Unified Theory of Local Quantum Many-Body Dynamics: Eigenoperator Thermalization Theorems

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Explaining quantum many-body dynamics is a long-held goal of physics. A rigorous operator algebraic theory of dynamics in locally interacting systems in any dimension is provided here in terms of time-dependent equilibrium (Gibbs) ensembles. The theory explains dynamics in closed, open, and time-dependent systems, provided that relevant pseudolocal quantities can be identified, and time-dependent Gibbs ensembles unify wide classes of quantum nonergodic and ergodic systems. The theory is applied to quantum many-body scars, continuous, discrete, and dissipative time crystals, Hilbert space fragmentation, lattice gauge theories, and disorder-free localization, among other cases. Novel pseudolocal classes of operators are introduced in the process: *projected-local*, which are local only for some states, *cryptolocal*, whose locality is not manifest in terms of any finite number of local densities, and *transient* ones, that dictate finite-time relaxation dynamics. An immediate corollary is proving saturation of the Mazur bound for the Drude weight. This proven theory is intuitively the rigorous algebraic counterpart of the weak eigenstate thermalization hypothesis and has deep implications for thermodynamics: Quantum many-body systems "out of equilibrium" are actually *always* in a time-dependent equilibrium state for any natural initial state. The work opens the possibility of designing novel out-of-equilibrium phases, with the newly identified *scarring* and *fragmentation* phase transitions being examples.

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I. INTRODUCTION

In recent decades, the eigenstate thermalization hypothesis (ETH) [1] has become the cornerstone of our understanding of nonequilibrium quantum many-body dynamics. It deals with how an isolated quantum many-body system when prepared in a far-from-equilibrium initial state can relax to a state that is effectively in equilibrium for the purpose of determining expectation values of local observables. The *weak* ETH in canonical form concerns the time-averaged dynamics of observables, and states [1],

$$\lim_{T\to\infty}\frac{1}{T}\int_0^T dt\langle\psi(t)|O|\psi(t)\rangle = \frac{1}{N}\sum_j\langle j|O|j\rangle = \frac{1}{Z}\mathrm{tr}\left[O\,\exp\left(-\beta H+\sum_\alpha\mu_\alpha Q_\alpha\right)\right], \quad (1)$$

where the sum in the first line goes over an appropriate *microcanonical* window of the joint eigenstates $|j\rangle$ of the system's Hamiltonian H and a set of conservation laws Q_α $[Q_\alpha, H]=0$ and Z and N are normalization constants. The expression is usually written as a sum over the eigenstates and should be replaced with an integral in the thermodynamic limit provided that the eigenstates are well defined there. Physically, the hypothesis states that the long-time expectation values of all observables is set only by the initial expectation values of H and Q_α in terms of a (generalized) Gibbs ensemble via the Lagrange multiplies β and μ_α. In this form, the ETH also accounts for quantum integrable models that have an infinite set of conservation laws [2]. These principles of local

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equilibration have also been fundamental in the recent successful approach of generalized hydrodynamics [3–5]. The strong version of the ETH [1] is postulated as holding also without the time averaging in Eq. (1).

However, since its formulation, there has been an unmet desire to prove the ETH. Only in recent years have systems been identified, beyond quantum integrable ones, that seemingly violate the ETH. These include quantum many-body scars [6–14], Hilbert space fragmented models [15–17], time crystals [18–20], and others [21–23]. Naturally, they draw lots of attention due to their nonergodic and nonmixing dynamics seemingly defying the ETH.

Indeed, even though the ETH may possibly be provable for certain quantum systems, there is no reason to expect that it will hold in full generality. The reason for this is that, since the ETH is a statement about thermodynamically large systems, eigenstates may be singular, or they may not exist [24,25]. Moreover, even eigenvalues and the spectra of H and Q_α may display singularities. Such nonanalyticities are important and cannot be ignored; e.g., they are the reason behind phase transitions [26,27].

In order to therefore "prove" the ETH, an equivalent formulation in terms of a rigorous theory is needed. The standard choice is to describe local observables by C^* algebras [24,26]. In other words, one needs to move from eigenstates to eigenoperators. Here, I focus on locally interacting lattice models on hypercubic graphs in arbitrary dimension. I prove that, for a sufficiently low-entangled initial state, i.e., clustering (including physically realistic equilibrium states), the general long-time dynamics of all local observables in all such systems is described by a time-dependent generalized Gibbs ensemble determined by pseudolocal quantities [even without time averaging in Eq. (1)]. In particular, two immediate corollaries are the operator equivalent of Eq. (1) for the time average and saturation of the Mazur bound for the Drude weight. Moreover, it is shown that a time-dependent Gibbs ensemble describes dynamics for all times (not just in the long-time limit). Hence, remarkably, a quantum many-body system is always in a state similar to an equilibrium state that contains exponentially decaying nonconserved quantities. In other words, the time-dependent generalized Gibbs ensemble is the effective state describing the dynamics of all local observables (the full state of the system, if defined, may be pure provided that it was initially pure). The reason for this is the bijection (one-to-one correspondence) between pseudolocal quantities and local observables [28]. It turns out that knowing the dynamics of pseudolocal quantities gives all the information about the dynamics of local quantities, but the dynamics of pseudolocal quantities is much easier to find analytically. For instance, the Hamiltonian itself H is a pseudolocal quantity and has trivial dynamics for an isolated system $(d/dt)H = i[H, H] = 0$. Indeed, *a priori*, to find dynamics of local observables, one needs to find all the eigenstates of H. Here, I rather show how to find solutions for the dynamics of *all* local observables provided one knows a *much* smaller (usually finite) set of special pseudolocal quantities.

Later on, the theory is explicitly used for recent topical nonergodic and nonmixing cases, including quantum many-body scars, time crystals, projected Hamiltonians, Hilbert space fragmented models, disorder-free localized models, lattice gauge theories, and others. In the process, I introduce several kinds of pseudolocal quantities, including projected-local (local only for some states) and cryptolocal whose locality is not manifest in terms of sums of local quantities. This eigenoperator thermalization theory holds for open (dissipative), time-dependent, and independent quantum systems and does not rely on integrability, holds in any dimension, and relies only on locality of interactions and clustering of the initial state. Note that the theory for open systems is nontrivial. One might assume that if we have a D-dimensional system, we may treat any environment as a "fictive" enlargement of the system to $D + k$ and then study only observables in the original system, i.e., perform a Stinespring dilation [29], but such dilations are, in general, nonlocal, and, hence, our theory would not apply without explicit generalizations.

A. Outline of the article.

In Sec. II, I discuss the relation between this article and the previous work done on pseudolocality and thermalization, provide the main definitions needed, and demonstrate the importance of pseudolocality with a specific example often overlooked in the literature. The overview of the main results and discussion of the proof of the weak ETH, together with the descriptions of the long-time dynamics of several nonergodic models, is given in Sec. III. Section IV contains the main theorems, which are proved in Appendix A. These are used to study various physical examples, including scars and fragmentation in Sec. V. Finally, the conclusion (Sec. VII) contains a list of possible immediate research directions stemming from the present work.

II. PRELIMINARIES AND PREVIOUS WORK

In order to rigorously study dynamics of quantum many-body systems in the thermodynamic limit, we need to move to the framework of C^* algebras. This is to be contrasted with the standard ETH, where one focuses on the eigenstates of the system's Hamiltonian H. However, eigenstates may not even exist in the thermodynamic limit, and, more broadly, an inner product on the corresponding Hilbert space $\mathcal{H}$ is not well defined [24]. This is more than just a mathematical curiosity, because related discontinuities are responsible for phase transitions [24,26]. Indeed, operator algebraic approaches to dynamics and thermalization of quantum many-body systems have a long history—from

the work of von Neumann on the ergodic theorem [30] and Robinson, Emch, Hume, Narnhofer, and others, e.g., Refs. [31–34].

The physical systems studied here are locally interacting D-dimensional lattice models on infinite hypercubic lattices, with a set of sites $\Gamma = \mathcal{Z}^D$, with each *site* x having a finite-dimensional space of matrices and finite subsets of *balls* Λ of the full lattice of size $V := |\Lambda|$. Correspondingly, local operators $O, P, Q\ldots$ form a C^* algebra $\mathfrak{U}_{\text{loc}} := \otimes_{x\in\Lambda} M_d(\mathbb{C})$, where d is the dimension of the local matrix of operators on site x. The algebra $\mathfrak{U}_{\text{loc}}$ is equipped with a norm that may be Cauchy completed to the full quasilocal algebra $\mathfrak{U}$ [26,27]. More specifically, defining a standard state ω as a positive linear functional on the algebra $\mathfrak{U}_{\text{loc}}$ [with $\omega(\mathbb{1}) = 1$], with the finite case having the familiar density matrix representation ρ, $\omega(O) = \text{tr}(\rho O)$. The standard Gelfand-Naimark-Segal (GNS) construction allows for Cauchy completion with respect to the norm induced by the (symmetrized) connected correlator inner product [28]:

$$\langle O, Q\rangle_\omega^c := \sum_{x\in\Gamma} \frac{1}{2}\omega(\{O_x^\dagger, Q\}) - \omega(O_x^\dagger)\omega(Q), \qquad (2)$$

where O_x is the displacement of O by x, $\dagger$ denotes the conjugate, and $\{x, y\} := xy + yx$.

A crucial notion is that of pseudolocal quantities, introduced in Refs. [35–37], defined rigorously by Doyon, and this is the framework I use in the article. Under Doyon's framework, Eq. (2) exists provided that the state ω is *p-clustering*, i.e., essentially $|\omega(OQ) - \omega(O)\omega(Q)| \leq C\text{dist}(O, Q)^{-p}$, where C is a constant that does not depend on distance and the operators O and Q. This defines a Hilbert space of local observables $\mathcal{H}_\omega$. Pseudolocal quantities [38] are linear functionals defined as limits of sequences of local operators $A_V \in \mathfrak{U}_{\text{loc}}$ satisfying the following conditions: (A) $\omega(A_V^\dagger A_V) \leq \gamma V$, for some γ and $\forall\ V$, and (B) the limit $\mathcal{A}_\omega(O) := \lim_{V\to\infty} \omega(A_V^\dagger O)$ exists for all $O \in \mathfrak{U}_{\text{loc}}$. Without loss of generality, we take $\omega(A_V) = 0$. One may also define two-sided pseudolocal quantities $\hat{\mathcal{A}}_\omega(O) := \lim_{V\to\infty} \frac{1}{2}\omega(\{A_V, O\})$ and right pseudolocal quantities $\mathcal{A}_\omega^\dagger(O) := \lim_{V\to\infty} \omega(OA_V)$. The results generalize directly for all these three types of quantities, and we consider them interchangeably. Doyon demands that the whole construction is translationally invariant. We relax this requirement partially and later fully (to be defined precisely later). It is important to note that pseudolocality is state dependent; a quantity may be pseudolocal for one state and not another. This is to be contrasted with the more standard and restrictive notion of (quasi)locality for which one is used to thinking of pseudolocal quantities as sums of local terms. The present work, apart from pseudolocal conserved charges that were studied extensively before (e.g., Ref. [37]), also relies on recently introduced pseudolocal dynamical symmetries (e.g., Refs. [39,40]). We see here that pseudolocal dynamical symmetries are enough to characterize essentially all nonergodic and ergodic dynamics.

One of Doyon's crucial results is that there is a bijection $\mathfrak{D}$ between the Hilbert space of local operators $\mathcal{H}_\omega$ and the set of $\mathcal{A}_\omega$ (denoted as $\mathfrak{A}_\omega$). More specifically, $\forall\ \mathcal{A}_\omega, O$ there exists a $Q \in \mathcal{H}_\omega$ such that

$$\mathcal{A}_\omega(O) = \langle Q^\dagger, O\rangle_\omega^c, \qquad (3)$$

and similarly for the two-sided and right pseudolocal quantities. Physically, Q is the local density of the quantity $\mathcal{A}_\omega$. Indeed, one may physically think of the limit $A = \lim_{V\to\infty}\sum_x Q_{x,V}$ as the pseudolocal quantity, as well as mathematically provided that $Q_{x,V}$ is a Cauchy sequence, which means intuitively that as $V \to \infty$ the "support" of $Q_{x,V}$ grows with V in a well-defined way.

Another important notion is that of a *pseudolocal state* $\omega := \omega_1$, defined via a set of $\{\mathcal{A}_s]\}$ and its corresponding flow over *p-clustering* states:

$$\omega_{s_+}(O) - \omega_{s_-}(O) = \int_{s_-}^{s_+} ds\mathcal{A}_s(O), \qquad (4)$$

$\forall\ 0 \leq s_- < s_+ \leq 1$ and $\forall\ O$. In the very useful case when $\omega_s(O)$ is analytic, we have simply

$$\frac{d}{ds}\omega_s(O) = \mathcal{A}_s(O). \qquad (5)$$

The corresponding solution then may be thought of as a path-ordered exponential of A_s provided that it exists.

The rest of Doyon's framework concerns the long-time limit of closed many-body systems (with time propagator τ_t) and deals with cases when the limit $\lim_{t\to\infty}\omega(\tau_t(O))$ exists for all local O, showing that the system relaxes to a linear functional counterpart of a generalized Gibbs ensemble [2]. Here, we drop this requirement, allowing for the study of the general dynamics $\forall\ t$. This allows us to give analytical solutions to general nonergodic dynamics, provided that relevant pseudolocal quantities can be identified for a given model.

In Refs. [41,42], the weak ETH in the sense of typical eigenstates of the Hamiltonian being equal to the canonical ensemble is proven. However, this does not immediately imply the weak ETH in the canonical dynamical sense of Eq. (1), because atypical eigenstates may play an important role in the dynamics (as discussed in Ref. [42]), leading to violations of Eq. (1). Furthermore, in the thermodynamic limit, eigenstates and eigenvalues do not provide full information about the physics of a system, because eigenstates may not even exist and the spectrum of an operator need not equal its eigenvalues.

A rigorous theory for open quantum systems (quantum Markovian semigroups) and their long-time properties has

been established [43–46] even for unbounded generators of the time evolution [47]. Much work has been devoted to showing that $w-\lim_{t\to\infty}\tau_t(O)=e^{iHt}Oe^{-iHt}$, where H is the closed system Hamiltonian. This by itself does not imply anything for local observables or thermalization, because for thermodynamically large H local observables may thermalize independently of any external bath, like in closed systems.

A. Lack of relation between (restricted) spectrum-generating algebras and weak ergodicity breaking

Before moving on to the main results of the article, I wish to discuss relations between (restricted) spectrum-generating algebras (projectors to eigenstates of a Hamiltonian H) and dynamical symmetries (conservation laws).

A spectrum-generating algebra (SGA) [48] is defined as the existence of an operator R satisfying

$$[H,R]=\lambda R. \tag{6}$$

It is clear that for any Hamiltonian we have such operators $H|E_\alpha\rangle=E_\alpha|E_\alpha\rangle$, i.e., $R=|E_\alpha\rangle\langle E_\beta|$ with $\lambda=E_\alpha-E_\beta$. These R are *not* pseudolocal in the sense from the previous subsection and play no direct role in the dynamics of physically relevant observables. In the literature [49], sometimes one extends the requirement demanding that $R^k|E_0\rangle\neq 0,\ \forall\ k<V$ (or specializes to *commutant* algebras [50,51]). This is then equivalent to the *restricted* SGA [49] and is linked with a phenomenon in *quantum many-body scarred* models, called weak ergodicity breaking [6], wherein local observables show persistent oscillations from special (but nonequilibrium and clustering) initial states. More precisely, this is a form of nonstationary dynamics, because weak ergodicity breaking can also occur for large random fluctuations around a thermal expectation value. However, the restricted SGA requirement is neither sufficient nor necessary for R to have implications for dynamics of observables. This is easiest to observe in the $V\to\infty$ limit. A generic many-body H has a dense and extensive spectrum. Thus, we may always find such an R for any λ. More specifically,

$$R=\lim_{V\to\infty}\int_{|E_\alpha-E_\beta-\lambda|<\varepsilon(V)} d\mu_\alpha d\mu_\beta |E_\alpha\rangle\langle E_\beta|, \tag{7}$$

with $\lim_{V\to\infty}\varepsilon(V)\to 0$ being a suitable function used in taking the thermodynamic limit to avoid issues with the existence of eigenstates corresponding to the continuous spectrum and where the integral is taken over some suitable spectral measure $d\mu$. Then $R^k|E_0\rangle\neq 0$ will be fulfilled for some $|E_0\rangle$ and $\forall\ k$.

The fact that this requirement is not strictly necessary can be observed by considering a $D=1$ quadratic fermionic lattice model with periodic boundary conditions:

$$H=\sum_{j=0}^{n} Jc_j^\dagger c_{j+1}+\mu c_j^\dagger c_j+\text{H.c.}, \tag{8}$$

in the $J\to 0$ limit. Clearly, we have a SGA $\lim_{J\to 0}[H,c_k^\dagger]=\mu c_k^\dagger$ with $c_k=\sum_j e^{ikj}c_j$, with $(c_k^\dagger)^2|\psi\rangle=0$. However, trivially, in the $J\to 0$ limit all local observables (in the fermion picture) that are off-diagonal in the number basis $[O,c_j^\dagger c_j]\neq 0$ persistently oscillate for arbitrarily long times for all (clustering) initial states that are not eigenstates of the total particle number operator.

The restricted SGA, i.e., with $R^k|E_0\rangle\neq 0$, does *not* imply weak ergodicity breaking, nor the converse; rather, we see that what is needed is a SGA with a pseudolocal R at select frequencies λ, i.e., a pseudolocal *dynamical symmetry* [40,52].

This should not be so surprising. Indeed, setting $\lambda=0$ in Eq. [48] implies that R commutes with H. Formally, every thermodynamically large Hamiltonian has a infinite number of such R (projectors to its eigenstates), but this does not mean that H is integrable or even that R is a physically relevant conservation law. What is actually needed to have physically relevant conservation laws is locality.

III. OVERVIEW AND MAIN RESULTS

The assumptions of the article are listed here.

Assumption 1.—The system under question is either an open (Markovian) or closed, time-independent, or time-dependent system with local (finite-range) interactions on some D-dimensional hypercubic lattice.

