XY* Transition and Extraordinary Boundary Criticality from Fractional Exciton Condensation in Quantum Hall Bilayer

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XY* transitions represent one of the simplest examples of unconventional quantum criticality, in which fractionally charged excitations condense into a superfluid and display novel features that combine quantum criticality and fractionalization. Nevertheless, their experimental realization is challenging. Here we propose to study the XY* transition in quantum Hall bilayers at filling $(\nu_1, \nu_2) = (\frac{1}{3}, \frac{2}{3})$ where the exciton condensate (EC) phase plays the role of the superfluid. Supported by exact diagonalization calculation, we argue that there is a continuous transition between an EC phase at small bilayer separation to a pair of decoupled fractional quantum Hall states at large separation. The transition is driven by condensation of a fractional exciton, a bound state of a Laughlin quasiparticle and quasihole, and is in the XY* universality class. The fractionalization is manifested by unusual properties including a large anomalous exponent and fractional universal conductivity, which can be conveniently measured through interlayer tunneling and counterflow transport, respectively. We also show that the edge is likely to realize the newly predicted extraordinary log boundary criticality. Our work highlights the promise of quantum Hall bilayers as an ideal platform for exploring exotic bulk and boundary critical behaviors that are amenable to immediate experimental exploration in dual-gated bilayer systems. The XY* critical theory can be generalized to a bilayer system with an arbitrary Abelian state in one layer and its particle-hole partner in the other layer. Therefore, we anticipate many distinct XY* transitions corresponding to the different Laughlin states and Jain sequences in the single-layer case.

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I. INTRODUCTION

The study of quantum phase transitions and universal critical behaviors is one of the major focuses in condensed matter physics [1,2]. Although many quantum critical points (QCPs) are described by the well-established Landau-Ginzburg theory, exceptions arise due to fractionalization beyond the conventional symmetry-breaking order. One category is the deconfined quantum critical points (DQCPs) between two different symmetry-breaking phases [3]. Another category is phase transitions between one phase with fractionalization or topological order and another conventional phase. One simple example is the XY* transition, initially discussed between a Z_2 topologically ordered insulator (or quantum spin liquid) and a superfluid (or XY ferromagnetism) phase [4–7]. The critical theory of such a transition is well understood [4,7], and its existence in lattice models has been numerically verified [5,8]. However, experimental observation of the XY* transition is still elusive. Given that even the unambiguous experimental realization of a Z_2 spin liquid phase is a great challenge, and that recent progress in synthetic quantum systems targets topological order in the absence of global U(1) symmetry [9–11], the experimental study of an XY* QCP adjacent to a quantum spin liquid phase remains challenging for the near future.

Here we turn to quantum Hall systems, where fractionalization itself has been well established at fractional fillings [12]. It is natural to imagine that experimental realization of a QCP with fractionalization in quantum Hall systems is easier, though such a possibility has not been

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well explored except on plateau transitions [2]. Here we consider the quantum Hall bilayer system with the electron gases in two layers separated by an insulating barrier, giving rise to two separate Landau levels coupled together through the Coulomb repulsion [13–16]. The fillings in the two layers ν_1, ν_2 can be controlled separately. In addition, one can tune d/l_B experimentally to study the possible phase transitions. Here, d is the distance between the two layers and l_B is the magnetic length. At small d/l_B, it is known that the ground state is an exciton condensation phase [15–20] for the whole line of $\nu_1+\nu_2=1$. There have been many theoretical discussions on other possible phases at larger d/l_B at $(\nu_1,\nu_2)=(\frac{1}{2},\frac{1}{2})$ [21–38].

Recently, the evolution under tuning d/l_B was experimentally investigated for this filling $(\nu_1,\nu_2)=(\frac{1}{2},\frac{1}{2})$ [39]. There, one finds only a crossover between the Bose-Einstein-condensation (BEC) regime to the Bardeen-Cooper-Schrieffer (BCS) regime all within a single-exciton-condensation (EC) phase. There has been theoretical discussion of superfluid-to-insulator transition at $(1/2,1/2)$ filling [40], but we are not aware of any experimental observation so far. In contrast, for filling $(\frac{1}{3},\frac{2}{3})$ [or relatedly, $(\nu_1,\nu_2)=(\frac{1}{3},-\frac{1}{3})$, where by $\nu<0$ we mean the system is hole doped relative to the charge neutrality], a phase transition is bound to happen. At small d/l_B, the ground state is still an exciton condensation phase. In the large-d/l_B limit, the two layers decouple, and the top layer is in the $\nu=\frac{1}{3}$ Laughlin state [41], while the bottom layer is in the $\nu=-\frac{1}{3}$ (or $\nu=\frac{2}{3}$) Laughlin state. This large-d/l_B phase can be viewed as a fractional quantum spin Hall insulator (FQSH) with a K matrix $K=\begin{pmatrix}3&0\\0&-3\end{pmatrix}$ if we view the layer as a pseudospin. Given the recent experimental progress in tuning d/l_B at $(\nu_1,\nu_2)=(\frac{1}{2},\frac{1}{2})$, experimental measurements at filling $(\nu_1,\nu_2)=(\frac{1}{3},-\frac{1}{3})$ or $(\nu_1,\nu_2)=(\frac{1}{3},\frac{2}{3})$ should be straightforward. Note, while conceptually one can think of changing the separation d, in experiment one can tune the ratio d/l_B more conveniently by simultaneously changing the magnetic field and density to keep the filling constant. Thus, the transition is well within experimental reach [42]. Actually, there already exists some experimental evidence of a direct transition between the exciton superfluid and FQSH phase at filling $(\nu_1,\nu_2)=(\frac{1}{3},\frac{2}{3})$ [43] in a GaAs quantum-well system. However, the nature of the phase transition is not clear from the existing experimental data. A previous theoretical work already studied similar transition in a model with a hard-core interaction and suggested the transition is in the XY universality class [44]. Here we provide numerical evidence for a continuous transition in a realistic model with Coulomb interactions. More importantly, through a more careful treatment of the global U(1) symmetry, we point out that this critical point is actually an XY* transition with the critical boson carrying a fractional physical charge and thus not gauge invariant. The physical exciton operator is a composite operator in the critical theory, an important point that is overlooked in Ref. [44]. The fractionalized nature of the transition leads to many experimentally verifiable physical consequences including a large anomalous scaling dimension and fractional counterflow conductivity compared to the familiar superfluid-to-Mott transition in the usual XY class.

We perform exact diagonalization (ED) [45] for the Coulomb coupled quantum Hall bilayer at filling $(\nu_1,\nu_2)=(\frac{1}{3},\frac{2}{3})$ and find a direct transition between the EC phase at small d/l_B and the FQSH phase at large d/l_B. The transition appears continuous in the finite-size calculation, suggesting the possibility of a continuous QCP. Motivated by the numerical calculation, we propose a critical theory between the EC and FQSH phases in the universality class of XY* transition. Starting from the FQSH phase, the Laughlin electron and Laughlin hole in the two layers bind to form a fractional exciton, with bosonic statistics whose condensation then confines all the anyons and leads to the EC phase at small d/l_B. The critical theory is described by the superfluid-to-insulator transition of the fractional exciton, which carries an exciton charge $1/3$ compared to the ordinary exciton. We also discuss the realization of an extraordinary boundary criticality [46] in the edge at this QCP. The XY* transition here can be easily generalized to the case with an arbitrary Abelian FQHE phase in one layer and its particle-hole partner in the other layer. Thus, we anticipate many different XY* transitions in the $(D,d/l_B)$ parameter space, where D is the displacement field to tune the exciton density.

II. MODEL AND SYMMETRY

We consider the quantum Hall bilayer at filling $(\nu_1,\nu_2)=(x,-x)$ illustrated in Fig. 1. Here, $\nu_a=(N_e/N_\Phi)$, where N_Φ is the number of the magnetic flux in the system. $\nu_2=-x$

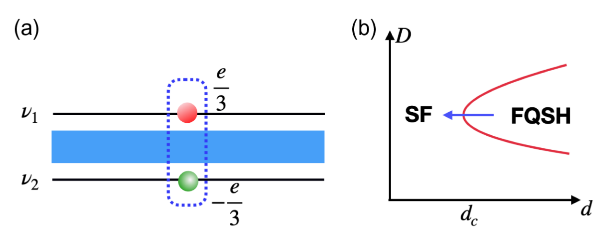

FIG. 1. (a) Illustration of quantum Hall bilayer, with an insulating layer (blue) in between the two 2DEGs. (b) Schematic phase diagram in terms of (D,d) while fixing $\nu_1+\nu_2=0$. The displacement field D is the chemical potential for excitons. The distance d tunes interlayer Coulomb interaction strength. We are interested in the quantum phase transition (indicated by the blue arrow) tuned by d at fixed exciton density $(\nu_1,\nu_2)=(\frac{1}{3},-\frac{1}{3})$. SF denotes the exciton condensation phase with $\langle c_1^\dagger c_2\rangle\neq 0$, and FQSH denotes the fractional quantum spin Hall insulator formed by two decoupled Laughlin states with opposite chiralities.

means that the system is hole doped with hole density at x per flux. We are mainly focused on $x = \frac{1}{3}$, but similar physics can happen for other rational x with an incompressible Abelian FQHE state in the decoupling limit. x here is the exciton density and can be tuned through the displacement field D, while the total filling $\nu_1 + \nu_2$ is fixed to be 0. Up to a stacking of an integer quantum Hall state at layer 2, it is also equivalent to consider the filling $(\nu_1, \nu_2) = (x, 1-x)$ with $\nu_1 + \nu_2 = 1$. Thus, we also consider the filling $(\nu_1, \nu_2) = (\frac{1}{3}, \frac{2}{3})$.

We have the Hamiltonian

$$\mathcal{H} = \frac{1}{2} \sum_{a,b=1,2} V_{ab}(\mathbf{q}) \colon \rho_a(\mathbf{q}) \rho_b(-\mathbf{q}) \colon, \tag{1}$$

where $\rho_a(\mathbf{q}) = \int d^2\mathbf{q} \rho_a(\mathbf{r}) e^{-i\mathbf{q}\cdot\mathbf{r}}$, and $\rho_a(\mathbf{r})$ is the charge density at layer a projected to the lowest Landau level. We have the Coulomb interaction $V_{11}(q) = V_{22}(q) = (e^2/\varepsilon q)$ and $V_{12}(q) = V_{21}(q) = (e^2/\varepsilon q) e^{-qd}$. d represents the distance between two layers in the unit of magnetic length $l_B = \sqrt{\hbar c / eB}$.

The Hamiltonian considered above has an antiunitary symmetry $\mathcal{MCT}$ for the quantum Hall bilayer at filling $(\nu_1, \nu_2) = (x, -x)$. The symmetry is a combination of layer exchange symmetry $\mathcal{M}$, charge conjugation $\mathcal{C}$, and time reversal $\mathcal{T}$. We define electron operators in layers 1 and 2 as $c_1(\mathbf{r})$ and $c_2(\mathbf{r})$. The symmetry $\mathcal{MCT}$ acts as $c_1(\mathbf{r}) \to c_2^\dagger(\mathbf{r})$, $c_2(\mathbf{r}) \to c_1^\dagger(\mathbf{r})$ combined with complex conjugate $\mathcal{K}$. Under $\mathcal{MCT}$, we have $\rho_1(\mathbf{r}) \to -\rho_2(\mathbf{r})$ and $\rho_1(\mathbf{q}) \to -\rho_2(-\mathbf{q})$. One can check that the Hamiltonian satisfies this symmetry. The $\mathcal{MCT}$ can be weakly broken due to the asymmetry of the two layers such as different interaction strengths. We discuss the effect of weak $\mathcal{MCT}$ symmetry breaking later.

