued together along the trajectory of the six thin shells (red lines). The black holes are the boundaries of the geometry. The second black hole is behind the figure in this perspective, as indicated by the dashed arrows.

$$I[X_n] = 2 \left(\sum_{i=1}^n \tilde{\beta}_{m_i} \right) F(\beta_n) + \sum_{i=1}^n I[X_{\text{shell}}^{m_i}]. \quad (\text{B4})$$

Including the contributions from the normalizations Z_1 and Z'_1 , we obtain the semiclassical overlap squared,

$$\overline{\langle \Psi_{m_1} | \Psi_{m_2} \rangle \langle \Psi_{m_2} | \Psi_{m_3} \rangle \dots \langle \Psi_{m_n} | \Psi_{m_1} \rangle}_c = e^{-2(\sum_i \tilde{\beta}_{m_i} \Delta F - \sum_i \Delta I[X_{\text{shell}}^{m_i}]}, \quad (\text{B5})$$

where $\Delta F = F(\beta_n) - F(\beta)$ and $\Delta I[X_{\text{shell}}^{m_i}] = I[X_{\text{shell}}^{m_i}]|_{\beta_n} - I[X_{\text{shell}}^{m_i}]|_{\beta}$.

1. Large mass limit and universality

In the limit $m_i \gg M$ for all of the shells, the preparation temperature coincides with the physical temperature, $\tilde{\beta}_{m_i} \approx \beta$. Equation (B3) gives $\beta_n \approx n\beta$, and the moment (B5) reduces to

$$\overline{\langle \Psi_{m_1} | \Psi_{m_2} \rangle \langle \Psi_{m_2} | \Psi_{m_3} \rangle \dots \langle \Psi_{m_n} | \Psi_{m_1} \rangle}_c \approx \frac{Z(n\beta)^2}{Z(\beta)^{2n}}, \quad (\text{B6})$$

which is the main result for the overlap that we use in Sec. IV. It is straightforward to show that Eq. (B6) holds also for higher moments of the overlaps between general multishell states in the large mass limit.

APPENDIX C: SPREAD COMPLEXITY

In this appendix, we briefly review the notion of spread complexity recently introduced in Ref. [41]. Consider a time-independent Hamiltonian H . Time evolution of a initial state $|\psi\rangle$ is determined by the Schrödinger equation

$$i\partial_t |\psi(t)\rangle = H|\psi(t)\rangle. \quad (\text{C1})$$

A natural notion of quantum complexity arises from quantification of the spread of $|\psi(t)\rangle$ over the Hilbert space. To this end, we can define a cost function relative to a complete, orthonormal, ordered basis $\mathcal{B} = \{|B_n\rangle : n = 0, 1, 2, \dots\}$ for the Hilbert space

$$C_{\mathcal{B}}(t) = \sum_n n |\langle \psi(t) | B_n \rangle|^2 \equiv \sum_n n p_{\mathcal{B}}(n, t). \quad (\text{C2})$$

Unitarity of time evolution implies that the cost of a wave function increases if it spreads deeper into the basis. The question then arises as to which basis should be used to measure the spread. Inspired by other notions of complexity, such as Kolmogorov complexity [144], we now define spread complexity as the minimum over a finite time interval of this cost function over all bases \mathcal{B} ,

$$C(t) = \min_{\mathcal{B}} C_{\mathcal{B}}(t). \quad (\text{C3})$$

Reference [41] then shows that, under some assumptions, there is an essentially unique basis minimizing Eq. (C3) across a finite time domain. This basis follows the expansion

$$|\psi(t)\rangle = \sum_{n=0}^{\infty} \frac{(-it)^n}{n!} |\psi_n\rangle, \quad (\text{C4})$$

where we have defined $|\psi_n\rangle = H^n |\psi\rangle$. These states are neither orthogonal nor normalized. However, they generate an ordered, orthonormal basis $\mathcal{K} = \{|K_n\rangle : n = 0, 1, 2, \dots\}$, by means of the Gram-Schmidt procedure with the initial condition $|\psi(0)\rangle \equiv |K_0\rangle$. The basis \mathcal{K} is called the Krylov basis in the recent literature.

