Autonomous Quantum Devices: When Are They Realizable without Additional Thermodynamic Costs?

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The resource theory of quantum thermodynamics has been a very successful theory and has generated much follow-up work in the community. It requires energy-preserving unitary operations to be implemented over a system, bath, and catalyst as part of its paradigm. So far, such unitary operations have been considered a "free" resource in the theory. However, this is only an idealization of a necessarily inexact process. Here, we include an additional auxiliary control system which can autonomously implement the unitary by turning an interaction "on or off." However, the control system will inevitably be degraded by the backaction caused by the implementation of the unitary. We derive conditions on the quality of the control device so that the laws of thermodynamics do not change and prove—by utilizing a good quantum clock—that the laws of quantum mechanics allow the backreaction to be small enough so that these conditions are satisfiable. Our inclusion of nonidealized control into the resource framework also raises interesting prospects, which were absent when considering idealized control. Among other things, the emergence of a third law without the need for the assumption of a light cone. Our results and framework unify the field of autonomous thermal machines with the thermodynamic quantum resource-theoretic one, and lay the groundwork for all quantum processing devices to be unified with fully autonomous machines.

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I. INTRODUCTION

Thermodynamics has been tremendously successful in describing the world around us. It has also been at the heart of developing new technologies, such as heat engines which powered the Industrial Revolution, and jet and space rocket propulsion, just to name a few. In more recent times, scientists have been developing a theoretical understanding of thermodynamics for tiny systems for which often quantum effects cannot be ignored. These ongoing developments are influential in optimizing current quantum technologies or understanding important physical processes. Take, for example, molecular machines or nanomachines such as molecular motors [1], which are important in biological processes [2], or distant technologies such as nanorobots [3], where quantum effects on the control mechanism and the backreaction they incur are likely to be significant due to their small size.

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The modern quantum thermodynamics literature tends to be about two types of processes: those which are fully autonomous (i.e., the processes described by time-independent Hamiltonians) and those which assume implicit external control at no extra cost (i.e., the processes described by time-dependent Hamiltonians). An example of processes described by a constant Hamiltonian is the Brownian ratchet popularized by Feynman *et al.* [4], which simply sits between two thermal baths and extracts work *in situ*. There are many autonomous quantum thermal machines built on similar principles [5–16]. However, there are a number of processes, such as quantum Carnot cycles, that are described by time-dependent Hamiltonians and thus require external control. This is true both in theory [17–22] and in experiment [23]. See Fig. 1 for a comparison of autonomous and nonautonomous processes.

The nonautonomous engines of the kind depicted in Fig. 1 require an external agent that makes the changes. This does not happen in the engines used in our daily life. E.g., car engines do not require any external control; the passage via different strokes during the cycle is caused by suitable feedback mechanisms. An example of a thermal machine that requires switching between the strokes by an external agent is the quantum heat engine of Ref. [23], where alternating coupling to the hot and the cold bath is implemented by switching between two lasers—one producing

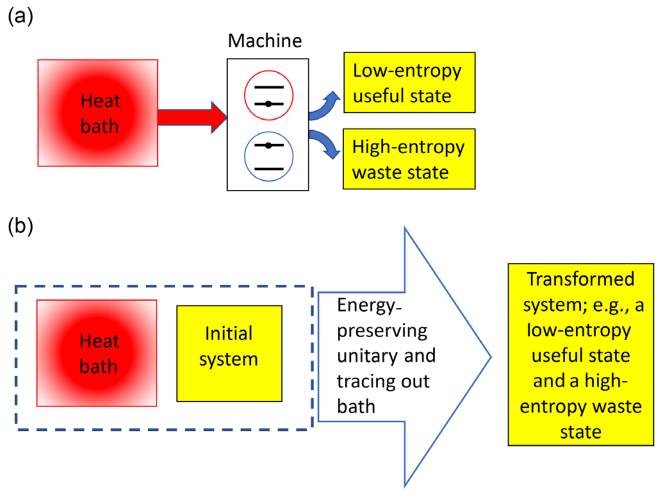

FIG. 1. Fully autonomous thermal machines vs a type of nonautonomous cycle-based machine. (a) Depiction of a quantum thermal-absorption machine. These devices do not need external control to operate; i.e., they are governed by time-independent Hamiltonians. Given enough time, they settle into a functioning steady state where heat from a heat bath is converted via a machine (composed of fine-tuned energy levels and couplings) into a low-entropy useful state (such as a charged battery) and a high-entropy "waste" state (such as a room-temperature thermal state). See Refs. [5,24] for reviews. (b) Schematic of a nonautonomous thermal machine. In this resource-based framework [25], an energy-preserving unitary is performed over a heat bath and initial system state. The unitary is chosen so that the transformed system state is of high value (e.g., it could represent a charged battery). The control required to perform the energy-preserving unitary necessitates a time-dependent Hamiltonian and may not be thermodynamically cost-free.

thermal light of high temperature while the other one producing light at low temperature.

In this context, the following problem appears. While the nonautonomous machines involve additional systems responsible for making the changes, those additional systems are by definition not considered explicitly. For microscopic engines, such systems might actually be a place where a significant amount of entropy and/or energy is being deposited. Such entropy production is actually likely to occur in microscopic regimes due to the quantum backreaction occurring between the controlling unit and the controlled system. Thus, there may be hidden thermodynamic costs which are not accounted for. Hence, the following question can be posed: Given a nonautonomous thermal machine, is it possible to provide an explicit control scheme, such that the overall (now autonomous) machine will exhibit no additional cost?

This question is especially relevant in the context of the recently developed resource theory of thermodynamics [26], where any process is supposed to arise from the concatenation of basic operations which are energy-preserving unitary transformations over a microscopic system of interest and a thermal bath. Thus, here we deal with external control represented by a time-dependent Hamiltonian that implements the subsequent unitaries. In such a microscopic regime, the hidden costs acquired by the control system may be indeed high, as is indicated by the phenomenon of so-called embezzling [27,28] (see Sec. II B).

The problem of the cost of making the resource-theoretic thermal machines autonomous was considered in Ref. [29]. The control device was implemented by means of an idealized momentum clock. Actually, any conceivable control system that enables one to go from a time-dependent Hamiltonian description to a time-independent one must involve a clock as part of the control unit. I.e., a device for which the change in its state, due to time evolution, allows one to predict time.

E.g., in a car engine the role of the clock is played by periodic motion of the piston (arising via so-called self-oscillation [30]), or in the already mentioned single-ion heat engine of Ref. [23], the timing involved in the changing of the lasers is ultimately due to an external electronic device, which is a kind of clock.

Unfortunately, the clock used in Ref. [29] requires infinite energy. It was first noted by Pauli that such clocks are unphysical [31], and we provide more weight to Pauli's argument in this paper.

In this paper, the question of whether one can make the resource-theoretic thermal machines autonomous without incurring an extra thermodynamic cost is reconsidered and positively answered.

Namely, we start with nonautonomous scenario, where an external agent performs energy-preserving unitary on system plus bath. We then examine the clock which turns on and off the interactions implementing the unitaries (as per Fig. 2) and derive conditions so that the change in the clock's state due to the backreaction on it has a vanishingly small thermodynamic cost. We then show that clocks exist which satisfy our criterion. In particular, we find a family of

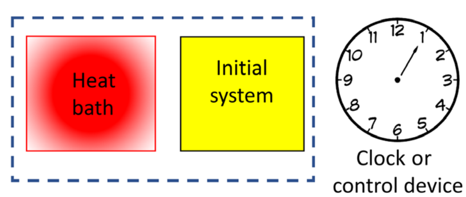

FIG. 2. Schematic of the autonomous quantum devices we focus on: a nonautonomous thermal machine complemented by a quantum clock. The system in the dotted-line box is the same as in Fig. 1(b). It depicts the standard systems involved in the resource-theoretic approach to thermodynamics. If no other systems are involved, its dynamics are described by a time-dependent Hamiltonian. If one includes an additional quantum system whose state changes in a predictable fashion with the passing of time, i.e., a clock, then it can turn on and off interaction terms at specific times, leading to an autonomous implementation of the resource-theoretic approach to thermodynamics. Hence, we use a clock as a control device.

clocks with different dimensions, for which there is no change in energy while the difference in entropy relative to before and after the unitary has been performed is vanishingly small as the clock increases in size. Importantly, since our clocks use finite energy, they avoid the issues of the clock of Ref. [29]. Our work thus demonstrates that the control needed to implement thermodynamic transformations in the resource-theoretic paradigm can indeed be neglected under certain achievable circumstances. In this way, we show that nonautonomous resource-theoretic thermal machines can be recast into autonomous ones without additional cost.

As a by-product, our necessary conditions for the change in the clock to not have a significant additional thermodynamic cost reveal the emergence of a third law: If the clock implements the unitary too quickly (relative to the free dynamics of the system and clock), it will suffer a large backreaction and will represent a significant additional thermodynamic cost in addition to failing to implement correctly the required unitary. The minimum time interval in which the unitary can be implemented without the clock suffering significant backreaction is limited by the dimension of the clock. This demonstrates the emergence of a third law without the need to impose a light cone or locality condition on how the unitary is implemented [32].

The rest of this paper is divided into five main sections: Setting II, Results III, Discussion IV, and Conclusions V. In the Setting section, we start by describing the thermodynamic transformations under the convention of idealized control. This is summarized in Definition 1. Then, in Sec. II B, we describe how to explicitly implement the control via time-independent dynamics on the system of interest and an additional system called a "clock." Finally, before moving to the Results section, we show why the cost of control can be counterintuitive by showing how it is related to the established phenomenon of catalytic embezzlement and how idealized control requires infinite energy (see Proposition 3). Our results discussed in Sec. III start with the simplest case possible: the control of so-called noisy operations, in which baths are a source of entropy but not heat. The result is quantified in Theorems 1 and 2. The core of Theorem 1 is what we can call "no-embezzling conditions." Namely, for the first time, we give a lower bound for the value of error on the catalyst that does not cause deviation from the second laws, i.e., from the limitations for transitions via noisy operations at zero error on the catalyst. We then move on to consider the full paradigm of control of thermodynamic operations in which the baths are a source of entropy and heat—the so-called thermal operations. This case is summarized in Theorems 3 and 4. In both cases (i.e., noisy and thermal operations), we allow for catalysts and provide conditions under which the cost of control is neglectable. The situation is more nuanced in the case of thermal operations and has unforeseen consequences which we discuss. Finally, in the last two sections (Discussion IV and Conclusions V), we discuss in more detail the implications of our work followed by a summary.

The proofs of our results are given in the Appendix. Additional technical details required for the proofs are relegated to the Supplemental Material [33].

II. SETTING

A. Types of thermodynamic transformations

1. Background: Thermal operations and variants

Resource theories have been applied to the study of quantum thermodynamics. In this setting, one considers transformations from a state ρ_A^0 to ρ_A^1 for which there exists a unitary U_{AG} over system A and a Gibbs state τ_G such that $\rho_A^1 = \mathrm{tr}_G[U_{AG}(\rho_A^0 \otimes \tau_G)U_{AG}^\dagger]$. This setup is entropy preserving since it is a unitary transformation. In order to call it a thermal operation (TO), we further require the process to be energy preserving, namely, $[U_{AG}, \hat{H}_A + \hat{H}_G] = 0$, where $\hat{H}_A$ is the local Hamiltonian of the A system and $\hat{H}_G$ that of the thermal bath [54]. These operations can be extended to the strictly larger class of catalytic TOs (CTOs) by considering additional "free" objects called catalysts ρ_{Cat}^0. In this case, the A system is bipartite with the requirement that the catalyst is returned to its initial state after the transformation $\rho_S^1 \otimes \rho_{Cat}^0 = \mathrm{tr}_G[U_{SCatG}(\rho_S^0 \otimes \rho_{Cat}^0 \otimes \tau_G)U_{SCatG}^\dagger]$, with a Hamiltonian $\hat{H}_A$ of the form $\hat{H}_S + \hat{H}_{Cat}$. The bath provides a source of entropy and heat. In the special case in which its Hamiltonian is completely degenerate, its Gibbs state τ_G becomes the maximally mixed state $\tau_G \propto \mathbb{1}_G$, and the bath can now provide only entropy. These are known as catalytic noisy operations (CNOs), or simply, noisy operations (NOs) when there is no catalyst involved [55,56]. It is known that CNOs allow for transitions that are not possible by NOs [57,58].