III. PHASE DIAGRAM

We fix the filling to be $(\nu_1, \nu_2) = (\frac{1}{3}, \frac{2}{3})$ and study the phase diagram of tuning d/l_B through ED. At small d/l_B, the system is in an exciton condensation phase with an order parameter $\langle c_1^\dagger c_2 \rangle \neq 0$. Here, c_1, c_2 are annihilation operators of electrons in layers 1 and 2, respectively. At large d/l_B, the two layers decouple, and we have a FQSH phase (up to stacking an integer quantum Hall state at layer 2) if viewing layers 1 and 2 as spin up and spin down. The question is whether there is a direct phase transition or an intermediate phase in between.

Figure 2(a) shows the flow of low-lying energies with layer distance d/l_B. For simplicity, we set $l_B = 1$ in the following discussion. We use a torus geometry and the Landau gauge. The evolution of energy spectra indicates a direct transition at $d_c \approx 1.7$. When $d > d_c$, we can identify a ninefold near degeneracy expected for decoupled $\nu = \pm 1/3$ Laughlin states in the two layers. When approaching d_c from large d, the topological order indicated by the ground-state degeneracy disappears at d_c.

The phase at $d < d_c$ is an exciton superfluid with order parameter $\langle S^y(\mathbf{r}) \rangle \neq 0$, where $S^y(\mathbf{r}) = i(c_1^\dagger(\mathbf{r}) c_2(\mathbf{r}) - \text{H.c.})$. In the lowest Landau level, we have operators $c_{1;m}$ and $c_{2;m}$ where m is the Landau index and labels the position along the x direction in our gauge. So we can define $S_m^y = i(c_{1;m}^\dagger c_{2;m} - \text{H.c.})$, where $m = 1, 2, \ldots, N_\Phi$. Then we calculate the correlation function $\langle S_i^y S_j^y \rangle$ which is a function of $|i-j|$. In the inset of Fig. 2(b), we show that $\langle S_i^y S_j^y \rangle$ is almost a constant with $|i-j|$ at small d, but decays fast at large d. In particular, we can use $\langle S_i^y S_{i+N_\Phi/2}^y \rangle$ to characterize the exciton condensation. In Fig. 2(b), it is clear that $\langle S_i^y S_{i+N_\Phi/2}^y \rangle$ is nonzero at $d < d_c$ and almost vanishes at $d > d_c$. When approaching d_c from small d, the exciton condensation order parameter disappears smoothly across d_c.

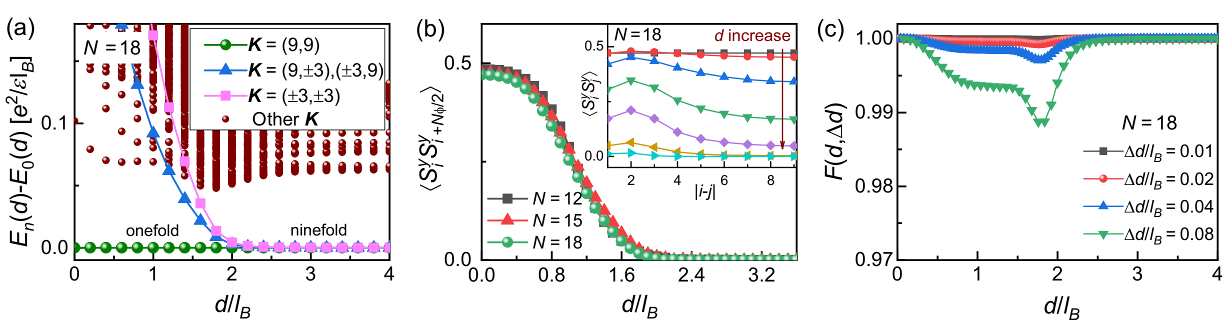

FIG. 2. (a) The low-lying energy spectra as a function of the layer distance d/l_B for the filling $(\nu_1, \nu_2) = (\frac{1}{3}, -\frac{1}{3})$. Here we show only the inequivalent momentum sectors. There are ninefold degenerate states at larger d/l_B. (b) The correlator $\langle S_i^y S_{i+N_\phi/2}^y \rangle$ versus layer distances d/l_B. The inset shows $\langle S_i^y S_j^y \rangle$ as a function of the orbital distance $|i-j|$. Here, $i, j = 1, \ldots, N_\phi$ are orbital indices and the corresponding distance is $|i-j| L / N_\phi$. From top to bottom, d/l_B ranges from 0 to 2.4 with interval 0.4. (c) The fidelity $F(d, \Delta d)$ as a function of the layer distances d/l_B taking different intervals of parameters $\Delta d/l_B$. Note that with decreased Δd the dip is weakened.

To further probe the nature of the transition at d_c, we compute the ground-state fidelity, which is defined by the wave-function overlap between the ground state at $d - \Delta d$ and d, i.e., $F(d, \Delta d) = |\langle \Psi(d - \Delta d)|\Psi(d)\rangle|$. The fidelity has been shown to be a good indicator to distinguish the continuous transition from the first-order transition for both symmetry-breaking and topological phase transitions [47,48]. As shown in Fig. 2(c), the ground-state fidelity displays a single weak dip at the critical distance d_c instead of showing a sudden jump. Meanwhile, the dip is further weakened with the decrease of Δd. Thus, the numerical evidence indicates the transition might be continuous, though one cannot rule out a weak first-order transition in a finite-size calculation. In the following, we propose a critical theory for this QCP in the universality class of XY*. The XY* transition is well established to be continuous in other contexts, which further supports the continuous transition scenario of the QCP at d_c from the theoretical side.

IV. FIELD THEORY OF AN XY* TRANSITION

We turn to the filling $(\nu_1, \nu_2) = (\frac{1}{3}, -\frac{1}{3})$ for simplicity. The FQSH phase at the large-d limit is described by the following effective field theory:

$$\mathcal{L} = -\frac{3}{4\pi} a_1 da_1 + \frac{3}{4\pi} a_2 da_2 + \frac{1}{2\pi} A_1 da_1 - \frac{1}{2\pi} A_2 da_2, \quad (2)$$

where adb is an abbreviation of $\epsilon_{\mu\nu\sigma} a_\mu \partial_\nu b_\sigma$. Here, $a_{1;\mu}$ and $a_{2;\mu}$ are emergent dynamical gauge fields, while $A_{1;\mu}$ and $A_{2;\mu}$ are probe fields of the two layers. For example, $\vec{E}_a = -\vec{\nabla} A_{a;0} - (\partial \vec{A}_a/\partial t)$ is the electric field applied to layer a. Note in the experiment, one can apply $\vec{E}_1$ and $\vec{E}_2$ separately and measure currents in a layer-resolved fashion. We then define physical charge (Q_1, Q_2) under (A_1, A_2). We can also label anyon excitations in terms of their charges $l = (l_1, l_2)$ under (a_1, a_2). The physical charge of the anyon l is $(Q_1, Q_2) = (\frac{1}{3} l_1, \frac{1}{3} l_2)$. Its statistics is $\theta = (l_1^2 - l_2^2/3)\pi$. We also make a basis change to define $A_{c;\mu} = (A_{1;\mu} + A_{2;\mu}/2)$ and $A_{s;\mu} = A_{1;\mu} - A_{2;\mu}$. The corresponding charge is $Q_c = Q_1 + Q_2$, and $Q_s = (Q_1 - Q_2/2)$. Q_s is the layer pseudospin viewed as a spin $1/2$, S_z.

The elementary anyon is $l = (\pm 1, 0)$ and $l = (0, \pm 1)$ with charge $\pm 1/3$ at each layer. When we decrease d, the interlayer Coulomb interaction increases. Then an anyon with charge $1/3$ at layer 1 tends to bind with an anyon with charge $-1/3$ at layer 2 into an exciton. When d is further decreased, the binding energy increases, and this exciton of anyon can condense and lead to the exciton condensation phase at small d. This fractional exciton is labeled by $l = (1, -1)$ with physical charge $(Q_1, Q_2) = (\frac{1}{3}, -\frac{1}{3})$ or $Q_c = 0, Q_s = \frac{1}{3}$. We label the creation operator of this fractional exciton as $\varphi^\dagger$. The condensation of φ is captured by the following critical theory (see the Appendix for derivation):

$$\mathcal{L}_c = \left| \left(\partial_\mu - i\frac{1}{3} A_{s;\mu} \right) \varphi \right|^2 - s|\varphi|^2 - g|\varphi|^4 + \frac{1}{6\pi} A_c dA_s. \quad (3)$$

When $s < 0$, this is a superfluid phase of A_s. When $s > 0$, we have the correct response of $(1/6\pi) A_c dA_s$ for the FQSH phase. In principle, φ is also coupled to a gauge field, which, however, does not affect the critical properties we discuss here due to a Chern-Simons term (see the Appendix). Note further that tuning the transition at fixed layer density eliminates the single-time-derivative chemical potential term $\varphi^* i\partial_t \varphi$. Note that this is unrelated to the $\mathcal{MCT}$ symmetry which maps φ to $-\varphi$. We need to fine-tune to the tip of the parabola in Fig. 1 to get a critical theory with dynamical exponent $z = 1$, a feature shared with the familiar Bose-Hubbard model. Fortunately, we see that control over gate voltages makes this tuning feasible. φ may feel a background flux $dA_s(\mathbf{r}) = B_1(\mathbf{r}) - B_2(\mathbf{r})$, where $B_a(\mathbf{r})$ is the magnetic field in the layer $a = 1, 2$ along the z direction. In real experiments, we expect $B_1(\mathbf{r}) = B_2(\mathbf{r})$ given the small interlayer distance d. Therefore, we conclude that φ does not feel any background magnetic flux. It is then clear that the critical theory is the usual "relativistic" XY transition driven by the condensation of a boson which carries charge $1/3$ under A_s. A counterintuitive feature that is shared with other XY* transitions is that despite the condensation of a fractionally charged boson, the superfluid itself is conventional. One can readily check that the only gauge-invariant order parameter is the usual one for integer charge, and all anyons are confined. Alternatively, one can show the vortex quantization is the conventional one despite the fractional charge, as a result of attaching an anyon to the fundamental vortex [49].

One can also describe the transition using a vortex field through the familiar particle-vortex duality. In the Appendix, we derive the dual theory to be

$$\mathcal{L} = |(\partial_\mu - i3a_\mu)\tilde{\varphi}|^2 - r|\tilde{\varphi}|^2 - \tilde{g}|\tilde{\varphi}|^4 + \frac{1}{e^2} f_{\mu\nu} f^{\mu\nu} + \frac{1}{2\pi} A_s da + \frac{1}{6\pi} A_s dA_c, \quad (4)$$

where $\tilde{\varphi}$ is the vortex field of the superfluid phase and a_μ is an internal U(1) gauge field with $f_{\mu\nu} = \partial_\mu a_\nu - \partial_\nu a_\mu$ as its flux. One important distinction from the theory derived in Ref. [44] is that the critical vortex field $\tilde{\varphi}$ carries charge 3 under the U(1) gauge field a in our theory, while the charge is 1 in Ref. [44]. Note this factor of 3 cannot be removed by a naive redefinition $a_\mu \to \frac{1}{3} a_\mu$, which will lead to a fractional mutual Chern-Simons term $(1/6\pi) A_s da$. Here we always fix the coefficient of the mutual Chern-Simons term so that the monopole operator of a_μ carries charge 1 under A_s and can be identified as the elementary physical exciton operator. As a result, $\tilde{\varphi}$ sees the physical exciton (monopole of a) as a 6π flux and needs to be identified as a triple vortex instead of an elementary vortex in the superfluid phase. In contrast to Ref. [44], our treatment includes the probe field A_s which allows us to carefully treat charge quantization and note this important point. In the dual viewpoint, starting from the superfluid phase, the triple vortex becomes gapless and condenses, leading to a fractional insulator. However, the elementary vortex of the superfluid phase remains gapped across the QCP and becomes the anyon in the FQSH phase.