To calculate this notion of spread complexity, we must derive the Krylov basis \mathcal{K} . This is achieved via the Lanczos algorithm [145,146], which recursively applies the Gram-Schmidt procedure to $|\psi_n\rangle = H^n |\psi(0)\rangle$ to generate $\mathcal{K} = \{|K_n\rangle : n = 0, 1, 2, \dots\}$ in the following way:

$$|A_{n+1}\rangle = (H - a_n)|K_n\rangle - b_n|K_{n-1}\rangle, \quad |K_n\rangle = b_n^{-1}|A_n\rangle, \quad (\text{C5})$$

where a_n and b_n are dubbed the Lanczos coefficients

$$a_n = \langle K_n | H | K_n \rangle, \quad b_n = \langle A_n | A_n \rangle^{1/2}. \quad (\text{C6})$$

The initial conditions for this recursive generation are $b_0 \equiv 0$ and $|K_0\rangle = |\psi(0)\rangle$ as the initial state.

It is immediate that the Lanczos algorithm (C5) implies

$$H|K_n\rangle = a_n|K_n\rangle + b_{n+1}|K_{n+1}\rangle + b_n|K_{n-1}\rangle. \quad (\text{C7})$$

Equivalently, the Hamiltonian becomes a tridiagonal matrix in the Krylov basis. For finite-dimensional systems, like the

ones we consider in this paper, this representation is known as the ‘‘Hessenberg form’’ of the Hamiltonian:

$$H = \begin{pmatrix} a_0 & b_1 & & & & \\ b_1 & a_1 & b_2 & & & \\ & b_2 & a_2 & b_3 & & \\ & & \ddots & \ddots & \ddots & \\ & & & b_{N-2} & a_{N-2} & b_{N-1} \\ & & & & b_{N-1} & a_{N-1} \end{pmatrix}. \quad (\text{C8})$$

This finding leads to the interesting observation that every quantum dynamical evolution can be framed in terms of an appropriate one-dimensional hopping Hamiltonian.

To compute the spread complexity, we must expand the state in the basis in which the spread is minimized:

$$|\psi(t)\rangle = \sum_n \psi_n(t) |K_n\rangle. \quad (\text{C9})$$

Finally, given $\psi_n(t)$, we apply the definition of complexity in Eqs. (C2) and (C3):

$$C(t) = C_{\mathcal{K}}(t) = \sum_n n p_n(t) = \sum_n n |\psi_n(t)|^2. \quad (\text{C10})$$

This notion of spread complexity can be seen as a generalization of the notion of Krylov operator complexity, introduced in Ref. [147], to quantum states, now understood in terms of a minimization of the spread of the wave function over all choices of basis.