Assumption 2.—The whole construction is space-translation invariant in some generalized sense; i.e., there exists an automorphism on the lattice $\mathcal{Z}^D$ denoted as ι_x for which we have $\iota_x\circ\iota_y=\iota_{x+y}$. Note that this does *not* necessarily mean that the system is directly translationally invariant and includes cases with (Bloch) translation invariance at finite momentum. This allows us to treat, e.g., modulated pseudolocal quantities [53]. In any case, later on, we fully drop this requirement and allow that no such automorphism exists; i.e., we allow for arbitrary disorder. In this setup, the set of pseudolocal quantities must be reduced to the set of *pseudolocalized* quantities that have subextensive growth.

Assumption 3.—The expectation values $\langle O\rangle_t=\omega(\tau_t(O))$, $\forall\ O\in\mathfrak{U}_{\text{loc}}$, is bounded $\forall\ t$ and in the $t\to\infty$ limit. This is a physically reasonable assumption for most lattice models, except for perhaps bosonic ones (with infinite-dimensional local Hilbert spaces) at infinite density, but such systems can be treated with standard semiclassical approaches [54].

Assumption 4.—The system is initially prepared in a *pseudolocal* state, which essentially means that it does not have correlations that are *too* long ranged (thermal states in $D>1$ at high temperatures, ground states of

gapped chains, etc., satisfy even stronger exponential clustering).

The main technical contribution of this article compared to the framework in Ref. [28] is dropping the requirement of the existence of the long-time limit and closed quantum many-body dynamics. In fact, we discuss dynamics for *general* times. Dropping this seemingly innocuous requirement allows us to give *analytical* solutions for the long-time limit of many nonergodic and *chaotic* systems (provided that all the pseudolocal quantities can be identified). This includes, but is not limited to, quantum many-body scars, Hilbert space fragmented models, time crystals, and lattice gauge theories. I emphasize that, even though quantum integrable systems are covered under the presented theory, the theory in no way relies on integrability.

First, let us overview the results for closed time-independent systems with Hamiltonian H.

A. Far-from-equilibrium states are *always* in equilibrium for local observables

Assume that the system is initially $t = 0$ in a pseudolocal state given by the pseudolocal flow $\omega_s \coloneqq \omega_{s,t=0}$. These kinds of states can be written as exponentials of local extensive operators, for instance, $\omega_{s,t=0}(O) = \mathrm{tr}[\exp(-sH_{t=0})O]$ for some initial Hamiltonian $H_{t=0}$ that we quench from. As we see later, this implies that, essentially barring issues with path ordering and existence, the system is *always* (from $t = 0$ to any other t) in a *time-dependent Gibbs ensemble* given by

$$\rho(t) = \frac{\exp(-\beta H + \int_0^1 du \mu_u e^{\lambda(u)t} A_u)}{\mathrm{tr}[\exp(-\beta H + \int_0^1 du \mu_u e^{\lambda(u)t} A_u)]}, \tag{9}$$

where $\mathrm{Re}[\lambda(u)] \leq 0$ and we explicitly write the thermal part of the state. Note that the only time dependence is in the $e^{\lambda(u)t}$ term inside the exponential [$\lambda(u)$ does not depend on time]. Although this result is rather formal, it gives us physical insight into the nature of equilibration as illustrated in Fig. 1.

A quantum many-body system relaxes by starting and remaining in a Gibbs state, given by time-dependent, possibly exponentially decaying, chemical potentials $\mu_u e^{\lambda(u)t}$ with corresponding pseudolocal quantities $A_u = \sum_x Q_x(u)$. The values of μ_u are set by the initial state with flow ω_s. Thus, even a "far-from-equilibrium" state is a time-dependent equilibriumlike state as far as local observables are concerned.

I also conjecture and provide evidence later that the *transient* pseudolocal quantities with $\mathrm{Re}[\lambda(u)] < 0$ are responsible for local diffusive relaxation. Note that these quantities are not the only relaxation part of Eq. (9). In particular, the system may also relax algebraically in t by dephasing of the purely imaginary $\mathrm{Re}[\lambda(u)] = 0$ as happens for quadratic models [55].

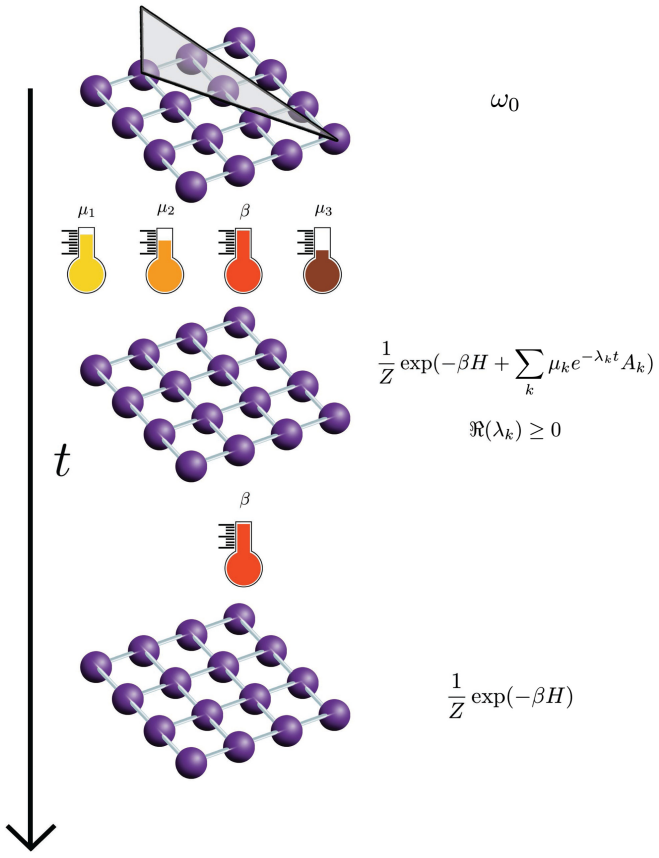

FIG. 1. An illustration of generic nonequilibrium quantum many-body dynamics. The system starts off in a clustering state (with finite power-law correlations at most) and then proceeds through an Gibbs-like state with transient pseudolocal quantities and corresponding temperatures (or chemical potentials) before thermalizing to a Gibbs ensemble.

B. Long-time dynamics

Assuming that clustering holds for long enough times and defining the Fourier transform of the expectation value $\langle O\rangle_{s,t}$ in the time evolved state $\omega_{s,t}$,

$$\langle O\rangle_{s,\lambda} = \lim_{T\to\infty} \frac{1}{T}\int_0^T dt e^{i\lambda t}\langle O\rangle_{s,t}, \tag{10}$$

the system is in a time-dependent *generalized* Gibbs ensemble, which is a pseudolocal state, defined via pseudolocal quantities $\hat{\mathcal{A}}_{s,\lambda}$ that oscillate at frequencies λ, i.e., satisfying

$$\hat{\mathcal{A}}_{s,\lambda}([H,O]) = \lambda \hat{\mathcal{A}}_{s,\lambda}(O), \quad \forall\, O \in \mathfrak{U}_{\mathrm{loc}}, \quad \lambda \in \mathfrak{R}, \tag{11}$$

where we define $\hat{\mathcal{A}}_{s,\lambda}(O) \coloneqq \lim_{T\to\infty}(1/T)\int_0^T e^{i\lambda t}\hat{\mathcal{A}}_{s,t}(O)$ and recall that $\hat{\mathcal{A}}_{s,t}(O) = \sum_x \langle A_x^\dagger O\rangle_{s,t}$ where we can subtract a constant from A_x always such that $\langle A_x^\dagger\rangle_{s,t} = 0$. Recall also that the subscript x denotes translation across

the lattice. In particular, this implies the existence of corresponding (nonunique) pseudolocal sequences

$$[H_V, A_V] = \lambda_V A_V, \qquad \lambda = \lim_{V\to\infty} \lambda_V, \tag{12}$$

where we drop the explicit dependence on s and λ. I call these sequences *dynamical symmetries* in analogy with previous work on such operators [40,52].

Let us assume that the following representation of the state is well defined in the $\Lambda \to \infty$ limit:

$$\rho^\Lambda(t) = \frac{\exp[-\beta(H_\Lambda + \sum_k e^{i\lambda_k t}\mu_k A_k^\Lambda)]}{\mathrm{tr}_\Lambda\{\exp[-\beta(H_\Lambda + \sum_k e^{i\lambda_k t}\mu_k A_k^\Lambda)]\}}, \tag{13}$$

and the set (12) is countable. We discuss when this is true later, but, essentially, the state is well defined in the thermodynamic limit provided that $|\beta|$ is large enough in $D > 1$ and always in $D = 1$. Then the following *weak eigenoperator thermalization* result holds:

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T dt \langle O\rangle_t = \lim_{\Lambda\to\infty} \mathrm{tr}_\Lambda(\rho^\Lambda_{\lambda=0} O), \tag{14}$$

where $\rho^\Lambda_{\lambda=0}$ is the zero-frequency component, defined as before, i.e., as in Eq. (1). This proves the weak ETH in the canonical form (1).

Likewise, it is shown that the Mazur bound [56] for the Drude weight is saturated at all frequencies.

1. No translation invariance

If we drop translation invariance even in the generalized sense, we may apply the theory provided that, instead of pseudolocal quantities, we focus on pseudolocalized ones. We replace Doyon's sesquilinear form with displaced operators with the more typical inner product [27]

$$\langle O, Q\rangle^{\mathrm{loc}}_\omega := \langle O^\dagger Q\rangle_\omega, \tag{15}$$

with quasilocal $O, Q \in \mathfrak{U}$. The algebra may be extended to a Hilbert space $\mathcal{H}^{\mathrm{loc}}_\omega$ and its dual $(\mathcal{H}^{\mathrm{loc}}_\omega)^\dagger$. Pseudolocalized quantities $\mathcal{A}^{\mathrm{loc}}_\omega(O)$ are now formed as $\lim_{\Lambda\to\infty}$ limits of $A^\dagger \in (\mathcal{H}^{\mathrm{loc}}_\omega)^\dagger$, which always exist according to Assumption 3. The space of all $\mathcal{A}^{\mathrm{loc}}_\omega \in \mathfrak{A}^{\mathrm{loc}}_\omega$ is trivially in bijection with $\mathcal{H}^{\mathrm{loc}}_\omega$, and the entire construction proceeds as before.

This allows us to treat Hilbert space fragmented systems that may not be translationally invariant in any sense (e.g., Ref. [57]).

2. Open and time-dependent systems

Curiously, the preceding discussions work with very little modification for open quantum systems described by continuous quantum Markovian semigroups, i.e., the Lindblad master equation. Analogously to the Hamiltonian case, we look at *local* quantum Liouvillians on D-dimensional hypercube lattices described by a generator of time evolution:

$$\mathcal{L}O = i[H, O] + \sum_x \int d\eta [2L_x^\dagger(\eta) O L_x(\eta) - \{L_x(\eta)L_x^\dagger(\eta), O\}], \tag{16}$$

where $L_x(\eta), O \in \mathfrak{U}_{\mathrm{loc}}$ and the operator $\mathcal{L}: \mathfrak{U}^\dagger_{\mathrm{loc}} \otimes \mathfrak{U}_{\mathrm{loc}} \to \mathfrak{U}^\dagger_{\mathrm{loc}} \otimes \mathfrak{U}_{\mathrm{loc}}$. The corresponding time evolution is given by a power series defined by the exponential map $\tau_t(O) := \exp(\mathcal{L}t)(O)$. Physically, the Lindblad jump operators model the action of some external (memoryless) environment on the system and are applicable to a wide range of physical systems that have bath degrees of freedom that are much faster than the systems ones, e.g., quantum optics, cold atoms, etc.

As we see, the results in Secs. III A and III B extend directly to the quantum Markovian semigroup case if we replace ad_H with $\mathcal{L}$. More remarkably, we see that, under very mild assumptions, the long-time dynamics is described by *open* time-dependent (generalized) Gibbs ensembles (tGGEs) generated by a flow given by pseudolocal quantities satisfying

$$\hat{\mathcal{A}}_{s,\lambda}([H, O]) = \lambda \hat{\mathcal{A}}_{s,\lambda}(O), \qquad \lambda \in \mathfrak{R},$$
$$\hat{\mathcal{A}}_{s,\lambda}([L_x(\eta), O]) = \hat{\mathcal{A}}_{s,\lambda}([L_x^\dagger(\eta), O]) = 0, \quad \forall\ O \in \mathfrak{U}_{\mathrm{loc}}. \tag{17}$$

In other words, the long-time dynamics of open quantum systems is determined exclusively by the Hamiltonian *even for baths that drive the system inherently out of equilibrium*. The role of the Lindblad operators is mainly to select a smaller subset of pseudolocal dynamical symmetries that do so [i.e., those that commute with them in the sense of Eq. (17)].

For time-dependent systems, we likewise have an equivalent extension with $H \to H_\lambda$, where H_λ is defined in the standard extended space framework [58,59].

C. Physics: Scars, fragmentation, etc

It turns out that, to the best of my knowledge, for all known cases of quantum nonergodic dynamics, excluding integrability, it is sufficient to consider a *finite* set of dynamical symmetries (12) A^Λ_μ that close a finite algebra with H_Λ under commutation. This also includes the widely topical cases of quantum many-body scarred models [6–9,60–63], Hilbert space fragmented models [15,17,64–66], disorder-free localization models [23], lattice gauge theories [22,67–74], and all types of time crystals [39,75–80]. This allows us to use the representation (11) to describe the long-time dynamics of these models

provided $|\beta|$ is small enough. Moreover, Eq. (11) defines the unique $\omega_{s,\lambda}$. Specifically, the β and μ_k are the only unknown parameters, and these are set by the initial state. Thus, Eq. (11) provides an *exact* solution to all such (chaotic) dynamics.

In order to find the tGGE for such models, we need to, however, introduce two new classes of pseudolocal quantities.

Quantum many-body scars and the scarring phase transition.—It is shown that quantum many-body scarred models are defined by a novel class of pseudolocal dynamical symmetries that I call projected-local. More specifically, in Eq. (13), $A_k^\Lambda = \mathcal{P}_\Lambda(\sum_{x\in\Lambda} Q_x)$, where Q_x is a translation of a local operator and $\mathcal{P}_\Lambda$ is a map $\mathcal{P}_\Lambda\colon \mathfrak{U}_\Lambda \to \mathfrak{U}_\Lambda$. In all the cases studied, this is simply a projector $\mathcal{P}_\Lambda x = PxP$, $P^2 = \mathbb{1}$. These dynamical symmetries, as sequences, satisfy the requirements of pseudolocality; i.e., they are pseudolocal only for certain initial states ω_0 and, therefore, lead to persistent oscillations and nonstationary dynamics only when the system is prepared in these initial states. This provides likewise an unambiguous definition of quantum many-body scars in line with the single-body definition [54]. Crucially, the representation (13) does not have clustering of correlations (and is not a pseudolocal state) for certain values of $|\mu_k|$. In fact, there exists a critical value when this happens. This indicates the presence of a novel type of *scarring* phase transition, distinct from standard thermodynamic phase transitions: In standard thermodynamic phase transitions, the Gibbs ensemble no longer admits a unique representation above some value of $|\beta|$, indicating symmetry breaking. By contrast, the scarring phase transitions happens because a pseudolocal quantity stops being pseudolocal. These quantities are also responsible for nonergodicity in embedded Hamiltonians.

Hilbert space fragmentation, induced localization, and fragmentation phase transitions.—Models with true Hilbert space fragmentation turn out to be described by pseudolocalized quantities, like the ones we define for models without assuming translation invariance $\mathcal{A}_\omega^{\mathrm{loc}}$ (Sec. III B 1). These lead to memory of the initial conditions *locally*, i.e., a form of localization. They may be derived from the statistically localized integrals of motions [81] or nonlocal commutant algebras of the models [17] but are distinct from them. Curiously, they cannot be written in a manifestly local manner as sums of translated local operators. Hence, we call them cryptolocal. Similar pseudolocalized charges are responsible for nonergodic behavior in disorder-free localization and lattice gauge theories. It can happen, in contrast to both thermodynamic phase transitions and the scarring phase transition above, that, for certain continuous changes of the chemical potentials corresponding to cryptolocal quantities, the tGGE state abruptly stops being clustered. The cryptolocal quantities remain pseudolocal, however, unlike in the scarring phase transition. This still signals a change of phase and emergence of long-range order. This should be contrasted with phase transitions between thermalization (weak fragmentation) and nonergodicity (strong fragmentation) [82,83], because in the present work the phase transition is between two distinct nonergodic phases. We study an explicit example later.

IV. DYNAMICS OF QUANTUM MANY-BODY SYSTEMS

We now turn to stating the main theorems and lemmas beginning with the ones we need to prove the main results from the previous section. The proofs are in Appendix A. Assume that the dynamics is provided by a time-dependent Markovian *closed and dense* generator:

$$\mathcal{L}_{\Lambda,t} = \sum_{x\in\Lambda} \mathcal{L}_{x,t}, \tag{18}$$

where $\mathcal{L}_{x,t}$ is the time-dependent local Hamiltonian density (local Lindblad jump operator) translated by x as in Eq. (16). The dynamics for $O \in \mathfrak{U}_{\mathrm{loc}}$ is given as ($\mathcal{L}_t \coloneqq \lim_{\Lambda\to\infty} \mathcal{L}_{\Lambda,t}$)

$$\tau_t(O) \coloneqq \mathcal{T} \int_0^t dp\, \exp(dp\mathcal{L}_p)(O), \tag{19}$$

where $\mathcal{T}$ is the time ordering operator. We implicitly define a doubled C^* algebra $\mathcal{L}\colon \mathfrak{U}_{\mathrm{loc}}^\dagger \otimes \mathfrak{U}_{\mathrm{loc}} \to \mathfrak{U}_{\mathrm{loc}}^\dagger \otimes \mathfrak{U}_{\mathrm{loc}}$. This defines the equation of motion

$$\frac{d\omega_t}{dt}(O) = \omega_t(\mathcal{L}_t(O)), \tag{20}$$

where the time evolved state is $\omega_t \coloneqq \omega_0 \circ \tau_t$. Note that the map is generally contractive and we have $||\tau_t(O)|| \le ||O||$ and dissipative $||\tau_t(O^\dagger Q) - \tau_t(O^\dagger)\tau_t(Q)|| \ge 0$ [44,84]. For closed systems, however, the equality in the relations holds; i.e., the map is an isometry and preserves the algebra.