V. EXPERIMENTAL SIGNATURES

We then move to the possible experimental signatures of this unusual QCP. In terms of φ, Eq. (3) is the standard critical theory for the XY transition describing interaction tuned superfluid to Mott insulator transition. The critical exponents for thermodynamic quantities are the same as the XY transition. However, the critical boson φ here is a nonlocal field and does not correspond to the microscopic order parameter. Hence, the transition is usually called XY* to highlight its difference from the conventional XY transition, which will be manifested in exciton correlation functions and conductivity.

A. Exciton correlation function

First, at the QCP, the critical boson has a power-law correlation function: $\langle\varphi^\dagger(\mathbf{x})\varphi(\mathbf{y})\rangle \sim (1/|\mathbf{x}-\mathbf{y}|^{1+\eta})$ with $\eta \approx 0.038$. However, the fractional exciton order φ is not measurable. The physical order parameter is the conventional exciton operator $\Phi^\dagger = c_1^\dagger c_2$. It is a composite operator in the critical theory: $\Phi = \varphi^3$ and its correlation function has a large decaying exponent $\langle\Phi^\dagger(\mathbf{x})\Phi(\mathbf{y})\rangle \sim (1/|\mathbf{x}-\mathbf{y}|^{1+\eta^*})$ with $\eta^* \approx 3.2$ estimated from the scaling dimension of the φ^3 mode of the 3D XY universality class [50]. The same exponent appears in the correlation function along the time direction, which leads to $\langle\Phi^\dagger(\mathbf{r},\omega)\Phi(\mathbf{r},-\omega)\rangle \sim \omega^{\eta^*}$ at position $\mathbf{r}$.

Interestingly, this exponent can be measured through a local interlayer tunneling experiment at position $\mathbf{r}$. Considering a local tunneling term $H' = \Gamma\int d^2\mathbf{r}\delta(\mathbf{r}) \times c_1^\dagger(\mathbf{r})c_2(\mathbf{r}) + \text{H.c.}$, linear response theory derives $I = 2e\Gamma^2 \text{Im}\chi_R(\omega = eV, \mathbf{r})$ with I and V as the current and voltage in the z direction. $\chi_R(\omega,\mathbf{r})$ is the Fourier transformation of $\chi_R(t,\mathbf{r}) = -i\theta(t)\langle[\Phi^\dagger(\mathbf{r},t),\Phi(\mathbf{r},0)]\rangle$ in the time direction [51]. So we expect that $(dI/dV) \sim V^{\eta^*-1} \approx V^{2.2}$ and is nonlinear to V at zero temperature at $d = d_c$. On the other hand, when $d < d_c$, we expect dI/dV to have a zero bias peak [52], and when $d > d_c$, it should have a threshold gap. A nonlinear I-V curve is expected at the edge of the FQHE phase with fractional charge [51]. Here we offer an example of the bulk tunneling at the QCP, and the large exponent η^* is a manifestation of the fractional charge carried by the critical boson. Sometimes it is more convenient to measure a global tunneling [53] from the term $H' = \Gamma\int d^2\mathbf{r} c_1^\dagger(\mathbf{r})c_2(\mathbf{r}) + \text{H.c.}$ In this case, we expect $I = 2e\Gamma^2\text{Im}\chi_R(\omega = eV, \mathbf{q} = 0)$ [54,55]. We have $(dI/dV) \sim V^{\eta^*-3} \approx V^{0.2}$ at the critical point.

B. Universal conductivity

The XY criticality is known to exhibit a universal conductivity. For our system, we define a 4×4 conductivity tensor in the direct-current limit as $\sigma = \begin{pmatrix} \sigma_{xx} & \sigma_{xy} \\ -\sigma_{xy} & \sigma_{xx} \end{pmatrix}$, where σ_{xx} and σ_{xy} are both a symmetric 2×2 matrix in layer space. $\sigma_{xx;ab} = (J_{x;a}/E_{x;b})$, where a, $b = 1, 2$ labels the two layers and $\sigma_{xy;ab}$ is defined similarly.

From Eq. (3), we get the longitudinal conductivity tensor at the QCP to be

$$\sigma_{xx} = \frac{\sigma_b}{9}\frac{e^2}{h}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}, \tag{5}$$

where σ_b is the universal conductivity for the ordinary XY transition, a number of order one in units of e^2/h. The factor of $\frac{1}{9}$ is because the critical boson carries only 1/3 of the ordinary exciton. Thus, the conductivity at this XY* transition is of order approximately $0.1(e^2/h)$. Further, σ_{xy} is purely from the background Chern-Simons term in Eq. (3). For the filling $(\nu_1,\nu_2) = (\frac{1}{3}, -\frac{1}{3})$, we have $\sigma_{xy} = \begin{pmatrix} \frac{1}{3} & 0 \\ 0 & -\frac{1}{3} \end{pmatrix}(e^2/h)$. For $(\nu_1,\nu_2) = (\frac{1}{3}, \frac{2}{3})$, we have $\sigma_{xy} = \begin{pmatrix} \frac{1}{3} & 0 \\ 0 & \frac{2}{3} \end{pmatrix}(e^2/h)$.

The inverse of σ gives the resistivity tensor $\rho = \begin{pmatrix} \rho_{xx} & -\rho_{yx} \\ \rho_{yx} & \rho_{xx} \end{pmatrix}$. For $(\nu_1,\nu_2) = (\frac{1}{3}, -\frac{1}{3})$, we have $\rho_{xx} = (h/e^2)\sigma_b\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ and $\rho_{yx} = (h/e^2)\begin{pmatrix} 3 & 0 \\ 0 & -3 \end{pmatrix}$. For $(\nu_1,\nu_2) = (\frac{1}{3}, \frac{2}{3})$, we have $\rho_{xx} = (h/e^2)(1/4+\sigma_b^2)\begin{pmatrix} 4\sigma_b & -2\sigma_b \\ -2\sigma_b & \sigma_b \end{pmatrix}$ and $\rho_{yx} = (h/e^2)(1/4+\sigma_b^2)\times \begin{pmatrix} 12+\sigma_b^2 & \sigma_b^2 \\ \sigma_b^2 & 6+\sigma_b^2 \end{pmatrix}$.

The above discussion is exactly at the QCP and zero temperature. In practice, the experiments are always at finite temperature, and one expects critical scaling $\rho(T,\delta) = F(T/\delta^{\nu z})$, where F is a universal function and $\delta = d - d_c$ is the deviation from the critical point. We have $z = 1$ and $\nu \approx 0.67$ as the known critical exponents for the XY transition. From collapsing the data of $(T, d - d_c)$, one can extrapolate the exponent νz and the universal conductivity. Such a scaling has been performed for the superconductor-to-insulator transitions [8,56–60].

VI. EXTRAORDINARY LOG BOUNDARY CRITICALITY

For the FQSH phase at $d > d_c$, there are helical edge modes. At $d < d_c$, the helical edge modes will be gapped out by the long-range exciton order. Here let us decide the fate of these edge modes at the critical point. The edge theory of the FQSH phase is

$$S_0 = \int dtdx \frac{1}{2\pi\tilde{v}_F\lambda}[(\partial_t\tilde{\theta})^2 - \tilde{v}_F^2(\partial_x\tilde{\theta})^2], \tag{6}$$

where $\tilde{\theta}$ represents the helical edge modes of the FQSH phase. Here, $e^{i\tilde{\theta}}$ creates a fractional exciton with charge 1/3 under A_s at the edge. $\lambda = \frac{4}{3}$ in the decoupling limit and becomes smaller when including interlayer repulsing at finite d, so we have $\lambda < \frac{4}{3}$. At the QCP, it is further coupled to the bulk critical boson through

$$S_{\mathrm{boundary}} = S_0 - s\int dxdt(e^{i\tilde{\theta}}\varphi^* + e^{-i\tilde{\theta}}\varphi). \tag{7}$$

We assume the antiunitary layer exchange symmetry $\mathcal{MCT}$ to guarantee that $e^{i\tilde{\theta}}$ carries zero momentum. Without the $\mathcal{MCT}$ symmetry, the above term is absent due to momentum mismatch and disorder needs to be involved, which we leave to future analysis.

The scaling dimension of the coupling s is $[s] = 2 - \Delta_\varphi - \frac{1}{4}\lambda \approx 0.78 - \frac{1}{4}\lambda > 0$, where we use $\Delta_\phi = 1.22$ as the boundary scaling dimension of the order parameter, so the coupling is relevant and flows to infinity (see the Appendix). It is thus very likely that it flows to the extraordinary-log-boundary critical point [46] recently proposed for the 3D XY transition. At this fixed point, the exciton order is almost long-range ordered at the edge: $\langle\Phi^\dagger(x)\Phi(y)\rangle \sim [1/\log(|x-y|)^{\tilde{q}}]$. This is in contrast to the large power-law decaying exponent for the correlation function in the bulk, manifested in the interlayer tunneling I-V curves as illustrated in Fig. 3(a). Besides, the exciton transport at the edge is still superfluidlike with an infinite conductance $G = (1/9\lambda)$ [61], which should be infinite at zero temperature, dramatically different from the metallic bulk transport. As a result, transport measurements in Hall bar geometry with edge and in the Corbino geometry without edge are very different at the QCP. The flow of $1/\lambda$ to zero is only logarithmic, so at finite temperature we expect $G \sim \log(1/T)$, which may be tested in experiment.

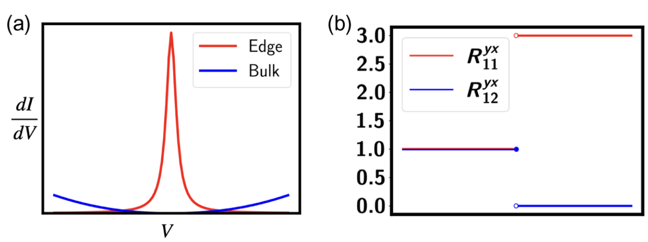

FIG. 3. (a) Illustration of the different behaviors of interlayer tunneling I-V curves between the bulk and edge at the critical point. (b) Hall resistivity R_{11}^{xy} and R_{21}^{xy} (in units of h/e^2) in the Hall bar geometry as tuning the distance d for the filling $(\nu_1,\nu_2) = (\frac{1}{3},\frac{2}{3})$. At d_c, because of the extraordinary boundary criticality, the exciton still behaves like a superfluid at the edge, which guarantees that $R_{11}^{yx} = R_{12}^{yx} = 1$. This is very different from the bulk values $R_{11}^{yx} = (12 + \sigma_b^2/4 + \sigma_b^2)$ and $R_{12}^{yx} = (\sigma_b^2/4 + \sigma_b^2)$ at the QCP, which are at intermediate values between the $d < d_c$ and $d > d_c$ phases.