-
- [1] J. D. Bekenstein, *Black holes and entropy*, *Phys. Rev. D* **7**, 2333 (1973).
- [2] S. W. Hawking, *Particle creation by black holes*, *Commun. Math. Phys.* **43**, 199 (1975); **46**, 206(E) (1976).
- [3] S. W. Hawking, *Euclidean Quantum Gravity* (Springer, Boston, MA, 1979), pp. 145–173.
- [4] J. M. Maldacena, *The large N limit of superconformal field theories and supergravity*, *Adv. Theor. Math. Phys.* **2**, 231 (1998).
- [5] E. Witten, *Anti-de Sitter space and holography*, *Adv. Theor. Math. Phys.* **2**, 253 (1998).
- [6] S. S. Gubser, I. R. Klebanov, and A. M. Polyakov, *Gauge theory correlators from noncritical string theory*, *Phys. Lett. B* **428**, 105 (1998).
- [7] S. Ryu and T. Takayanagi, *Holographic derivation of entanglement entropy from AdS/CFT*, *Phys. Rev. Lett.* **96**, 181602 (2006).
- [8] A. Lewkowycz and J. Maldacena, *Generalized gravitational entropy*, *J. High Energy Phys.* **08** (2013) 090.
- [9] A. Strominger and C. Vafa, *Microscopic origin of the Bekenstein-Hawking entropy*, *Phys. Lett. B* **379**, 99 (1996).
- [10] I. Bena, E. J. Martinec, S. D. Mathur, and N. P. Warner, *Fuzzballs and microstate geometries: Black-hole structure in string theory*, [arXiv:2204.13113](https://arxiv.org/abs/2204.13113).
- [11] L. Bombelli, R. K. Koul, J. Lee, and R. D. Sorkin, *A quantum source of entropy for black holes*, *Phys. Rev. D* **34**, 373 (1986).
- [12] M. Srednicki, *Entropy and area*, *Phys. Rev. Lett.* **71**, 666 (1993).
- [13] G. ’t Hooft, *On the quantum structure of a black hole*, *Nucl. Phys.* **B256**, 727 (1985).
- [14] T. Jacobson and A. Satz, *Black hole entanglement entropy and the renormalization group*, *Phys. Rev. D* **87**, 084047 (2013).
- [15] J. H. Cooperman and M. A. Luty, *Renormalization of entanglement entropy and the gravitational effective action*, *J. High Energy Phys.* **12** (2014) 045.
- [16] In the case of the AdS/CFT correspondence, Ref. [17] has shown an equivalence between the Bekenstein-Hawking entropy in AdS and an entanglement entropy of a type-II algebra defined in the dual CFT. Here, we will follow a different logic and work directly with the underlying discrete Hilbert space of the black hole.
- [17] V. Chandrasekaran, G. Penington, and E. Witten, *Large N algebras and generalized entropy*, *J. High Energy Phys.* **04** (2023) 009.
- [18] L. Susskind, *Computational complexity and black hole horizons*, *Fortsch. Phys.* **64**, 24 (2016); **64**, 44 (2016).
- [19] D. Stanford and L. Susskind, *Complexity and shock wave geometries*, *Phys. Rev. D* **90**, 126007 (2014).
- [20] S. Aaronson, *The complexity of quantum states and transformations: From quantum money to black holes*, [arXiv:1607.05256](https://arxiv.org/abs/1607.05256).
- [21] V. Balasubramanian, M. Decross, A. Kar, and O. Parrikar, *Quantum complexity of time evolution with chaotic Hamiltonians*, *J. High Energy Phys.* **01** (2020) 134.
- [22] The authors of Ref. [23] put forward an interesting approach to the interior volume of black holes in 2D gravity, where they defined a path integral definition of ‘‘volume’’ that saturates at exponentially large times. Here, we will follow a different approach that applies in any dimension.
- [23] L. V. Iliesiu, M. Mezei, and G. Sárosi, *The volume of the black hole interior at late times*, *J. High Energy Phys.* **07** (2022) 073.
- [24] In this work, we define ‘‘black hole microstate’’ to be any gravitational state with the same exterior geometry as the black hole.
- [25] M. Sasieta, *Wormholes from heavy operator statistics in AdS/CFT*, *J. High Energy Phys.* **03** (2023) 158.
- [26] Our results recall the idea that semiclassical gravity contains an enormously many null states [27,28] and the proposal that the AdS/CFT encoding map is highly nonisometric because many classically distinct states are mapped to nonorthogonal quantum states [29–31]. Several authors have also pointed out that a finite-dimensional Hilbert space, like a microcanonical sector of the CFT dual to AdS, can host a large number of approximately orthogonal states. In fact, the number of states with a fixed small overlap is exponentially large in the Hilbert