In these frameworks, the operations (NOs, CNOs, TOs, and CTOs) are considered to be free from the resource perspective, since they preserve entropy and energy over system A and the bath G—the two resources in thermodynamics. However, note that there is the assumption that the external control (i.e., the ability to apply energy-preserving unitaries over the setup) is "perfect." In order to challenge this perspective, we now introduce an auxiliary system to represent explicitly the system which implements the external control, while aiming to show to what extent it can be free, from the resource-theory perspective.

2. t-catalytic thermal operations

If the control system is a thermodynamically free resource, its final state after the transition must be as useful as the state it would have been in had it not implemented the unitary, and instead evolved unitarily according to its free Hamiltonian. One way to realize this within the resource-theoretic paradigm is to choose a control device whose free evolution is periodic and let the time taken to apply the unitary be an integer multiple of its period. In this scenario, the control device fits nicely within the resource-theory framework, since when viewed at integer multiples of the period, the control device is a catalyst according to CTOs.

The downside with this approach is that the times corresponding to multiples of the period are a measure zero of all possible times. Consequently, not only would one need an idealized clock which can tell the time with zero uncertainty to discern these particular times, but one would like to be able to say whether the transition was thermodynamically allowed during proper intervals of time. Fortunately, there is a simple generalization of CTOs [59] which naturally resolves this issue. We introduce t-CTOs which take into account that the transition is not instantaneous, but moreover occurs over a finite time interval. In the following definition, one should think of the catalyst system as playing the role of the external control device.

Definition 1. (t-CTO and t-CNO) A transition from $\rho_S^0(t_1)$ to $\rho_S^1(t_2)$ with $t_1 \leq t_2$ is possible under t-CTO if and only if there exists a finite-dimensional quantum state ρ_{Cat} with Hamiltonian $\hat{H}_{\mathrm{Cat}}$ such that

$$\rho_S^0(0) \otimes \rho_{\mathrm{Cat}}^0(0) \underset{\mathrm{TO}}{\longrightarrow} \bar{\sigma}_S(t) \otimes \rho_{\mathrm{Cat}}^0(t), \tag{1}$$

where

$$\bar{\sigma}_S(t) = \begin{cases} \rho_S^0(t) & \text{if } t \in [0, t_1], \\ \rho_S^1(t) & \text{if } t \in [t_2, t_3], \end{cases} \tag{2}$$

$\rho_D^n(t) \coloneqq e^{-it\hat{H}_D}\rho_D^n e^{it\hat{H}_D}$, $D \in \{S, \mathrm{Cat}\}$, $n \in \{0, 1\}$, and t_1 is called "the time when the TO began," while t_2 "the time at which the TO was finalized." $[0, t_1)$ and $(t_2, t_3]$ are both proper intervals called "the time before the TO began" and "the time after the TO was finalized," respectively. In the special cases where the bath can be only maximally mixed $\tau_G \propto \mathbb{1}_G$, it is denoted $\tilde{\tau}_G$ and we call the transition a t-CNO.

Unless stated otherwise, we always use the notation $\rho_D^n(t)$, $n \in \{0, 1\}$ to denote the free evolution of a normalized quantum state ρ_D^n on some Hilbert space $\mathcal{H}_D$ according to its free Hamiltonian $\hat{H}_D$, namely, $\rho_D^n(t) = e^{-it\hat{H}_D}\rho_D^n e^{it\hat{H}_D}$.

Definition 1 captures two notions: on the one hand, that the individual subsystems are effectively noninteracting before and after the transition has taken place, and on the other hand, that during the time interval (t_1, t_2) in which the transition occurs, arbitrarily strong interactions could be realized. Note that there are two special cases for which t-CTOs reduce to CTOs at times t_1, t_2: when the Hamiltonian of the catalyst is trivial (i.e., if $\hat{H}_{\mathrm{Cat}} \propto \mathbb{1}_{\mathrm{Cat}}$), and when the catalyst is periodic with t_1, t_2 integer multiples of its period T_0 [i.e., if $\rho_{\mathrm{Cat}}^0(t_1) = \rho_{\mathrm{Cat}}^0(t_2) = \rho_{\mathrm{Cat}}^0(T_0)$].

From the resource-theoretic perspective, the characterization of t-CTOs is the same as CTOs as the following proposition shows.

Proposition 2. (t-CTO and CTO operational equivalence) A t-CTO from $\rho_S^0(t_1)$ to $\rho_S^1(t_2)$ using a catalyst $\rho_{\mathrm{Cat}}^0(0)$ exists if and only if a CTO from ρ_S^0 to ρ_S^1 exists using catalyst $\rho_{\mathrm{Cat}}^0(0)$. In other words,

$$\rho_S^0(0) \otimes \rho_{\mathrm{Cat}}^0(0) \underset{\mathrm{TO}}{\longrightarrow} \bar{\sigma}_S(t) \otimes \rho_{\mathrm{Cat}}^0(t), \tag{3}$$

where $\bar{\sigma}_S(t)$ is defined in Eq. (2) if and only if

$$\rho_S^0 \otimes \rho_{\mathrm{Cat}}^0(0) \underset{\mathrm{TO}}{\longrightarrow} \rho_S^1 \otimes \rho_{\mathrm{Cat}}^0(0). \tag{4}$$

Proof.—It is simple. For $t \in [0, t_1]$, Eq. (3) always holds since the lhs and rhs differ only by an energy-preserving unitary on the catalyst, which is a valid TO. Therefore, the only nontrivial instance of Eq. (3) is for $t \in [t_2, t_3]$. Let us now compare Eqs. (3) and (4) for $t \in [t_2, t_3]$: The only difference is an energy-preserving unitary transformation on the catalyst state on the rhs. However, all energy-preserving unitary translations are TOs. Therefore, one can always go from the rhs of Eq. (4) to the rhs of Eq. (3) via a TO. This proves the "if" part of the proposition. Conversely, since the inverse of an energy-preserving unitary is another energy-preserving unitary, one can always go from the rhs of Eq. (3) to the rhs of Eq. (4) via a TO. ■

While the generalization to t-CTOs is admittedly quite trivial in nature, it is nevertheless important when considering the autonomous implementation of CTOs. So far, the t-CTOs only allow us to include the external control mechanism explicitly into the CTOs paradigm in such a way that they constitute a free resource. In the next section, we see how this free resource unfortunately corresponds to unphysical time evolution governed by an idealized clock. It, however, sets the benchmark for what we should be aiming to achieve, if only approximately, with a more realistic control device.

B. Idealized control, clocks, and embezzling catalysts

When a dynamical catalyst in a t-CTO is responsible for autonomously implementing the transition, it must have its own internal notion of time in order to implement the unitary between times t_1 and t_2. While in practice, the clock part may form only a small part of the full dynamical catalyst system, for convenience of expression, we refer to such dynamical catalysts as a clock and denote the state of the clock with the subscript Cl. Specifically, we require the clock to induce dynamics on a system A which corresponds to a t-CTO on A. In other words, evolution of the form $\rho_{\mathrm{AClG}}^F(t) = e^{-it\hat{H}_{\mathrm{AClG}}}(\rho_A^0 \otimes \rho_{\mathrm{Cl}}^0 \otimes \tau_G)e^{it\hat{H}_{\mathrm{AClG}}}$ where $\rho_{\mathrm{AClG}}^F(t)$ satisfies [60]

$$\rho_{\mathrm{ACl}}^F(t) = \rho_A^F(t) \otimes \rho_{\mathrm{Cl}}^0(t), \quad \rho_A^F(t) = \begin{cases} \rho_A^0(t) & \text{if } t \in [0, t_1], \\ \rho_A^1(t) & \text{if } t \in [t_2, t_3]. \end{cases} \tag{5}$$

Here, $\rho^0_{\rm Cl}(t)$ denotes the free evolution of the clock,

$$\rho^0_{\rm Cl}(t) = e^{-it\hat{H}_{\rm Cl}}\rho^0_{\rm Cl}e^{it\hat{H}_{\rm Cl}}. \tag{6}$$

In the case in which the clock aims to implement autonomously a TO, we have that the rhs of Eq. (5) satisfies $\rho^0_{\rm A}(t) = \rho^0_{\rm S}(t)$ and $\rho^1_{\rm A}(t) = \rho^1_{\rm S}(t)$, while in the case of a CTO, $\rho^0_{\rm A}(t) = \rho^0_{\rm S}(t) \otimes \rho^0_{\rm Cat}(t)$ and $\rho^1_{\rm A}(t) = \rho^1_{\rm S}(t) \otimes \rho^0_{\rm Cat}(t)$. In this latter case, we see that we have two catalysts. The first one $\rho^0_{\rm Cat}$ simply allows for a transition on S which would otherwise be forbidden under TOs, while the second one $\rho^0_{\rm Cl}$ is the clock which implements the transition autonomously. Furthermore, note that while the rhs of Eq. (5) is evolving according to the free Hamiltonian $\hat{H}_{\rm A} + \hat{H}_{\rm Cl}$, the Hamiltonian $\hat{H}_{\rm AClG}$ can, in principle, be of any form such that Eq. (5) holds.

The following rules out the possibility of dynamics of the form Eq. (5) for a wide class of clock Hamiltonians even when Eq. (5) is relaxed to include correlations between system A and the clock.

Proposition 3. (Idealized control no-go) Consider a time-independent Hamiltonian $\hat{H}_{\rm AClG}$ on $\mathcal{H}_{\rm AG} \otimes \mathcal{H}_{\rm Cl}$ where $\mathcal{H}_{\rm AG}$ is finite dimensional and $\mathcal{H}_{\rm Cl}$ arbitrary, which, without loss of generality, we expand in the form $\hat{H}_{\rm AClG} = \hat{H}_{\rm AG} \otimes \mathbb{1}_{\rm Cl} + \sum_{l,m=1}^{d_{\rm A}d_{\rm G}} |E_l\rangle\langle E_m|_{\rm AG} \otimes \hat{H}^{(l,m)}_{\rm Cl}$, where $\{|E_l\rangle_{\rm AG}\}_{l=1}^{d_{\rm A}d_{\rm G}}$ are the energy eigenstates of $\hat{H}_{\rm AG} = \hat{H}_{\rm A} + \hat{H}_{\rm G}$, the free Hamiltonian on $\mathcal{H}_{\rm A}$ and the bath. Both of the following two assertions cannot simultaneously hold:

Case 1. For all $k, l = 1, 2, \ldots, d_{\rm A}d_{\rm G}$; $k \neq l$, the power-series expansion in t,

$$\mathrm{tr}[e^{-it\hat{H}^{(k,k)}_{\rm Cl}}\rho^0_{\rm Cl}e^{it\hat{H}^{(l,l)}_{\rm Cl}}] \tag{7}$$

$$= \sum_{n,m=0}^{\infty} \mathrm{tr}\left[\frac{(-i\hat{H}^{(k,k)}_{\rm Cl})^n}{n!}\rho^0_{\rm Cl}\frac{(i\hat{H}^{(l,l)}_{\rm Cl})^m}{m!}\right]t^{n+m} \tag{8}$$

has a radius of convergence $r > t_2$.