So far, we have focused on the filling $(\nu_1,\nu_2) = (\frac{1}{3},-\frac{1}{3})$. For the filling $(\nu_1,\nu_2) = (\frac{1}{3},\frac{2}{3})$, the bulk behavior is exactly the same. But there is an additional edge mode from the integer quantum Hall effect. At the clean sample or weak disorder regime, this integer quantum Hall edge model cannot be hybridized with the FQSH edge modes and can be ignored. Thus, we expect the same extraordinary critical behavior. For example, at filling $(\nu_1,\nu_2) = (\frac{1}{3},\frac{2}{3})$, we show that ρ_{11}^{yx} and ρ_{21}^{yx} in the bulk are at certain fractional values depending on the universal conductivity σ_b. However, because of the extraordinary boundary behavior, we expect $\rho_{11}^{yx} = \rho_{21}^{yx} = 1$ in the Hall bar measurement, as illustrated in Fig. 3(b). The distinction between edge and bulk transport can be a direct verification of the proposed extraordinary boundary criticality. If there is strong disorder, then the two edge modes in layer 2 may be coupled together and flow to the Kane-Fisher-Polchinski fixed point [62] for the filling $(\nu_1,\nu_2) = (\frac{1}{3},\frac{2}{3})$. If this happens, we expect the coupling of the bulk exciton order parameter to the edge to be irrelevant, and we have an ordinary boundary critical behavior [46]. It is interesting to study the transition between extraordinary boundary criticality and the ordinary boundary criticality tuned by the disorder strength, which we leave to future work.

VII. XY* TRANSITION FOR GENERAL ABELIAN STATES

In the previous sections we focus on the filling $(\nu_1,\nu_2) = (\frac{1}{3},-\frac{1}{3})$. Here we point out that the XY* transition exists for any rational filling $(\nu_1,\nu_2) = (x,-x)$ as long as there is an Abelian FQHE phase at the filling x.

Let us consider a bilayer with an arbitrary Abelian FQHE state in one layer and its particle-hole partner in the other layer. We still have the $\mathcal{MCT}$ symmetry. Any Abelian FQHE phase can be captured by a K matrix with dimension N. The low-energy theory in the decoupled limit is

$$\mathcal{L} = -\frac{1}{4\pi}\mathbf{a}_1^T\mathbf{K}d\mathbf{a}_1 + \frac{1}{4\pi}\mathbf{a}_2^T\mathbf{K}d\mathbf{a}_2 + \frac{1}{2\pi}A_1\mathbf{q}^Td\mathbf{a}_1 - \frac{1}{2\pi}A_2\mathbf{q}^Td\mathbf{a}_2, \tag{8}$$

where $\mathbf{K}$ is an $N\times N$ matrix. $\mathbf{q}$ is an $N\times 1$ vector. Similarly, $\mathbf{a}_1, \mathbf{a}_2$ are emergent U(1) gauge fields with N components in the two layers. As before, A_1, A_2 are probe fields in the two layers with only one component. An Abelian FQHE phase is specified by $\mathbf{K}$ and $\mathbf{q}$. The analysis below applies to any Abelian FQHE phase.

The $\mathcal{MCT}$ transforms in the following way: $[a_1^0(t,\mathbf{r}), \vec{\mathbf{a}}_1(t,\mathbf{r})] \to [-\mathbf{a}_2^0(-t,\mathbf{r}), \vec{\mathbf{a}}_2(-t,\mathbf{r})]$, $[A_1^0(t,\mathbf{r}), \vec{A}_1(t,\mathbf{r})] \to [-A_2^0(-t,\mathbf{r}), \vec{A}_2(-t,\mathbf{r})]$.

Then we redefine $A_c = \frac{1}{2}(A_1 + A_2), A_s = A_1 - A_2$, $\mathbf{a}_c = \mathbf{a}_1 + \mathbf{a}_2$, $\mathbf{a}_s = \mathbf{a}_1 - \mathbf{a}_2$, the action for the decoupled phase is

$$\mathcal{L} = -\frac{1}{4\pi}\mathbf{a}_1^T\mathbf{K}d\mathbf{a}_1 + \frac{1}{4\pi}\mathbf{a}_2^T\mathbf{K}d\mathbf{a}_2 + \frac{1}{2\pi}A_c\mathbf{q}^T d\mathbf{a}_s + \frac{1}{4\pi}A_s\mathbf{q}^T d\mathbf{a}_c. \tag{9}$$

Suppose the lowest charged excitation at each layer is generated by the vector $\mathbf{l}_0$. Then, $\mathbf{l} = (\mathbf{l}_0, -\mathbf{l}_0)^T$ generates a boson with $Q_c = 0$ and $Q_s = \mathbf{q}^T\mathbf{K}^{-1}\mathbf{l}_0$. Let us label this boson as φ, then the critical theory is

$$\begin{aligned}\mathcal{L} = &|(\partial_\mu - i\mathbf{l}_0^T(\mathbf{a}_{1;\mu} - \mathbf{a}_{2;\mu}))\varphi|^2 - s|\varphi|^2 - g|\varphi|^4\\ &-\frac{1}{4\pi}\mathbf{a}_1^T\mathbf{K}d\mathbf{a}_1 + \frac{1}{4\pi}\mathbf{a}_2^T\mathbf{K}d\mathbf{a}_2\\ &+\frac{1}{2\pi}A_c\mathbf{q}^T d\mathbf{a}_s + \frac{1}{4\pi}A_s\mathbf{q}^T d\mathbf{a}_c,\end{aligned} \tag{10}$$

which can be rewritten as

$$\begin{aligned}\mathcal{L} = &|[\partial_\mu - i\mathbf{l}_0^T\mathbf{a}_{s;\mu})]\varphi|^2 - s|\varphi|^2 - g|\varphi|^4\\ &-\frac{1}{4\pi}\mathbf{a}_c^T\mathbf{K}d\mathbf{a}_s + \frac{1}{2\pi}A_c\mathbf{q}^T d\mathbf{a}_s + \frac{1}{4\pi}A_s\mathbf{q}^T d\mathbf{a}_c.\end{aligned} \tag{11}$$

With the assumption that $\det \mathbf{K} \neq 0$, we can integrate $\mathbf{a}_c$, which locks $\mathbf{a}_s = \mathbf{K}^{-1}\mathbf{q}A_s$. Then the final critical theory is

$$\mathcal{L} = |[\partial_\mu - iQ_sA_{s;\mu})]\varphi|^2 - s|\varphi|^2 - g|\varphi|^4 + \frac{\sigma_{xy}^{cs}}{2\pi}A_c dA_s, \tag{12}$$

where $Q_s = \mathbf{q}^T\mathbf{K}^{-1}\mathbf{l}_0$. $\sigma_{xy}^{cs} = \mathbf{q}^T\mathbf{K}^{-1}\mathbf{q}$.

The above action clearly describes an XY* transition with a condensation of a fractional exciton of exciton charge $Q_s = \mathbf{q}^T\mathbf{K}^{-1}\mathbf{l}_0$. Similar to our discussion for $(\nu_1, \nu_2) = (\frac{1}{3}, -\frac{1}{3})$, there is a fractional counterflow conductivity:

$$\sigma_{xx} = Q_s^2\sigma_b\frac{e^2}{h}\begin{pmatrix}1 & 1\\ 1 & 1\end{pmatrix}, \tag{13}$$

where σ_b is again the universal conductivity of the usual XY transition.

One simple example is $(\nu_1, \nu_2) = (\frac{1}{5}, -\frac{1}{5})$. Then the K matrix is one dimensional with $K = 5$, $l = 1$, $q = 1$. We simply reach that $Q_s = \frac{1}{5}$, indicating a fractional exciton with $1/5$ exciton charge at the XY* transition. A more nontrivial example is to consider $(\nu_1, \nu_2) = (\frac{2}{5}, -\frac{2}{5})$, or equivalently, $(\frac{2}{5}, \frac{3}{5})$. Now in the decoupled phase, we should use $K = \binom{3\ 1}{1\ 2}$ and $q = (1, 0)^T$. The smallest charged anyon is generated by $\mathbf{l}_0 = (1, 1)^T$, with charge $1/5$ and statistics $\frac{3}{5}\pi$. In the bilayer setup, $\mathbf{l} = (\mathbf{l}_0, -\mathbf{l}_0)^T$ generates a bosonic fractional exciton with charge $Q_s = \frac{1}{5}$. Again, we expect an XY* transition with $Q_s = \frac{1}{5}$. In both of these cases, the universal counterflow conductivity is $\frac{1}{25}\sigma_b e^2/h$. The physical exciton order parameter is $\Phi = \varphi^5$ and should have an even larger anomalous scaling dimension than the $(\nu_1, \nu_2) = (\frac{1}{3}, \frac{2}{3})$ case. For this case, a finite interlayer tunneling destroys the superfluid phase, but the XY* QCP is stable because $-\varphi^5 - \text{H.c.}$ is known to be irrelevant for the XY transition. In contrast, for the filling $(\nu_1, \nu_2) = (\frac{1}{3}, \frac{2}{3})$, an interlayer tunneling term acts as $-\varphi^3 - \text{H.c.}$ at low energy and drives the QCP to be a first-order transition.

One piece of direct evidence of the fractional exciton in the XY* transition is a fractional universal conductivity. However, this requires the value of σ_b, the universal conductivity of the ordinary XY transition. Unfortunately, there is no measurement or well-established prediction of the universal conductivity σ_b so far despite some theoretical and numerical calculations [63,64]. However, in a quantum Hall bilayer we can find XY* transitions at different fillings $(x, -x)$ to independently measure both σ_b and Q_s. For example, there is an XY* transition at filling $(\nu_1, \nu_2) = (\frac{1}{3}, -\frac{1}{3})$ with $Q_s = \frac{1}{3}$, and another XY* transition at filling $(\nu_1, \nu_2) = (\frac{2}{5}, -\frac{2}{5})$ with $Q_s = \frac{1}{5}$. A clear prediction is that the ratio of the universal counterflow conductivity at the QCP of these two fillings should be $\frac{25}{9}$. There should be many different XY* transitions corresponding to different rational fillings $x = (m/2pm \pm 1)$ with the charge of the elementary anyon known from well-established theory; thus, one can even do scaling between the counterflow conductivity at the QCP and the expected value of Q_s to test the picture that the critical exciton boson is formed by a pair of anyons.

VIII. DISCUSSION

Here we discuss the implications of asymmetry in the layer space, disorder, and interlayer tunneling which are neglected in our analysis of the ideal model.

First, the $\mathcal{MCT}$ symmetry can be weakly broken by asymmetry in the two layers. For example, different gate differences can lead to different intralayer interaction strengths. However, the stability of the XY* transition does not really need the $\mathcal{MCT}$ symmetry, which maps φ to $-\varphi$ in our critical theory. The relativistic nature of our theory relies on an emergent particle-hole symmetry $\varphi \to \varphi^\dagger$, which is due to the absence of the $i\varphi^*\partial_t\varphi$ term. This is purely from fine-tuning to the tip of the parabola phase boundary in Fig. 1 and does not rely on the $\mathcal{MCT}$ symmetry. The only worry we have is the asymmetry of the magnetic fields in the two layers which can lead to an effective background flux for φ. However, we expect that the difference of the magnetic field between the two layers is negligible in experiments given a small interlayer distance d. Actually, if we assume the magnetic field is always along the z direction in the experimental setup, then the familiar Gauss law $\vec{\nabla}\cdot\mathbf{B}(\mathbf{r}) = 0$ leads to $B_1(\mathbf{r}) - B_2(\mathbf{r}) = 0$. Hence, we do not need to worry about the effective flux of the critical boson φ. We conclude that the XY* transition is stable to other $\mathcal{MCT}$ asymmetries unless there is a gradient of magnetic field between the two layers. However, it is important to tune the transition at fixed exciton density. Away from this fine-tuning point, the chemical-potential-induced transition is nonrelativistic as in the familiar boson Hubbard model. Fortunately, it is easy to tune both the interlayer distance d/l_B and exciton chemical potential D (see Fig. 1) in the experiment, so there is no obstacle to tune to the XY* transition.