The convergence properties of the series in Eq. (19) has been extensively studied [43], but for our purposes what is relevant is that $\lim_{\Lambda\to\infty} \tau_t^\Lambda(O)$ exists in the Hilbert space of local observables $\mathcal{H}_\omega$ defined previously in Sec. II.

Locality of time evolution for time-dependent quantum Liouvillians has been established [85–87] using Lieb-Robinson bounds [88] generalized for such dynamics [89,90]. Define $()_\Lambda$ to be the projection to some sublattice (ball) Λ. There exist some $\varphi > 0, v > 0$ such that, for $\Delta > 2D - 1$,

$$||\tau_t(O) - [\tau_t(O)]_\Lambda|| \le \varphi ||O_\Lambda|| \Delta^{D-1} \exp(-\Delta + v|t|), \tag{21}$$

where $\Delta = \mathrm{dist}[\mathrm{supp}(O), \mathcal{Z}^D \backslash \Lambda]$, dist is the metric (distance) on the lattice $\mathcal{Z}^D$, and supp is the support of the operator O. The value v is called the Lieb-Robinson

velocity. Note that, in contrast to the result for the isolated system [91], there is an extra polynomial dependence on distance Δ^{D-1}.

Despite this extra dependence, it is possible to generalize a theorem by Doyon to driven-dissipative dynamics.

Theorem 1.—Let τ_t be the time evolution and ω be a pseudolocal state with flow ω_s; then $\forall\ O, Q \in \mathfrak{U}_{\mathrm{loc}}$ and $t \in \mathbb{R}$,

(i) the limit $\lim_{\Lambda\to\infty}[\tau_t(O)]_\Lambda$ exists in $\mathcal{H}_\omega$ and $\lim_{\Lambda\to\infty}||[\tau_t(O)]_\Lambda||_{\mathcal{H}_\omega}$ exists and is uniformly bounded with an induced form $\langle O, Q\rangle^c_{\omega\circ\tau_t}$ and

(ii) the state $\omega_t = \omega\circ\tau_t$ is pseudolocal.

If the Lindblad jump operator terms are not zero, then it is important to note the following.

Remark 1.—In general, we have for the form $\langle O, Q\rangle^c_{\omega\circ\tau_t} \neq \langle \tau_t(O), \tau_t(Q)\rangle^c_\omega$.

A. Eternal equilibrium

In this section, we describe the finite time nonequilibrium dynamics. Although the construction is involved, it provides physical insight. For the sake of presentation, for the moment we specialize to time-independent cases and discuss generalizations in later sections.

Theorem 2.—If we initialize the system in a pseudolocal (nonequilibrium state) ω with flow ω_s, then the state of the system $\forall\ t \in \mathbb{R}^+$ is given by the pseudolocal state ω_t with flow:

$$\omega_{s,t}(O) = \omega_{0,t}(O) + \int_0^s du \mathcal{A}_{u,t}(O), \tag{22}$$

with $\mathcal{A}_{u,t} \coloneqq \mathcal{A}_{\omega_u\circ\tau_t}$. The pseudolocal quantities solve for almost all u the following well-defined Cauchy problem:

$$\frac{d}{dt}\mathcal{A}_{u,t} = \mathfrak{L}(\mathcal{A}_{u,t}) = \mathcal{A}_{u,t}\circ\mathcal{L}. \tag{23}$$

There exists a $M \geq 0$ such that $\mathfrak{L}: \mathcal{H}^\dagger_{u,t} \to \mathcal{H}^\dagger_{u,t}$ generates an strongly continuous contracting semigroup $\mathfrak{T}_t \coloneqq e^{(\mathfrak{L})t}$ solving Eq. (23):

$$\mathcal{A}_{u,t} = e^{Mt}\mathfrak{T}_t\mathcal{A}_{u,0} \tag{24}$$

admitting a resolvent and solving Eq. (23):

$$\mathfrak{T}_t = \int_\Gamma d\lambda e^{\lambda t}(\lambda - \mathfrak{L})^{-1}, \tag{25}$$

where $\mathrm{Re}(\lambda) \leq 0$ and Γ is an appropriate path.

Note that $\mathfrak{L}$ is not self-adjoint, in general, even in the purely Hamiltonian case $L_x(\eta) = 0$. This shows that the dynamics of local operators in even an isolated many-body system has a natural arrow of time induced by the semigroup $\mathfrak{T}_t$, i.e., $t \geq 0$. Intuitively, the generator $\mathcal{L}$ "intertwines" infinitesimally between $\mathcal{H}_{u,t}$ and $\mathcal{H}_{u,t+dt}$.

Remark 2.—The requirement about $M \geq 0$ in the generator of the semigroup is purely technical and may be dropped (i.e., set $M = 0$) provided the infinite time limit $\lim_{t\to\infty}||\tau_t O||_{\mathcal{H}_\omega}$ exists $\forall\ O \in \mathfrak{U}_{\mathrm{loc}}$. This should be the case in all physically reasonable examples. Otherwise, we could have unbounded values of local observables. Alternatively, we may set $M = 0$ provided we are interested only in dynamics for all finite $t \in \mathbb{R}$.

The result as given looks complicated even for the purely Hamiltonian case. However, if we assume that $\omega_{s,t}$ is analytic,

$$\frac{d}{ds}\omega_{s,t}(O) = \mathcal{A}_{s,t}(O) \tag{26}$$

$\forall\ s \in [0,1]$ and that it admits a well-defined matrix representation, it directly follows from the linearity of the functional $\mathcal{A}_{u,t}$ that the thermodynamic state of the system is the limit of a time-dependent Cauchy sequence in Λ, which closely resembles a Gibbs ensemble; i.e., the state is of the form

$$\rho^\Lambda(t) = \mathcal{P}\frac{\exp[-\beta(H_\Lambda + \int_0^1 du e^{\lambda(u)t}\mu_u A^\Lambda_u)]}{\mathrm{tr}_\Lambda\{\mathcal{P}\exp[-\beta(H_\Lambda + \int_0^1 du e^{\lambda(u)t}\mu_u A^\Lambda_u)]\}}, \tag{27}$$

where $\mathcal{P}$ is a suitable path ordering along the flow of the pseudolocal state.

Physically, the system proceeds from a state ω_0 which is an eigenstate or thermal state of a local Hamiltonian that is distinct from the Hamiltonian driving the time evolution of the system. This state admits a decomposition in terms of pseudolocal quantities A_u for the given generator $\mathfrak{L}$ that may be divided into two main classes:

(1) those for which $\mathrm{Re}[\lambda(u)] < 0$, i.e., *exponentially decaying* transient ones that disappear from $\rho(t)$ exponentially quickly and correspond to exponential decay of expectation values of local observables, and

(2) those for which $\mathrm{Re}(\lambda(u)) = 0$, which may be further subdivided:

(a) *Those for which the spectrum is continuous around* $\lambda(u)$.—These may correspond to polynomial decay of expectation values of local observables, which may be seen by, e.g., invoking the stationary phase approximation.

(b) *Those* $\lambda(u)$ *around which the spectrum of* $\mathfrak{L}$ *is isolated.*—These are either $\lambda(u) = 0$, i.e., these are the pseudolocal conservation laws and must include the Hamiltonian H itself, or $\mathrm{Im}[\lambda(u)] \neq 0$ and these are pseudolocal dynamical symmetries that are studied in the next section. The latter correspond to persistent oscillations at fundamental frequencies $\mathrm{Im}[\lambda(u)]$.

Further intuition about Theorem 2 can be gained for the closed infinitely large system $\Lambda = \infty$ case, by observing that the main statement of the theorem in Eq. (23) can be

formally and unrigorously written as $[H_\infty, A_u^\infty] = \lambda(u) A_u^\infty$, where λ_u need not be real. This is because H_∞ need not be self-adjoint on the entire Hilbert space. By contrast, H_Λ for finite system Λ always has real and countable eigenstates $H_\Lambda|E_n^\Lambda\rangle = E_n^\Lambda|E_n^\Lambda\rangle$. However, most of the corresponding $A_{n,m}^\Lambda = |E_n^\Lambda\rangle\langle E_m^\Lambda|$ and $[H_\Lambda, A_{n,m}^\Lambda] = (E_n^\Lambda - E_m^\Lambda)A_{n,m}^\Lambda$ become thermodynamically irrelevant for *all* local observables O, i.e., $\lim_{\Lambda\to\infty} \langle A_{n,m}^\Lambda O\rangle_t = 0$, $\forall$ O (similarly to a well-known assumption of the ETH [1]). The pseudolocal A_u^Λ are precisely the linear combinations of such $A_{n,m}^\Lambda$ (and the only linear combinations) that have finite overlap with (at least some) local observables in the thermodynamic limit. Moreover, they are such that they grow most extensively with system size *for the given state* and, hence, are well defined in the thermodynamic limit. In other words, a naive solution of the dynamics of local observables requires knowing all the eigenstates and energies of H, but most of these are not thermodynamically relevant. The same information can be gained about dynamics of local observables if one knows a much smaller subset in terms of A_u^Λ.

The physical relevance, if any, of the residual spectrum of $\mathfrak{L}$ is not immediately clear. Cases 2(a) and 2(b) have been studied previously and are discussed in the next section. To the best of my knowledge, there are no known constructions of the transient pseudolocal quantities in case 1.

However, here, I give a physically motivated conjecture that they are responsible for diffusive relaxation that may be studied by quantum hydrodynamics [92].

To see this, consider a 1D lattice with a (unrigorous) local conservation law $[H, Z_0] = 0$ at finite momentum $Z_k = \sum_x e^{ikx} z(x)$, where $z(x)$ is the translated local density. The diffusion equation for this charge is

$$\frac{\partial}{\partial t} z(x,t) = \kappa \frac{\partial^2}{\partial x^2} z(x,t), \tag{28}$$

where $z(x,t) \coloneqq \tau_t[z(x)]$. The finite-momentum solution is $Z(k,t) = Z(k,0)e^{-k^2\kappa t}$. Studying $Z(k,t) = \mathrm{tr}\rho(t)Z_k$ and comparing with Eq. (27) indicates that the exponential decay of Z_k at finite momentum proceeds because the overlap between the transient pseudolocal quantities $\mathrm{tr}(A_u Z_k) \neq 0$ for $k \neq 0$. Hence, it is reasonable to assume that the transient pseudolocal quantities are responsible for diffusion.

The relation between the pseudolocal quantities and transport is more complicated for other types of relaxation. Consider the 1D convection equation, solving for, e.g., ballistic transport in integrable models [3,4] for the simplest linear case:

$$\frac{\partial}{\partial t} z(x,t) = v \frac{\partial}{\partial x} z(x,t). \tag{29}$$

For an initial condition $z(x,0) = z_0(x)$, the equation is solved by any $z(x,t) = z_0(x+vt)$. So, depending on the initial condition, it could correspond to a local faster-than-exponential relaxation [e.g., for a Gaussian wave packet $z_0(x) = C\exp(-ax^2)$] to persistent oscillations for an initial condition with finite momentum k.

B. Asymptotic dynamics

In this section, we develop a general theory in terms of pseudolocal quantities for the long-time dynamics.

Theorem 3 (general eigenoperator thermalization).—Assume that the system is in (nonequilibrium) pseudolocal state ω_0 with flow ω_s. In the long-time limit, the state of the system is a pseudolocal (open) tGGE ω_t with flow $\omega_{s,t}$.

(1) The tGGE satisfies

$$\omega_{s,\lambda}(\mathcal{L}_t(O)) = -i\lambda\omega_{s,\lambda}(O) \quad \forall\ O \in \mathfrak{U}_{\mathrm{loc}}, \tag{30}$$

where $\omega_{s,\lambda}$ is the component of the state at frequency λ, i.e., $\omega_{s,\lambda} \coloneqq \lim_{T\to\infty}(1/T)\int_0^T dt e^{i\lambda t}\omega_{s,t}$ with $\lambda \in \mathbb{R}$.

(2) Provided that the dual map $\tau_t^\dagger$ has a faithful stationary state and a time-independent generator, then we also have

$$\omega_{s,\lambda}([H, O]) = -\lambda\omega_{s,\lambda}(O), \tag{31}$$

$$\omega_{s,\lambda}([L_x(\eta), O]) = \omega_{s,\lambda}([L_x^\dagger(\eta), O]) = 0, \ \forall\, O \in \mathfrak{U}_{\mathrm{loc}}. \tag{32}$$

In both cases 1 and 2, the corresponding quantities $\hat{\mathcal{A}}_{s,\lambda}(O) \coloneqq \lim_{T\to\infty}(1/T)\int_0^T e^{i\lambda t}\hat{\mathcal{A}}_{s,t}(O)$ satisfy the same relations as the state [with $\omega_{s,\lambda} \to \hat{\mathcal{A}}_{s,\lambda}$ in Eq. (30), respectively, (31) and (32)] for almost all s; i.e., in case 2, they satisfy Eq. (17). The quantities $\hat{\mathcal{A}}_{s,t}(O)$ are called *pseudolocal dynamical symmetries*.

Naturally, for the Hamiltonian case, the requirement of the faithful stationary state is trivial (e.g., the tracial state is always a faithful stationary state) and $L_x(\eta) = 0$ so only the Hamiltonian condition is relevant.

Note that the functional ω_λ is a pseudolocal functional [in the sense of Eq. (4)] with the flow $\omega_{s,t}$ from above, but it is not necessarily positive for $\lambda \neq 0$. But we abuse terminology and still refer to it as a "state." The flow $\omega_{s,t}$ is two dimensional, and we may reduce it to a single parameter flow $\omega_{\lambda,s}$ with flow along the time direction being infinite and deformed by $e^{i\lambda t}$. The pseudolocal quantities $\hat{\mathcal{A}}_{s,\lambda}(O)$ are defined across a family of pseudolocal states parametrized by t. Equations (31) and (32) need not be finite. Indeed, if there is no frequency λ in the dynamics, they give $0 = 0$ identically.

These results may appear to be daunting to apply to any given Hamiltonian, but as we see, they have necessary and sufficient interpretations in terms of standard theoretical physics concepts—eigenoperators and equilibrium states. Physical intuition about Theorem 3 may be gained by

observing that the criteria of the theorem essentially state that for finite system Λ there exist A_λ^Λ such that $[H_\Lambda, A_\lambda^\Lambda] = \lambda A_\lambda^\Lambda$, where A_λ^Λ are pseudolocal, in the sense discussed previously, for the finite-frequency state $\omega_{s,\lambda}$, i.e., such that $\lim_{\Lambda\to\infty}\langle A_\lambda^\Lambda O\rangle$ exists and is nonzero for at least some local O. Moreover, the Lindblad part states that these A_λ^Λ are invisible to the dissipation (unaffected by it) when the dynamics of local observables at frequency λ is described by the same $\omega_{s,\lambda}$ state.

Indeed, any operator sequence A_Λ that remains pseudolocal during the time evolution, i.e., there exists $\gamma > 0$ such that it satisfies $\omega_t(A_\Lambda^\dagger A_\Lambda) \le \gamma|\Lambda|$ (i.e., extensive) and $\lim_{\Lambda\to\infty}\omega_t(A_\Lambda^\dagger O)$ exists $\forall\ O \in \mathfrak{U}_{\rm loc}$, then $A_\Lambda - \omega_t(A_\Lambda)$ defines a pseudolocal dynamical symmetry. A very convenient and general case is the following.

Corollary 1.—Assume that there exist a pseudolocal sequence A_Λ and a clustering initial state ω in the sense defined previously and assume that they satisfy under time evolution (where $\omega_t = \omega\circ\tau_t$)

$$[\tau_t(A_\Lambda)]_\Lambda = e^{-i\lambda t}A_\Lambda + Z_\Lambda(t), \qquad \lambda \in \mathbb{R}, \tag{33}$$

$$\exists\,\gamma > 0, \qquad |\omega_t(A_\Lambda^\dagger A_\Lambda)| \le \gamma|\Lambda|, \quad \forall\ t, \Lambda, \tag{34}$$

$$\lim_{\Lambda\to\infty}\omega_t(A_\Lambda^\dagger O) \in \mathbb{C}, \quad \forall\ t, \qquad O \in \mathfrak{U}_{\rm loc}. \tag{35}$$

This defines a (left) pseudolocal dynamical symmetry if $\omega(e^{-i\lambda t}A_\Lambda + Z_\Lambda(t)) = 0$; if not, then we may use the zero-average sequence $e^{-i\lambda t}A_\Lambda + Z_\Lambda(t) - \omega(e^{-i\lambda t}A_\Lambda + Z_\Lambda(t))$ to define a (left) pseudolocal dynamical symmetry.

This follows directly from Theorem 3 and the definition of pseudolocal quantities. We may analogously construct right and two-sided pseudolocal dynamical symmetries.

This type of pseudolocal dynamical symmetry is actually more general than needed to study all topical examples of quantum many-body nonergodicity from the literature, so we further specialize.

Definition 1 (simple pseudolocality).—If a pseudolocal sequence in Corollary 1 has $Z_\Lambda(t) = 0$, then we call such a sequence *simple*.

Several well-known and newly introduced (in later sections) examples are given in Fig. 2.

We are now able to fully characterize the long-time dynamics in terms of the finite frequencies of the tGGE. Let us look at *local* dynamical symmetries, i.e., those whose operator sequences may be written as $A_\Lambda = \sum_{x\in\Lambda} a_x$, where a_x is the translation by x of the local density $a \in \mathfrak{U}_{\rm loc}$.