Case 2. For some $0 < t_1 < t_2 < t_3$, there exists a TO from $\rho^0_{\rm A}(t)$ to

$$\rho^F_{\rm A}(t) = \begin{cases} \rho^0_{\rm A}(t) & \text{for } t \in [0, t_1] \\ \mathrm{tr}_{\rm G}[U_{\rm AG}(\rho^0_{\rm A}(t) \otimes \tau_{\rm G})U^\dagger_{\rm AG}] & \text{for } t \in [t_2, t_3], \end{cases} \tag{9}$$

which is implementable via unitary dynamics of the form

$$\rho^F_{\rm A}(t) = \mathrm{tr}_{\rm GCl}[e^{-it\hat{H}_{\rm AClG}}(\rho^0_{\rm A} \otimes \rho^0_{\rm Cl} \otimes \tau_{\rm G})e^{it\hat{H}_{\rm AClG}}], \tag{10}$$

where $U_{\rm AG}$ in Eq. (9) has a nondegenerate spectrum, and it is an energy-preserving unitary, namely, $[U_{\rm AG} \otimes \mathbb{1}_{\rm Cl}, \hat{H}_{\rm AGCl}] = 0$.

See the Appendix Sec. A 1 for a proof by contradiction. The requirement of nondegenerate spectrum in Case 2 for $U_{\rm AG}$ allows for exclusion of the trivial cases $U_{\rm AG} \propto \mathbb{1}_{\rm AG}$ for which Cases 1 and 2 can simultaneously hold [61]. Furthermore, the no-go proposition also covers the more relaxed setting in which the clock (or any catalyst included in A) is allowed to become correlated with the system. The correlated scenario is also important and studied within the context of idealized control in Refs. [62–64].

Physical intuition suggests that if the Hamiltonian $\hat{H}_{\rm AClG}$ is infinite dimensional, the dynamics it induces can be arbitrarily well approximated by replacing it by a projection onto an arbitrarily large finite-dimensional subspace. However, such a projection would imply that the terms $\hat{H}^{(l,m)}_{\rm Cl}$ found in the Hamiltonian $\hat{H}_{\rm AClG}$ are replaced with finite-dimensional matrices, and the series in line (7) would converge. Therefore, according to the above proposition, if Case 1 holds, the Hamiltonian $\hat{H}_{\rm AClG}$ cannot be approximated as one would expect.

On the other hand, Case 2 includes the desirable scenario of idealized control discussed at the beginning of Sec. II B. Therefore, the no-go proposition tells us that if idealized control is possible, it requires infinite-dimensional Hamiltonians which cannot be approximated in the way one might expect.

It can also be seen that the contradicting statements, Cases 1 and 2 in Proposition 3 are not due to a necessity to implement Case 2 with "abruptly changing" dynamics, since the unitary $U_{\rm AG}$ facilitating the TO from $\rho^0_{\rm A}$ to $\rho^1_{\rm A}$ can be implemented via a smooth function of t, namely, $U_{\rm AG}(t) = \exp[-i\hat{H}_u \int_{t_1}^t \bar{\delta}(x)dx]$, with $\bar{\delta}(t)$ a normalized bump function with support on some interval $\subseteq [t_1, t_2]$ and $\hat{H}_u$ an appropriately chosen time-independent Hamiltonian.

The no-go proposition thus rules out physical implementation of idealized control for a number of cases. We now give some examples in which Case 1 or 2 holds. Proposition 3, Case 1 holds when $\rho^0_{\rm Cl}$ is an analytic vector [65]. The simplest example of this is when $\rho^0_{\rm Cl}$ has bounded support on the spectral measures of the Hamiltonians $\{\hat{H}^{(k,k)}_{\rm Cl}\}_{k=1}^{d_{\rm A}d_{\rm Cl}}$, such as in the finite-dimensional clock case. One can, however, find examples for Proposition 3 in which Case 2 is fulfilled while Case 1 is not. This corresponds to the case of the idealized momentum clock used for control in Ref. [29]. In this case, the Hamiltonian $\hat{H}_{\rm AGCl}$ from Proposition 3 can be written in the form $\hat{H}_{\rm AGCl} = \hat{H}_{\rm AG} \otimes \mathbb{1}_{\rm Cl} + \sum_{n=1}^{d_{\rm A}d_{\rm G}} \Omega_n |E_n\rangle\langle E_n|_{\rm AG} \otimes g(\hat{x}_{\rm Cl}) + \mathbb{1}_{\rm AG} \otimes \hat{p}_{\rm Cl}$, with $\hat{x}_{\rm Cl}$, $\hat{p}_{\rm Cl}$ canonical position and momentum operators of a particle on a line. When g and the initial clock state have bounded support in position, Case 2 in Proposition 3 is satisfied, but Case 1 is not. Unfortunately, such a clock state is so spread out in momentum, the power-series expansion $\mathrm{Exp}[-it\hat{p}_{\rm Cl}] = \sum_{n=0}^\infty (-it\hat{p}_{\rm Cl})^n/n!$ diverges in norm when evaluated on it. This is closely related to another unphysical property of such clock states, namely, that the Hamiltonian has no ground state, as first pointed out by Pauli [31]. We also see how this idealized control allows one to violate the third law or thermodynamics in Sec. III B—something which should not be possible with control coming from a physical system. We thus refer to dynamics for which $\rho^F_{\mathrm{ACl}}(t)$ satisfies Eq. (5) as idealized dynamics.

Take home message from Proposition 3: Control devices which do not suffer any backreaction when implementing a thermodynamic transition, arguably necessitate unphysical Hamiltonians.

At first sight, these observations may appear to be of little practical relevance, since indeed, one does not care about implementing the transition from ρ^0_{S} to ρ^1_{S} exactly, but only to a good approximation. Furthermore, for a sufficiently large clock, one might reasonably envisage being able to implement all transformations whose final states $\rho^F_{\mathrm{S}}(t)$ are in an epsilon ball of those reachable under t-CNO (and not a larger set) to arbitrary small epsilon as long as the final clock state becomes arbitrarily close in trace distance to the idealized case, namely, if $\|\rho^F_{\mathrm{Cl}}(t)-\rho^0_{\mathrm{Cl}}(t)\|_1$ tends to zero as the dimension of the clock becomes large and approaches an idealized clock of infinite energy. Unfortunately, this intuitive reasoning may be false due to a phenomenon known as embezzlement. Indeed, when Eq. (5) is not satisfied, the clock is disturbed by the act of implementing the unitary. As such, it is no longer a catalyst, but only an inexact one. Inexact catalysis has been studied in the literature with some counterintuitive findings. In Ref. [28], an inexact catalysis pair ρ^0_{Cat}, ρ^1_{Cat} of dimension d_{Cat} was found such that for any d_{S}-dimensional system, their trace distance vanished in the large-d_{Cat} limit:

$$\|\rho^0_{\mathrm{Cat}}-\rho^1_{\mathrm{Cat}}\|_1=\frac{d_{\mathrm{S}}}{1+(d_{\mathrm{S}}-1)\log_{d_{\mathrm{S}}} d_{\mathrm{Cat}}}. \tag{11}$$

Yet the noisy operation $\rho^0_{\mathrm{S}}\otimes\rho^0_{\mathrm{Cat}}\underset{\mathrm{NO}}{\longrightarrow}\rho^1_{\mathrm{S}}\otimes\rho^1_{\mathrm{cat}}$ becomes valid for all states ρ^0_{S}, ρ^1_{S} in the large-d_{Cat} limit. In other words, they showed that the actual transition laws for the achievable state ρ^1_{S} given an initial state ρ^0_{S} cannot be approximated by those of CNOs; they are completely trivial, since all transformations are allowed. This paradoxical phenomenon is known as work embezzlement [66] and stems from the concept of entanglement embezzlement [27].

By virtue of Proposition 2, the above example shows that simply finding a clock satisfying $\|\rho^F_{\mathrm{Cl}}(t)-\rho^0_{\mathrm{Cl}}(t)\|_1\to 0$ as $d_{\mathrm{Cl}}\to\infty$ is not sufficient to conclude that the set of allowed transformations generated by t-CNOs (and thus, CTOs) corresponds to the set of transformations which can actually be implemented with physical control systems. A thought experiment illustrating such phenomena can be found at the classical level in Fig. 3.

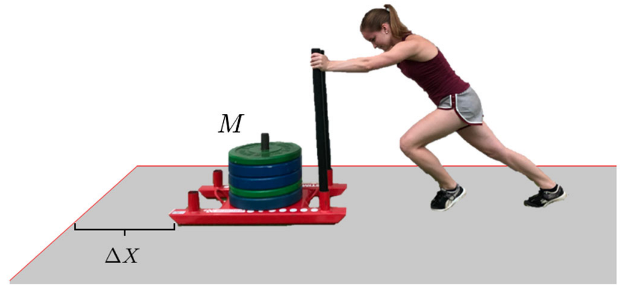

FIG. 3. The counterintuitive phenomenon of embezzlement. Consider a thought experiment in which an athlete who has to push a mass M a distance ΔX against a resistive force $F=Mg$ due to gravity pushing down on the weight. Suppose the distance the athlete has to push the weight is given by $\Delta X=f(M)$, where $f(M)\to 0$ as $M\to\infty$. The work done by the athlete pushing the weight is $W=\mu_0 F\Delta X=\mu_0 gMf(M)$, for some coefficient of resistance μ_0. One might be inclined to reason that the amount of work the athlete has to do in the limit of infinite mass M is zero, since the distance ΔX the weight has to be pushed is zero in this limit. However, a closer analysis would reveal that this is only correct if $f(M)$ decays sufficiently quickly—quicker than an inverse power. An analogous phenomenon is at play in our control setting. There, in the case of the idealized clock, Eq. (5) holds, yet this is unachievable since it requires infinite energy. However, all finite clocks suffer a minimal backreaction, and even though this backreaction can vanish in the large-dimension or -energy limit [cf. Eq. (11)], this is not sufficient to conclude that the set of implementable transformations are close to those implementable via the idealized clock. Moreover, the rate at which the error needs to vanish and whether this is physically achievable were (prior to this work) completely unknown.

III. RESULTS

We start with the easier case of CNOs in Sec. III A before moving on to the more demanding setting of CTOs in Sec. III B.

A. Autonomous control for catalytic noisy operations

In this section, we provide two theorems which together show that there exist clocks which are sufficiently accurate to allow the full realization of t-CNOs to arbitrarily high precision. Our first result gives a sufficient condition on the clock so as to be guaranteed that the achieved dynamics of the system are close to a transition permitted under t-CNOs. It can be viewed as a converse theorem to the result in Ref. [28] discussed at the end of Sec. II B.

In the following theorem, let $V_{\mathrm{SCatClG}}(t)=e^{-it\hat{H}_{\mathrm{SCatClG}}}$ be an arbitrary unitary implemented via a time-independent Hamiltonian $\hat{H}_{\mathrm{SCatClG}}$ over $\rho^0_{\mathrm{S}}\otimes\rho^0_{\mathrm{Cat}}\otimes\rho^0_{\mathrm{Cl}}\otimes\tilde{\tau}_{\mathrm{G}}$ and suppose that the final state at time $t\geq 0$,

$$\rho^F_{\mathrm{SCatClG}}(t) = V_{\mathrm{SCatClG}}(t)(\rho^0_{\mathrm{S}} \otimes \rho^0_{\mathrm{Cat}} \otimes \rho^0_{\mathrm{Cl}} \otimes \tilde{\tau}_{\mathrm{G}})V^{\dagger}_{\mathrm{SCatClG}}(t) \tag{12}$$

deviates from the idealized dynamics by an amount

$$\|\rho^F_{\mathrm{SCatCl}}(t) - \rho^F_{\mathrm{S}}(t) \otimes \rho^0_{\mathrm{Cat}}(t) \otimes \rho^0_{\mathrm{Cl}}(t)\|_1 \leq \epsilon_{\mathrm{emb}}(t; d_{\mathrm{S}}, d_{\mathrm{Cat}} d_{\mathrm{Cl}}), \tag{13}$$

where recall that $\rho^0_{\mathrm{Cl}}(t)$ is the free evolution of the clock according to its free, time-independent Hamiltonian $\hat{H}_{\mathrm{Cl}}$ [Eq. (6)] and likewise for $\rho^0_{\mathrm{Cat}}(t)$ with arbitrary Hamiltonian $\hat{H}_{\mathrm{Cat}}$.