- space dimension; see, e.g., Refs. [32,33]. In our case, the overlaps and the Hilbert space dimension will be determined by the Euclidean gravity path integral.
- [27] D. Marolf and H. Maxfield, *Transcending the ensemble: Baby universes, spacetime wormholes, and the order and disorder of black hole information*, *J. High Energy Phys.* **08** (2020) 044.
- [28] V. Balasubramanian, A. Kar, S. F. Ross, and T. Ugajin, *Spin structures and baby universes*, *J. High Energy Phys.* **09** (2020) 192.
- [29] V. Balasubramanian, A. Kar, Yue, C. Li, and O. Parrikar, *Quantum error correction in the black hole interior*, *J. High Energy Phys.* **07** (2023) 189.
- [30] C. Akers, N. Engelhardt, D. Harlow, G. Penington, and S. Vardhan, *The black hole interior from non-isometric codes and complexity*, [arXiv:2207.06536](https://arxiv.org/abs/2207.06536).
- [31] A. Kar, *Non-isometric quantum error correction in gravity*, *J. High Energy Phys.* **02** (2023) 195.
- [32] J. Chakravarty, *Overcounting of interior excitations: A resolution to the bags of gold paradox in AdS*, *J. High Energy Phys.* **02** (2021) 027.
- [33] R. Chao, B. W. Reichardt, C. Sutherland, and T. Vidick, *Overlapping qubits*, [arXiv:1701.01062](https://arxiv.org/abs/1701.01062), [10.4230/LIPIcs.ITCS.2017.48](https://arxiv.org/abs/10.4230/LIPIcs.ITCS.2017.48).
- [34] J. S. Cotler, G. Gur-Ari, M. Hanada, J. Polchinski, P. Saad, S. H. Shenker, D. Stanford, A. Streicher, and M. Tezuka, *Black holes and random matrices*, *J. High Energy Phys.* **05** (2017) 118; **09** (2018) 002(E).
- [35] P. Saad, S. H. Shenker, and D. Stanford, *A semiclassical ramp in SYK and in gravity*, [arXiv:1806.06840](https://arxiv.org/abs/1806.06840).
- [36] P. Saad, S. H. Shenker, and D. Stanford, *JT gravity as a matrix integral*, [arXiv:1903.11115](https://arxiv.org/abs/1903.11115).
- [37] T. G. Mertens and G. J. Turiaci, *Solvable models of quantum black holes: A review on Jackiw-Teitelboim gravity*, *Living Rev. Relativity* **26**, 4 (2023).
- [38] J. Wheeler, *Relativity, Groups and Topology*, edited by B. S. DeWitt and C. M. DeWitt (Gordon and Breach, New York, 1964).
- [39] A. Almheiri, T. Hartman, J. Maldacena, E. Shaghoulian, and A. Tajdini, *The entropy of Hawking radiation*, *Rev. Mod. Phys.* **93**, 035002 (2021).
- [40] A. R. Brown, H. Gharibyan, G. Penington, and L. Susskind, *The python's lunch: Geometric obstructions to decoding Hawking radiation*, *J. High Energy Phys.* **08** (2020) 121.
- [41] V. Balasubramanian, P. Caputa, J. M. Magan, and Q. Wu, *Quantum chaos and the complexity of spread of states*, *Phys. Rev. D* **106**, 046007 (2022).
- [42] V. Balasubramanian, J. M. Magan, and Q. Wu, *Tridiagonalizing random matrices*, *Phys. Rev. D* **107**, 126001 (2023).
- [43] See Refs. [29–31,44] for recent progress in this direction.
- [44] I. Kim, E. Tang, and J. Preskill, *The ghost in the radiation: Robust encodings of the black hole interior*, *J. High Energy Phys.* **06** (2020) 031.
- [45] L. Susskind, L. Thorlacius, and J. Uglum, *The stretched horizon and black hole complementarity*, *Phys. Rev. D* **48**, 3743 (1993).
- [46] Black hole like solutions in general relativity which differ only in the interior geometry have appeared in, e.g., Refs. [47–49].