Theorem 4 (local tGGE).—Let the initial state be pseudolocal as before. Assume that there is a countable finite set of local dynamical symmetries parametrized by $k = 1, 2, \ldots, M$ with sequences $\{(A_k)_\Lambda\}$ with corresponding frequencies $\lambda_l \in \mathbb{R}$ [i.e., $Z_k(t) = 0$] and assume that this set forms a basis for an algebra closed under commutation (i.e., any element of the algebra may written as a linear combination in this basis). Assume that the dynamics τ_t has no pseudolocal dynamical symmetries except those generated by this set. Then the long-time dynamics is given by a normal state $\omega_t^{\rm tGGE}$ with matrix representation

$$\omega_t^{\rm tGGE}(O) = \lim_{\Lambda\to\infty}\mathrm{tr}_\Lambda[(\rho_t^{\rm tGGE})_\Lambda O], \quad \forall\ O \in \mathfrak{U}_\Lambda,$$
$$(\rho_t^{\rm tGGE})_\Lambda = \frac{\exp\left[\sum_k \mu_k e^{i\lambda_k t}(A_k)_\Lambda\right]}{\mathrm{tr}_\Lambda\{\exp\left[\sum_k \mu_k e^{i\lambda_k t}(A_k)_\Lambda\right]\}}, \tag{36}$$

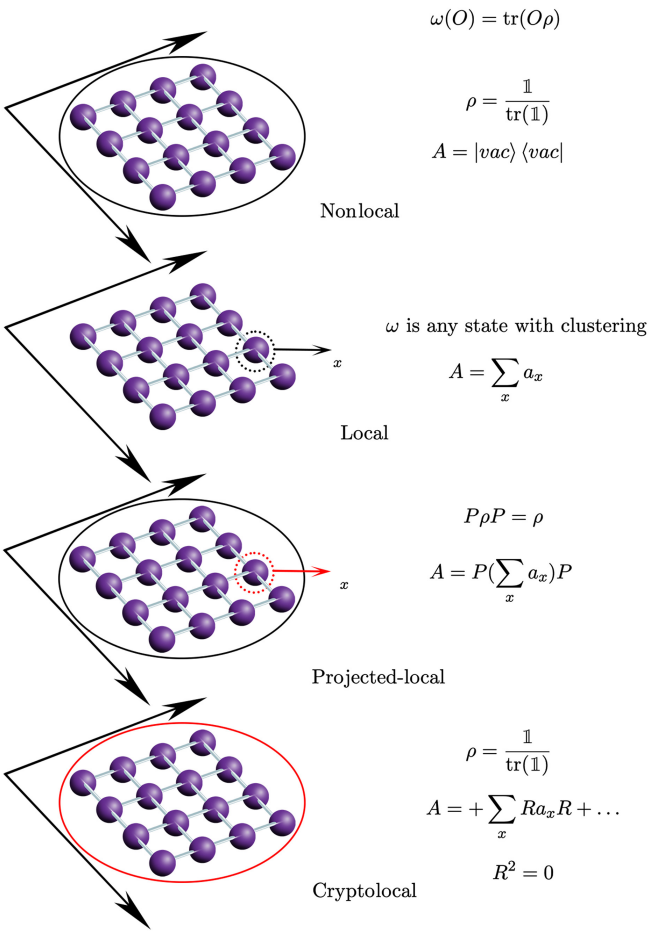

FIG. 2. An illustration of nonlocality vs different kinds of pseudolocality. The shown subsystem is of arbitrary size Λ. First case (nonlocal): A has support on the entire subsystem for all sizes of the subsystem (e.g., a projector to vacuum state). It is not pseudolocal for the infinite-temperature state. Second case (local): A is a simple sum of operators that act only on one site (translated to site x). It is pseudolocal for any state with clustering. Third case (projected-local): A local operator (second case) is sandwiched between a projector P (that acts on the entire subsystem), and the clustering state is entirely inside the subspace P projects to. It is pseudolocal for such a state because the state does not "see" the projector. Fourth case (cryptolocal): A contains terms (denoted R) that act on the entire subsystem for all size of the subsystem, but they cancel and they are not visible in the size of A for, e.g., the infinite-temperature state.

which exists, is a tGGE in the sense above with exponential clustering, and is analytic in μ_k, t for $|\mu_k| < \mu_*$ and $\forall\ t$. In 1D, $\mu_* = \infty$.

The same holds for open quantum systems if $[L_x(\eta), A_k] = [L_x^\dagger(\eta), A_k] = 0$. Note that we view conservation laws as special cases of dynamical symmetries with $\lambda_k = 0$. In particular, for any system, we always have $A_0 = H$, $\lambda_0 = 0$, and $\mu_0 = \beta$.

Taking $\lambda = 0$, which is must be finite for at least some O, we arrive at the operator form of the (dynamical) weak ETH immediately for both open and closed systems. This also proves the ETH for open systems [93–95].

Corollary 2 (weak eigenoperator thermalization).—Let the system satisfy Assumptions 1–4. Let ω be a pseudolocal initial state, and let the set $\{(A_k)_\Lambda, \mu_k, \lambda_k\}$ be as in Theorem 4. Pick a subset with $\lambda_k = 0$. Let $\kappa \coloneqq \{k|\lambda_k = 0\}$. We then have

$$\lim_{T\to\infty}\frac{1}{T}\int_0^T dt\omega(\tau_t(O)) = \lim_{\Lambda\to\infty}\mathrm{tr}_\Lambda\left[\frac{\exp\left[\sum_{k\in\kappa}\mu_k(A_k)_\Lambda\right]}{\mathrm{tr}_\Lambda\{\exp\left[\sum_{k\in\kappa}\mu_k(A_k)_\Lambda\right]\}}O\right],\quad \forall\ O\in\mathfrak{U}_{\mathrm{loc}}, \tag{37}$$

where $\mu_0 \coloneqq \beta$ and $(A_0)_\Lambda = H_\Lambda$.

Intuitively, the canonical weak ETH is equivalent to the zero-frequency case of Theorem 3, and, hence, the corresponding pseudolocal dynamical symmetries are only pseudolocal conservation laws and the corresponding state is a standard (generalized) Gibbs state.

Remark 3.—If an additional physically reasonable assumption that $\exists \lim_{t\to\infty}||\tau_t(O)||_{\mathcal{H}_\omega}$ holds, then the $\lim_{t\to\infty}\omega\circ\tau_t = \omega_t^{\mathrm{tGGE}}$; i.e., ω_t^{tGGE} is the asymptotic state in the strong sense in Theorems 3 and 4, not just at given frequencies.

In order to avoid confusion, let me now remark that in Ref. [96] it is shown that local quantum Liouvillians may be used to engineer unique stationary dark states with long-range order (i.e., not pseudolocal states). I emphasize that there is no contradiction with the work presented here, because it deals with time-averaged dynamics rather than the stationary state dynamics. As the unique stationary dark state is not clustered, using Lieb-Robinson bounds, it follows that the relaxation time of such models cannot be independent of system size [89]. In cases where the relaxation time diverges, the long-time average need not equal the exact stationary state average; cf. Refs. [97,98]. Moreover, Ref. [96] shows uniqueness only for the exact 0 eigenvalue, and it would be interesting to check what happens for purely imaginary eigenvalues, as they also contribute to the long-time dynamics.

Let us now step away from far-from-equilibrium dynamics for the moment and see what we can say about dynamics near equilibrium. Let us assume that the system is in a clustering equilibrium state in the sense that $\omega\circ\tau_t = \omega$. A very useful concept in that case is the Mazur bound [56,99] for the Drude weight [100], which provides a lower bound for the susceptibilities and ballistic transport. It is given in terms of quasilocal charges, and its finite-frequency version [101], which is sufficient for our purposes, reads

$$\lim_{T\to\infty}\frac{1}{NT}\int_0^T dte^{-i\lambda t}\omega[O^\dagger O(t)] \geq \sum_j \frac{|\omega(O^\dagger A_j)|^2}{N\omega(A_j^\dagger A_j)}, \tag{38}$$

where $A_j = \sum_x a_x^{(j)}$, $O = \sum_x o_x$, $[H, A_j] = \lambda A_j$, and we assume that A_j are orthogonal in the sense $\omega(A_j^\dagger A_k) = 0$ if $k \neq j$. It has long been conjectured that the Mazur bound saturates if all the A_j are known [37,102]. A partial result for finite systems called the Suzuki equality exists [103,104], but this conjecture remains unproven in the thermodynamic limit where it would have deep implications for, e.g., superconducting transport in the linear response regime [105].

In the setup discussed here, the Mazur bound becomes a straightforward equality.

Corollary 3 (Mazur equality).—Let $\omega = \omega\circ\tau_t$ be a pseudolocal equilibrium state. We have the following identity:

$$\lim_{T\to\infty}\frac{1}{T}\int_0^T dte^{-i\lambda t}\omega(O^\dagger\tau_t(o)) = \hat{\mathcal{A}}_{1,\lambda}(o), \tag{39}$$

where $\hat{\mathcal{A}}_{1,\lambda}$ is pseudolocal dynamical symmetries, i.e., satisfying $\hat{\mathcal{A}}_{1,\lambda}([H, q]) = \lambda\hat{\mathcal{A}}_{1,\lambda}(q)$, $\forall\ q\in\mathfrak{U}_{\mathrm{loc}}$, which has maximal overlap with o in the sense that

$$\hat{\mathcal{A}}_{1,\lambda}(q) \coloneqq \lim_{T\to\infty}\left\langle \mathfrak{D}\left(\frac{1}{T}\int_0^T dte^{i\lambda t}\tau_t(o)\right), q\right\rangle_\omega^c, \tag{40}$$

where $\mathfrak{D}:\mathcal{H}_\omega \to \mathfrak{U}_\omega$ is the bijection between the Hilbert space of local observables and the pseudolocal quantities.

This follows directly from the definition of the inner product on $\mathcal{H}_\omega$, pseudolocal dynamical symmetries, Theorem 1, and the existence of the bijection. By construction, if we know all the pseudolocal dynamical symmetries, we also know the ones having maximal overlap with o. In this setup, normalization is not needed, as it is present by construction.

This is likewise consistent with similar results obtained for fully extensive operators of closed Hamiltonians at zero frequency and finite momentum [106] and finite frequency [107].

Before we turn to studying examples, let us state the following simple result in anticipation of its use for pseudolocal quantities later on.

Corollary 4.—Assuming that a sequence $A_\Lambda \in \mathfrak{U}_{\mathrm{loc}}$ is a simple pseudolocal sequence defining *left* pseudolocal quantities $\mathcal{A}_s(O)$ with respect to some state ω with

flow ω_s, additionally satisfying $\tau_t(A_\Lambda^\dagger A_\Lambda) = \tau_t(A_\Lambda^\dagger)\tau_t(A_\Lambda)$, i.e., a sequence satisfying

(1) $\omega_s(A_\Lambda^\dagger A_\Lambda) \leq \gamma|\Lambda|^D$, for some γ and $\forall\ \Lambda$, and
(2) the limit $\mathcal{A}_s(O) \coloneqq \lim_{\Lambda\to\infty} \omega(A_\Lambda^\dagger O)$ exists for all $O \in \mathfrak{U}_{\text{loc}}$,

then it remains a simple pseudolocal sequence with respect to $\omega\circ\tau_\tau$, $\forall\ \tau$. The similar holds for two-sided and right pseudolocal sequences.

This immediately follows from Theorem 1 and Definition 1. Note that the extra requirement $\tau_t(A_\Lambda^\dagger A_\Lambda) = \tau_t(A_\Lambda^\dagger)\tau_t(A_\Lambda)$ is trivially satisfied for closed systems.

C. Generalizations to time-dependent systems and systems without translational invariance

In the previous section, I specialize to time-independent systems at certain points for the sake of clarity of presentation. Likewise, I also specialize to systems that have translational invariance in some sense (e.g., at finite momentum). Let us now discuss how to generalize the main results.

Time-dependent systems.—The first part of Theorem 3 applies to time-dependent systems. The second part holds for periodic dynamics provided we work in the extended space representation [58]. To sketch the idea, let $H(t) = H(t+P)$, we extend the equations of motion

$$\frac{\partial}{\partial t}\psi(\theta,t) = \left(\frac{\partial}{\partial\theta} - iH(\theta)\right)\psi(\theta,t), \tag{41}$$

and then $\phi(t) = \psi(t,t)$ solves the original time-dependent equations of motion. Provided that $\psi(\theta,t)$ is sufficiently well defined (see Refs. [58,59] for details and examples), we may then perform a Fourier transformation and obtain

$$\frac{\partial}{\partial t}\psi(k,t) = \sum_{k'}\left(\tilde{H}(k-k') - \frac{2\pi k}{P}\right)\psi(k',t). \tag{42}$$

The total generator $(H_\omega)_{kk'} \coloneqq \tilde{H}(k-k') - k(2\pi k/P)$ then may be used in the rest of the theorem.

No translation invariance.—Disordered systems are examples of systems that have no translation invariance or any generalized automorphism that may replace it. In order to treat such systems, we must reformulate the framework of Ref. [28] from the beginning. We begin by introducing an alternative definition of the inner product. We begin with the sesquilinear form

$$\langle O,Q\rangle_\omega^{\text{loc}} \coloneqq \omega(O^\dagger Q) - \omega(O^\dagger)\omega(Q), \qquad O,Q \in \mathfrak{U}_{\text{loc}}. \tag{43}$$

Note that we do not sum over the sites in the first term. The form is degenerate, i.e., define $\mathcal{N}_\omega^{\text{loc}} \coloneqq \{\langle O,O\rangle_\omega^{\text{loc}} = 0\}$. An example is $O = \mathbb{1}$. We define the quotient $H_\omega = \mathfrak{U}/\mathcal{N}_\omega^{\text{loc}}$ that we then Cauchy complete to a Hilbert space $\mathcal{H}_\omega^{\text{loc}}$. This is the standard GNS construction [26].

Define pseudolocalized operator sequences that satisfy $\langle A_\Lambda, A_\Lambda\rangle_\omega^{\text{loc}} \leq \gamma$ for some $\gamma > 0$. As $A_\Lambda \in \mathfrak{U}_{\text{loc}}$ and $\mathfrak{U}_{\text{loc}}$ is dense in $\mathcal{H}_\omega^{\text{loc}}$, we may always find a pseudolocalized Cauchy sequence (with respect to $||||_{\mathcal{H}_\omega^{\text{loc}}}$) $j \mapsto A_j \in \mathfrak{U}$ that converges to $\hat{\mathcal{A}}_\omega^{\text{loc}}(O) = \lim_{j\to\infty}\langle A_j O\rangle_\omega^{\text{loc}}$. The space of such pseudolocalized quantities $\mathfrak{U}_\omega^{\text{loc}}$ is in a trivial bijection with $\mathcal{H}_\omega^{\text{loc}}$ being its dual. We then may repeat the entire construction of Ref. [28], as well as the present work.

The only major differences are (i) that the *localized* tGGE (Theorem 4) with pseudolocalized quantities is defined for *any* μ_k in all dimensions; (ii) the Mazur equality is finite for strictly local operators without integrating over space; and (iii) pseudolocalized states (i.e., pseudolocal states defined via pseudolocalized quantities) are exponentially clustering, and this property is preserved under the time evolution.

Note that we can apply the above pseudolocalized construction to translationally invariant systems, but the construction is not, in general, useful, and the one used in the previous sections is more powerful for such systems. This is because most systems do not have pseudolocalized dynamical symmetries. Two notable exceptions are, as we see, lattice gauge theories and systems with disorder-free localization.

Note that the local integrals of motion [108] (or l-bits [109]) of many-body localized models fall into the category of pseudolocalized dynamical symmetries.

V. APPLICATIONS

The idea for exact solutions is the following. One needs to identify the pseudolocal quantities the model poses; then one may straightforwardly construct the tGGE giving the exact solution for the long-time dynamics. A practical outline of this procedure, given by the theory here in the previous section, for time-independent closed systems is the following.

(1) First, identify all potentially pseudolocal operators A_u^V (with respect to a desired initial state ρ_0) that satisfy $[H_V, A_u^V] = \lambda_u A_u^V$ for a finite size Hamiltonian H_V of size V.
(2) Propose the solution as $\rho_{\text{tGGE}}(t) = (1/Z)\exp\times(\sum_u \mu_u e^{i\lambda_u t}A_u)$, where $Z = \text{tr}[\exp(\sum_u \mu_u e^{i\lambda_u t}A_u)]$ is the normalization (or time-dependent partition function) and where we drop the size V superscript. If subsets $\{A_u\}$ have the same frequency λ_u (i.e., in case of degeneracy), it may be necessary to orthonormalize them. The tGGE is the correct solution to the long-time dynamics of local observables, as proven in the previous section, provided that one knows all the relevant A_u, (i) that A_u are pseudolocal with respect to the initial state ρ_0, and (ii) that the

state $\rho(t)$ has clustering. This needs to be separately checked as discussed below in steps 4 and 5.

(3) One needs to compute the chemical potentials μ_u that are fixed by the initial state. This is done by solving for μ_u the set of equations $\mathrm{tr}[\rho(0)A_u] = \mathrm{tr}[\rho_{\mathrm{tGGE}}(0)A_u]$. This is, in principle, a highly complicated set of nonlinear equations requiring numerical solutions, but analytical solutions can be found for wide classes of initial state ρ_0. For instance, those that are symmetric or antisymmetric with respect to some discrete symmetry S whereas an A_u is the opposite (antisymmetric or symmetric), e.g., $S\rho(0)S^\dagger = \pm\rho(0)$ and $SA_uS^\dagger = \mp A_u$, then $\mu_u = 0$. Similarly, if $[\rho(0), A_u] = \kappa_u A_u$, the solution may be found with some transfer matrix approach. We utilize these approaches later in the examples. Likewise, one may perform high and low μ_u expansions and then solve a reduced nonlinear set.

(4) Define the sequences for increasing V, $\tilde{A}_u^V = A_u^V - \langle A_u^V\rangle_t$, where $\langle O\rangle_t = \mathrm{tr}[\rho_{\mathrm{tGGE}}(t)O]$ for the state at time t. We compute how the size of the operators grows with system size $\mathrm{tr}[\rho_{\mathrm{tGGE}}(t)(\tilde{A}_u^V)^\dagger\tilde{A}_u^V]$. Because of Corollary 1, it is sufficient to check only for one value of t or in the initial state $\rho(0)$, which may be a simpler calculation analytically. One approach is if the set $\{A_u^V\}$ in $\rho_{\mathrm{tGGE}}(t)$ is in involution and is simple enough for Z to admit, e.g., a transfer matrix form, then we can compute the relevant expectation values in a way that is standardly done for equilibrium partition functions $(1/Z)(\partial^2/\partial\mu_u\partial\mu_u^*)Z = \langle(A_u^V)^\dagger A_u^V\rangle_t$, etc. In the case $\mathrm{tr}[\rho_{\mathrm{tGGE}}(t)(\tilde{A}_u^V)^\dagger\tilde{A}_u^V] > CV$ (C does not depend on V), then rescale the sequence $\tilde{A}_u^V \to \tilde{A}_u^V/f(V)$ by some appropriate function $f(V)$ so that $\mathrm{tr}[\rho_{\mathrm{tGGE}}(t)(\tilde{A}_u^V)^\dagger\tilde{A}_u^V] \le CV$. In both cases, one needs to check that $\lim_{V\to\infty}\langle A_u^V O\rangle$ exists for all local observables O (this likely is the case) and is nonzero for at least some local observables. Because of the rescaling, overlap with all local observables can be zero, and in that case the sequence A_u^V does not correspond to any pseudolocal sequence and must be discarded from the *Ansatz* $\rho_{\mathrm{tGGE}}(t)$. It is sufficient to check this using the densities of A_u^V as the local observables, by, e.g., using the time-dependent partition function.