Theorem 1. (Sufficient conditions for t-CNOs). For all states ρ^0_{S} not of full rank, and for all catalysts ρ^0_{Cat}, clocks ρ^0_{Cl}, and maximally mixed states $\tilde{\tau}_{\mathrm{G}}$, there exists a state $\sigma_{\mathrm{S}}(t)$ which is ϵ_{res} close to $\rho^F_{\mathrm{S}}(t)$,

$$\|\sigma_{\mathrm{S}}(t) - \rho^F_{\mathrm{S}}(t)\|_1 \leq \epsilon_{\mathrm{res}}(d_{\mathrm{S}}, d_{\mathrm{Cat}} d_{\mathrm{Cl}}, \epsilon_{\mathrm{emb}}(t; d_{\mathrm{S}}, d_{\mathrm{Cat}} d_{\mathrm{Cl}})), \tag{14}$$

such that for all times $t \geq 0$, a transition from

$$\rho^0_{\mathrm{S}} \otimes \rho^0_{\mathrm{Cat}} \otimes \rho^0_{\mathrm{Cl}} \quad \text{to} \quad \sigma_{\mathrm{S}}(t) \otimes \rho^0_{\mathrm{Cat}}(t) \otimes \rho^0_{\mathrm{Cl}}(t) \tag{15}$$

is possible via a NO [i.e., ρ^0_{S} to $\sigma_{\mathrm{S}}(t)$ via t-CNO]. Specifically, for fixed d_{S} and in the limit that $d_{\mathrm{Cat}} d_{\mathrm{Cl}}$ and $1/\epsilon_{\mathrm{emb}}$ tend to infinity:

$$\epsilon_{\mathrm{res}}(d_{\mathrm{S}}, d_{\mathrm{Cat}} d_{\mathrm{Cl}}, \epsilon_{\mathrm{emb}}) = 15\sqrt{\frac{d_{\mathrm{S}} \ln(d_{\mathrm{Cat}} d_{\mathrm{Cl}})}{\ln(1/\epsilon_{\mathrm{emb}})}}(1 + d_{\mathrm{Cat}} d_{\mathrm{Cl}} \epsilon^{1/7}_{\mathrm{emb}}). \tag{16}$$

Explicitly, one possible choice for $\sigma_{\mathrm{S}}(t)$ is

$$\sigma_{\mathrm{S}}(t) = \begin{cases} \mathbb{1}_{\mathrm{S}}/d_{\mathrm{S}} & \text{if } \|\rho^F_{\mathrm{S}}(t) - \mathbb{1}_{\mathrm{S}}/d_{\mathrm{S}}\|_1 < \epsilon_{\mathrm{res}}, \\ (1-\epsilon_{\mathrm{res}})\rho^F_{\mathrm{S}}(t) + \epsilon_{\mathrm{res}} \mathbb{1}_{\mathrm{S}}/d_{\mathrm{S}} & \text{if } \|\rho^F_{\mathrm{S}}(t) - \mathbb{1}_{\mathrm{S}}/d_{\mathrm{S}}\|_1 \geq \epsilon_{\mathrm{res}}. \end{cases}$$

See Appendix Sec. A 2 for a proof and an expression for ϵ_{res} which holds when $d_{\mathrm{Cat}} d_{\mathrm{Cl}}$ and ϵ_{emb} are finite. Note that this theorem also holds more generally if one replaces $\hat{H}_{\mathrm{SCatClG}}$ with any time-dependent Hamiltonian. However, the time-independent Hamiltonian case is better physically motivated.

Before we move on, let us understand the physical meaning of the terms $\epsilon_{\mathrm{emb}}, \epsilon_{\mathrm{res}}$. By comparing the definition of ϵ_{emb} in Eq. (13) with that of Eq. (5), we see that it is the difference in trace distance between the dynamics achieved with the idealized clock and the actual dynamics achieved by the clock. Thus, the quantity ϵ_{emb} upper bounds how much one can embezzle from the resulting unavoidable inexact catalysis of the clock. Then, ϵ_{res} (which is a function of ϵ_{emb}) characterizes the resolution, i.e., how far from a t-CNO transition one can achieve due to embezzlement from the inexact catalysis. For example, consider a hypothetical clock for which ϵ_{emb} decays as an inverse power with d_{Cl}. Then, ϵ_{res} would diverge with increasing d_{Cl} and Theorem 1 would not tell us anything useful. On the other hand, if we have a more precise clock with, for example, ϵ_{emb} exponentially small in d_{Cl}, then Theorem 1 would tell us that ϵ_{res} converges to zero as d_{Cl} increases.

Take home message from Theorem 1: There is a threshold on the amount of backreaction the control system can incur, above which the laws of thermodynamics have to be modified to include the thermodynamics of the control system. Theorem 1 provides a bound on this threshold when the bath transfers entropy but not heat.

Whether ϵ_{emb} and ϵ_{res} can both be simultaneously small depends on both the quality of the clock used and the transition one wishes to implement. Two examples at opposite extremes are as follows. Both ϵ_{emb} and ϵ_{res} are trivially arbitrarily small (zero in fact), and the conditions in Theorem 1 are satisfied when the t-CTO transition is the identity transition (i.e., ρ^0_{S} to ρ^0_{S}). At the opposite extreme, both ϵ_{emb} and ϵ_{res} cannot be small or vanishing when one attempts a nontrivial t-CNO transition which occurs instantaneously, i.e., one for which $\rho^F_{\mathrm{S}}(t) = \rho^0_{\mathrm{S}}$ for $t \in [0, t_1]$ and $\rho^F_{\mathrm{S}}(t) = \rho^1_{\mathrm{S}}$ for $t \in (t_1, t_3]$.

Our next theorem shows how one can implement to arbitrary approximation all t-CTO transitions, over any fixed time interval (t_1, t_2), yet without allowing for a larger class, as the examples in Eq. (11) and Fig. 4(b) do. To achieve this, one must choose the time-independent Hamiltonian $\hat{H}_{\mathrm{SCatClG}}$ and initial clock state ρ^0_{Cl} appropriately. The theorem will use

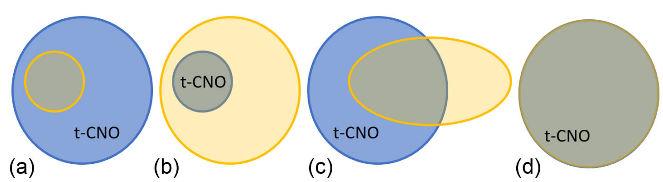

FIG. 4. Possible scenarios resulting from the physical implementation of t-CNOs. Given a state ρ^0_{S}, the above blue Venn diagrams represent the set of states ρ^1_{S} which can be reached under t-CNOs. The orange Venn diagrams represent possible scenarios of reachable states when attempting to implement a t-CNO, while gray represents the intersection of the two sets. Because of the apparent impossibility of perfect control and that embezzlement can occur [see Eq. (11)], all options (a) to (d) are, in principle, open. Theorem 1 gives sufficient conditions on the control (clock) so that either (a) or (d) occur. Theorem 2 shows that transitions implemented via the quasi-ideal clock can achieve (d) under reasonable circumstances.

the quasi-ideal clock [67] discussed in detail in the Appendix Sec. A 3 for the clock system on $\mathcal{H}_{\mathrm{Cl}}$. The quasi-ideal clock has been proven to be optimal for some tasks related to reference frames [68–70] and clocks [71,72], and it is also believed to be optimal for others [73]. In the following, T_0 denotes the period of the quasi-ideal clock (when evolving under its free evolution), i.e., $\rho^0_{\mathrm{Cl}}(T_0) = \rho^0_{\mathrm{Cl}}(0)$.

Theorem 2. (Achieving t-CNOs). Consider the quasi-ideal clock [67] detailed in Sec. A 3 a with a time-independent Hamiltonian of the form $\hat{H}_{\mathrm{SCatClG}} = \hat{H}_{\mathrm{S}} + \hat{H}_{\mathrm{Cat}} + \hat{H}_{\mathrm{G}} + \hat{I}_{\mathrm{SCatClG}} + \hat{H}_{\mathrm{Cl}}$ giving rise to unitary dynamics

$$\rho^F_{\mathrm{SCatClG}}(t) = V_{\mathrm{SCatClG}}(t)(\rho^0_{\mathrm{S}} \otimes \rho^0_{\mathrm{Cat}} \otimes \rho^0_{\mathrm{Cl}} \otimes \tilde{\tau}_{\mathrm{G}}) V^\dagger_{\mathrm{SCatClG}}(t).$$

For every pair $\rho^0_{\mathrm{S}}, \rho^1_{\mathrm{S}}$ for which there exists a t-CNO from ρ^0_{S} to ρ^1_{S} using a catalyst ρ^0_{Cat}, there exists an interaction term $\hat{I}_{\mathrm{SCatClG}}$ such that the following hold.

(1) $\sigma_{\mathrm{S}}(t)$ satisfies Eq. (15) and is of the form

$$\sigma_{\mathrm{S}}(t) = \begin{cases} \rho^0_{\mathrm{S}}(t) & \text{for times } t \in [0, t_1] \text{ (i.e., ``before'' the transition)}, \\ \rho^1_{\mathrm{S}}(t) & \text{for times } t \in [t_2, T_0] \text{ (i.e., ``after'' the transition)}. \end{cases}$$

(2) ϵ_{emb} ([satisfying Eq. (13)] is given by

$$\epsilon_{\mathrm{emb}} = (2 + 3\sqrt{d_{\mathrm{S}} d_{\mathrm{Cat}}}) \sqrt{\varepsilon_{\mathrm{Cl}}(d_{\mathrm{Cl}})}, \qquad (17)$$

for all $t \in [0, t_1] \cup [t_2, T_0]$, where $\varepsilon_{\mathrm{Cl}}(\cdot)$ is independent of d_{S}, d_{Cat}, d_{G} and is of order

$$\varepsilon_{\mathrm{Cl}}(d_{\mathrm{Cl}}) = \mathcal{O}(\mathrm{poly}(d_{\mathrm{Cl}}) \exp[-c d_{\mathrm{Cl}}^{1/4}]), \qquad (18)$$

as $d_{\mathrm{Cl}} \to \infty$, with $c = c(t_1, t_2, T_0) > 0$ for all $0 < t_1 < t_2 < T_0$, and it is independent of d_{Cl}.

See the Appendix Sec. A 4 a for a proof.

As a direct consequence of Theorem 1, in the scenario described in Theorem 2, ϵ_{res} is of power-law decay in d_{Cl} as $d_{\mathrm{Cl}} \to \infty$, and thus, both ϵ_{emb} and ϵ_{res} are simultaneously small. Therefore, the quasi-ideal clock allows all t-CNOs to be implemented without additional costs not captured by the resource theory.

Take home message from Theorem 2: There exist control systems whose incurred backreaction is small enough that one is below the threshold mentioned in the previous box. Hence, in conjunction with Theorem 1, it implies that the laws of thermodynamics (for baths that transfer only entropy and not heat) do not need to be modified by taking into account the control device.