- [47] V. P. Frolov and I. D. Novikov, *Black Hole Physics: Basic Concepts and New Developments*, *Fundamental Theories of Physics* (Springer Dordrecht, 1998), [10.1007/978-94-011-5139-9](https://doi.org/10.1007/978-94-011-5139-9).
- [48] S. D. H. Hsu and D. Reeb, *Unitarity and the Hilbert space of quantum gravity*, *Classical Quantum Gravity* **25**, 235007 (2008).
- [49] Z. Fu and D. Marolf, *Bag-of-gold spacetimes, Euclidean wormholes, and inflation from domain walls in AdS/CFT*, *J. High Energy Phys.* **11** (2019) 040.
- [50] I. Kourkoulou and J. Maldacena, *Pure states in the SYK model and nearly-AdS₂ gravity*, [arXiv:1707.02325](https://arxiv.org/abs/1707.02325).
- [51] A. Goel, H. T. Lam, G. J. Turiaci, and H. Verlinde, *Expanding the black hole interior: Partially entangled thermal states in SYK*, *J. High Energy Phys.* **02** (2019) 156.
- [52] J. Chandra and T. Hartman, *Coarse graining pure states in AdS/CFT*, [arXiv:2206.03414](https://arxiv.org/abs/2206.03414).
- [53] The reason for the notation Z_1 will become clearer as we proceed.
- [54] We keep the notation $M_{+,-}$ since it is conventional in the analysis of thin shells in general relativity. The same will follow for the associated manifolds.
- [55] V. Balasubramanian, P. Kraus, A. E. Lawrence, and S. P. Trivedi, *Holographic probes of anti-de Sitter space-times*, *Phys. Rev. D* **59**, 104021 (1999).
- [56] T. Banks, M. R. Douglas, G. T. Horowitz, and E. J. Martinec, *AdS dynamics from conformal field theory*, [arXiv:hep-th/9808016](https://arxiv.org/abs/hep-th/9808016).
- [57] V. Balasubramanian, S. B. Giddings, and A. E. Lawrence, *What do CFTs tell us about Anti-de Sitter space-times?*, *J. High Energy Phys.* **03** (1999) 001.
- [58] A. Hamilton, D. N. Kabat, G. Lifschytz, and D. A. Lowe, *Holographic representation of local bulk operators*, *Phys. Rev. D* **74**, 066009 (2006).
- [59] V. Balasubramanian and P. Kraus, *A stress tensor for anti-de Sitter gravity*, *Commun. Math. Phys.* **208**, 413 (1999).
- [60] M. Henningson and K. Skenderis, *The holographic Weyl anomaly*, *J. High Energy Phys.* **07** (1998) 023.
- [61] K. Skenderis, *Lecture notes on holographic renormalization*, *Classical Quantum Gravity* **19**, 5849 (2002).
- [62] In the AdS/CFT context, this means a number scaling with the central charge of the CFT.
- [63] W. Israel, *Singular hypersurfaces and thin shells in general relativity*, *Nuovo Cim. B* **44**, 1 (1966); **48**, 463(E) (1967).
- [64] Basically, the total Euclidean time, say, β_L , should equal the sum of $\Delta\tau$ and the preparation temperature $\tilde{\beta}_L$ as in Fig. 3.
- [65] It is also possible to find classical trajectories for the shell separating X^\pm , where the elapsed Euclidean time (2.14) is too large to satisfy Eqs. (2.15) and (2.16). In this case, the shell will lie to the “right” of both Euclidean horizons indicated in Fig. 2; i.e., in the Lorentzian continuation, it will lie outside the horizon of the right-hand black hole. We will not consider this class of gravitational states.
- [66] Notice the marked differences of the present construction of black hole microstates, in comparison with the fuzzball program [10]. The fuzzball program aims to quantize a particular space of smooth supergravity solutions with no horizons. Here, we are identifying an infinite set of black