(5) Clustering of $\rho_{\mathrm{tGGE}}(t)$ needs to be also checked; i.e., for two local observables on sites $x(y)$, we must have $\lim_{||x-y||\to\infty}\lim_{V\to\infty}\langle O_x O_y\rangle_t = \langle O_x\rangle_t\langle O_y\rangle_t$. This is again sufficient to do for the local densities of A_u and may be done like in step 4 by computing Z. In case $\rho_{\mathrm{tGGE}}(t)$ is *not* clustered, then it cannot be used as the correct *Ansatz*. This signals formation of long-range order. In that case, symmetry breaking of the A_u must be considered as in equilibrium [24,26].

(6) Finally, expectation values of local observables can be computed from the time-dependent partition function similarly to equilibrium, by, e.g., adding a small field α corresponding to desired observable O, $Z \to Z(\alpha O)$ and then $\langle O\rangle_t = [dZ(\alpha O)/d\alpha]|_{\alpha=0}$. This may be done fully analytically provided that O in some sense closes an algebra with the A_u or can be again done in the low or high chemical potential expansion perturbatively, in general.

Fortunately, in certain cases, structures known from the previous literature can be used to construct the pseudolocal quantities, and in others they may be straightforwardly identified from the requirements in Corollary 1.

A. Projected-local quantities: Quantum many-body scars and embedded Hamiltonians

Quantum many-body scars [6,110,111] and embedded Hamiltonians [112,113] are two different manifestations of the same underlying pseudolocal algebra, as we now see. First, we need a definition.

Definition 2 (projected-local quantity).—A projected-local quantity is one satisfying the dynamical symmetry volume growth condition (34) from Corollary 1 for some clustering initial states ω, but not *all* of them. In particular, it does *not* satisfy it for the tracial state $\omega(O) = \mathrm{Tr}(O)$, i.e., the infinite-temperature state.

Specifically, let $A'_\Lambda = \sum_{x\in\Lambda} a_x$ be a pseudolocal sequence and let

$$P_\Lambda = \sum_{k,j} |\psi_j\rangle\langle\psi_k| \tag{44}$$

be a generalized projector to the eigenspaces of H_Λ, i.e.,

$$[H_\Lambda, P_\Lambda] = \nu P_\Lambda. \tag{45}$$

Assume that $A = P_\Lambda^\dagger A' P_\Lambda$ satisfies the condition for a pseudolocal dynamical symmetry from Corollary 1, with a corresponding pseudolocal initial state with flow ω_s for which

$$\omega_s(P_\Lambda O P_\Lambda) = \omega_s(O), \tag{46}$$

$$\omega_s[Z(t)O] = \omega_s[OZ(t)] = 0, \quad \forall\, s\in[0,1], \quad O\in\mathfrak{U}_{\mathrm{loc}}. \tag{47}$$

The long-time dynamics is then given by a tGGE according to Theorem 3. Moreover, the local form of the tGGE from Theorem 4 is the unique normal representation provided that

$$P_\Lambda \rho_t^{\mathrm{tGGE}} P_\Lambda = \rho_t^{\mathrm{tGGE}}. \tag{48}$$

Scarring phase transition.—For the sake of simplicity, assume that H_Λ, A_Λ, $A_\Lambda^\dagger$, and $[A_\Lambda, A_\Lambda^\dagger]$ generate the $SU(2)$ algebra when acted on by ω_s, i.e., $\omega_s([H_\Lambda, A_\Lambda] - \lambda A_\Lambda) = 0$,

etc. The local tGGE has $(A_0)_\Lambda = H_\Lambda$ ($\lambda_0 = 0$), $(A_1)_\Lambda = A'_\Lambda$ ($\lambda_1 = \lambda$), $(A_2)_\Lambda = (A_1^\dagger)_\Lambda$ ($\lambda_2 = -\lambda$), and $(A_3)_\Lambda = [(A_1)_\Lambda, (A_2)_\Lambda]$ ($\lambda_3 = 0$). Equation (48) is true only for certain values of μ_k. For other values, we cannot use the matrix representation, because the state does not have clustering and, hence, is not the valid pseudolocal state.

This is intimately related to the initial state property (47). Indeed, if we modify the flow $\omega_{s(v)}$ continuously such that for some critical v_* value of $\omega_{s(v_*)}$ the property (47) does not hold, then projected-local quantities are no longer pseudolocal with respect to the initial state and are not to be included in the long-time tGGE. For instance, we may vary the inverse temperature of a thermal initial state of another Hamiltonian H' $\omega_{\beta(s)}$ such that for some value of $\beta(s)$ Eq. (47) no longer holds. As this variation of initial temperature can be done continuously and the property (47) is discontinuous—i.e., it either does or does not hold—this demonstrates a novel kind of phase transition between ergodic behavior and scarred dynamics. This phase transition is to be contrasted with the standard thermodynamic phase transitions, which occur because the thermal state is no longer the valid representation above some inverse temperature $|\beta| > \beta_*$ so that it no longer has a certain symmetry. Here, the eigenoperators themselves stop being local. This is consistent with the very recent numerical observation of the dynamical phase diagram of the PXP model [114]. Because of the property (48), the scarring phase is stable to local perturbations, as all the terms in the exponent of ρ_{tGGE} are local operators for all values of t, and, hence, the same arguments as for thermodynamic phases can be applied [24,26].

In the rest of this subsection, for the sake of notation, we drop the subscript Λ and implicitly deal with the finite system case and its thermodynamic limit.

Embedded Hamiltonians.—Let P_x be a set of strictly local projectors, i.e., projectors with finite support $\Lambda_x \subset \Lambda$. Let also $[H', P_x] = 0, \ \forall \ x$. Embedded Hamiltonians are defined as [112]

$$H = \sum_x P_x h_x^0 P_x + H', \tag{49}$$

where h_x^0 is some local Hamiltonian density translated by x. Clearly, any operator for which $[H', A'] = \lambda A'$ may be used to construct a simple projected-local quantity of the form $A = \sum_x (\mathbb{1} - P_x) a_x (\mathbb{1} - P_x)$, $Z(t) = 0$.

Restricted spectrum-generating algebras.—Restricted spectrum-generating algebras are formulation of quantum many-body scarred models (Supplement in Ref. [52] and Refs. [49,61]) for which

$$\begin{aligned} H|\psi_0\rangle &= E_0|\psi_0\rangle, \\ [H, Q^+]|\psi_k\rangle &= \lambda Q^+|\psi_k\rangle, \quad \forall \ n, (Q^+)^k|\psi_0\rangle \neq 0. \end{aligned} \tag{50}$$

There are several equivalent formulations [62,115,116] to this one. As discussed in Sec. II A, the existence of such a structure *a priori* does not imply anything for quantum many-body dynamics in the thermodynamic limit. However, if Q^+ is itself pseudolocal, then it clearly defines a restricted local quantity with $P = \sum_k |k\rangle\langle k|$ and $Z(t) = 0$. Numerous models studied in the literature satisfy this requirement; e.g., see Ref. [49] for a review.

The PXP model.—Consider the original model of quantum many-body scarring [6,117], the one-dimensional PXP model:

$$H_{\mathrm{PXP}} = \sum_x \frac{1}{4}(\mathbb{1} - \sigma_{x-1}^z)\sigma_i^x(\mathbb{1} - \sigma_{x+1}^z), \tag{51}$$

where σ_x^α is the $\alpha = x, y, z$ Pauli matrix on site x. Curiously, this model does not have projected-local dynamical symmetries for all times, but it has them for finite times. To see what this means, recall [110] (see also Ref. [118]) that

$$\begin{aligned} [H_{\mathrm{PXP}}, S_\pi^+] &= S_\pi^+ + O_{ZZZ}, \\ S_\pi^+ &= \frac{1}{2}\sum_x (-1)^x \left[\sigma_x^z - \frac{i}{2}(\mathbb{1} - \sigma_{x-1}^z)\sigma_x^y(\mathbb{1} - \sigma_{x+1}^z)\right], \\ O_{ZZZ} &= \sum_x (-1)^x \sigma_{x-1}^z \sigma_x^z \sigma_{x+1}^z. \end{aligned} \tag{52}$$

Clearly, $\tau_t(S_\pi^+) = e^{it}S_\pi^+ + Z(t)$, but, comparing with the volume growth (34) from Corollary 1, we get for $t > 0$

$$|\omega\{Z^\dagger(t)[e^{it}S_\pi^+ + Z(t)]\}| \leq \phi e^{|v|t}|\Lambda|, \tag{53}$$

which we get from the proof of Theorem 1 (more specifically, the special case in Ref. [28]). In other words, the γ in Eq. (34) is time dependent. This does not allow us to define $e^{it}S_\pi^+ + Z(t)$ as a pseudolocal dynamical symmetry for all t, but fixing some maximal time the conditions are still satisfied. Physically, this reflects the decay of oscillations of local observables [6]. Moreover, for special initial states, such as the Neel state, the growth in Eq. (53) is smaller than for other initial states. This confirms numerical results on weak ergodicity breaking [6].

Hence, the PXP and the other quantum many-body scarred models come from different classes of models, but they both have the projected-local quantities as a common feature explaining physically relevant dynamics of local observables. The existence of such quantities should, therefore, be taken as defining quantum many-body scars.

B. Cryptolocal quantities: Statistically localized integrals of motion and Hilbert space fragmentation

In Refs. [15–17,119,120], fragmented models that do not possess (explicitly) quasilocal conservation laws but do possess finite autocorrelation functions are introduced.

This behavior is explained through statistically localized integrals of motion (SLIOM) in Ref. [81] and, alternatively, commutant algebras in Ref. [50]. Using such algebraic structures, it is possible to provide Mazur bounds on autocorrelation functions of local observables. However, it remains unclear whether these Mazur bounds saturate, and the far-from-equilibrium dynamics of fragmented models remain inaccessible to analytical study. Using the theory developed in the previous section, it is possible to give far-from-equilibrium dynamics in terms of the tGGEs and show that the Mazur bound is saturated. In order to do so, we must introduce new types of pseudolocal (pseudolocalized) quantities.

Definition 3 (cryptolocality).—Cryptolocal quantities are those that satisfy the pseudolocality conditions from Corollary 1 but cannot be written as manifestly translation-invariant sums of local densities (not even with diverging quasilocal support). Likewise, cryptolocalized quantities are those that meet the pseudolocalized conditions from Sec. IV C but cannot be written as manifestly localized objects.

We study the prototypical one-dimensional $t-J_z$ model [50,81]:

$$H_{t-J_z} = \sum_{x,\sigma\in\{\uparrow,\downarrow\}} -t_{x,x+1}(d_{x,\sigma}d^{\dagger}_{x+1,\sigma} + \text{H.c.}) + \sum_{x,\sigma\in\{\uparrow,\downarrow\}} J^{z}_{x,x+1}S^{z}_{x}S^{z}_{x+1} + \sum_{x} h_x S^{z}_{x} + g_x (S^{z}_{x})^2, \tag{54}$$

where $t_{x,x+1}$, $J^{z}_{x,x+1}$, h_x, and g_x are arbitrary and

$$S^{z}_{x} = d^{\dagger}_{x,\uparrow}d_{x,\uparrow} - d^{\dagger}_{x,\downarrow}d_{x,\downarrow}, \tag{55}$$

$$d_{x,\sigma} = c_{x,\sigma}(1 - c^{\dagger}_{x,-\sigma}c_{x,-\sigma}), \tag{56}$$

where $-\sigma{:}\uparrow(\downarrow)\to\downarrow(\uparrow)$ means taking opposite spin of σ and $c^{\dagger}_{x,\sigma}$ and $c_{x,\sigma}$ are fermionic creation and annihilation operators on site x with spin σ.

Consider the "left" and "right" SLIOMs [81]

$$\mathcal{Q}^{(l,r)}_{k} = \sum_{x=1}^{L} \mathcal{P}^{(l,r)}_{k,x}(N^{\uparrow}_{x} - N^{\downarrow}_{x}), \tag{57}$$

where $\mathcal{P}^{(l,r)}_{k,x}$ is the projector onto configurations where the kth charge from the left (right) is on site x and $N^{\sigma}_{x} = d^{\dagger}_{x,\sigma}d_{x,\sigma}$ (see Ref. [81] for details).

Using these, we may construct cryptolocalized and cryptolocal quantities:

$$A_{\vec{\alpha}} = L^{\nu}\sum_{k,j=l,r} \frac{\alpha^{j}_{k}}{(\sum_{k',j'=l,r}\alpha^{j'}_{k'})}\frac{\mathcal{Q}^{(j)}_{k}}{\omega(\mathcal{Q}^{(j)}_{k}\mathcal{Q}^{(j)}_{k})}, \tag{58}$$

where $\nu = 0, 1/4, 1/2$. The $\nu = 0$ case corresponds to cryptolocalized cases and the other two to cryptolocal. The reader may be surprised that the quantity growing as $\omega(A_{\vec{\alpha}}A_{\vec{\alpha}}) \propto L^{1/2}$ is pseudolocal, but it is according to the general definition in Corollary 1. Note also that we are free to "tune" the sequence (58) between a pseudolocal one (extensive) and pseudolocalized by changing ν. If in doing so we promote a sequence that is pseudolocal to a pseudolocalized one, the corresponding pseudolocalized quantity simply gives vanishing functionals $\mathcal{A}^{\text{loc}}_{\omega}(O) = 0$ for all $O \in \mathfrak{U}_{\text{loc}}$. Fragmented models are special, because they have cryptolocalized quantities with respect to the infinite-temperature state, which is directly implied by the present work and by the finite values of the corresponding Mazur bounds identified previously [e.g., Eq. (11) in Ref. [81]]. Naturally, the reader may be concerned about the presence of infinitely long (nonlocal) strings $\mathcal{P}^{(l,r)}_{k,x}$ in the cryptolocal quantities and how their presence affects the clustering of the corresponding ρ_{tGGE}. It turns out, as discussed in the example below, that most of these strings cancel and the remaining ones are subextensive in number (thermodynamically irrelevant) for most initial states. Interestingly, they can be thermodynamically relevant for some initial states with clustering. The corresponding long-time limit is not, therefore, a $\rho_{\text{tGGE}}(t)$ state containing cryptolocal charges.

Furthermore, these operator sequences may provide Mazur bounds (or equalities according to Corollary 3) and tGGEs, thus completing the picture of nonequilibrium dynamics for fragmented models. Other fragmented models (e.g., Ref. [121]) may be treated analogously.

C. Strictly localized quantities: Disorder-free localization and lattice gauge theories

Now we deal with strictly localized quantities that should be contrasted from cryptolocalized cases associated with fragmentation.

A prototypical model with disorder-free localization is one with spin-fermion coupling [23]:

$$H_{\text{sf}} = -J\sum_{x}\sigma^{z}_{x,x+1}c^{\dagger}_{x}c_{x} - h\sum_{x}\sigma^{x}_{x-1,x}\sigma^{x}_{x,x+1}, \tag{59}$$

where the c_x ($c^{\dagger}_{x}$) are spinless fermion lowering (raising) operators acting on sites x and $\sigma^{\alpha}_{x,x+1}$ are spin-1/2 Pauli matrices, as before, acting on the bonds between the sites.

Very much related to disorder-free localization models are lattice gauge theories [22,68,69], such as the simple $\mathbb{Z}_2$ lattice gauge theory [67]:

$$H_{\mathbb{Z}_2} = \sum_{x} J(a^{\dagger}_{x}\sigma^{z}_{x,x+1}a_{x+1} + \text{H.c.}) - h\sigma^{x}_{x,x+1}, \tag{60}$$

where a_x ($a_x^\dagger$) are *hard-core* bosonic annihilation (creation) operators with $n_x = a_x^\dagger a_x$ representing matter occupation on site x.

Both types of models are characterized by full sets of strictly local (or pseudolocalized) symmetries G_x, where $G_x \in \Lambda_x \subset \Lambda$, i.e., G_x has finite support.

For instance, the generator of the $\mathbb{Z}_2$ symmetry is

$$G_x^{\mathbb{Z}_2} = (-1)^{n_j} \sigma_{x-1,x}^x \sigma_{x,x+1}^x. \tag{61}$$

Understanding nonequilibrium dynamics of these models has attracted lots of interest recently. Using the theory developed here, exact solutions can be given in terms of tGGEs given with pseudolocalized (or strictly localized) quantities generated by the corresponding symmetries of the models. More specifically, the full set consists of projectors to the eigenspaces of these generators. Similar holds for non-Abelian lattice gauge theories [122], in which case we need to be mindful that the generators close some algebra and then we may use the tGGE solution.

Analogous results hold for theories that have fragmentation due to strictly local quantities rather than cryptolocalized ones [65–67,69,123–126], as well as pseudolocalized ones [127–130].

D. Other cases

Let us briefly discuss other cases for which *Ansätze* similar to the tGGE have been previously conjectured. The added benefit of the theory from the previous sections is giving the correct forms of the tGGE and proving that these are the exact solutions.

Discrete time crystals in closed systems.—Many-body localized models have been employed for several years for study of discrete time crystals [19,131,132], i.e., many-body systems that display parametric down-conversion in the sense of breaking the period of an external drive $T \to nT$. They are conjectured to go into cryptoequilibrium states [133] that maximize entropy. The present work proves this in the form of the tGGE state. Moreover, the correct pseudolocalized dynamical symmetries are the l-bits identified in Refs. [134,135].

Discrete and continuous dissipative time crystals.—Time crystals in locally interacting systems induced or stabilized by dissipation have been studied, both the discrete version (described above) [136–142] and the continuous version for which the time-translation symmetry breaking occurs without any external time-dependent drive, in terms of dynamical symmetries [20,39,52,143–147]. The present work shows that the correct form of the long-time limit is the tGGE containing the dynamical symmetries.