The property that $\tilde{\tau}_{\mathrm{G}}$ is a maximally mixed state for CNOs is at the heart of two important aspects involved in proving Theorems 1 and 2. On the one hand, all CNOs (and hence, all t-CNOs by virtue of Proposition 2), which are implemented via an arbitrary finite-dimensional catalyst ρ_{Cat} can be done so with maximally mixed states $\tilde{\tau}_{\mathrm{G}}$ of finite dimension [74,75]. The other relevant aspect is that they are the only states which are not "disturbed" by the action of a unitary, namely, $U_{\mathrm{G}} \tilde{\tau}_{\mathrm{G}} U^\dagger_{\mathrm{G}} = \tilde{\tau}_{\mathrm{G}}$ for all unitaries U_{G}. Together, these mean that the clock needed only to control a system of finite size, and thus, the backreaction it experiences is limited and independent of the dimension d_{G} [76].

One would like to prove analogous theorems to Theorems 1 and 2 for t-CTOs. Unfortunately, their Gibbs states satisfy neither of these two aforementioned properties. Indeed, there exist CTOs on finite-dimensional systems $\mathcal{H}_{\mathrm{S}}$ which require infinite-dimensional Gibbs states of infinite mean energy to implement them [32,75,77]. This observation combined with the fact that Gibbs states are also generally disturbed by the CTO in the sense that $U_{\mathrm{G}} \tau_{\mathrm{G}} U^\dagger_{\mathrm{G}} \neq \tau_{\mathrm{G}}$ for some U_{G} suggests that a theorem like Theorem 2 for which ϵ_{res} from Theorem 1 vanishes is not possible since the backreaction on any finite energy or dimensional clock would be infinite in some cases. Furthermore, there is a technical problem which prevents such theorems. The proof of Theorem 1 uses the known, necessary, and sufficient transformation laws for noisy operations (the nonincrease of the so-called Rényi α entropies). However, only necessary (but not sufficient) second laws are known for CTOs (the most well-known of which are the nonincreases of the so-called Rényi α divergences [25]).

B. Autonomous control for catalytic thermal operations

In order to circumvent the dilemma explained at the end of the previous section, we now examine how well the energy-preserving unitary of t-CTOs can be implemented when one restricts to attempting to implement t-CTOs which can be implemented with finite baths. We also allow for some uncertainty in our knowledge, or ability to prepare, the time-independent Hamiltonian which implements the transition. Specifically, we consider

$$\hat{H}_{\mathrm{SCatClG}} = \hat{H}_{\mathrm{S}} + \hat{H}_{\mathrm{Cat}} + \hat{H}_{\mathrm{G}} + \hat{H}^{\mathrm{int}}_{\mathrm{SCatG}} \otimes \hat{H}^{\mathrm{int}}_{\mathrm{Cl}} + \hat{H}_{\mathrm{Cl}}, \qquad (19)$$

where

$$[\hat{H}_{\mathrm{S}} + \hat{H}_{\mathrm{Cat}} + \hat{H}_{\mathrm{G}}, \hat{H}^{\mathrm{int}}_{\mathrm{SCatG}}] = 0, \qquad (20)$$

and normalization chosen such that the interaction term has eigenvalues bounded by pi: $\|\hat{H}^{\mathrm{int}}_{\mathrm{SCatG}}\|_\infty \leq \pi$. With the interaction term $\hat{H}^{\mathrm{int}}_{\mathrm{SCatG}}$ in the Hamiltonian Eq. (19), and the aid of the thermal bath and clock, we are targeting to implement the joint system-catalyst state

$$\sigma^1_{\mathrm{SCat}} \coloneqq \mathrm{tr}_{\mathrm{G}}[e^{-i\hat{H}^{\mathrm{int}}_{\mathrm{SCatG}}}(\rho^0_{\mathrm{S}} \otimes \rho^0_{\mathrm{Cat}} \otimes \tau_{\mathrm{G}})e^{i\hat{H}^{\mathrm{int}}_{\mathrm{SCatG}}}]. \tag{21}$$

From Eq. (21), we can observe that the interaction term $\hat{H}^{\mathrm{int}}_{\mathrm{SCatG}}$ already allows for potential Hamiltonian engineering imperfections, since ideally, the interaction term should leave the final state σ^1_{SCat} in Eq. (21) in a product state of the form $\rho^1_{\mathrm{S}} \otimes \rho^0_{\mathrm{Cat}}$. To capture these imperfections in $\hat{H}^{\mathrm{int}}_{\mathrm{SCatG}}$, we introduce $\hat{I}^{\mathrm{int}}_{\mathrm{SCatG}}$ which, for the initial state $\rho^0_{\mathrm{S}} \otimes \rho^0_{\mathrm{Cat}} \otimes \tau_{\mathrm{G}}$, implements an uncorrelated system-catalyst state:

$$\rho^1_{\mathrm{S}} \otimes \rho^0_{\mathrm{Cat}} = \mathrm{tr}_{\mathrm{G}}[e^{-i\hat{I}^{\mathrm{int}}_{\mathrm{SCatG}}}(\rho^0_{\mathrm{S}} \otimes \rho^0_{\mathrm{Cat}} \otimes \tau_{\mathrm{G}})e^{i\hat{I}^{\mathrm{int}}_{\mathrm{SCatG}}}]. \tag{22}$$

Here, ρ^1_{S} is an arbitrary state that can be produced by such a transformation; i.e., it is an arbitrary state that can be obtained from ρ^0_{S} via a CTO. Note that the evolution according the total Hamiltonian in Eq. (19) will not produce such a transformation through time evolution even if we have the term $\hat{I}^{\mathrm{int}}_{\mathrm{SCatG}}$ instead of $\hat{H}^{\mathrm{int}}_{\mathrm{SCatG}}$ since the clock is not ideal.

If we denote the difference between the states in Eqs. (21) and (22) by

$$\epsilon_H \coloneqq \|\sigma^1_{\mathrm{SCat}} - \rho^1_{\mathrm{S}} \otimes \rho^0_{\mathrm{Cat}}\|_1, \tag{23}$$

then Proposition 45 states that ϵ_H is upper bounded by

$$\epsilon_H \leq 2\|\delta\hat{I}^{\mathrm{int}}_{\mathrm{SCatG}}\|_\infty + \|\delta\hat{I}^{\mathrm{int}}_{\mathrm{SCatG}}\|^2_\infty, \tag{24}$$

where $\|\delta\hat{I}^{\mathrm{int}}_{\mathrm{SCatG}}\|_\infty$ denotes the largest eigenvalue in magnitude of the imperfection in the Hamiltonian preparation: $\delta\hat{I}^{\mathrm{int}}_{\mathrm{SCatG}} \coloneqq \hat{H}^{\mathrm{int}}_{\mathrm{SCatG}} - \hat{I}^{\mathrm{int}}_{\mathrm{SCatG}}$. Note that there is also some freedom in the definition of $\hat{I}^{\mathrm{int}}_{\mathrm{SCatG}}$ in Eq. (22) since the final state of the bath is traced out and hence irrelevant. One can minimize $\|\delta\hat{I}^{\mathrm{int}}_{\mathrm{SCatG}}\|_\infty$ over this degree of freedom, reducing the control requirements over the bath degrees of freedom and improving the bounds on ϵ_H.

We now introduce a state $\rho^{\mathrm{target}}_{\mathrm{SCatG}}(t)$, which we call the target state. It is the state which we would be able to implement with the Hamiltonian in Eq. (19) if we had access to an idealized clock. Hence, any deviations from this will be due to using physical clocks in the control. It is given by

$$\rho^{\mathrm{target}}_{\mathrm{SCatG}}(t) \coloneqq U^{\mathrm{target}}_{\mathrm{SCatG}}(t)[\rho^0_{\mathrm{S}}(t) \otimes \rho^0_{\mathrm{Cat}}(t) \otimes \tau_{\mathrm{G}}]U^{\mathrm{target}\dagger}_{\mathrm{SCatG}}(t), \tag{25}$$

where $U^{\mathrm{target}}_{\mathrm{SCatG}}(t) = e^{-i\theta(t)\hat{H}^{\mathrm{int}}_{\mathrm{SCatG}}}$ with

$$\theta(t) = \begin{cases} 0 & \text{for } t \in [0, t_1], \\ 1 & \text{for } t \in [t_2, t_3]. \end{cases} \tag{26}$$

(Recall that the physical meaning of t_1, t_2, and t_3 can be found in Definition 1.) Therefore, tracing out the bath, we have for $t \in [0, t_1]$,

$$\rho^{\mathrm{target}}_{\mathrm{SCat}}(t) = \rho^0_{\mathrm{S}}(t) \otimes \rho^0_{\mathrm{Cat}}(t), \tag{27}$$

while for $t \in [t_2, t_3]$,

$$\rho^{\mathrm{target}}_{\mathrm{SCat}}(t) = e^{-it(\hat{H}_{\mathrm{S}}+\hat{H}_{\mathrm{Cat}})}\sigma^1_{\mathrm{SCat}}e^{it(\hat{H}_{\mathrm{S}}+\hat{H}_{\mathrm{Cat}})}. \tag{28}$$

We now define a quantity $\Delta(t; x, y)$ which depends only on properties of the clock system:

$$\Delta(t; x, y) \coloneqq \langle\rho^0_{\mathrm{Cl}}|\hat{\Gamma}^\dagger_{\mathrm{Cl}}(x, t)\hat{\Gamma}_{\mathrm{Cl}}(y, t)|\rho^0_{\mathrm{Cl}}\rangle, \tag{29}$$

$$\hat{\Gamma}_{\mathrm{Cl}}(x, t) \coloneqq e^{-it\hat{H}_{\mathrm{Cl}}+ix(\theta(t)\mathbb{1}_{\mathrm{Cl}}-t\hat{H}^{\mathrm{int}}_{\mathrm{Cl}})}, \qquad x, t \in \mathbb{R}. \tag{30}$$

The following theorem states that if $\Delta(t; x, y)$ is small for all $x, y \in [-\pi, \pi]$ and the dimension of the bath d_{G} is not too large, then the clock can implement a unitary over the system, catalyst, and clock which is close to a t-CTO using the time-independent Hamiltonian in Eq. (19). Furthermore, the clock itself is not disturbed much during the process.

Theorem 3. (Sufficient conditions for t-CTOs) For all states ρ^0_{S} and ρ^0_{Cat}, consider unitary dynamics $V_{\mathrm{SCatClG}}(t) = e^{-it\hat{H}_{\mathrm{SCatClG}}}$ implemented via any Hamiltonian of the form Eq. (19), with an initial pure clock state $\rho^0_{\mathrm{Cl}} = |\rho^0_{\mathrm{Cl}}\rangle\langle\rho^0_{\mathrm{Cl}}|$. Namely, $\rho^F_{\mathrm{SCatClG}}(t) = V_{\mathrm{SCatClG}}(t)(\rho^0_{\mathrm{S}} \otimes \rho^0_{\mathrm{Cat}} \otimes \rho^0_{\mathrm{Cl}} \otimes \tau_{\mathrm{G}}) \times V^\dagger_{\mathrm{SCatClG}}(t)$. Then, the following hold.