- hole microstates with horizons and singularities. The fact that this should not be considered as a problem is already clear in the construction of the Hartle-Hawking state itself (i.e., the state with no dust shell inserted), which has horizons and singularities and corresponds to a perfectly well-defined state in the dual CFT, namely, the TFD state [67].
- [67] J. M. Maldacena, *Eternal black holes in anti-de Sitter*, *J. High Energy Phys.* **04** (2003) 021.
- [68] Recall that the shell can be formed semiclassically, for example, in general relativity coupled to a scalar field.
- [69] These normalizations arise from the standard procedure of filling in the bulk geometry while fixing the boundary conditions determined by the computation of the norm of the state. The geometries X and X' are each given by two copies of the Euclidean part of Fig. 2 glued on the $\tau = 0$ surface.
- [70] R. Emparan, C. V. Johnson, and R. C. Myers, *Surface terms as counterterms in the AdS/CFT correspondence*, *Phys. Rev. D* **60**, 104001 (1999).
- [71] In the $m \rightarrow 0$ limit with fixed β , the effective temperature $\tilde{\beta}_m \rightarrow \beta/2$, in which the limit X_{shell} collapses and $I[X_{\text{shell}}] \rightarrow 0$.
- [72] Recent work, in 2D JT gravity especially, has focused on the effect of wormholes on the factorization of gravitational amplitudes and interpretations of this phenomenon in terms of a fundamental averaging over theories (see, e.g., Refs. [36,73–80]. Alternative uses of wormholes and their relation to factorization have appeared recently in Ref. [81].
- [73] G. Penington, S. H. Shenker, D. Stanford, and Z. Yang, *Replica wormholes and the black hole interior*, *J. High Energy Phys.* **03** (2022) 205.
- [74] D. Stanford, *More quantum noise from wormholes*, [arXiv:2008.08570](https://arxiv.org/abs/2008.08570).
- [75] J. Pollack, M. Rozali, J. Sully, and D. Wakeham, *Eigenstate thermalization and disorder averaging in gravity*, *Phys. Rev. Lett.* **125**, 021601 (2020).
- [76] A. Maloney and E. Witten, *Averaging over Narain moduli space*, *J. High Energy Phys.* **10** (2020) 187.
- [77] J. Cotler and K. Jensen, *AdS₃ gravity and random CFT*, *J. High Energy Phys.* **04** (2021) 033.
- [78] A. Altland and J. Sonner, *Late time physics of holographic quantum chaos*, *SciPost Phys.* **11**, 034 (2021).
- [79] A. Altland, D. Bagrets, P. Nayak, J. Sonner, and M. Vielma, *From operator statistics to wormholes*, *Phys. Rev. Res.* **3**, 033259 (2021).
- [80] J. Chandra, S. Collier, T. Hartman, and A. Maloney, *Semiclassical 3D gravity as an average of large- c CFTs*, *J. High Energy Phys.* **12** (2022) 069.
- [81] P. Betzios, E. Kiritsis, and O. Papadoulaki, *Interacting systems and wormholes*, *J. High Energy Phys.* **02** (2022) 126.
- [82] J. M. Deutsch, *Quantum statistical mechanics in a closed system*, *Phys. Rev. A* **43**, 2046 (1991).
- [83] M. Srednicki, *Chaos and quantum thermalization*, *Phys. Rev. E* **50**, 888 (1994).
- [84] This hypothesis is developed in the context of CFTs in Ref. [85], and a reformulation in terms of multimatrix models has appeared in Ref. [86].
- [85] A. Belin and J. de Boer, *Random statistics of OPE coefficients and Euclidean wormholes*, *Classical Quantum Gravity* **38**, 164001 (2021).
- [86] D. L. Jafferis, D. K. Kolchmeyer, B. Mukhametzhanov, and J. Sonner, *Matrix models for eigenstate thermalization*, *Phys. Rev. X* **13**, 031033 (2023).
- [87] J. Maldacena, S. H. Shenker, and D. Stanford, *A bound on chaos*, *J. High Energy Phys.* **08** (2016) 106.
- [88] From the perspective of the dual theory, even if the shell operator is not spatially small, i.e., it is fully delocalized, it can still be considered as simple in the internal large- N space of the CFT. The operator corresponds to a product of approximately local single-trace operators on each spatial site, and it displays “Lyapunov growth” for different holographic measures of operator size. The Euclidean time evolution transforms this simple asymptotic operator into a “complex” operator describing a shell behind the horizon of the Lorentzian black hole.
- [89] This randomness recalls an argument in the AdS/CFT context that typical microstates of black holes can be created by operators that are random polynomials in the CFT fields [90,91] and that the consequence of this randomness will be a universality of correlators computed in such states.
- [90] V. Balasubramanian, V. Jejjala, and J. Simon, *The Library of Babel*, *Int. J. Mod. Phys. D* **14**, 2181 (2005).
- [91] V. Balasubramanian, J. de Boer, V. Jejjala, and J. Simon, *The Library of Babel: The origin of gravitational thermodynamics*, *J. High Energy Phys.* **12** (2005) 006.
- [92] Classically different shells, of mass difference scaling parametrically with ℓ^{d-1}/G , are expected to have uncorrelated ETH coefficients. This expectation is consistent with the semiclassical calculation.
- [93] P. Saad, *Late time correlation functions, baby universes, and ETH in JT gravity*, [arXiv:1910.10311](https://arxiv.org/abs/1910.10311).
- [94] J.-M. Schlenker and E. Witten, *No ensemble averaging below the black hole threshold*, *J. High Energy Phys.* **07** (2022) 143.
- [95] T. Guhr, A. Muller-Groeling, and H. A. Weidenmuller, *Random matrix theories in quantum physics: Common concepts*, *Phys. Rep.* **299**, 189 (1998).
- [96] K. Papadodimas and S. Raju, *Local operators in the eternal black hole*, *Phys. Rev. Lett.* **115**, 211601 (2015).
- [97] H. Verlinde, *Deconstructing the wormhole: Factorization, entanglement and decoherence*, [arXiv:2105.02142](https://arxiv.org/abs/2105.02142).
- [98] A. del Campo, J. Molina-Vilaplana, and J. Sonner, *Scrambling the spectral form factor: Unitarity constraints and exact results*, *Phys. Rev. D* **95**, 126008 (2017).
- [99] D. Stanford and Z. Yang, *Firewalls from wormholes*, [arXiv:2208.01625](https://arxiv.org/abs/2208.01625).
- [100] P. Saad, D. Stanford, Z. Yang, and S. Yao, *A convergent genus expansion for the plateau*, [arXiv:2210.11565](https://arxiv.org/abs/2210.11565).
- [101] A. Blommaert, J. Kruthoff, and S. Yao, *An integrable road to a perturbative plateau*, *J. High Energy Phys.* **04** (2023) 048.
- [102] G. 't Hooft, *Dimensional reduction in quantum gravity*, *Conf. Proc. C* **930308**, 284 (1993).
- [103] L. Susskind, *The world as a hologram*, *J. Math. Phys. (N.Y.)* **36**, 6377 (1995).