Continuous time crystals in isolated systems.—In Ref. [40], the tGGE *Ansatz* has been previously conjectured for the XXZ spin chain containing quasilocal dynamical symmetries. The present work shows that this is the exact solution to the long-time dynamics.

Semilocal charges.—Very recently, the notion of pseudolocality has been extended to include semilocal operators, i.e., operators whose densities commute with distant operators on one side only [148,149] (see also Refs. [150–153]). These operators are sums of densities of the form $o_x^{\mathrm{sl}} = \lim_{N\to\infty} \prod_{k=-N}^{x} \sigma_k^z o_x$, where $o_x \in \Lambda_x \subset \Lambda$ is local. Note that the operator contains a *string* of, e.g., Pauli z operators. In Ref. [148], the algebra of quasilocal observables is extended to include semilocal operators. Although this is perfectly correct, the work above shows that semilocal operators are indeed pseudolocal with respect to specific states. That is they fall into the projected-local class (an initial state that does not see the Pauli string).

VI. EXAMPLES

We now study explicit examples from the previous section applying the procedure outlined there and go beyond existing techniques by studying general far-from-equilibrium quenches in cases where solutions are available only either from very special initial states for certain observables or near infinite temperature in the linear response regime. We compute the (finite-frequency) time-averaged expectation values of local observables:

$$\overline{\langle O \rangle}_{\lambda=\kappa} = \lim_{T\to\infty} \frac{1}{T} \int_0^T dt e^{-i\kappa t} \langle O \rangle_t, \tag{62}$$

and if not otherwise written $\langle O \rangle$ implies the $\lambda = 0$ zero-frequency case.

A. Spin-1 model with quantum many-body scars

Here, we look into the scarred spin-1 model on a D-dimensional hypercubic lattice studied in Ref. [111]:

$$H = \sum_{\langle xy \rangle} (S_x^x S_y^x + S_x^y S_y^y) + h \sum_x S_x^z + d \sum_x (S_x^z - 2)^2, \tag{63}$$

where $\langle xy \rangle$ means sum over nearest neighbors and S_x^α ($\alpha = x, y, z$) are spin-1 operators on site x. The lattice number is $V = N^d$. As shown in Ref. [111], the model has quantum many-body scarred eigenstates

$$|n\rangle = \mathcal{N} (J^+)^n |-1\rangle, \tag{64}$$

where $n = 0, \ldots, V$, $|-1\rangle$ is the fully polarized down state, the normalization is $\mathcal{N} = \sqrt{(V-n)!/n!V!}$, and

$$J^\pm = \frac{1}{2} \sum_x e^{\pm i \vec{x}\cdot\vec{\pi}} (S_x^\pm)^2, \tag{65}$$

where $\vec{x}$ is the lattice site position vector and $\vec{\pi}$ is a vector of the same dimensions whose all components are π. The model also has a $U(1)$ symmetry $J^z = \sum_x S_x^z$.

Define the projector to the scarred subspace $P = \sum_n |n\rangle\langle n|$. It is not difficult to see that $[P, J^\alpha] = 0$ and that $A_{\pm 1} = f(V) P J^\pm$ fulfill the conditions for simple dynamical symmetries with H from Definition 1, i.e., $[H, A_{\pm 1}] = \pm 2h A_{\pm 1}$ and where we anticipate that $f(V)$ is a system-size-dependent normalization that is required. Thus, $\lambda_{\pm 1} = \pm 2h$ and $\lambda = 2h$. We consider a simple initial state as an example:

$$\rho(0) = \frac{1}{Z_0}\exp(\mu_0 J^x), \tag{66}$$

where Z_0 is the normalization. As shown in Ref. [111], exact solutions are possible in the case when $\mu_0 \to \pm\infty$ for certain observables. Here, we compute the general case. We begin with an *Ansatz* for the tGGE:

$$\rho_{\mathrm{tGGE}} = \frac{1}{Z}\exp(-\beta H + \mu_z J^z + \mu e^{i\lambda t} A_1 + \mathrm{H.c.}), \tag{67}$$

where $Z(t) = \exp(-\beta H + \mu_z J^z + \mu e^{i\lambda t} A_1 + \mathrm{H.c.})$ is the time-dependent partition function. For the choice of initial state (66), we have $\mu_z = 0$. Using the fact that $\rho(0)$ is spin-flip symmetric and $H(d = 0)$ is spin-flip antisymmetric, we immediately get

$$\langle H\rangle_0 = \mathrm{tr}[\rho(0)H] = dV\left(\frac{1}{3} - \frac{1}{2\cosh(2|\mu_0|) + 1}\right). \tag{68}$$

We need to find for what μ_0 the $A_{\pm 1}$ are pseudolocal, however, according to the procedure outlined in the beginning of Sec. V. In order to do so, without loss of generality, but for the purposes of easing orthonormalization, we set $d = 0$. Because $[P, J^\alpha] = 0$ and $P^2 = \mathbb{1}$, we get that $\mu_0 = \mu$. The partition function for the initial state can be immediately computed:

$$Z_0 = [2\cosh(\mu_0) + 1]^V. \tag{69}$$

Likewise, as discussed in Appendix A, the partition function of the tGGE can also be computed:

$$\begin{aligned} Z = {} & \mathrm{csch}[\mu\cos(2ht)]\sinh[\mu(V+1)\cos(2ht)] - V \\ & + \sinh[V\log(3)] + \cosh[V\log(3)] - 1. \end{aligned} \tag{70}$$

As shown in Appendix A, it is straightforward to compute that in the initial state (note again that due to Corollary 1 it is sufficient to check pseudolocality for the initial state)

$$\begin{aligned} \langle A_1 A_{-1}\rangle_0 = {} & -\frac{f(V)^2}{4Z}\mathrm{csch}^3(\mu_0)\big[2(V^2 + 2V - 1)\sinh(\mu_0[V+1]) \\ & - (V+1)\{V\sinh(\mu_0[V+3]) \\ & + (V+2)\sinh(\mu_0[V-1])\}\big], \end{aligned} \tag{71}$$

$$\begin{aligned} \langle A_1\rangle_0 = {} & \frac{f(V)}{Z}\mathrm{csch}^2(\mu_0)\{V\sinh[\mu_0(V+2)] \\ & - (V+2)\sinh(\mu_0 V)\}, \end{aligned} \tag{72}$$

$$\langle J_x^+\rangle_0 = \frac{2\sinh(\mu_0)}{2\cosh(\mu_0) + 1}. \tag{73}$$

Our method is to fix $f(V)$ by demanding that $\lim_{V\to\infty}\langle \tilde{A}_1 O_x\rangle_0$ is not zero for at least some local O_x. The most convenient choice is the density of $J^\pm$, i.e., $J_x^\pm = e^{\pm\vec{x}\cdot\vec{\pi}}(S_x^\pm)^2$ (because the $\tilde{A}_{\pm 1}$ have overlap with it). We find that

$$\langle A_1 J_x^-\rangle_0 = \frac{1}{V}\langle A_1 A_{-1}\rangle_0 - \langle A_1\rangle_0\langle J_x^-\rangle_0, \tag{74}$$

where we use the fact that the tGGE and $J^\pm$ are Bloch translationally invariant with momentum π. Now we fix $f(V)$ by demanding that $\langle A_1 J_x^-\rangle_0 = 1$ (the actual value of the constant does not matter, only that it is finite) and compute $\langle \tilde{A}_1\tilde{A}_{-1}\rangle_0$. For $A_{\pm 1}$ to be pseudolocal, $\langle \tilde{A}_1\tilde{A}_{-1}\rangle_0 \le CV$, where C does not depend on V. We find, by expanding in $1/V$, that for finite μ and large V, $\langle \tilde{A}_1\tilde{A}_{-1}\rangle_0 \approx e^{4|\mu| + V\{\log[\cosh(|\mu|)+1] - |\mu|\}}$. For $\mu \to \pm\infty$ we find, on the other hand, that the $A_{\pm 1}$ are pseudolocal as $\langle \tilde{A}_1\tilde{A}_{-1}\rangle_0 = V$ for diverging μ. Therefore, there is no scarring phase transition for the initial state chosen here for finite μ in the thermodynamic limit. However, as we see, there is similar behavior to a phase transition for finite system size. Persistent oscillations (i.e., nonstationarity) in local observables are present only for diverging μ. However, even though the theory given in this paper is strictly speaking for thermodynamically large systems, we may still gain insight into finite size behavior of the models. First note that the growth of $\langle \tilde{A}_1\tilde{A}_{-1}\rangle_0/V$ can be almost negligible for a given μ up to some system size after which it grows quickly. To illustrate this, we plot $\langle \tilde{A}_1\tilde{A}_{-1}\rangle_0/V$ in Fig. 3.

It is known that translationally invariant systems with translationally invariant initial states can be expected to reach their asymptotic dynamics in times that $t_{\mathrm{relax}} = \mathcal{O}(1)$ [1]. This dynamics persists at least until finite size effects for local observables in the bulk start at times that are $t^* = \mathcal{O}(V)$. This is due to finite Lieb-Robinson velocity (i.e., it takes at least time V for quantum information to reach the end of the system and come back to the bulk before an observable there can see that the system is finite). Hence, even a finite size system can be expected to be described by a tGGE for $t_{\mathrm{relax}} \ll t \ll t^*$. The growth of $\langle \tilde{A}_1\tilde{A}_{-1}\rangle_0/V$ essentially represents that chemical potentials in the tGGEs must be rescaled in order for the expectation values of $A_{\pm 1}$ to be equal in the initial state $\rho(0)$ and the $\rho_{\mathrm{tGGE}}(0)$, i.e., $\mu = (V/e^{4|\mu_0| + V\{\log[\cosh(|\mu_0|)+1] - |\mu_0|\}})\mu_0$. For a given chemical potential μ_0, this stays almost constant, and

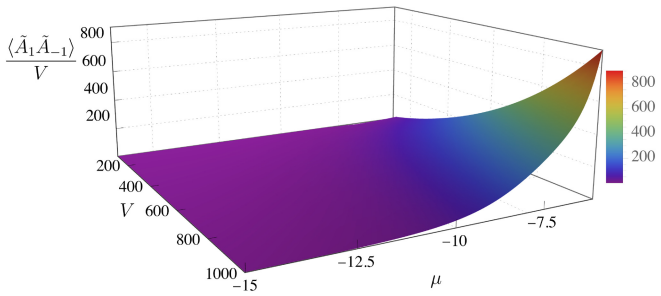

FIG. 3. The growth of $\langle \tilde{A}_1 \tilde{A}_{-1} \rangle / V$ with $1/\mu$ and V which can be quite slow, indicating that oscillations can persist beyond the scarring phase for very large systems and times even for finite μ. After some system size that depends on μ, $\langle \tilde{A}_1 \tilde{A}_{-1} \rangle / V$ grows exponentially with V, and the oscillations are no longer present after that system size. Note that results are the same for $\mu \to -\mu$.

then after reaching an almost critical system size V decays abruptly, and, hence, so does the contribution of $A_{\pm 1}$ to the expectation values of local observables at finite frequency λ. Interestingly, this decay is not at all visible for either numerical or experimental simulations up to some large system size V^* and corresponding long time (cf. the PXP discussions in the previous section). We show this later explicitly for local observables.

We may now calculate zero- and finite-frequency expectation values of local observables that have overlap with H (e.g., S^z_x), straightforwardly and, using the same techniques as before, those that commute with $J^x := \frac{1}{2}(J^+ J^-)$ (e.g., J^x_x):

$$J^x_x(t) = \frac{1}{Z} \sum_{k=0}^{\lfloor \frac{V}{2} \rfloor} 2(V-k) \sinh[(V-k)\cos(\lambda t)\mu]. \tag{75}$$

In Fig. 4, we illustrate finite-frequency expectation values and show that we can reproduce the known exact solution for $\mu \to \pm\infty$ in Ref. [111]. The decay of the finite-frequency amplitudes is doubly exponential with system size for finite μ (cf. the growth of the pseudolocal dynamical symmetries in Fig. 3). Essentially, the system behaves as if it were in the scarred phase (with finite-frequency amplitude close to 1) and then abruptly decays at some almost critical value of system size. Physically, this happens because for a given finite μ some of the initial state is not contained in the ground state of J^x (which is inside the scarred subspace) and the proportion of the state that is not in the scarred subspace grows with V which at some value is large enough to lead to exponential growth with system size of the (previously) pseudolocal dynamical symmetry.

To compute the zero-frequency values, we need the inverse temperature β for H. This can be accomplished for small μ_0 (β) by means of high-temperature expansion (truncating to the second order); we obtain in the thermodynamic limit

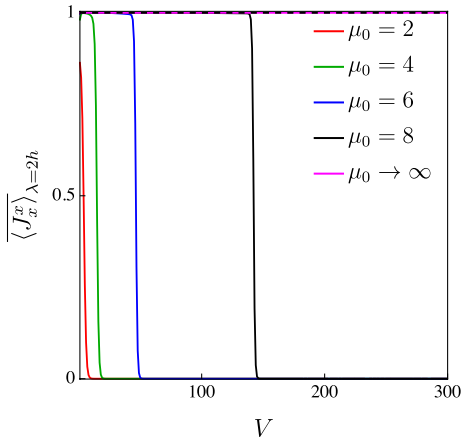

FIG. 4. Finite-frequency averages of J^x_x. For finite μ, the results are valid at times $1 \ll t \ll V$. The magenta line at infinite μ_0 agrees with the exact solution (dashed black line) from previous literature [111]. We see that the oscillation amplitudes display an almost discontinuous dependence on system size V—they are constant up to some "critical" system size, after which they decay abruptly to 0. In other words, for a fixed system size, the system is effectively in the scarred phase up to some value of the initial chemical potential, after which the oscillations abruptly decay.

$$\beta = \frac{2d \sinh^2(|\mu_0|)}{[d^2 + 3(h^2+4)][2\cosh(2|\mu_0|) + 1]}. \tag{76}$$

We may now easily compute expectation values of observables that have overlap with H. This is done in Fig. 5(a) [Fig. 5(b)] for S^z_x ($S^y_x S^y_{x+1}$) as an example. The initial chemical potential is $\mu_0 = 0.1$. Note that the expectation values have a nonlinear dependence on h and d, which implies that there is preference toward antiferromagnetic ordering even close to infinite temperature in the system; i.e., the induced magnetic field in the system is not maximized by maximizing the external fields.

B. $t-J_z$ model with fragmentation

Previous approaches [17,81] could only analytically treat the $t-J_z$ model in linear response and at infinite temperature. We now compute a far-from-equilibrium quench case.

As discussed in the previous Sec. V, the $t-J_z$ chain has cryptolocalized quantities that may be constructed from the SLIOMs. We assume that those and the Hamiltonian are the only relevant pseudolocal quantities of the model. Remembering that the tGGE is only the effective state governing the dynamics of local observables, we, for the sake of simplicity, focus only on the left half of the chain and, hence, can consider only the left SLIOMs that we now call $A_k := \mathcal{Q}^{(l)}_k$, for $k = 1, \ldots$. Therefore, the conjectured tGGE *Ansatz* giving the long-time (equivalently, zero-frequency) dynamics contains $A_0 = H_{t-J_z}$, the total spin $A_{-1} = S^z = \sum_x S^z_x$, and the SLIOMs for $k = 1, \ldots$.

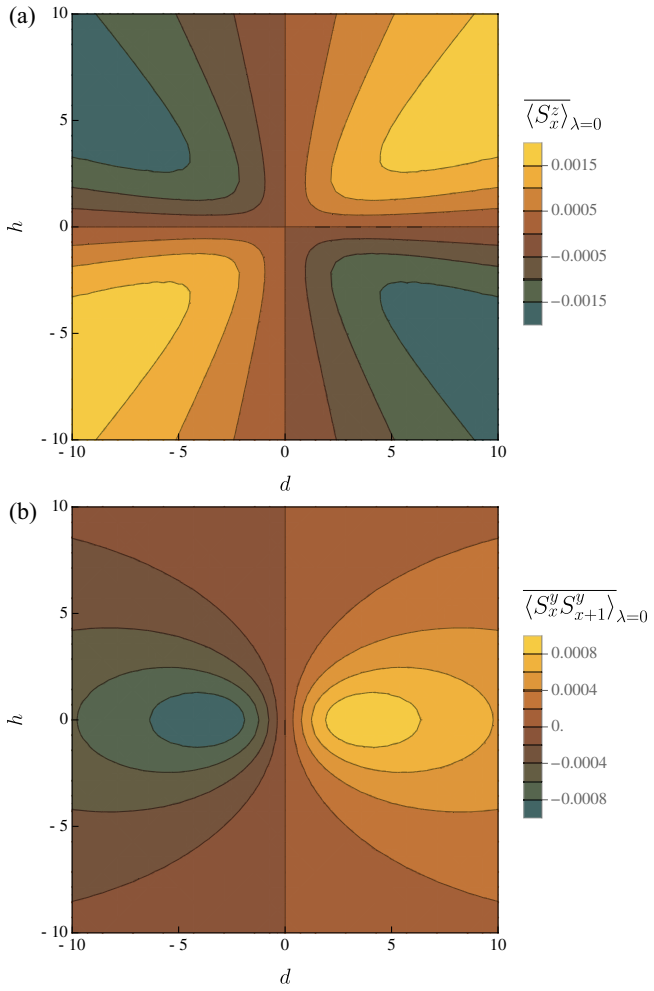

FIG. 5. The time-averaged value of S^z_x ($S^y_x S^y_{x+1}$) are given in (a) [(b)] as a function of the external fields h, d. We are close to infinite temperature because the chemical potential in the initial state is $\mu_0 = 0$. Hence, the values are small, but there is still a manifest nonlinear dependence.