As for disorder, it is known that the XY transition is unstable to disorder and must flow to a new fixed point with different exponents with possibly a Bose glass phase in between [65,66]. It is natural to conjecture that with disorder our XY* critical point flows to a disordered version of an XY* fixed point. Theoretical analysis of such a fixed point is challenging and left to future work.

Then we turn to interlayer tunneling. Obviously, it will destroy the superfluid phase because now the U(1) symmetry corresponding to the layer pseudospin S_z rotation is explicitly broken. However, the XY* critical point can be stable depending on the filling. For $(\nu_1, \nu_2) = (\frac{1}{3}, -\frac{1}{3})$, the interlayer tunneling introduces a term $\lambda\varphi^3 + \mathrm{H.c.}$, which is known to be relevant and will drive the transition to be first order. In contrast, for filling $(\nu_1, \nu_2) = (\frac{1}{5}, -\frac{1}{5})$, the presence of weak interlayer tunneling is expected to introduce a $(\lambda\varphi^5 + \mathrm{H.c.})$ anisotropy which is expected to be dangerously irrelevant [67]. Hence, the XY* critical point remains stable, though now it is between a FQSH phase and a trivial insulator without any symmetry breaking.

IX. SUMMARY

In conclusion, we propose a route to accessing the XY* QCP by tuning magnetic field (and hence, the d/l_B ratio) while keeping the filling fixed at $(\nu_1, \nu_2) = (\frac{1}{3}, -\frac{1}{3})$ or $(\nu_1, \nu_2) = (\frac{1}{3}, \frac{2}{3})$ in a quantum Hall bilayer. At such a QCP, two anyons from the FQSH phase in the $d > d_c$ side form a fractional exciton and condense, leading to the exciton condensation phase in the $d < d_c$ side. The fractional charge of the exciton is manifested in the large anomalous exponent of the exciton correlation function and a $1/9$ factor in the universal conductivity. We also argue that the edge at this XY* transition shows extraordinary boundary criticality behavior. At criticality, the exciton behaves like a superfluid at the edge despite it showing metallic transport in the bulk. The relation between anyons in the topological phase and vorticity on the ordered side has led to the proposal of interesting "memory" effects in other contexts [68] which can also be explored here. These directions will be worth exploring in the future. In summary, our work suggests a new approach to studying exotic quantum phase transitions with fractionalization and unusual boundary critical behaviors in the highly tunable quantum Hall bilayer systems.

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Note added.—Recently, we became aware of another preprint [69] discussing continuous quantum phase transition in a quantum Hall bilayer system but in a different setup with different physics.

APPENDIX A: NUMERICAL DETAILS

We consider $\nu_T = 1/3 + 2/3$ quantum Hall bilayers subject to a perpendicular magnetic field on a torus, which is spanned by length vectors $\mathbf{L_x}$ and $\mathbf{L_y}$, and thus, the orbital number (or flux number) in each layer N_ϕ is determined by the area of the torus, i.e.,

$$|\mathbf{L_x} \times \mathbf{L_y}| = 2\pi N_\phi. \tag{A1}$$

Here, the magnetic length $l_B \equiv \sqrt{\hbar c/eB} \equiv 1$ (the unit of length). We choose the Landau gauge $\mathbf{A} = (By, 0, 0)$ and consider the torus with aspect ratio to be 1. The single-particle wave functions in the lowest Landau level as basis reads

$$\psi_k(x, y) = \frac{1}{\sqrt{L_x\sqrt{\pi}}} \sum_{n=-\infty}^{+\infty} e^{[i(k+nL_y)x-(y+nL_y+k)^2/2]}, \tag{A2}$$

where $k \equiv 2\pi j/L_x$ with $j = 0, 1, \ldots, N_\phi - 1$ due to the periodical boundary condition along the x direction. The single-particle states ψ_k are centered at $y = -k$ with a distance $2\pi/L_x$ apart along the y direction, while they are extended in the x direction. Then, the N_ϕ states can be mapped into a one-dimensional (1D) lattice with each site representing a single-particle orbital ψ_k. One can then perform numerical simulation on such a 1D lattice in momentum space with the number of sites equal to the number of orbitals. The relationship between the area of the torus and the size of 1D lattice is determined by Eq. (A1). In order to realize the numerical diagonalization on a larger system size, one needs to reduce the dimension of the Hamiltonian block by taking advantage of magnetic translational symmetries along the x and/or y directions. The symmetry analysis was first provided by Haldane [45] with introducing two translation operators T_α ($\alpha = 1, 2$) with eigenvalues $e^{2\pi i K_\lambda/N_\phi}$ ($\lambda = x$, y and $K_\lambda = 0, \ldots, N_\phi - 1$). T_1 corresponds to the magnetic translation in the x direction, where $K_x = \sum_{k=0}^{N_\phi - 1} k n_k$ (mod N_ϕ) is the total momentum (in the unit of $2\pi/L_x$) of electrons taken modulo N_ϕ. T_2 translates the entire lattice configuration one step $L_y/N_\phi = 2\pi/L_x$ to the right along the y direction. Taking advantage of one or both symmetries, one can numerically diagonalize the Hamiltonian efficiently. Different from the sphere geometry, there is no orbital number shift on the torus, and the states are uniquely determined by their filling factor.

In the present work, we consider the physical systems with two identical 2D layers (with zero width) in the absence of electron interlayer tunneling while spins of electrons are fully polarized due to strong magnetic fields. Such a system can be described by the projected Coulomb interaction,

$$V = \frac{1}{N_\phi} \sum_{i<j,\alpha,\beta} \sum_{\mathbf{q},\mathbf{q}\neq 0} V_{\alpha\beta}(q) e^{-\frac{q^2}{2}} L_n^2 \left[\frac{q^2}{2}\right] e^{i\mathbf{q}\cdot(\mathbf{R}_{\alpha,i} - \mathbf{R}_{\beta,j})}. \quad \text{(A3)}$$

Here, $\alpha(\beta) = 1, 2$ denote two layers or, equivalently, two components of a pseudospin-1/2. $q = |\mathbf{q}| = \sqrt{q_x^2 + q_y^2}$, $V_{11}(q) = V_{22}(q) = e^2/(\varepsilon q)$, and $V_{12}(q) = V_{21}(q) = e^2/(\varepsilon q) e^{-qd}$ are the Fourier transformations of the intralayer and interlayer Coulomb interactions, respectively. d represents the distance between two layers in the unit of magnetic length l_B. $L_n(x)$ is the Laguerre polynomial with Landau-level index n, and $\mathbf{R}_{\alpha,i}$ is the guiding center coordinate of the ith electron in layer α. Here we consider rectangular unit cells with $L_x = L_y = L$ and set magnetic length $l_B \equiv \sqrt{\hbar c/eB}$ as the unit of length and $e^2/\varepsilon l_B$ as the unit of energy. Numerically, one needs to use the second-quantization form

$$V = \sum_{j_1 j_2 j_3 j_4} V^{\alpha\beta}_{j_1 j_2 j_3 j_4} c^\dagger_{j_1} c^\dagger_{j_2} c_{j_3} c_{j_4}, \quad \text{(A4)}$$

with

$$V^{\alpha\beta}_{j_1 j_2 j_3 j_4} = \delta'_{j_1+j_2, j_3+j_4} \frac{1}{4\pi N_\phi} \sum_{\mathbf{q},\mathbf{q}\neq 0} \delta'_{j_1 - j_4, q_y L_y/2\pi} V_{\alpha\beta}(q) \exp\left[-q^2/2 - i(j_1 - j_3) q_x L_x/N_\phi\right] L^2_{n=0}[-q^2/2]. \quad \text{(A5)}$$

Here, the Kronecker delta with the prime means that the equation is defined modulo N_ϕ. We also consider a uniform and positive background charge so that the Coulomb interaction at $q = 0$ is canceled out.

APPENDIX B: SYMMETRY OF THE MODEL

In this appendix, we point out a symmetry $\mathcal{MCT}$ for the quantum Hall bilayer at filling $(\nu_1, \nu_2) = (x, -x)$. Here, $\nu_a = (N_e/N_\Phi)$, where N_Φ is the number of the magnetic flux in the system. $\nu_2 < 0$ means that the system is hole doped with hole density at x per flux. We are mainly interested in the $x = 1/3$ point.

We define the electron operators in layers 1 and 2 as $c_1(\mathbf{r})$ and $c_2(\mathbf{r})$. Because layer 2 is hole doped, it is convenient to use the hole operator $h_2(\mathbf{r}) = c_2^\dagger(\mathbf{r})$. The Hamiltonian is

$$\mathcal{H} = \int d^2\mathbf{r} c_1^\dagger(\mathbf{r}) \frac{(-i\vec{\nabla} - e\vec{A}_1)^2}{2m} c_1(\mathbf{r}) + \int d^2\mathbf{r} h_2^\dagger(\mathbf{r}) \frac{(-i\vec{\nabla} + e\vec{A}_2)^2}{2m} h_2(\mathbf{r}) - D \int d^2\mathbf{r} \big(c_1^\dagger(\mathbf{r}) c_1(\mathbf{r}) + h_2^\dagger(\mathbf{r}) h_2(\mathbf{r})\big) + H_{\text{int}}, \quad \text{(B1)}$$

where e is the electron charge, which is negative. $\vec{A}_1(\mathbf{r})$ and $\vec{A}_2(\mathbf{r})$ are vector fields in the two layers. For the quantum Hall bilayer system, we have the magnetic field $\vec{A}_a(\mathbf{r}) = \frac{1}{2} B\hat{z} \times \mathbf{r} + \delta\vec{A}_a(\mathbf{r})$. Here, $\delta\vec{A}_a(\mathbf{r})$ is the probing field in each layer applied to measure the response of the system. D is the displacement field and can be viewed as the chemical potential to tune the exciton density x.

The interaction term is

$$\mathcal{H}_{\text{int}} = \frac{1}{2} \int d^2\mathbf{r} \int d^2\mathbf{r}' V_{ab}(|\mathbf{r} - \mathbf{r}'|) :\, \rho_a(\mathbf{r}) \rho_b(\mathbf{r}') :, \quad \text{(B2)}$$

where

$$\rho_1(\mathbf{r}) = e c_1^\dagger(\mathbf{r}) c_1(\mathbf{r}) \tag{B3}$$

and

$$\rho_2(\mathbf{r}) = -e h_2^\dagger(\mathbf{r}) h_2(\mathbf{r}). \tag{B4}$$

The Coulomb interaction is in the form $V_{ab}(\mathbf{r}) = [1/(2\pi)^2] \int d^2\mathbf{q} V_{ab}(\mathbf{q}) e^{i\mathbf{q}\cdot\mathbf{r}}$ with $V_{11}(q) = V_{22}(q) = (e^2/\varepsilon q)$ and $V_{12}(q) = V_{21}(q) = (e^2/\varepsilon q)e^{-qd}$. d represents the distance between two layers in the unit of magnetic length $l_B = \sqrt{\hbar c/eB}$.

Let us ignore the probing field $\delta\vec{A}_a(\mathbf{r})$ for now. Then, the Hamiltonian has the following $\mathcal{MCT}$ antiunitary symmetry:

$$\mathcal{MCT}:\ c_1(\mathbf{r}) \leftrightarrow h_2(\mathbf{r}). \tag{B5}$$

Note that this is an antiunitary, and we need to include a complex conjugate $\mathcal{K}$. Under $\mathcal{MCT}$, $\rho_1(\mathbf{r}) \to -\rho_2(\mathbf{r})$ and $\rho_1(\mathbf{q}) \to -\rho_2(-\mathbf{q})$. Under $\mathcal{MCT}$, an electron in layer 1 is transformed to a hole in layer 2. We can also include the vector potential $\delta\vec{A}_a(\mathbf{r})$ and also the electric potential term $\delta H = -\rho_1(\mathbf{r})A_1^0(\mathbf{r}) - \rho_2(\mathbf{r})A_2^0(\mathbf{r})$. Thus, under $\mathcal{MCT}$, we have $A_1^0(\mathbf{r}) \to -A_2^0(\mathbf{r})$ and $\delta\vec{A}_1(\mathbf{r}) \to \delta\vec{A}_2(\mathbf{r})$. In the following, we use $\vec{A}_a(\mathbf{r})$ to denote $\delta\vec{A}_a(\mathbf{r})$. The spacetime coordinates transform as $(t, \mathbf{r}) \to (-t, \mathbf{r})$ under $\mathcal{MCT}$.