- [104] R. Bousso, *A covariant entropy conjecture*, *J. High Energy Phys.* **07** (1999) 004.
- [105] In modern jargon, going from one shell to another by adding one particle effectively constitutes a motion in the code subspace of the original shell. Overlaps in the code subspace need not be exponentially suppressed in G . Within those subspaces, we can neglect the dynamical effects of gravity such as backreaction in the semiclassical limit.
- [106] A similar situation was encountered in Ref. [107] when trying to argue against the existence of global symmetries in two-dimensional models of quantum gravity describing near extremal black holes. In that context, wormholes provide the necessary global-symmetry-violating amplitudes to ensure that the number of linearly independent states remains below the Bekenstein-Hawking entropy of the black hole.
- [107] P.-S. Hsin, L. V. Iliesiu, and Z. Yang, *A violation of global symmetries from replica wormholes and the fate of black hole remnants*, *Classical Quantum Gravity* **38**, 194004 (2021).
- [108] P. Cvitanović, *Planar perturbation expansion*, *Phys. Lett. B* **99**, 49 (1981).
- [109] R. Speicher, *Free probability theory*, arXiv:0911.0087.
- [110] Notice that for any family of black hole microstates we choose, most states will have sufficiently large mass in this sense since the mass can increase without bound. Otherwise, one can choose a family whose first state starts already at sufficiently high mass.
- [111] Notice that different energy bands are orthogonal to each other, so the inner products of states in the different bands just partition.
- [112] Notice that all eigenvalues are positive semidefinite, as follows from the positive semidefinite nature of the Gram matrix.
- [113] G. Akemann, J. Baik, and P. Di Francesco, *The Oxford Handbook of Random Matrix Theory* (Oxford University Press, New York, 2015).
- [114] D. Stanford and E. Witten, *Fermionic localization of the Schwarzian theory*, *J. High Energy Phys.* **10** (2017) 008.
- [115] L. V. Iliesiu and G. J. Turiaci, *The statistical mechanics of near-extremal black holes*, *J. High Energy Phys.* **05** (2021) 145.
- [116] J. Boruch, L. V. Iliesiu, and C. Yan, *Constructing all BPS black hole microstates from the gravitational path integral*, arXiv:2307.13051.
- [117] A. Climent, R. Emparan, J. Magan, M. Sasieta, and A. V. Lopez, *Universal construction of black hole microstates*, arXiv:2401.08775.
- [118] A. Almheiri, T. Hartman, J. Maldacena, E. Shaghoulian, and A. Tajdini, *Replica wormholes and the entropy of Hawking radiation*, *J. High Energy Phys.* **05** (2020) 013.
- [119] G. Penington, *Entanglement wedge reconstruction and the information paradox*, *J. High Energy Phys.* **09** (2020) 002.
- [120] A. Almheiri, N. Engelhardt, D. Marolf, and H. Maxfield, *The entropy of bulk quantum fields and the entanglement wedge of an evaporating black hole*, *J. High Energy Phys.* **12** (2019) 063.
- [121] A. Almheiri, R. Mahajan, J. Maldacena, and Y. Zhao, *The Page curve of Hawking radiation from semiclassical geometry*, *J. High Energy Phys.* **03** (2020) 149.
- [122] D. N. Page, *Average entropy of a subsystem*, *Phys. Rev. Lett.* **71**, 1291 (1993).
- [123] D. N. Page, *Information in black hole radiation*, *Phys. Rev. Lett.* **71**, 3743 (1993).
- [124] The field of two-dimensional gravity has seen a tremendous rebirth in the last decade. See Refs. [37,125] for two good reviews.
- [125] G. Sárosi, *AdS₂ holography and the SYK model*, *Proc. Sci., Modave2017* (2018) 001.
- [126] V. Balasubramanian, A. Kar, O. Parrikar, G. Sárosi, and T. Ugajin, *Geometric secret sharing in a model of Hawking radiation*, *J. High Energy Phys.* **01** (2021) 177.
- [127] H. Geng, A. Karch, C. Perez-Pardavila, S. Raju, L. Randall, M. Riojas, and S. Shashi, *Inconsistency of islands in theories with long-range gravity*, *J. High Energy Phys.* **01** (2022) 182.
- [128] See, however, Ref. [129], which avoids this argument by considering entanglement between disjoint gravitating universes.
- [129] V. Balasubramanian, A. Kar, and T. Ugajin, *Entanglement between two gravitating universes*, *Classical Quantum Gravity* **39**, 174001 (2022).
- [130] L. Susskind, *Black holes at exp-time*, arXiv:2006.01280.
- [131] J. M. Magán, *Black holes, complexity and quantum chaos*, *J. High Energy Phys.* **09** (2018) 043.
- [132] J. M. Magán and J. Simón, *On operator growth and emergent Poincaré symmetries*, *J. High Energy Phys.* **05** (2020) 071.
- [133] P. Caputa, J. M. Magan, and D. Patramanis, *Geometry of Krylov complexity*, *Phys. Rev. Res.* **4**, 013041 (2022).
- [134] To be precise, the α_p coefficients increases at low energies, but the number of microscopic configurations grows with energy, so most of the support will be at the temperature β , as in standard statistical mechanics.
- [135] Notice that the geometric shell states are classically well defined. They should be understood quantum mechanically as some sort of coherent state. As such, they cannot really be eigenvectors of the true volume operator. Equivalently, the uncertainty in the dual variable in canonical quantum gravity, the extrinsic curvature, is small. In the classical limit, these states should be thought of as minimum uncertainty states in both volume and curvature, although, as discussed above, the volume of the classical solutions cannot be a linear operator of the quantum theory.
- [136] We remind the reader that, although this notion of volume is natural from the point of view of spread complexity, which involves a minimization over linear operators, it is still not clear whether the nonlinear notion of semiclassical volume of our microstates, viewed as a function in Hilbert space, displays saturation.
- [137] Here, we use the fact that the shell states can be chosen to form an approximately orthonormal basis. To be absolutely precise, one should slightly modify each shell state to make them exactly orthonormal, but we will not do that in this toy model.
- [138] In the context of spread complexity, one typically defines the first state to have position (and, therefore, spread complexity) zero. Here, we define it to have initial position 1, which is just a constant shift that accounts