(1) The deviation from the idealized dynamics is bounded by

$$\|\rho^F_{\mathrm{SCatCl}}(t) - \rho^F_{\mathrm{S}}(t) \otimes \rho^0_{\mathrm{Cat}}(t) \otimes \rho^0_{\mathrm{Cl}}(t)\|_1 \leq 2\epsilon_H\theta(t) + 6\sqrt{d_{\mathrm{S}}d_{\mathrm{Cat}}d_{\mathrm{G}}\mathrm{tr}[\tau^2_{\mathrm{G}}]}\sqrt{\max_{x,y\in[-\pi,\pi]}|1 - \Delta^2(t; x, y)|}. \tag{31}$$

(2) The final state $\rho^F_{\mathrm{S}}(t)$ is

$$\|\rho^F_{\mathrm{S}}(t) - \rho^{\mathrm{target}}_{\mathrm{S}}(t)\|_1 \leq \epsilon_H\theta(t) + \sqrt{d_{\mathrm{S}}d_{\mathrm{Cat}}d_{\mathrm{G}}\mathrm{tr}[\tau^2_{\mathrm{G}}]}\max_{x,y\in[-\pi,\pi]}|1 - \Delta^2(t; x, y)|, \tag{32}$$

close to one which can be reached via t-CTO: For all $t \in [0, t_1] \cup [t_2, t_3]$, the transition

$$\rho^0_{\mathrm{S}} \otimes \rho^0_{\mathrm{Cat}} \otimes \rho^0_{\mathrm{Cl}} \quad \text{to} \quad \rho^{\mathrm{target}}_{\mathrm{S}}(t) \otimes \rho^0_{\mathrm{Cat}}(t) \otimes \rho^0_{\mathrm{Cl}}(t) \tag{33}$$

is possible via a TO, i.e., ρ^0_{S} to $\rho^{\mathrm{target}}_{\mathrm{S}}$ via a t-CTO.

A proof can be found in the Appendix Sec. A 5.

Since the definition of the target state in Eq. (25) allows one to reach all t-CTOs which are implementable with a d_{G}-dimensional bath [78], Theorem 3 provides sufficient conditions for the implementation of all t-CTOs which are implementable via such baths. As long as the set of CTOs with finite bath size is a dense subset of the set of all

CTOs, Theorem 3 provides sufficient conditions for implementing a dense subset of CTOs. While the TO in Eq. (33) for $t \in [0, t_1]$ is "trivial" in the sense that it does not involve interactions between the subsystems or requires the thermal bath, it is nevertheless important since it captures the notion of "turning on" the unitary—an essential step in the implementation of any unitary operation.

Intuitively, in order for $\Delta(t; x, y) \approx 1$ for all $x, y \in [-\pi, \pi]$, we see from Eq. (30) that we want the initial clock state $|\rho^0_{\mathrm{Cl}}\rangle$ to be orthogonal to the interaction term $\hat{H}^{\mathrm{int}}_{\mathrm{Cl}}$ initially, and subsequently the dynamics of the clock according to its free Hamiltonian $\hat{H}_{\mathrm{Cl}}$ to "rotate" the initial clock state $|\rho^0_{\mathrm{Cl}}\rangle$ to a state which is no longer orthogonal to $\hat{H}^{\mathrm{int}}_{\mathrm{Cl}}$ after a time t_1 when the interaction starts to happen. Similarly, the evolution induced by $\hat{H}_{\mathrm{Cl}}$ should make the state $|\rho^0_{\mathrm{Cl}}\rangle$ orthogonal to $\hat{H}^{\mathrm{int}}_{\mathrm{Cl}}$ after time t_2. Meanwhile, the interaction term $\hat{H}^{\mathrm{int}}_{\mathrm{Cl}}$ should have imprinted a phase of approximately e^{-ix} onto the state $|\rho^0_{\mathrm{Cl}}\rangle$ during the time interval (t_1, t_2) to cancel out the phase factor $e^{ix\theta(t)}$ in Eq. (30). So we can think of the quantity $\Delta(t, x, y)$ as a formal mathematical expression which quantifies the intuitive physical picture of "turning on and off an interaction."

The quasi-ideal clock, which recall is of dimension d_{Cl} and period T_0 (when evolving under its free evolution), can realize the above intuition to a very good approximation. Indeed, the following theorem bounds the quantities on the rhs of Eqs. (31) and (32) up to engineering errors ϵ_H by setting $t_3 = T_0$ in Theorem 3.

Theorem 4. [Achieving t-CTOs] For the quasi-ideal clock, we have

$$\max_{x,y\in[-\pi,\pi]} |1 - \Delta^2(t; x, y)| \leq \mathcal{O}(\mathrm{poly}(d_{\mathrm{Cl}}) \exp[-c d_{\mathrm{Cl}}^{1/4}]) \tag{34}$$

as $d_{\mathrm{Cl}} \to \infty$ for all $t \in [0, t_1] \cup [t_2, T_0]$, where $\Delta^2(t; x, y)$ is defined in Eq. (30) and where $c = c(t_1, t_2, T_0) > 0$ for all $0 < t_1 < t_2 < T_0$ and is independent of d_{Cl}.

See the Appendix for the proof. On the other hand, it turns out that the idealized momentum clock discussed in Sec. II B satisfies $\Delta(t; x, y) = 1$ for all $x, y \in [-\pi, \pi]$ for an appropriate parameter choice in which Case 1 in Proposition 3 fails (see Sec. A 6 in the Appendix). Thus, the rhs of Eqs. (31) and (32) is exactly zero for all $t_1 < t_2$ in this case. This observation highlights another point of failure for this clock: It allows for the violation of the third law of thermodynamics. The third law states that any system cannot be cooled to absolute zero (its ground state) in finite time. In Refs. [75,77], it was shown that under CTOs, both the mean energy and dimension d_{G} of the bath need to diverge in order to cool a d_{S}-dimensional system to the ground state. The inability to do this in finite time by any realistic control system on $\mathcal{H}_{\mathrm{Cl}}$ manifests itself in that $\max_{x,y\in[-\pi,\pi]} |1 - \Delta^2(t; x, y)|$ cannot be exactly zero in this case, so that the rhs of Eq. (32) becomes large due to the factor $d_{\mathrm{G}}\mathrm{tr}[\tau^2_{\mathrm{G}}]$ diverging [79]. However, for the idealized momentum clock, the rhs of Eqs. (31) and (32) is exactly zero even in the limit $d_{\mathrm{G}}\mathrm{tr}[\tau^2_{\mathrm{G}}] \to \infty$, thus allowing one to cool the system on $\mathcal{H}_{\mathrm{S}}$ to absolute zero in any finite time interval $[t_1, t_2]$. Finally, it is also worth noting that the change in von Neumann entropy of the clock between before and after the unitary is implemented is vanishingly small for the quasi-ideal clock as its dimension increases. This follows from applying the Fannes inequality to the results of Theorems 3 and 4. This is because the Fannes inequality implies that the change in von Neumann entropy between two states approaches zero when the trace distance between said states decreases faster than $1/\log(d)$, where d is the dimension of the system in question.

Take home message from Theorem 3: The result provides bounds which characterize the backreaction incurred on any control device implementing an arbitrary thermodynamic transition, i.e., with baths which transfer both entropy and heat. It includes and quantifies engineering imperfections and has important physical consequences for nonequilibrium physics and the third law.

Take home message from Theorem 4: There exists a control device, such that the bounds in Theorem 3 for the incurred backreaction are, up to engineering inaccuracies, exponentially small in the device's dimension. Thus, Theorems 3 and 4 together imply the existence of control devices such that the laws of thermodynamics are not modified for baths that can transmit both entropy and heat.

IV. DISCUSSION

Other than the fact that Theorem 1 provides the necessary conditions for implementation of t-CNOs while Theorem 3 for implementation of t-CTOs, there are two main differences between them. The first is that Theorem 1 applies to any time-independent Hamiltonian, while Theorem 3 to Hamiltonians of a particular form. The other main difference is that Theorem 1 provides bounds in terms of how close the catalyst and clock are in trace distance to their desired states, while Theorem 3 provides bounds in terms of how close $\Delta(t;x,y)$ is to unity. While the latter condition implies small trace distance between the clock and its free evolution, the converse is not necessarily true. Fortunately, while $\Delta(t;x,y)\approx 1$ is a stronger constraint, we show that it can be satisfied by the quasi-ideal clock (this is Theorem 4). However, from a practical point of view, its fulfillment is likely harder to verify experimentally since quantum measurements can be used to evaluate trace distances, while the ability to experimentally determine $\max_{x,y\in[-\pi,\pi]}\Delta(t;x,y)$ is less clear.

Observe how the bounds in Theorems 1 and 3 increase with d_{Cat}, the dimension of the catalyst. This aspect of the bound is also relevant in some important cases. Most exemplary is the setting of the important results of Ref. [62] which show that if one allows the catalysts to become correlated, then, up to an arbitrarily small error ϵ, there exists a catalyst and energy-preserving unitary which achieves any TO between states block diagonal in the energy basis if and only if the second law (nonincrease of von Neumann free energy) is satisfied. Here, the dimension of the catalyst diverges as ϵ converges to zero. The setting considered is that of idealized control, and thus the divergence of the catalyst does not affect the implementation of transitions. However, if one were to consider realistic control such as in our paradigm, the rate at which the catalyst diverges would be an important factor in determining how much backreaction the clock would receive and consequently how large it would have to be to counteract this effect and achieve small errors in the implementation of the control.

There are various results regarding the costs of implementing unitary operations [80–88]. These all have in common the assumption of implicit external control, while restricting only the set of allowed unitaries which is implemented by the external control. The allowed set of unitaries is motivated physically by demanding that they obey conservation laws (such as energy conservation) or by comparing unitaries which allow for coherent vs incoherent operations. So while these works consider interesting paradigms, the questions they can address are of a very different nature from those posed and answered in this paper. In particular, the assumption of perfect control on the allowed set of unitaries means that effects such as backreaction or degradation of the control device are neglected.

While other bounds do impose limitations arising from dynamics, these bounds are not of the right form to address the problem at hand in this paper. Perhaps one of the most well-known results in this direction is the so-called quantum speed limit which characterizes the minimum time required for a quantum state to become orthogonal to itself or more generally, to within a certain trace distance of itself. Indeed, such results have been applied to thermodynamics, metrology, and the study of the rate at which information can be transmitted from a quantum system to an observer [89,90]. In our context, the promise is of a different form, namely, rather than the final state being a certain distance away from the initial state, we need it to be a state which is close to one permissible via the transformation laws of the resource theory (t-CNOs or t-CTOs). Similar difficulties arise when aiming to apply other results from the literature. Perhaps most markedly is Ref. [91]. Here, the necessary conditions in terms of bounds on the fidelity to which a unitary can be performed on a system via a control device are derived. Unfortunately, this result is unsuitable for our purposes for two reasons. First, their bounds become trivial in the case that the unitary over the system to be implemented commutes with the Hamiltonian of the system (as is the case in this paper). Second, since catalysis is involved in our setting, bounds in trace distance for deviations in the state of the clock due to the implementation of a unitary, are not meaningful due to the embezzling problem discussed in Sec. II B. The latter problem is also why one cannot arrive at the conclusions of this paper from Ref. [67] alone.

This work opens up interesting new questions for future research. In macroscopic thermodynamics, the second law applies to transitions between states which are in thermodynamic equilibrium. Such a notion is not present in the CTOs, since the second laws governing transitions apply always, regardless of the nature of the state. One intriguing possibility which comes into view with the results in this paper is that the CTOs actually hold only in equilibrium, and the apparent absence of this property has been hidden in the unrealistic assumption of idealized control. To see why, observe that we prove only that the transition laws for t-CTOs hold for times $t\in[0,t_1]\cup[t_2,t_3]$ where the unitary implementing the transition occurs within the time interval (t_1,t_2). It appears that CTOs are not satisfied for the state during the transition period (t_1,t_2). If this can be confirmed and proven to hold in general, then this suggests that the CTOs actually hold only in equilibrium. A potential physical mechanism explaining this could be that at times around t_1 the clock sucks up entropy from the system it is controlling, allowing it to become more pure after finally releasing entropy back around the t_2 time so that the system can then become mixed enough to satisfy the second laws.

Another aspect which the introduction of the paradigm of physical control into the paradigm of CTOs has given rise to naturally is the variant of the third law of thermodynamics stating that one cannot cool to absolute zero in finite time. It is noticeably absent from the CTO formalism. Future work could now investigate this property in more depth. Previous characterizations of the third law [32] had to assume that the spatial area which the unitaries in the idealized control could act upon satisfied a light-cone bound. While this is indeed a realistic assumption, it did not arise from the mathematics. Here it arises naturally even without the need for a light-cone bound assumption.