One can diagonalize the kinetic part and project the interaction in the lowest Landau levels. Within the lowest Landau level, the Hamiltonian is

$$\mathcal{H} = \frac{1}{2} \sum_{a,b=1,2} V_{ab}(\mathbf{q}) :\ \rho_a(\mathbf{q})\rho_a(-\mathbf{q}) :. \tag{B6}$$

Here,

$$\rho_a(\mathbf{q}) = \int d^2\mathbf{q} \rho_a(\mathbf{r}) e^{-i\mathbf{q}\cdot\mathbf{r}} \tag{B7}$$

with $\rho_a(\mathbf{r})$ as the charge-density operator projected to the lowest Landau level:

$$\rho_1(\mathbf{r}) = e \sum_{m,n} c_{1;m}^\dagger c_{1;n} \varphi_{+;m}^*(\mathbf{r}) \varphi_{+;n}(\mathbf{r}) \tag{B8}$$

and

$$\rho_2(\mathbf{r}) = -e \sum_{m,n} h_{2;m}^\dagger h_{2;n} \varphi_{-;m}^*(\mathbf{r}) \varphi_{-;n}(\mathbf{r}). \tag{B9}$$

In the above, $\varphi_{+,m}(\mathbf{r})$ and $\varphi_{-;m}$ are the wave functions of the state labeled by the Landau index m for the electron and hole, respectively. We have $\varphi_{-;m}(\mathbf{r}) = \varphi_{+;m}(\mathbf{r})^*$.

Under the symmetry $\mathcal{MCT}$, $c_{1;m} \to h_{2;m}$ and $\rho_1(\mathbf{r}) \to -\rho_2(\mathbf{r})$ still hold in the Hamiltonian projected to the lowest Landau level.

APPENDIX C: RELATION TO THE FQSH TO PAIRED-SUPERFLUID TRANSITION AND AN ESTIMATE FOR d_c

Thus far, we have phrased the discussion in terms of exciton condensation on the $(\nu_1, \nu_2) = (1/3, 2/3)$ insulator. Now, we consider performing a particle-hole conjugation on layer 2, as described above, i.e., $h_2(r) = c_2^\dagger(r)$. This leads to the two layers now having the same density but experiencing opposite magnetic fields as in Eq. (B1). This is nothing but the fractional version of the FQSH [70]. Furthermore, as a result of the particle-hole transformation in the bottom layer, the repulsive interlayer interaction now becomes an attractive interaction and leads to the formation of bound states between the layers of Laughlin quasiparticles, i.e., Laughlin-Cooper pairs of charge $2e/3$. The condensation of these fractional Cooper pairs leads to a paired superconductor [71] (which is smoothly connected to the paired condensate of electrons). Note, the Laughlin-Cooper pair is a boson and has mutual statistics with all the other anyons in the problem; hence, its condensation leads to a conventional superconductor.

The following simple energetic argument gives an estimate for the critical distance d_c. Consider creating a Laughlin quasielectron + quasihole in one layer and a corresponding pair in the other layer. The energy for each pair is just the gap: $\Delta_{1/3} \approx 0.1 e^2/\epsilon l_B$ [72]. Now consider the strength of the attractive interaction between the quasiparticle and quasihole in the two layers $\Delta E = e^{*2}/\epsilon d$, where $e^* = e/3$. A simple estimate for the critical point is when the binding energy overcomes the cost of creating the quasiparticles and hence, $e^{*2}/\epsilon d = \Delta_{1/3}$ or $(d_c^*/l_B) = 1.1$. In practice, the excitons will have a dispersion that further lowers their energy, and we would expect that the true transition occurs earlier, i.e., $d_c > d_c^*$. Our numerical calculations give $d_c \approx 1.7$. Similarly, one can estimate the critical distance for the $(\nu_1, \nu_2) = (1/5, 4/5)$ using $\Delta_5 = 0.024 e^2/l_B$ [73] and $e^* = e/5$.

APPENDIX D: CRITICAL THEORY

We consider the filling $(\nu_1, \nu_2) = (\frac{1}{3}, -\frac{1}{3})$. In the FQSH phase at large d/l_B, the effective low-energy theory is captured by a Chern-Simons theory with K matrix $K = \begin{pmatrix} 3 & 0 \\ 0 & -3 \end{pmatrix}$. There are two emergent gauge fields $a_{1;\mu}, a_{2;\mu}$ whose charges are labeled as $l = (l_1, l_2)$. We consider a fractional exciton labeled by $l = (1, 1)$. It carries physical charge $Q_c = 0$ and physical spin $Q_s = \frac{1}{3}$. The transition between the FQSH phase and the exciton condensation phase is described by the condensation of this fractional exciton whose creation operator is labeled as $\varphi^\dagger$. Then, the critical theory is

$$\begin{aligned}\mathcal{L} = &|[\partial_\mu - i(a_{1;\mu} - a_{2;\mu})]\varphi|^2 - s|\varphi|^2 - g|\varphi|^4 \\ &-\frac{3}{4\pi}a_1 da_1 + \frac{3}{4\pi}a_2 da_2 + \frac{1}{2\pi}A_c d(a_1 - a_2) \\ &+\frac{1}{4\pi}A_s d(a_1 + a_2), \end{aligned} \tag{D1}$$

where s is the tuning parameter. $A_{c;\mu}$ and $A_{s;\mu}$ are probing fields coupled to the total charge and the spin. The $\mathcal{MCT}$ symmetry requires that $\vec{A}_s = 0$. Physically, the magnetic fields of the two layers are the same, and the exciton does not feel any net magnetic field. From symmetry analysis, one can, in principle, include a term $\varphi^*(i\partial_t - A_{s;0})\varphi$. This term is fine-tuned to be zero because we are considering an interaction-driven transition through d/l_B at fixed exciton density. On the other hand, one can also tune a FQSH-to-SF transition through tuning the displacement field, which corresponds to a chemical-potential-tuned transition. The dynamical exponent for such a transition will be $z = 2$ and is not our interest.

We then make a redefinition: $a_c = a_1 + a_2, a_s = a_1 - a_2$. The above actions change to

$$\begin{aligned}\mathcal{L} = &|(\partial_\mu - ia_{s;\mu})\varphi|^2 - s|\varphi|^2 - g|\varphi|^4 - \frac{3}{4\pi}a_c da_s \\ &+\frac{1}{2\pi}A_c da_s + \frac{1}{4\pi}A_s da_c. \end{aligned} \tag{D2}$$

The redefinition of the gauge field changes the charge quantization rules. The charge under a_c is labeled as q_c and the charge of a_s is q_s. We have the charge transformation $q_c = (q_1 + q_2/2)$ and $q_s = (q_1 - q_2/2)$, where q_1, q_2 are charges under a_1, a_2. The elementary charge configuration is $(q_1, q_2) = (\pm 1, 0), (0, \pm 1)$. Thus, in terms of (q_c, q_s), the elementary charge is $(q_c, q_s) = (\pm\frac{1}{2}, \pm\frac{1}{2})$. In the FQSH phase with φ gapped, one can check that for the excitation $(q_c, q_s) = (\pm\frac{1}{2}, \pm\frac{1}{2})$, it has statistics $\theta = \pm(\pi/3)$, physical charge $Q_c = \pm\frac{1}{3}$, and physical spin $Q_s = \pm\frac{1}{6}$. This is exactly the elementary anyon on layer 1 or layer 2 in the FQSH phase. When φ is condensed, a_s is Higgsed, and we are left with the superfluid action $\mathcal{L}_{\mathrm{SF}} = (1/4\pi)A_s da_c$. We can see that the vortex charge q_v of the superfluid is $q_v = 2q_c$. Then, the elementary anyon with $(q_c, q_s) = \pm(\frac{1}{2}, \frac{1}{2})$ in the FQSH phase now becomes the elementary vortex with $q_v = \pm 1$ in the superfluid phase. Of course, now it costs infinite energy due to the coupling to the gapless gauge field a_c which represents the Goldstone mode of the superfluid. From this analysis, one can see that the elementary anyon in the FQSH phase becomes the vortex of the superfluid in the EC phase. Its energy cost is infinite in the EC phase and finite in the FQSH phase, but it remains gapped across the phase transition. It is known that the elementary vortex become gapless and then condensed in the usual superfluid-to-insulator transition. Later in the dual theory, we see that at the XY* critical point, what becomes gapless is a triple vortex, while the vortexes with $|q_v| = 1$, 2 remain gapped.

At the critical point, the topological property of the anyon does not matter, so we can integrate a_c, which simply locks $a_s = \frac{1}{3}A_s$. Then we reach the final critical theory:

$$\mathcal{L}_c = \left|\left(\partial_\mu - i\frac{1}{3}A_{s;\mu}\right)\varphi\right|^2 - s|\varphi|^2 - g|\varphi|^4 + \frac{1}{6\pi}A_c dA_s. \tag{D3}$$

When $s < 0$, this is a superfluid phase of A_s. When $s > 0$, we have the correct response of $(1/6\pi)A_c dA_s$ for the FQSH phase, though we have lost the information about the anyons by integrating a_c.

1. Dual theory

It is known that the XY critical theory such as in Eq. (D3) has a dual theory. Here we derive the dual critical theory. We start from Eq. (D2) and apply the standard particle-vortex duality for φ, and then we obtain a dual critical theory as

$$\begin{aligned}\mathcal{L} = &|(\partial_\mu - ib_\mu)\tilde{\varphi}|^2 - r|\tilde{\varphi}|^2 - \tilde{g}|\tilde{\varphi}|^4 + \frac{1}{e_c^2}f_{c;\mu\nu}f^{c;\mu\nu} \\ &+\frac{1}{e_s^2}f_{s;\mu\nu}f^{s;\mu\nu} - \frac{3}{4\pi}a_s da_c + \frac{1}{2\pi}bda_s \\ &+\frac{1}{2\pi}A_c da_s + \frac{1}{4\pi}A_s da_c, \end{aligned} \tag{D4}$$

where $f_{c;\mu\nu} = \partial_\mu a_{c;\nu} - \partial_\nu a_{c;\mu}$ is the field strength for a_c, and similarly, $f_{s;\mu\nu}$ is the field of a_s.

Integrating a_s, we lock $b = \frac{3}{2}a_c - A_c$. With a redefinition $a = \frac{1}{2}a_c - \frac{1}{3}A_c$, we get

$$\begin{aligned}\mathcal{L} = &|(\partial_\mu - i3a_\mu)\tilde{\varphi}|^2 - r|\tilde{\varphi}|^2 - \tilde{g}|\tilde{\varphi}|^4 + \frac{1}{e^2}f_{\mu\nu}f^{\mu\nu} \\ &+\frac{1}{2\pi}A_s da + \frac{1}{6\pi}A_s dA_c, \end{aligned} \tag{D5}$$

which is exactly the dual theory of Eq. (D3). Here, $f_{\mu\nu} = \partial_\mu a_\nu - \partial_\nu a_\mu$. $e^2 = e_c^2/4$. We choose the normalization of the gauge field a so we have the usual coupling $(1/2\pi)A_s da$. Hence, the monopole operator of a carries charge 1 under A_s and can be identified as the creation operator of the physical exciton $c_1^\dagger c_2$. When $\tilde{\varphi}$ is gapped in the $r > 0$ side, this describes a superfluid phase of A_s. Here, $\tilde{\varphi}$ here carries vortex charge $q_v = 3$ of the superfluid phase. Therefore, at the QCP the triple vortex instead of the elementary vortex of the superfluid becomes gapless.