- for the initial complexity of the shell operator. In any case, this shift is irrelevant at late times.
- [139] See Ref. [140] for a recent application in the context of supersymmetric black holes in theories with a high degree of supersymmetry. It would be interesting to understand to what extent our techniques can be extended to such scenarios.
- [140] L. V. Iliesiu, S. Murthy, and G.J. Turiaci, *Black hole microstate counting from the gravitational path integral*, [arXiv:2209.13602](https://arxiv.org/abs/2209.13602).
- [141] V. Balasubramanian, A. Lawrence, J.M. Magan, and M. Sasieta, *Microscopic origin of the entropy of astrophysical black holes*, [arXiv:2212.08623](https://arxiv.org/abs/2212.08623).
- [142] Notice a relative minus sign in this expression with respect to the Lorentzian version. This sign arises since the hypersurface \mathcal{W} is spacelike.
- [143] Here, we assume that the original temperature is so small that the n -boundary saddle is still dominated by a large AdS black hole.
- [144] A. N. Kolmogorov, *Three approaches to the quantitative definition of information*, *Int. J. Comput. Math.* **2**, 157 (1968).
- [145] C. Lanczos, *An iteration method for the solution of the eigenvalue problem of linear differential and integral operators*, *J. Res. Natl. Bur. Stand.* **45**, 255 (1950).
- [146] V. S. Viswanath and G. Müller, *The Recursion Method: Application to Many-Body Dynamics* (Springer-Verlag, 1994), ISBN-10: 354058319X, ISBN-13: 978-3540583196.
- [147] D. E. Parker, X. Cao, A. Avdoshkin, T. Scaffidi, and E. Altman, *A universal operator growth hypothesis*, *Phys. Rev. X* **9**, 041017 (2019).