Introducing similar nonidealized control for other resource theories [92,93] could allow us to understand the requirements of these paradigms.

V. CONCLUSIONS

The resource-theory approach to quantum thermodynamics has been immensely popular over the last few years. However, to date the conditions under which its underlying assumptions of idealized external control can be fulfilled have not been justified. While it is generally appreciated that they cannot be achieved perfectly, to what extent and under what circumstances they can be approximately achieved remained elusive. Our paper addresses this issue, providing sufficient conditions which we prove are satisfiable. In doing so, our work unites two very popular yet starkly different paradigms: fully autonomous thermal machines and resource-theoretic nonautonomous ones. Our approach and methods set the groundwork for future unifications of generic quantum processing machines, of which resource-theoretic thermal machines can be seen as a particular example, with generic autonomous quantum processes.

Not only could these results be instrumental for future experimental realizations of the second laws of quantum thermodynamics, but they can also open up new avenues of research into the third law of thermodynamics and the role of nonequilibrium physics.

In particular, we introduce a paradigm in which the cost of control in the resource-theory approach of quantum thermodynamics using CNOs and CTOs can be characterized. This is achieved via the introduction of t-CNOs and t-CTOs in which control devices fit naturally into this thermodynamic setting as dynamic catalysts.

We then derive sufficient conditions on how much the global dynamics including the control device can deviate from the idealized case in order for the achieved state transition to be close to one permissible via CNOs. This is followed by examples of a control device which achieves this level of precision.

Finally, we introduce Hamiltonians which lead us to a criterion for all CTOs with a finite-dimensional bath. The bound captures the requirement of better quality control, as the bath size needed to implement the CTO gets larger.

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Unless stated otherwise, the below commonly used notation has the indicated meaning.

(i) Abbreviations for transformations: NO, CNO, TO, and CTO. The prefix "t-" can be added to any of these abbreviations and stands for time. See Sec. II A for their definitions.

(ii) Subscripts: The following subscripts are added to states to indicate the subsystem they belong to. Subscript $_{\mathrm{S}}$ is the system, $_{\mathrm{Cat}}$ is the catalyst, $_{\mathrm{Cl}}$ is the clock, $_{\mathrm{G}}$ is the bath. A result with a subscript $_{\mathrm{A}}$ means the result holds for both cases A = S and A = SCat.

(iii) Partial trace: We use the quantum-information notation for partial trace. For a generic bipartite quantum state $\rho_{X_1X_2}$, we denote the state on subsystem x_1 after tracing out subsystem x_2 by ρ_{X_1}.

(iv) Time dependence: ρ_X^0 or σ_X^0 is the initial state on subsystem X. ρ_X^1 or σ_X^1 is the state on subsystem X after the application of a fixed transformation to the initial state ρ_X^1 or σ_X^1, respectively. $\rho_X^n(t)$ or $\sigma_X^n(t)$ for $n = 0, 1$ is the dynamically evolved state ρ_X^n or σ_X^n according to its local Hamiltonian: $\rho_X^n(t) = e^{it\hat{H}_X}\rho_X^n e^{-it\hat{H}_X}$ or $\sigma_X^n(t) = e^{it\hat{H}_X}\sigma_X^n e^{-it\hat{H}_X}$. These definitions are introduced in Sec. II A. The notation $\rho^F_{X_1,\ldots,X_l}(t)$ refers to a state on subsystems $X_1,\ldots,X_l$ at time t whose time evolution is not given (in general) by the sum of the local Hamiltonians $\hat{H}_{X_1}+,\cdots,+H_{X_l}$. Its exact definition is context dependent and given locally in the text.

(v) Dimensions: d_X is the Hilbert space dimension of subsystem X.

(vi) Thermal states: τ_X is the Gibbs state of subsystem X, i.e., $\tau_X = e^{-\hat{H}_X/T}/Z$, where Z is the partition function, and T is the temperature in appropriate units. The maximally mixed state denoted $\tilde{\tau}_X$ is a special Gibbs state corresponding to when $\hat{H}_X$ is proportional to the identity $\mathbb{1}_X$. It takes on the form $\tilde{\tau}_X = \mathbb{1}_X/d_X$.

APPENDIX: PROOF OVERVIEWS

In this appendix, we provide the proofs of the results in the maim text. Owing to the complexity of some of these proofs, with the exception of Proposition 3, the others have a high-level overview of the proof here, with details relegated to the Supplemental Material [33].

1. Proof of Proposition 3

Here we prove Proposition 3. We assume the assertions in both cases in the proposition and culminate in a contradiction, hence showing that the assertions cannot simultaneously hold. To start with, we denote the unitary transformation implementing the TO from $\rho^0_{\mathrm{AG}}(t)$ to $\rho^1_{\mathrm{AG}}(t)$ by $U_{\mathrm{AG}}(t) = e^{-i\delta(t)\hat{H}_u}$ where

$$\delta(t) = \begin{cases} 0 & \text{if } t \in [0, t_1], \\ 1 & \text{if } t \in [t_2, t_3]. \end{cases} \tag{A1}$$

By definition of TOs, $U_{\mathrm{AG}}(t)$ is an energy-preserving unitary which must commute with $\hat{H}_{\mathrm{AG}} = \hat{H}_{\mathrm{A}} \otimes \mathbb{1}_{\mathrm{G}} + \mathbb{1}_{\mathrm{A}} \otimes \hat{H}_{\mathrm{G}} = \sum_{n=1}^{d_{\mathrm{A}}d_{\mathrm{G}}} E_n |E_n\rangle\langle E_n|_{\mathrm{AG}}$ and can therefore be chosen to be of the form $\hat{H}_u = \sum_{n=0}^{d_{\mathrm{A}}d_{\mathrm{G}}} \Omega_n |E_n\rangle\langle E_n|_{\mathrm{AG}}$ with $\Omega_n \in [-\pi, \pi)$. In order to avoid trivial unitaries, we also assume that the phases are nondegenerate $\Omega_n \neq \Omega_p$ for $n \neq p$. It then follows from $[U_{\mathrm{AG}} \otimes \mathbb{1}_{\mathrm{Cl}}, \hat{H}_{\mathrm{AGCl}}] = 0$ that

$$\hat{H}^{(k,l)}_{\mathrm{Cl}} = 0 \tag{A2}$$

for $k \neq l$. Using the expansion of $\hat{H}_{\mathrm{AGCl}}$ from the preposition, it then follows that

$$\hat{H}_{\mathrm{AGCl}} = \hat{H}_{\mathrm{AG}} \otimes \mathbb{1}_{\mathrm{Cl}} + \sum_{n=1}^{d_{\mathrm{A}}d_{\mathrm{G}}} |E_n\rangle\langle E_n|_{\mathrm{AG}} \otimes \hat{H}^{(n,n)}_{\mathrm{Cl}}. \tag{A3}$$

Expanding the state ρ_{AG} in the energy basis $\rho_{\mathrm{AG}} = \sum_{l,m=1}^{d_{\mathrm{A}}d_{\mathrm{G}}} A_{l,m} |E_l\rangle\langle E_m|_{\mathrm{AG}}$, we find from the definition of $\rho^F_{\mathrm{AGCl}}(t)$,

$$\langle E_l|\rho^F_{\mathrm{AG}}(t)|E_m\rangle = A_{l,m}(t)\mathrm{tr}[e^{-it\hat{H}^{(l,l)}_{\mathrm{Cl}}} \rho_{\mathrm{Cl}} e^{it\hat{H}^{(m,m)}_{\mathrm{Cl}}}], \tag{A4}$$

where the time dependence of the coefficients $A_{l,m}(t)$ is defined via $\rho_{\mathrm{AG}}(t) = e^{-it\hat{H}_{\mathrm{AG}}} \rho_{\mathrm{AG}} e^{it\hat{H}_{\mathrm{AG}}} = \sum_{l,m=1}^{d_{\mathrm{A}}d_{\mathrm{G}}} A_{l,m}(t)|E_l\rangle\langle E_m|_{\mathrm{AG}}$. On the other hand,

$$\langle E_l|U_{\mathrm{AG}}(t)\rho_{\mathrm{AG}}(t)U^\dagger_{\mathrm{AG}}(t)|E_m\rangle = A_{l,m}(t)e^{-it(\Omega_m-\Omega_l)\delta(t)}. \tag{A5}$$

We now proceed to show the contradicting statement. Let us assume we can equate Eqs. (9) and (10) and furthermore assume that the power series in Eq. (8) is convergent in the neighborhood of either t_1 or t_2. Since Eq. (A5) holds in the case of Eq. (9), and Eq. (A4) holds in the case of Eq. (10), we find by equating these equations for all $m \neq l$, $m, l = 1, 2, \ldots, d_{\mathrm{A}}d_{\mathrm{G}}$:

$$e^{-it(\Omega_m-\Omega_l)\delta(t)} = \mathrm{tr}[e^{-it\hat{H}^{(l,l)}_{\mathrm{Cl}}} \rho^0_{\mathrm{Cl}} e^{it\hat{H}^{(m,m)}_{\mathrm{Cl}}}]. \tag{A6}$$

Hence, if the power-series expansion Eq. (8) holds,

$$e^{-it(\Omega_m-\Omega_l)\delta(t)} \tag{A7}$$

$$= \sum_{n,p=0}^{\infty} \mathrm{tr}\left[\frac{(-i\hat{H}^{(l,l)}_{\mathrm{Cl}})^n}{n!} \rho^0_{\mathrm{Cl}} \frac{(i\hat{H}^{(m,m)}_{\mathrm{Cl}})^p}{p!}\right] t^{n+p}. \tag{A8}$$

However, for $t \in [0, t_1]$ we have that $\delta(t) = 0$; thus, since $0 < t_1 < r$, with r the radius of convergence of the power series, for any $\tilde{t} \in (0, t_1)$, we find [94]

$$\frac{d^q}{dt^q} e^{-it(\Omega_m-\Omega_l)\delta(t)}|_{t=\tilde{t}} = 0 \quad \text{for } q \in \mathbb{N}^+. \tag{A9}$$

If we take derivatives of the rhs of Eq. (A8), evaluate at $t = \tilde{t}$, and set to zero, we find

$$\mathrm{tr}\left[\frac{(-i\hat{H}^{(l,l)}_{\mathrm{Cl}})^n}{n!} \rho^0_{\mathrm{Cl}} \frac{(i\hat{H}^{(m,m)}_{\mathrm{Cl}})^p}{p!}\right] = \delta_{0,n}\delta_{0,p}, \tag{A10}$$

where $\delta_{n,p}$ denotes the Kronecker delta function. Yet if we plug this solution into the rhs of Eq. (A8), we find a contradiction for $t \in [t_2, r) \neq \emptyset$.

2. Proof of Theorem 1

In this section we prove Theorem 1. We also need the results from Appendix Secs. A 1–A 5 to aid the proof.

The below theorem is a slightly more general version of Theorem 1 in three ways:

(1) In the below theorem, no time dependence is assumed, since while it is physically reasonable to do so, it is not necessary from a mathematical perspective to prove our theorem.

(2) We denote by ρ^0_{Cat} a generic catalyst of dimension D_{Cat}. To achieve the version of Theorem 1 in the main text, one makes the identification ρ^0_{Cat} in the below theorem with $\rho^0_{\mathrm{Cat}} \otimes \rho^0_{\mathrm{Cl}}$ in Theorem 1 and letting $D_{\mathrm{Cat}} = d_{\mathrm{Cat}}d_{\mathrm{Cl}}$. The motivation for this relabeling is that for the purposes of this proof, there is no point in distinguishing between the clock catalyst (which controls the interaction) and the other catalyst, which allows for thermodynamic transitions, which would otherwise not be permitted under TOs. In other words, it is only in later theorems that we care about actual dynamics where the distinction between the two types of catalysts is important.