Condensation of this triple vortex kills the superfluid phase and leads to an insulator. The elementary vortex remains gapped across the QCP and will become the anyon in the FQHE insulator after the superfluid is gone.

APPENDIX E: EXTRAORDINARY BOUNDARY CRITICALITY

We start from the edge theory for the FQSH phase at $d > d_c$. For now, let us consider the filling $(\nu_1, \nu_2) = (\frac{1}{3}, -\frac{1}{3})$. From the K matrix $K = \left(\begin{smallmatrix} 3 & 0 \\ 0 & -3 \end{smallmatrix}\right)$, we can write down the effective action for the helical edge modes:

$$S = \int dtdx \frac{3}{4\pi}\partial_t\varphi_1\partial_x\varphi_1 - \frac{3}{4\pi}\partial_t\varphi_2\partial_x\varphi_2 - \frac{3}{4\pi}v_F(\partial_x\varphi_1)^2 - \frac{3}{4\pi}v_F(\partial_x\varphi_2)^2 - g\frac{3}{2\pi}v_F(\partial_x\varphi_1)(\partial_x\varphi_2), \tag{E1}$$

where φ_1 and φ_2 represent the edge mode in layers 1 and 2, respectively. The $g < 0$ term is from interlayer repulsion. Note in our current convention the density operators are $\rho_1 = e(1/2\pi)\partial_x\varphi_1$ and $\rho_2 = -e(1/2\pi)\partial_x\varphi_2$.

We have the commutation relations

$$[\varphi_1(x), \partial_y\varphi_1(y)] = i\frac{2\pi}{3}\delta(x-y) \tag{E2}$$

and

$$[\varphi_2(x), \partial_y\varphi_2(y)] = -i\frac{2\pi}{3}\delta(x-y). \tag{E3}$$

Next, we do a linear combination and define

$$\varphi_1(x) = \frac{1}{\sqrt{3}}[\phi(x) + \theta(x)],$$
$$\varphi_2(x) = \frac{1}{\sqrt{3}}[\phi(x) - \theta(x)], \tag{E4}$$

so

$$[\theta(x), \partial_y\varphi(y)] = i\pi\delta(x-y). \tag{E5}$$

This leads to the action

$$S = \int dtdx \frac{1}{\pi}\partial_t\phi\partial_x\theta - \frac{\tilde{v}_F}{2\pi}\left[K(\partial_x\theta)^2 + \frac{1}{K}(\partial_x\phi)^2\right], \tag{E6}$$

where $\tilde{v}_F = \sqrt{1-g^2}v_F$ and $K = (1 - g/1 + g)$. So we have $K > 1$.

We can also integrate ϕ to reach

$$S = \int dtdx \frac{K}{2\pi\tilde{v}_F}[(\partial_t\theta)^2 - \tilde{v}_F^2(\partial_x\theta)^2]. \tag{E7}$$

This is the standard action for the Luttinger liquid, but the operator mapping is different. In particular, the fractional exciton $l = (1, -1)$ now corresponds to $\psi^\dagger \sim e^{i\varphi_R}e^{-i\varphi_L} = e^{i\frac{2}{\sqrt{3}}\theta(x)}$. Its scaling dimension is $[\psi] = \frac{1}{3K} < \frac{1}{3}$. We can make a redefinition $\tilde{\theta} = \frac{2}{\sqrt{3}}\theta$, so the action is

$$S_0 = \int dtdx \frac{1}{2\pi\tilde{v}_F\lambda}[(\partial_t\tilde{\theta})^2 - \tilde{v}_F^2(\partial_x\tilde{\theta})^2] \tag{E8}$$

with $\lambda = (4/3K)$. In this convention, $e^{i\tilde{\theta}}$ creates a fractional exciton with charge $1/3$ under A_s.

At the QCP, the boundary is described by the following action:

$$S_{\text{boundary}} = \int dtdx \frac{1}{2\pi\tilde{v}_F\lambda}[(\partial_t\tilde{\theta})^2 - \tilde{v}_F^2(\partial_x\tilde{\theta})^2] - s\int dxdt(e^{i\tilde{\theta}}\varphi^* + e^{-i\tilde{\theta}}\varphi). \tag{E9}$$

Following Ref. [46], we obtain the renormalization flow equation

$$\frac{ds}{dl} = \left(2 - \Delta_\varphi - \frac{1}{4}\lambda\right)s,$$
$$\frac{d\lambda}{dl} = -\pi^2 s^2\lambda^2. \tag{E10}$$

Given that $\Delta_\varphi \approx 1.219$ and initially $\lambda < \frac{4}{3}$, we have s flow to infinity and λ flows to zero when the renormalization-group flow l approaches infinity.

The ordinary exciton order creation operator is $e^{i3\tilde{\theta}}$. In the FQSH phase, its correlation function is

$$\langle e^{i3\tilde{\theta}(x)}e^{-i3\tilde{\theta}(y)}\rangle \sim \frac{1}{|x-y|^{9\lambda}}. \tag{E11}$$

Therefore, at the QCP, because λ flows to zero, the exponent of the above correlation function also flows to zero. In practice, it should have a log singularity [46] at the QCP:

$$\langle e^{i3\tilde{\theta}(x)}e^{-i3\tilde{\theta}(0)}\rangle \sim \frac{1}{(\log x)^q}, \qquad x \to \infty. \tag{E12}$$

This is the so-called extraordinary-log-boundary critical behavior. One can see that the exciton order has an almost long-range order at the edge, despite its correlation function having a large decaying exponent in the bulk.

The exciton current A_s couples in the following way: $\partial_\mu\tilde{\theta} \to (\partial_\mu - i\frac{1}{3}A_{s;\mu})\tilde{\theta}$. In the FQSH phase, it is known that the conductance under A_s is $G = \frac{1}{9}(1/\lambda)(e^2/h)$ [61]. At the

QCP, the conductance G then flows to infinity. This is expected because the exciton has almost long-range order, so its transport should still be superfluidlike.

We also want to comment on the local density of states of one layer probed by the scanning tunneling microscope (STM). Consider layer 1, the single-electron creation operator is $c_1^\dagger(x) \sim e^{i3\varphi_1(x)} = e^{i\sqrt{3}[\phi(x)+\theta(x)]}$. Then we expect that the STM of layer 1 has $(dI/dV) \sim V^\alpha$ with $\alpha = \frac{3}{2}[K + (1/K)] - 1$. In the decoupled phase at large d/l_b, we have $\alpha \approx 2$ as $K \approx 1$. However, at the QCP, α goes to infinity in the extraordinary criticality.

[1] Subir Sachdev, *Quantum Phase Transitions*, Phys. World **12**, 33 (1999).

[2] Shivaji Lal Sondhi, S. M. Girvin, J. P. Carini, and Dan Shahar, *Continuous Quantum Phase Transitions*, Rev. Mod. Phys. **69**, 315 (1997).

[3] Todadri Senthil, Ashvin Vishwanath, Leon Balents, Subir Sachdev, and Matthew P. A. Fisher, *Deconfined Quantum Critical Points*, Science **303**, 1490 (2004).

[4] Andrey V. Chubukov, T. Senthil, and Subir Sachdev, *Universal Magnetic Properties of Frustrated Quantum Antiferromagnets in Two Dimensions*, Phys. Rev. Lett. **72**, 2089 (1994).

[5] Sergei V. Isakov, Roger G. Melko, and Matthew B. Hastings, *Universal Signatures of Fractionalized Quantum Critical Points*, Science **335**, 193 (2012).

[6] Yan-Cheng Wang, Meng Cheng, William Witczak-Krempa, and Zi Yang Meng, *Fractionalized Conductivity and Emergent Self-Duality near Topological Phase Transitions*, Nat. Commun. **12**, 1 (2021).

[7] Michael Schuler, Louis-Paul Henry, Yuan-Ming Lu, and Andreas M. Läuchli, *Emergent XY* Transition Driven by Symmetry Fractionalization and Anyon Condensation*, SciPost Phys. **14**, 001 (2023).

[8] Fang Wang, Johan Biscaras, Andreas Erb, and Abhay Shukla, *Superconductor-Insulator Transition in Space Charge Doped One Unit Cell* $\mathrm{Bi_{2.1}Sr_{1.9}CaCu_2O_{8+x}}$, Nat. Commun. **12**, 1 (2021).

[9] Ruben Verresen, Mikhail D. Lukin, and Ashvin Vishwanath, *Prediction of Toric Code Topological Order from Rydberg Blockade*, Phys. Rev. X **11**, 031005 (2021).

[10] K. J. Satzinger *et al.*, *Realizing Topologically Ordered States on a Quantum Processor*, Science **374**, 1237 (2021).

[11] G. Semeghini, H. Levine, A. Keesling, S. Ebadi, T. T. Wang, D. Bluvstein, R. Verresen, H. Pichler, M. Kalinowski, R. Samajdar, A. Omran, S. Sachdev, A. Vishwanath, M. Greiner, V. Vuletić, and M. D. Lukin, *Probing Topological Spin Liquids on a Programmable Quantum Simulator*, Science **374**, 1242 (2021).

[12] Horst L. Stormer, Daniel C. Tsui, and Arthur C. Gossard, *The Fractional Quantum Hall Effect*, Rev. Mod. Phys. **71**, S298 (1999).

[13] Sankar Das Sarma and Aron Pinczuk, *Perspectives in Quantum Hall Effects: Novel Quantum Liquids in Low-Dimensional Semiconductor Structures* (John Wiley & Sons, New York, 2008).

[14] Bertrand I. Halperin, *Theory of the Quantized Hall Conductance*, Helv. Phys. Acta **56**, 75 (1983).

[15] J. P. Eisenstein and A. H. MacDonald, *Bose-Einstein Condensation of Excitons in Bilayer Electron Systems*, Nature (London) **432**, 691 (2004).

[16] J. P. Eisenstein, *Exciton Condensation in Bilayer Quantum Hall Systems*, Annu. Rev. Condens. Matter Phys. **5**, 159 (2014).

[17] K. Moon, H. Mori, Kun Yang, S. M. Girvin, A. H. MacDonald, L. Zheng, D. Yoshioka, and Shou-Cheng Zhang, *Spontaneous Interlayer Coherence in Double-Layer Quantum Hall Systems: Charged Vortices and Kosterlitz-Thouless Phase Transitions*, Phys. Rev. B **51**, 5138 (1995).

[18] Kun Yang, K. Moon, Lotfi Belkhir, H. Mori, S. M. Girvin, A. H. MacDonald, L. Zheng, and D. Yoshioka, *Spontaneous Interlayer Coherence in Double-Layer Quantum Hall Systems: Symmetry-Breaking Interactions, In-Plane Fields, and Phase Solitons*, Phys. Rev. B **54**, 11644 (1996).

[19] Xiaomeng Liu, Kenji Watanabe, Takashi Taniguchi, Bertrand I. Halperin, and Philip Kim, *Quantum Hall Drag of Exciton Condensate in Graphene*, Nat. Phys. **13**, 746 (2017).

[20] J. I. A. Li, T. Taniguchi, K. Watanabe, J. Hone, and C. R. Dean, *Excitonic Superfluid Phase in Double Bilayer Graphene*, Nat. Phys. **13**, 751 (2017).