Naturally, $\mu_k = 0$ for $\lambda_k \neq 0$, as there are no finite-frequency pseudolocal dynamical symmetries. Let us write the basis for one site as $|\uparrow\downarrow, \uparrow, \downarrow, \mathrm{vac}\rangle$, where the arrows denote the spin of the fermions on that site. As an example, let us take the following far-from-equilibrium initial product state:

$$\rho(0) = \mathcal{N} \prod_{\otimes x=1}^{n} |\alpha_x, \beta_x, \gamma_x, 0\rangle\langle \alpha_x, \beta_x, \beta_x, 0| \otimes \mathbb{1}_{N-n-1}, \tag{77}$$

where we demand that $\gamma_x = \beta_x$ if x is an even site and $\mathcal{N}$ is the normalization. The state is, thus, a general pure product state with singlets and doublons on sites $1 \ldots n$ and an infinite-temperature state (identity) on the rest of the sites. For the sake of convenience, let us set $h_x = g_x = 0$ in Eq. (54) (the external on-site field does not influence the physics significantly). The initial state $\rho(0)$ is parity antisymmetric with respect to spin flip, while H_{t-J_z} is parity symmetric. Hence, $\langle H_{t-J_z} \rangle_0 = \beta = \lambda_0 = 0$, and the tGGE does not contain the Hamiltonian. Moreover, for any finite n, the expectation value of the extensive operator S^z is finite, but the expectation value of S^z is extensive in the tGGE for any finite μ_{-1}; hence, $\mu_{-1} = 0$. Thus, the tGGE contains only the cryptolocalized quantities coming from the SLIOMs. The SLIOMs mutually commute and are diagonal in the particle number basis. As discussed in Appendix A, it is thus a matter of straightforward combinatorics to calculate the partition function:

$$Z = 2^N \left[1 + \sum_{k=1}^{N} \binom{N}{k} \prod_{j=1}^{k} \cosh(\mu_j) \right]. \tag{78}$$

We may now show that the SLIOMs are pseudolocal by computing their norm with respect to the tGGE. It is sufficient to check for large system size N:

$$\langle \tilde{A}_j \tilde{A}_j \rangle = \frac{1}{Z} \frac{d^2}{d\mu_j^2} Z - \left(\frac{1}{Z} \frac{d}{d\mu_j} Z \right)^2, \tag{79}$$

which for sufficiently large N is $\langle \tilde{A}_j \tilde{A}_j \rangle = C_j + \mathcal{O}(1/N)$, where $0 < C_j \leq 1$ are constants independent of system size.

By introducing small fields in the tGGE, we may calculate expectation values of local observables O_x in the tGGE:

$$Z(\alpha_1, \alpha_2) = \mathrm{tr}\left(\sum_j \mu_j A_j + \alpha_1 O_x + \alpha_2 O_y \right), \tag{80}$$

and, hence,

$$\langle O_x \rangle = \frac{d}{d\alpha_1} Z(\alpha_1, \alpha_2)|_{\alpha_{1,2}=0}, \tag{81}$$

$$\langle O_x O_y \rangle = \frac{\partial^2}{\partial\alpha_1 \partial\alpha_2} Z(\alpha)|_{\alpha_{1,2}=0}. \tag{82}$$

For example, for the diagonal and commuting observables $Z_x := N_x^{\uparrow} - N_x^{\downarrow}$ and $N_x = N_x^{\uparrow} + N_x^{\downarrow} - 2$, we have

$$\langle Z_1 \rangle = \frac{2^N}{Z} \sinh(\mu_1) \left[1 + \sum_{k=2}^{N} \binom{N-1}{k-1} \prod_{j=2}^{k} \cosh(\mu_j) \right], \tag{83}$$

$$\langle N_x \rangle = \frac{2^N}{Z} \left[1 + \sum_{k=1}^{N} \binom{N-1}{k} \prod_{j=1}^{k} \cosh(\mu_j) \right], \tag{84}$$

$$\langle N_x N_y \rangle = 2\langle N_x \rangle. \tag{85}$$

In order to finish proof that the SLIOMs are pseudolocalized, we must compute their overlap with local

observables and show that it is nonvanishing for at least some. It is sufficient to consider $\langle Z_1 \tilde{A}_j \rangle$ for large enough N. It is not difficult to verify that this is finite $\langle Z_1 \tilde{A}_j \rangle$ for finite j. Hence, the SLIOMs define cryptolocalized quantities and can potentially go into the tGGE. However, the SLIOMs have strings of operators of the type $\prod_{y=1}^{x}(N_y + 2)Z_x$. But, as is discussed in Appendix A, these strings can contribute subextensively to the connected correlator and, hence, are thermodynamically irrelevant.

Finally, we need to show clustering of the tGGE state itself. We find that the state is not clustered for all μ_k, even though the SLIOM quantities are pseudolocalized for all μ_k. This signals a fragmentation phase transition for which the pseudolocal quantities are always the same, but the state acquires long-range order and cannot be represented as a matrix exponential. This is similar to thermodynamic phase transitions [26] and should be contrasted to the scarring phase transition above for which the pseudolocal quantities themselves stop being pseudolocal for certain values of the chemical potentials. As the nonlocal strings have overlap only with N_x, it is sufficient to check the long-range connected correlator for $|\langle N_x N_y \rangle - \langle N_x \rangle \langle N_y \rangle|$. We illustrate this with an example by parametrizing the chemical potentials as $\prod_{j=1}^{k} \cosh(\mu_j) \to x^{ak} + 1$, where, for the purposes of keeping the result valid for the simpler case of $\mu_{-1} = 0$, the chemical potentials μ_j should have a cutoff such that $\mu_j = 0$ for some very large $j > \kappa$, but with κ still being much smaller than N when taking the thermodynamic limit. We then find that (as may be verified by means of, e.g., computer algebra)

$$\lim_{N\to\infty} |\langle N_x N_y \rangle - \langle N_x \rangle \langle N_y \rangle| = \begin{cases} \frac{(x^a-1)(3x^a+1)}{4(x^a+1)^2}, & \text{if } \log(x^a+1) > \log(2), \\ 0, & \text{otherwise.} \end{cases} \tag{86}$$

Note that, curiously, unlike thermodynamic phase transition, the order is either completely nonlocal (the same for all x, y) or completely absent. This proves the fragmentation phase transition in the model, and the phase diagram is given in Fig. 6(a).

Finally, of course, we must relate the chemical potentials to the initial state. This may be done for arbitrary choices of the initial state parameters for large system sizes numerically, but, in order to give closed form expressions, we consider the case when $|\beta_x|^2 - |\gamma_x|^2$ is small. In particular, for, e.g., $\alpha_x = \beta_x = \gamma_x = 0$ for $x > 3$, we have that

$$\mu_1 = \frac{|\alpha_1|^2(|\beta_2|^2 - |\gamma_2|^2)}{(|\alpha_1|^2 + 2|\beta_1|^2)(|\alpha_2|^2 + |\beta_2|^2 + |\gamma_2|^2)},$$
$$\mu_2 = \frac{2|\beta_1|^2}{|\alpha_1|^2}\mu_1, \tag{87}$$

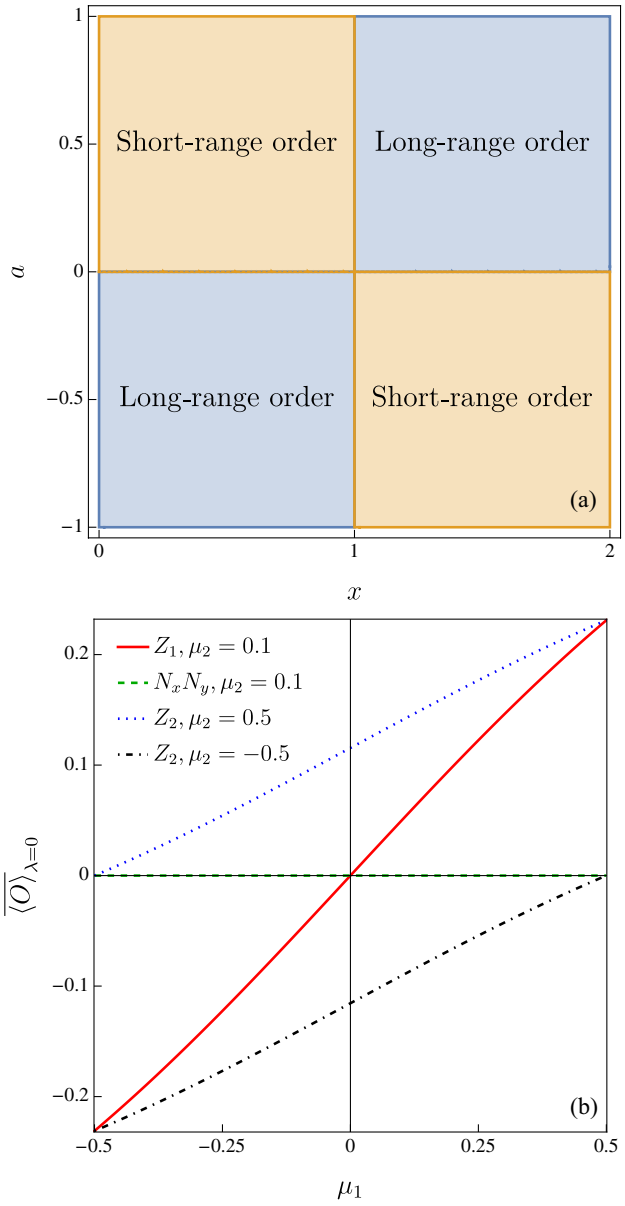

FIG. 6. (a) The fragmentation phase diagram showing the phases where the strings are thermodynamically irrelevant (short-range order) and where they are not. (b) Time average of certain local observables for various values of the initial state parameter [see Eq. (87) for the initial state parameters].

and $\mu_j = 0$ for $j > 2$. Thus, the time averages of local observables are quite complicated functions of the initial state parameters even for this simple product state. We illustrate some of them in Fig. 6(b).

VII. CONCLUSION

The main goal of nonequilibrium quantum many-body theoretical physics is computing the dynamics of systems out of equilibrium. Locality is what crucially unifies dynamical properties of quantum many-body systems

providing a framework applicable to isolated, driven, and dissipative quantum many-body systems. The theory presented here allows for exact solutions of quantum many-body dynamics for all locally interacting systems with finite local degrees of freedom on hypercubic lattices of arbitrary dimensions. This constitutes a very wide class of quantum many-body systems and includes paradigmatic models such as spin models and fermionic lattice models. The theory provides the solution in terms of a tGGE and does not rely on integrability.

The basic "recipe" is the following. If one finds evidence of nonergodicity in a quantum many-body system (at either finite or zero frequency), the present work shows that it must be due to pseudolocal dynamical symmetries. Provided one can then identify these pseudolocal dynamical symmetries, one may find the solution to the dynamics of local observables immediately as a tGGE. The chemical potentials in the tGGE are set by the initial states. Conversely, proving the absence of any such symmetry [apart from, e.g., the Hamiltonian or some $U(1)$ charge] immediately proves ergodicity at zero frequency. The theory, thus, proves both the weak eigenstate thermalization hypothesis in dynamical form and saturation of the Mazur bound.

More generally, the theory is an important step toward solving the main goal of computing nonequilibrium quantum many-body dynamics, because it does so for wide classes of locally interacting systems. In the future, it can be applied to all such wide classes of systems and has the potential to provide analytical solutions where there were previously none.

A. Open problems

The work presented here opens numerous possible research directions. I list only a few below.

(i) *New forms of nonergodicity.*—The complete theory presented here allows not only for study of known forms of nonergodicity, it can help in classifying and generating models with novel types of nonergodicity. For instance, Hilbert space fragmentation is identified with the existence of cryptolocal conservation laws in such models and quantum many-body scars with projected-local dynamical symmetries. Can one have a model with quantities that are of both types, i.e., a projected cryptolocal dynamical symmetry? This would imply a fragmented scar, i.e., dynamics that has local oscillating memory for certain initial states.

(ii) *Constructing transient dynamical symmetries.*—Transient dynamical symmetries, identified here, may play an important role in the type of transport a system has, e.g., diffusive, superdiffusive, etc. To the best of my knowledge, transient dynamical symmetries dictating the finite time dynamics have not been identified. However, superficially similar structures are known. For instance, in few-body bosonic models, one may have quasinormal modes (e.g., Ref. [154]), i.e., metastable decaying eigenmodes of the Hamiltonian with complex energy (that is not self-adjoint due to being unbounded) [25]. One may attempt to find transient dynamical symmetries by adapting the procedure of Refs. [128,155,156] to imaginary frequency. Likewise, in Refs. [157,158], prethermalization is studied by having a prethermal Hamiltonian as a transient conservation law. This could be a starting point for a theory of prethermalization based on transient dynamical symmetries.

(iii) *Scarring phase transitions.*—The existence of a novel phase transition between weak ergodicity breaking (nonstationary dynamics) and ergodicity has been proven here. It is distinct from thermodynamic phase transitions, because it happens because projected-local operators stop being pseudolocal when one smoothly varies the chemical potential rather than being a discontinuity in the equilibrium state itself. What is the nature of this phase transition in terms of, e.g., universality classes?

(iv) *Long-range order in the fragmentation phase transitions.*—The fragmentation phase transition introduced here and shown for the $t-J_z$ chain has been studied only in the short-range correlated phase. Studying the long-range correlated phase will require going beyond the matrix representation of the tGGE and will entail generalizing techniques that are used for symmetry breaking in equilibrium [26], as it remains unclear how to treat the cryptolocalized quantities responsible for the fragmentation phase transition due to their nonlocal strings.

(v) *Generalized hydrodynamics for scars and fragmentation.*—Identifying scars and fragmentation in terms of pseudolocal quantities opens the possibility of constructing a generalized hydrodynamics theory for the integrable forms of these models. For instance, recent work in Refs. [107,159] using dynamical symmetries for hydrodynamics could be combined here with the integrable limit of a constrained scarred Hubbard model [61].

(vi) *Implications for quantum information processing.*—Many quantum algorithms can understood as local many-body dynamics acting on a system (i.e., a collection of qubits). For instance, the quantum Fourier transform is a local algorithm with the end result of the computation being stored locally. The present work fully classifies the long-time limit of such systems. Can it be used to identify possible quantum error correction algorithms? It could also be conceivably used to strengthen the quantum threshold theorem [160]. Indeed, locality in the form of the Lieb-Robinson bound

(crucial for the theory here) has been recently employed to study error correction and entanglement generation [161,162].

(vii) *tGGEs for deep thermalization.*—As the theory here applies to dissipative systems and the long-time limit is given in terms of tGGEs, they could offer an alternative way to study recently introduced *deep thermalization,* i.e., thermalization induced by projective measurements [163–166]. What is the connection between tGGEs in that case and the recently introduced deep GGE?

(viii) *Including unbounded densities.*—The theory presented here works when the local degrees of freedom are bounded. What happens for, e.g., bosons on a lattice, for which the densities can be infinite? Lieb-Robinson bounds hold for these systems, too [167], and, hence, one may conceivably upgrade the theory from this paper to account for them.

(ix) *Long-range interactions.*—The theory here crucially relies on Lieb-Robinson bounds for local dynamics. These bounds have been extended for long-range interacting systems [168]. Could these bounds be used to define dynamically relevant long-range quantities instead of pseudolocal ones? Alternatively, the above discussed connections with lattice gauge theories offers another possible way to treat long-range interactions (e.g., Ref. [169]) based on the theory in this paper—one may introduce unphysical gauge degrees of freedom in a local model and then, assuming that they are very fast, adiabatically eliminate them [29], reducing the local problem (which is treatable) to a long-range model (which one wants to study). This would allow for treatment of nonergodic dynamics in long-range models (e.g., Refs. [77,170–179]).

(x) *Pseudolocalized quantities for proving many-body localization.*—Many-body localization (MBL) in disordered systems has been proven under certain assumption on the spectrum of these systems [180]. It is curious, as shown in this paper, that if one drops any notion of translational invariance (i.e., including disorder), then the many-body dynamics must be based on pseudolocalized quantities. Of course, these quantities contain precisely the l-bits of MBL [181]. Can this approach be formalized and used to prove MBL without any assumptions?

(xi) *Toward a nonequilibrium Landau theory.*—The tGGE introduced here is a time-dependent version of the Gibbs ensemble. Can this similarity be exploited to formulate a nonequilibrium Landau theory, and the corresponding free energies, for strongly interacting systems out of equilibrium?

(xii) *Entropy oscillations.*—The present work deals purely with the dynamics of local observables. Can the same framework be upgraded to study entanglement entropy dynamics and other quantities [65,182–186]?

(xiii) *Quantum hydrodynamics.*—The transient pseudolocal quantities could be a starting point for a rigorous framework of quantum hydrodynamics beyond integrable models. Once these quantities are identified the kinds of transport they imply should follow immediately.

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APPENDIX A: PROOFS

Here, I give proofs of the statements in the main text.

Proof of Theorem 1.—We begin by noting that the Lieb-Robinson locality relation (21) can be weakened. Namely, for some $\mu > 0$ and $\phi > 0$,

$$||\tau_t(O) - [\tau_t(O)]_\Lambda|| \leq \phi|O|||O||\exp(-\mu\Delta + v|t|), \tag{A1}$$

which follows directly from faster exponential growth than polynomial growth and $|O|$ is the size of the support of the operator O.