[21] N. E. Bonesteel, I. A. McDonald, and C. Nayak, *Gauge Fields and Pairing in Double-Layer Composite Fermion Metals*, Phys. Rev. Lett. **77**, 3009 (1996).

[22] Y. B. Kim, C. Nayak, E. Demler, N. Read, and S. Das Sarma, *Bilayer Paired Quantum Hall States and Coulomb Drag*, Phys. Rev. B **63**, 205315 (2001).

[23] J. Schliemann, S. M. Girvin, and A. H. MacDonald, *Strong Correlation to Weak Correlation Phase Transition in Bilayer Quantum Hall Systems*, Phys. Rev. Lett. **86**, 1849 (2001).

[24] A. Stern and B. I. Halperin, *Strong Enhancement of Drag and Dissipation at the Weak-to Strong-Coupling Phase Transition in a Bilayer System at a Total Landau Level Filling* $\nu = 1$, Phys. Rev. Lett. **88**, 106801 (2002).

[25] S. H. Simon, E. H. Rezayi, and M. V. Milovanovic, *Coexistence of Composite Bosons and Composite Fermions in* $\nu = 1/2 + 1/2$ *Quantum Hall Bilayers*, Phys. Rev. Lett. **91**, 046803 (2003).

[26] D. N. Sheng, L. Balents, and Z. Wang, *Phase Diagram for Quantum Hall Bilayers at* $\nu = 1$, Phys. Rev. Lett. **91**, 116802 (2003).

[27] K. Park, *Spontaneous Pseudospin Spiral Order in Bilayer Quantum Hall Systems*, Phys. Rev. B **69**, 045319 (2004).

[28] Naokazu Shibata and Daijiro Yoshioka, *Ground State of* $\nu = 1$ *Bilayer Quantum Hall Systems*, J. Phys. Soc. Jpn. **75**, 043712 (2006).

[29] Gunnar Möller, Steven H. Simon, and Edward H. Rezayi, *Paired Composite Fermion Phase of Quantum Hall Bilayers at* $\nu = 1/2 + 1/2$, Phys. Rev. Lett. **101**, 176803 (2008).

[30] Gunnar Möller, Steven H. Simon, and Edward H. Rezayi, *Trial Wave Functions for* $\nu = 1/2 + 1/2$ *Quantum Hall Bilayers*, Phys. Rev. B **79**, 125106 (2009).

[31] M. V. Milovanović and Z. Papić, *Nonperturbative Approach to the Quantum Hall Bilayer*, Phys. Rev. B **79**, 115319 (2009).

[32] J. Alicea, O. I. Motrunich, G. Refael, and M. P. A. Fisher, *Interlayer Coherent Composite Fermi Liquid Phase in Quantum Hall Bilayers*, Phys. Rev. Lett. **103**, 256403 (2009).

[33] Z. Papić and M. V. Milovanović, *Disordering of the Correlated State of the Quantum Hall Bilayer at Filling Factor* $\nu = 1$, Mod. Phys. Lett. B **26**, 1250134 (2012).

[34] Inti Sodemann, Itamar Kimchi, Chong Wang, and T. Senthil, *Composite Fermion Duality for Half-Filled Multicomponent Landau Levels*, Phys. Rev. B **95**, 085135 (2017).

[35] Hiroki Isobe and Liang Fu, *Interlayer Pairing Symmetry of Composite Fermions in Quantum Hall Bilayers*, Phys. Rev. Lett. **118**, 166401 (2017).

[36] Z. Zhu, L. Fu, and D. N. Sheng, *Numerical Study of Quantum Hall Bilayers at Total Filling* $\nu_t = 1$*: A New Phase at Intermediate Layer Distances*, Phys. Rev. Lett. **119**, 177601 (2017).

[37] Biao Lian and Shou-Cheng Zhang, *Wave Function and Emergent SU(2) Symmetry in the* $\nu_t = 1$ *Quantum Hall Bilayer*, Phys. Rev. Lett. **120**, 077601 (2018).

[38] Glenn Wagner, Dung X. Nguyen, Steven H. Simon, and Bertrand I. Halperin, *S-Wave Paired Electron and Hole Composite Fermion Trial State for Quantum Hall Bilayers with* $\nu = 1$, Phys. Rev. Lett. **127**, 246803 (2021).

[39] Xiaomeng Liu, J. I. A. Li, Kenji Watanabe, Takashi Taniguchi, James Hone, Bertrand I. Halperin, Philip Kim, and Cory R. Dean, *Crossover between Strongly Coupled and Weakly Coupled Exciton Superfluids*, Science **375**, 205 (2022).

[40] Kun Yang, *Dipolar Excitons, Spontaneous Phase Coherence, and Superfluid-Insulator Transition in Bilayer Quantum Hall Systems at* $\nu = 1$, Phys. Rev. Lett. **87**, 056802 (2001).

[41] Robert B. Laughlin, *Anomalous Quantum Hall Effect: An Incompressible Quantum Fluid with Fractionally Charged Excitations*, Phys. Rev. Lett. **50**, 1395 (1983).

[42] Yihang Zeng, *Study of Two-Dimensional Correlated Quantum Fluid in Multi-Layer Graphene System*, Ph.D. thesis, Columbia University, 2021.

[43] A. R. Champagne, A. D. K. Finck, J. P. Eisenstein, L. N. Pfeiffer, and K. W. West, *Charge Imbalance and Bilayer Two-Dimensional Electron Systems at* $\nu_t = 1$, Phys. Rev. B **78**, 205310 (2008).

[44] Hua Chen and Kun Yang, *Interaction-Driven Quantum Phase Transitions in Fractional Topological Insulators*, Phys. Rev. B **85**, 195113 (2012).

[45] F. D. M. Haldane, *Many-Particle Translational Symmetries of Two-Dimensional Electrons at Rational Landau-Level Filling*, Phys. Rev. Lett. **55**, 2095 (1985).

[46] Max Metlitski, *Boundary Criticality of the* $\mathrm{O}(N)$ *Model in* $d = 3$ *Critically Revisited*, SciPost Phys. **12**, 131 (2022).

[47] Paolo Zanardi and Nikola Paunković, *Ground State Overlap and Quantum Phase Transitions*, Phys. Rev. E **74**, 031123 (2006).

[48] Shi-Jian Gu, *Fidelity Approach to Quantum Phase Transitions*, Int. J. Mod. Phys. B **24**, 4371 (2010).

[49] S. A. Kivelson, D. S. Rokhsar, and J. P. Sethna, $2e$ *or not* $2e$*: Flux Quantization in the Resonating Valence Bond State*, Europhys. Lett. **6**, 353 (1988).

[50] Martin Hasenbusch and Ettore Vicari, *Anisotropic Perturbations in Three-Dimensional* $\mathrm{O}(N)$*-Symmetric Vector Models*, Phys. Rev. B **84**, 125136 (2011).

[51] X. G. Wen, *Edge Excitations in the Fractional Quantum Hall States at General Filling Fractions*, Mod. Phys. Lett. B **05**, 39 (1991).

[52] J. P. Eisenstein, *Exciton Condensation in Bilayer Quantum Hall Systems*, Annu. Rev. Condens. Matter Phys. **5**, 159 (2014).

[53] I. B. Spielman, J. P. Eisenstein, L. N. Pfeiffer, and K. W. West, *Observation of a Linearly Dispersing Collective Mode in a Quantum Hall Ferromagnet*, Phys. Rev. Lett. **87**, 036803 (2001).

[54] Ady Stern, Steven M. Girvin, Allan H. MacDonald, and Ning Ma, *Theory of Interlayer Tunneling in Bilayer Quantum Hall Ferromagnets*, Phys. Rev. Lett. **86**, 1829 (2001).

[55] In this case, it is important to align the two graphene layers so the interlayer tunneling probes the spectral weight of the exciton order parameter at momentum $\mathbf{q} = 0$ and energy $\omega = eV$.

[56] A. F. Hebard and M. A. Paalanen, *Magnetic-Field-Tuned Superconductor-Insulator Transition in Two-Dimensional Films*, Phys. Rev. Lett. **65**, 927 (1990).

[57] Ali Yazdani and Aharon Kapitulnik, *Superconducting-Insulating Transition in Two-Dimensional a-MoGe Thin Films*, Phys. Rev. Lett. **74**, 3037 (1995).

[58] N. Marković, C. Christiansen, and A. M. Goldman, *Thickness–Magnetic Field Phase Diagram at the Superconductor-Insulator Transition in 2D*, Phys. Rev. Lett. **81**, 5217 (1998).

[59] E. Bielejec and Wenhao Wu, *Field-Tuned Superconductor-Insulator Transition with and without Current Bias*, Phys. Rev. Lett. **88**, 206802 (2002).

[60] Bar Hen, Xinyang Zhang, Victor Shelukhin, Aharon Kapitulnik, and Alexander Palevski, *Superconductor-Insulator Transition in Two-Dimensional Indium–Indium-Oxide Composite*, Proc. Natl. Acad. Sci. U.S.A. **118**, e2015970118 (2021).

[61] Thierry Giamarchi, *Quantum Physics in One Dimension* (Clarendon Press, Oxford, 2003), Vol. 121.

[62] C. L. Kane, M. P. A. Fisher, and J. Polchinski, *Randomness at the Edge: Theory of Quantum Hall Transport at Filling* $\nu = 2/3$, Phys. Rev. Lett. **72**, 4129 (1994).

[63] W. Chen, M. P. A. Fisher, and Y.-S. Wu, *Mott Transition in an Anyon Gas*, Phys. Rev. B **48**, 13749 (1993).

[64] William Witczak-Krempa, Erik S. Sørensen, and Subir Sachdev, *The Dynamics of Quantum Criticality Revealed by Quantum Monte Carlo and Holography*, Nat. Phys. **10**, 361 (2014).

[65] M. Wallin, E. S. Sorensen, S. M. Girvin, and A. P. Young, *Superconductor-Insulator Transition in Two-Dimensional Dirty Boson Systems*, Phys. Rev. B **49**, 12115 (1994).

[66] Igor F. Herbut, *Dual Superfluid-Bose-Glass Critical Point in Two Dimensions and the Universal Conductivity*, Phys. Rev. Lett. **79**, 3502 (1997).

[67] Jie Lou, Anders W. Sandvik, and Leon Balents, *Emergence of U(1) Symmetry in the 3D XY Model with* Z_q *Anisotropy*, Phys. Rev. Lett. **99**, 207203 (2007).

[68] T. Senthil and Matthew P. A. Fisher, *Fractionalization in the Cuprates: Detecting the Topological Order*, Phys. Rev. Lett. **86**, 292 (2001).

[69] Ying-Hai Wu, Hong-Hao Tu, and Meng Cheng, *Continuous Phase Transitions between Fractional Quantum Hall States and Symmetry-Protected Topological States*, arXiv:2302.06501.

[70] Michael Levin and Ady Stern, *Fractional Topological Insulators*, Phys. Rev. Lett. **103**, 196803 (2009).

[71] Pouyan Ghaemi, Jérôme Cayssol, D. N. Sheng, and Ashvin Vishwanath, *Fractional Topological Phases and Broken Time-Reversal Symmetry in Strained Graphene*, Phys. Rev. Lett. **108**, 266801 (2012).

[72] R. Morf and B. I. Halperin, *Monte Carlo Evaluation of Trial Wave Functions for the Fractional Quantized Hall Effect: Disk Geometry*, Phys. Rev. B **33**, 2221 (1986).

[73] N. E. Bonesteel, *Composite Fermions and the Energy Gap in the Fractional Quantum Hall Effect*, Phys. Rev. B **51**, 9917 (1995).