Moreover, we can use the contractivity $||\tau_t(O)|| \leq ||O||$ to get that

$$||[\tau_t(O)]_\Lambda|| \leq [1 + \phi|O|\exp(v|t|)]||A||. \tag{A2}$$

By pseudolocality of ω, it is p clustering [28]; i.e., there exist ν, $a > 0$ such that, for every ℓ,

$$|(O,Q)_\omega| := |\omega(O,Q) - \omega(O)\omega(Q)| \leq \frac{\nu\ell^a||O||||Q||}{\mathrm{dist}(O,Q)^p}, \tag{A3}$$

for some $p > D$. Likewise, the same holds for the flow ω_s for the same parameters p, ν, ℓ, and a. Using this and Eqs. (A1) and (A2), we can proceed along the same lines as the proof of Theorem 6.3 in Ref. [28] to conclude that there exist some a_1 and ν_1 such that

$$||[\tau_t(O)_\Lambda, Q]_\omega| \leq \frac{\nu_1\ell^{a_1}||O||||Q||}{\mathrm{dist}(O,Q)^q}, \tag{A4}$$

and some a_2 and ν_2

$$|[\tau_t(O)_\Lambda, \tau_t(Q)_{\Lambda'}]_\omega| \leq \frac{\nu_2 \ell^{a_2} ||O|| ||Q||}{\mathrm{dist}(O,Q)^q}, \tag{A5}$$

$\forall\ q < p$. However, this by itself is not enough to show that ω_t is q clustering, because the map is, in general, dissipative:

$$\tau_t(O^\dagger Q) \neq \tau_t(O^\dagger)\tau_t(Q). \tag{A6}$$

However, we may use another result based on the Lieb-Robinson bound obtained in Corollary 1 in Ref. [85] again in weaker form. Namely, there exist some $C' > 0$ and $\mu' > 0$ such that

$$||\tau_t(O^\dagger Q) - \tau_t(O^\dagger)\tau_t(Q)|| \leq C' ||O|| ||Q|| e^{v|t| - \mu' \mathrm{dist}(O,Q)}. \tag{A7}$$

We also have

$$(O,Q)_{\omega\circ\tau_t} = \omega[\tau_t(OQ) - \tau_t(O)\tau_t(Q)] - [\tau_t(O), \tau_t(Q)]_\omega. \tag{A8}$$

Therefore,

$$|(O,Q)_{\omega\circ\tau_t}| \leq ||\tau_t(OQ) - \tau_t(O)\tau_t(Q)|| + |[\tau_t(O), \tau_t(Q)]_\omega|. \tag{A9}$$

Clearly, we can bound the decaying exponential in Eq. (A7) by some (time-dependent constant C'') for any finite $q > 0$:

$$||\tau_t(OQ) - \tau_t(O)\tau_t(Q)|| \leq \frac{C'' \ell^{a_2} ||O|| ||Q||}{\mathrm{dist}(O,Q)^q}. \tag{A10}$$

From this, the q clustering of $|(O,Q)_{\omega\circ\tau_t}|$ immediately follows. This proves the first point of the theorem. The second follows from the proof of Theorem 6.5 in Ref. [28] if we observe that

$$||O||^2_{\mathcal{H}_{\omega\circ\tau_t}} = (O^\dagger, O)_{\omega\circ\tau_t} \geq |[\tau_t(O^\dagger), \tau_t(O)]_\omega| - 2||O||^2, \tag{A11}$$

which we find from the dissipative property of the map and Eq. (A8). ∎

Proof of Theorem 2.—From Theorem 1, the state is of the form

$$\omega_{s,t}(O) = \omega_{0,t}(O) + \int_0^s du \mathcal{A}_{u,t}(O) \quad \forall\ t. \tag{A12}$$

Hence, we may write, using the equations of motion (20),

$$\frac{d}{dt}\omega_{s,t}(O) = \int_0^s du \mathcal{A}_{u,t}(\mathcal{L}(O)) = \int_0^s du \frac{d}{dt}\mathcal{A}_{u,t}(O), \tag{A13}$$

where we use $\omega_0(\mathcal{L}(O)) = 0$ for the tracial state $\omega_{0,t}$. This implies

$$\int_I du \left(\mathcal{A}_{u,t}(O) - \frac{d}{dt}\mathcal{A}_{u,t}(\mathcal{L}(O)) \right) = 0 \tag{A14}$$

for every open interval $I \subset [0,1]$. By similar arguments as in the proof of Theorem 6.6 in Ref. [28], we conclude that

$$\frac{d}{dt}\mathcal{A}_{u,t}(O) = \mathcal{A}_{u,t}(\mathcal{L}(O)) \tag{A15}$$

for almost all u.

Consider now the map proven in Theorem 1 $\tau_t \colon \mathfrak{U}_{\mathrm{loc}} \to \mathcal{H}_u$ to be bounded. Using a straightforward generalization of the appendix of Doyon, we find that $\lim_{t\to 0^+} ||\tau_t O - \mathbb{1} O||_{\mathcal{H}_u} = 0$; hence, τ_t is strongly continuous. Moreover, as the proof Theorem 1 works if we replace $t \to e^{oi\phi}t$ and, using Theorem 4.6 [(a) and (b)] in Ref. [187], τ_t it is also analytic. Define the dual map $\tau_t^\oplus \colon \mathcal{H}_u^\dagger \to \mathcal{U}_{\mathrm{loc}}^\dagger$. Hence, $\lim_{t\to 0^+} ||\tau_t^\oplus \mathcal{A}_u - \mathcal{A}_u||_{\mathfrak{U}_{\mathrm{loc}}} = \lim_{t\to 0^+} ||\tau_t O - O||_{\mathcal{H}_u}$ by a well-known result for bounded operators [25]. As $\mathfrak{U}_{\mathrm{loc}}$ generates a (dense) subset of $\mathcal{H}_{u,t}$, we may "dilate" $\tau_t^\oplus$ to the operator $\mathfrak{T}'_t := (\tau_t^\oplus)_{u,t} \colon \mathcal{H}_u^\dagger \to \mathcal{H}_{u,t}^\dagger$, which is also bounded and strongly continuous. By construction, $\mathfrak{T}'_t$ is a strongly continuous semigroup that solves the Cauchy problem (A15) in the dual form:

$$\frac{d}{dt}\mathcal{A}_{u,t} = \mathfrak{L}'(\mathcal{A}_{u,t}) = \mathcal{A}_{u,t}\circ\mathcal{L}. \tag{A16}$$

The corresponding generator $\mathfrak{L}'$ is densely defined and closed by the Hille-Yosida theorem [187]. Moreover, by Proposition 1.4 in Ref. [187], there exists and M $\mathfrak{T}_t = e^{-Mt}\mathfrak{T}'_t := e^{\mathfrak{L}t}$ is contracting, and the claim about the spectral resolution also follows from the Hille-Yosida theorem. Likewise, an application of the Hille-Yosida theorem (in particular, Proposition 2.2 in Ref. [187]) shows that $\mathrm{Re}[\sigma(\mathfrak{L})] \leq 0$. ∎

Proof of Theorem 3.—From the equations of motion (20), by partial integration we immediately get

$$\int_0^T dt e^{i\lambda t}\omega_{s,t}(\mathcal{L}(O)) = e^{i\lambda t}\omega_{s,t}(O)|_0^T + i\lambda \int_0^T dt e^{i\lambda t}\omega_{s,t}(O), \tag{A17}$$

and taking the $T \to \infty$ limit, we immediately get the first statement of the theorem as $\omega_{s,t}(O)$ is bounded.

To proceed, let us recall two useful definitions from the literature. First, the integrated dissipation function [44,84]

$$D_t(A,B) \coloneqq \tau_t(A^\dagger B) - \tau_t(A^\dagger)\tau_t(B), \tag{A18}$$

which is sesquilinear and $D_t(A,A) \geq 0$ [44]. Second, the decoherence-free subalgebra $\mathcal{N}$ [43,47], which in our C^*-algebra case is

$$\mathcal{N} \coloneqq \{O \in \mathfrak{U}_{\mathrm{loc}} | D_t(O,O) = 0\}. \tag{A19}$$

Our first step is to generalize a theorem by Frigerio [43] for fixed points to long-time dynamics. Consider the faithful stationary state ω of τ_t, i.e., $\omega \circ \tau_t = \omega$. We have (Theorem 3.1 in Ref. [43])

$$\lim_{s\to\infty} \omega(D_t(\tau_s(O), Q)) = 0, \quad \forall\ O, Q \in \mathfrak{U}, \tag{A20}$$

including $Q = \tau_s(O)$. By the Cauchy-Schwarz inequality and using the fact that D_t is sesquilinear and ω faithful [$\omega(O^\dagger O) = 0$ iff $O = 0$], i.e., positive, we have

$$w^* - \lim_{t\to\infty} \tau_t(O) \in \mathcal{N}. \tag{A21}$$

Theorem 3.2 by Dhahri, Fagnola, and Rebolledo [47] that says

$$\tau_t(O) = e^{iHt} O e^{-iHt}, \quad \forall\ O \in \mathcal{N}, \tag{A22}$$

in applies in our case because we assumed that $L_x(\eta)$, $\forall\ x, \eta$ is bounded in the C^*-algebra norm and τ_t is strongly continuous [87].

Without loss of generality, assume $\lambda \neq 0$ and take a sequence $C = (2\pi n/\lambda)$:

$$\frac{1}{T}\int_0^T dt e^{i\lambda t} \frac{d}{dt}\omega_s(\tau_t(O)) = \frac{1}{T}\int_0^C dt e^{i\lambda t}\frac{d}{dt}\omega_s(\tau_t(O)) + \frac{1}{T}\int_0^{T-C} dt e^{i\lambda t}\frac{d}{dt}\omega_s(\tau_{t+C}(O)).$$

As $\omega(\mathcal{L}(O))$ is bounded, the first term on the rhs goes to 0 as $T \to \infty$. The second term may be estimated by Eqs. (A21) and (A22), and these say that, $\forall\ \varepsilon > 0$, there exists an n such that, for every $t > 0$,

$$\left| \frac{d}{dt}\omega_s(\tau_{t+C}(O)) - i\omega_{s,t}([H,O]) \right| < \varepsilon. \tag{A23}$$

Hence, as $\omega_{s,t}$ are bounded, we get, using Lebesgue's dominated convergence theorem, that

$$\lim_{T\to\infty}\frac{1}{T}\int_0^T dt e^{i\lambda t}\frac{d}{dt}\omega_s(\tau_t(O)) = i\lim_{T\to\infty}\frac{1}{T}\int_0^T dt e^{i\lambda t}\omega_{s,t}([H,O]).$$

Using this and Theorem 3.3 in Ref. [47] that says $[L_x(\eta), O] = [L_x^\dagger(\eta), O] = 0$, $\forall\ O \in \mathcal{N}$ we get the claim in the second case.

We have as before Eq. (A15) for the pseudolocal quantities for almost all u and

$$\omega_{s,\lambda} = \delta_{\lambda,0}\omega_0(A) + \int_0^s du \mathcal{A}_{u,\lambda}(O),$$

where we define Fourier transform $\mathcal{A}_{u,\lambda}(O) \coloneqq \lim_{T\to\infty}(1/T)\int_0^T e^{i\lambda t}\mathcal{A}_{u,t}(O)$ and where we interchange the order of integration, which we can do according to Fubini's theorem because of continuity of the time evolution, i.e., $(d/dt)\omega_{s,t}(O) = \omega_{s,t}(\mathcal{L}(O))$, boundedness of the linear functional $\mathcal{A}_{u,t}(O)$, and the fact that the functions are Lebesgue integrable (Definition 5.4 in Ref. [28]). Thus, we arrive to the final statement. ∎

Proof of Theorem 4.—Define $\beta(\mathcal{H})_\Lambda \coloneqq \sum_k \mu_k e^{i\lambda_k t}(A_k)_\Lambda +$ H.c. A theorem by Araki says that $\forall\ \beta$ in 1D the state is well defined in the $\Lambda \to \infty$ limit and is analytic [188]. A similar result holds for some critical value of $|\beta| > \beta_*$ (and by extension for $|\mu_k| > \mu_*$ and $\forall\ t$) in higher dimensions [189].

Let us show that $(A_k)_\Lambda \coloneqq \sum_{x\in\Lambda} a_x$ are pseudolocal dynamical symmetries with frequencies λ_k, i.e., that the corresponding $\mathcal{A}_{s,\lambda}(O)$ satisfy the conditions in Theorem 3. Without loss of generality, assume that $\omega_t(A_k) = 0$. Then we can explicitly check that

$$\int_0^T dt\left\{ e^{i\lambda_k t}\mathrm{tr}_\Lambda\left[\rho_\Lambda(t) e^{-i\lambda_k t}\sum_{x\in\Lambda}(a_{x,k}), [H_\Lambda, O]\right]\right\} = \int_0^T dt\left\{\mathrm{tr}_\Lambda\left[\left([H_\Lambda, \rho_\Lambda(t)]\sum_{x\in\Lambda}(a_{x,k}) + \rho_\Lambda(t)\left[H_\Lambda, \sum_{x\in\Lambda}(a_{x,k})\right]\right)O\right]\right\}.$$

Because $\rho(t)$ is analytic, we may express it as a unique uniformly converging Fourier series $\rho_\Lambda(t) = \sum_{n\in\mathbb{Z}} e^{itn(\theta/T)}\rho_n$. Moreover, by the dual equations of motion,

$$\frac{d}{dt}[\rho_\Lambda(t)]_\Lambda = -i\{[H_\Lambda, \rho_\Lambda(t)]\}_\Lambda. \tag{A24}$$

Hence,

$$\int_0^T dt\left\{\mathrm{tr}_\Lambda\left[\left([H_\Lambda, \rho_\Lambda(t)]\sum_{x\in\Lambda}(a_{x,k})\right)O\right]\right\} = 0,$$

using orthogonality of the Fourier coefficients. Now note that by assumption $(d/dt)\tau_t[(A_k)]_\Lambda = \mathcal{L}(A_k)_\Lambda = i\lambda(A_k)_\Lambda$, which immediately implies the desired result upon taking the thermodynamic limit and taking into account the proof of Theorem 1.

Parametrize the flow with a single parameter s, i.e., $\mu_k(s) = \mu_k s$. By analyticity, the equations for the flow of the tGGE satisfy

$$\frac{d}{dt}\omega_{s,t}(O) = \mu_k e^{i\lambda_k t}\mathcal{A}_{s,t=0}(O), \tag{A25}$$

which, in general, is solved by a path-ordered exponential. However, as we assume that the pseudolocal dynamical symmetries form a closed finite algebra under commutation, we may rewrite this solution in the form given in the theorem without path ordering by using the Baker-Campbell-Hausdorff formula. The rest of the claims follow directly from Theorems 6.1 and 6.2 and Corollary 6.7 in Ref. [28] with $\beta H_\Lambda \to \beta(\mathcal{H})_\Lambda$. ∎

APPENDIX B: DETAILS OF THE EXAMPLES

Here, we discuss some details of the calculations done in Sec. VI.

1. Spin-1 model with scars

As shown in Ref. [111], $J^\pm$ and J^z generate an $su(2)$ algebra, and the scarred states in Eq. (64) form a representation for this algebra with $J^\pm$ being the roots (i.e., raising and lowering operators). The initial state (66) is an element of this algebra. The eigenvalues of J^z in the scarring representation are $2n - V$, where $n = 0, \ldots, V$. By rotating the algebra with $\exp i(\pi/2)J^y$ to the x basis, the initial state is diagonal. This allows us to easily find the trace Z_0 as given in the main text. For the tGGE using the scar representation we may directly find that

$$Z = \frac{e^{-V\mu(t)}(e^{\mu(t)V+\mu(t)[2(V+1)-V]} - 1)}{e^{2\mu(t)} - 1} + 3^V - V - 1 \tag{B1}$$

by counting the dimension of the kernel P for the second part in the sum, and we abbreviate $\mu(t) = \cos(2ht)\mu$. Likewise, we find for the other results from the main text using the representation in the scarred subspace and rotating x to z

$$\langle A_1 A_{-1}\rangle_0 = 4\frac{f(V)^2}{Z_0}\sum_{m=-V/2}^{V/2} e^{2\mu_0 V}\left[\frac{m^2}{2} + \frac{V}{4}\left(\frac{V}{2}+1\right)\right], \tag{B2}$$

$$\langle A_1\rangle_0 = \frac{2f(V)}{Z_0}\sum_{m=-V/2}^{V/2} e^{2\mu_0 V} m. \tag{B3}$$

All of these can be simplified into the forms given in the main text. The initial average energy $\langle H\rangle_0$ is straightforwardly found using the spin-flip symmetry discussed in the main text. Similarly, to find the β at small μ_0, we simply expand $\exp(-\beta H) \approx \mathbb{1} + \beta H$ with the equation

$$\langle H\rangle_0 = \langle H\rangle = \mathrm{tr}(-\beta H^2), \tag{B4}$$

which is easily solved to be

$$\beta = \frac{2dN^D \sinh^2(|\mu_0|)}{[2\cosh(2|\mu_0|) + 1][(d^2 + 3h^2)N^D + 12(N-1)^D]}. \tag{B5}$$

Taking the thermodynamic limit gives the result in the main text.

2. $t-J_z$ model

As shown in Ref. [17], the SLIOMs can be conveniently written as

$$A_k = \sum_{j_1<\cdots<j_k} S_{1,j_1-1}\left(\prod_{m=1}^{k-1} P_{j_m} S_{j_m+1,j_{m+1}-1}\right) Z_{j_k}, \tag{B6}$$

where $P_x = N_x^\uparrow + N_x^\downarrow$, $S_{x,y} = \prod_{j=x}^{y}(\mathbb{1} - P_j)$, and the sum is defined as $\sum_{j_1<\cdots<j_k}(x) := \sum_{j_1=1}^{N}\sum_{j_2=j_1+1}^{N}\cdots\sum_{j_k=j_{k-1}+1}^{N}(x)$. Useful relations are also $Z_x^2 = P_x, P_x Z_x = Z_x$, and it is useful to see that all the operators in the SLIOMs commute and that P_x, $(\mathbb{1} - P_x)$, and $S_{x,y}$ are projectors. The A_k are diagonal in the Z_x basis. Hence, in order to compute the partition function of the tGGE (78), we note that P_x $(\mathbb{1} - P_x)$ have eigenvalues $\{1, 0\}$ (respectively, $\{0, 1\}$) on opposite subspaces and Z_x has eigenvalues $\{1, -1, 0\}$ ($1, -1$ correspond to the P_x eigenvalue 1 subspace). Hence, it becomes a matter of simple combinatorics and splitting of the expressions $\exp(\mu_k A_k)$ to evaluate

$$Z = \sum_{j=1}^{4^N} \exp\left(\prod_{m=1}^{N} \lambda_{j,m}\right), \tag{B7}$$

where $\lambda_{j,m}$ is the jth diagonal value of the mth operator from the left. Similarly, we can find $Z(\alpha_1, \alpha_2)$ for the examples given in the main text.

The SLIOMs contain nonlocal strings $S_{j,m}$. However, since $P_x S_{j,m} = 0$ for $j \le x \le m$ and recalling that $Z_x^2 = P_x, P_x Z_x = Z_x$ and that P_x, $(\mathbb{1} - P_x)$, and $S_{x,y}$ are projectors, it is easy to see that these strings can identically cancel in the expressions $\exp(\mu_k A_k)$ when expanded, apart from the leading order in μ_k the contribution of which in the thermodynamic limit is identically small. However, products of the different A_k in the full ρ_{tGGE} can, for different μ_k, render the contribution for the strings thermodynamically finite, and this is the origin of the fragmentation phase transition discussed in the main text.

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