(3) The bound on $\epsilon_{\mathrm{res}}(\epsilon_{\mathrm{emb}}, d_{\mathrm{S}}, D_{\mathrm{Cat}})$ in Eq. (A14) is a more general version than that stated in Theorem 1

in the main text. A proof that Eq. (A14) implies the version stated in the main text can be found in Corollary 33.

Theorem 5. [Sufficient conditions for implementing CNOs] Consider arbitrary initial state ρ_S^0 of not full rank and arbitrary catalyst ρ_{Cat}^0. Consider arbitrary unitary V_{SCatG} over $\rho_S^0\otimes\rho_{Cat}^0\otimes\tilde\tau_G$, and suppose that the final state $\rho_{SCatG}^F = V_{SCatG}(\rho_S^0\otimes\rho_{Cat}^0\otimes\tilde\tau_G)V_{SCatG}^\dagger$ satisfies

$$\|\rho_{SCat}^F - \rho_S^F\otimes\rho_{Cat}^0\|_1 \le \epsilon_{emb}. \tag{A11}$$

Then there exists a state σ_S which is close to ρ_S^F,

$$\|\sigma_S - \rho_S^F\|_1 \le \epsilon_{res}, \tag{A12}$$

such that

$$\rho_S^0\otimes\tilde\rho_{Cat} \succ \sigma_S\otimes\tilde\rho_{Cat} \tag{A13}$$

for some finite-dimensional catalyst $\tilde\rho_{Cat}$. Here, $\epsilon_{res} = \epsilon_{res}(\epsilon_{emb}, d_S, D_{Cat})$, where d_S, D_{Cat} are the dimensions of system ρ_S^0 and catalyst ρ_{Cat}^0, respectively. Specifically,

$$\epsilon_{res}(\epsilon_{emb}, d_S, D_{Cat}) = 5\sqrt{\frac{d_S^{5/3} + 4(\ln d_S D_{Cat})\ln d_S}{\ln(1/\epsilon_{emb})} + d_S D_{Cat}\epsilon_{emb}^{1/6} + 5\left((d_S D_{Cat})^2\sqrt{\frac{\epsilon_{emb}}{d_S D_{Cat}}}\ln\sqrt{\frac{d_S D_{Cat}}{\epsilon_{emb}}}\right)^{\frac{2}{3}}}. \tag{A14}$$

Explicitly, one possible choice for σ_S is

$$\sigma_S = \begin{cases} \mathbb{1}_S/d_S & \text{if } \|\rho_S^F - \mathbb{1}_S/d_S\|_1 < \epsilon_{res}, \\ (1-\epsilon_{res})\rho_S^F + \epsilon_{res}\mathbb{1}_S/d_S & \text{if } \|\rho_S^F - \mathbb{1}_S/d_S\|_1 \ge \epsilon_{res}. \end{cases} \tag{A15}$$

a. Overview of the proof

We show that catalytic majorization holds by using the Klimesh conditions given in Theorem 6. Since we assume that the initial state is not of full rank, and the final state σ_S by definition is of full rank, it is enough to show that for $\alpha > 0$, $g_\alpha(\rho_S^0)$ is strictly larger than $g_\alpha(\sigma_S)$. In terms of simpler functions f_α given by Eq. (A4), we need to show that

$$f_\alpha(\rho_S^0) > f_\alpha(\sigma_S) \quad \text{for } \alpha > 1,$$
$$f_\alpha(\rho_S^0) < f_\alpha(\sigma_S) \quad \text{for } \alpha\in(0,1]. \tag{A16}$$

In particular f_1 is the Shannon entropy, so the condition for $\alpha = 1$ can also be written as

$$S_1(\rho_S^0) < S_1(\sigma_S). \tag{A17}$$

There are other equivalent ways of writing the conditions using the Tsallis-Aczel-Daroczy entropy (in short, Tsallis entropy) defined in Eq. (A6), or Renyi entropy of Definition 8,

$$T_\alpha(\rho_S^0) < T_\alpha(\sigma_S) \quad \text{for } \alpha > 0, \tag{A18}$$
$$S_\alpha(\rho_S^0) < S_\alpha(\sigma_S) \quad \text{for } \alpha > 0. \tag{A19}$$

It is in one way more convenient than the condition in terms of f_α. Namely, the case $\alpha = 1$ is not given by a separate formula. Indeed, $T_1 = \lim_{\alpha\to1} T_\alpha$ (the same for S_α). Note here that for each single α, the inequality with Renyi entropy S_α is equivalent to inequality for T_α. Thus, for some α's we may show the inequality for T_α while for others for S_α. Now, let us sketch how we approach this problem.

(1) Showing the inequalities (A18) for states ρ_S^0 and ρ_S^F up to term η_α.

By assumption, the initial state $\rho_S^0\otimes\rho_{Cat}^0\otimes\rho_G^0$ is unitarily transformed into ρ_{SCatG}^F. This transformation does not change functions like f_α, T_α, S_α. Then, going back and forth between f_α's and T_α's, and using the continuity of T_α from Theorem 7, we obtain

$$T_\alpha(\rho_S^0) \le T_\alpha(\rho_S^F) + \eta_\alpha D_{Cat}^\alpha \quad \text{for } \alpha > 0 \tag{A20}$$

with η_α satisfying

$$\eta_\alpha \ge 6D\left(\frac{\epsilon_{emb}}{D}\right)^\alpha \quad \text{for } \alpha\in(0,1/2], \tag{A21}$$

$$\eta_\alpha \ge -32D\sqrt{\frac{\epsilon_{emb}}{D}}\ln\sqrt{\frac{\epsilon_{emb}}{D}} \quad \text{for } \epsilon_{emb}\le\frac{1}{32D^2},$$
$$\alpha\in\left(\frac{1}{2},2\right),$$
$$\eta_\alpha \ge 6\sqrt{D\epsilon_{emb}} \quad \text{for } \alpha\in[2,\infty), \tag{A22}$$

where $D = d_S D_{Cat}$. Anticipating that there may be problems with α around ∞ (this becomes clear later) we also obtain a similar inequality for the Renyi S_∞ entropy:

$$S_\infty(\rho_S^0) \le S_\infty(\rho_S^F) + \eta_\infty \tag{A23}$$

with

$$\eta_\infty = D_{Cat}\epsilon_{emb}. \tag{A24}$$

(2) Removing term η_α by replacing ρ_S^F with its approximated version σ_S.

The inequalities (A20) are not yet satisfactory, since we need strict inequalities, while the above ones are not only not strict, but also there are terms η_α. Fortunately, we want to show the strict inequality not for the state ρ_S^F itself, but for its approximated version σ_S. The state σ_S is just ρ_S^F with admixture of the maximally mixed state when it is far from it, and it is just the maximally mixed state when it is close to it.

The idea now is to show that due to this admixture, σ_S will have larger values of entropies than ρ_S^F by such an amount that it will allow one to bypass the η's and obtain the needed strict inequalities. A crucial step is done in Proposition 19, where for $\epsilon_{\rm emb} \le (1/32D^2)$ we obtain the following inequalities:

$$T_\alpha(\rho_S^0) < T_\alpha(\sigma_S^F(\tilde{\epsilon}_T(\alpha))) \quad \text{for } \alpha > 0, \tag{A25}$$

where

$$\tilde{\epsilon}_T(\alpha) \le \begin{cases} (96D\frac{\epsilon_{\rm emb}^\alpha}{\alpha})^{\frac{1}{3}} =: \bar{\epsilon}_{T\,\min}(\alpha) & \text{for } \alpha \in (0, 1/2], \\ (-1024D^2\sqrt{\frac{\epsilon_{\rm emb}}{D}}\ln\sqrt{\frac{\epsilon_{\rm emb}}{D}})^{\frac{1}{3}} =: \bar{\epsilon}_{T\rm mid} & \text{for all } \alpha \in (1/2, 2], \\ (96\sqrt{D\epsilon_{\rm emb}}D^\alpha)^{\frac{1}{3}} =: \bar{\epsilon}_{T\,\max}(\alpha) & \text{for } \alpha \in (2, \infty). \end{cases} \tag{A26}$$

It may appear like the end of the story is near. We need just to choose some $\epsilon_{\rm res}$ which is larger than all of the three values above. The Tsallis entropies on the rhs will then just grow (as the entropies grow when we increase the admixture with identity or if we replace with identity; see Lemma 20); hence, the inequalities will be still satisfied. Thus, for so-chosen $\epsilon_{\rm res}$ we obtain Eq. (A18) [where recall that σ_F depends on $\epsilon_{\rm res}$ as in Eq. (A15)]. However, there is a problem with α around 0 and around ∞. For the α's, the above bounds for $\tilde{\epsilon}_T$ become large, while we want them to tend to zero for $\epsilon_{\rm emb}$ going to zero. In other words, we do not have a uniform bound for $\tilde{\epsilon}_T$ for all α's at the moment.

For α lying in those regions, we turn to Renyi entropies and prove inequality (A19) rather than Eq. (A18). To deal with large values of α, we use Eq. (A23) in conjunction with Proposition 19 to show that for $\alpha > 1$,

$$S_\alpha(\rho_S^0) < S_\alpha(\sigma_S^F(\epsilon_\infty(\alpha))) \tag{A27}$$

with

$$\epsilon_\infty(\alpha) \le 4\sqrt{\frac{\ln d_S}{\alpha} + D\epsilon_{\rm emb}} =: \bar{\epsilon}_\infty(\alpha). \tag{A28}$$

To deal with values of α around zero, we prove in Proposition 19 that for $\alpha \in (0, 1)$,

$$S_\alpha(\rho_S^0) < S_\alpha(\sigma_S^F(\epsilon_0(\alpha))) \tag{A29}$$

for

$$\epsilon_0(\alpha) \le \left(\frac{d_S - 1}{d_S}\right)^{\frac{1}{2\alpha}} =: \bar{\epsilon}_0(\alpha). \tag{A30}$$

In this latter case, we use our assumption that the rank of the initial state ρ_S^0 is not full. Note that the above ϵ's behave reasonably for large (or small) values of α. The $\bar{\epsilon}_0(\alpha)$ goes to zero as α goes to zero, and $\bar{\epsilon}_\infty$ tends to $4\sqrt{D\epsilon_{\rm emb}}$. We then choose some $\alpha_{\min}$ and $\alpha_{\max}$, and below $\alpha_{\min}$ as well as above $\alpha_{\max}$ we use the inequalities for Renyi entropies (A19), while between $\alpha_{\min}$ and $\alpha_{\max}$, we use inequalities for Tallis entropy (A18). The rest of the proof is to choose $\alpha_{\min}$ and $\alpha_{\max}$ in such a way that the resulting common bound $\epsilon_{\rm res}$ for all five types of ϵ's [i.e., three coming from Eq. (A26) and the other two from Eqs. (A28) and (A30)] is the smallest possible. Finally, one may ask why we do not use the Renyi entropy everywhere. We do not use it because it is easier to deal with Tsallis entropies for this region of α in Proposition 19.

Now we are ready to present the full proof of the theorem, with most of the technical lemmas relegated to the Supplemental Material [33].

Proof.—Since ρ_S^0 is not of full rank, and the final state σ_S is by definition of full rank, we need only to consider Klimesh conditions from Theorem 6 for $\alpha > 0$. Consider first $\alpha > 0$, $\alpha \neq 1$. If for some unitary U we have

$$U\rho_S^0 \otimes \rho_{\rm Cat}^0 \otimes \tilde{\tau}_G U^\dagger = \rho_{\rm SCatG}^F, \tag{A31}$$

then

$$f_\alpha(\rho_S^0 \otimes \rho_{Cat}^0 \otimes \tilde{\tau}_G) = f_\alpha(\rho_{SCatG}^F), \qquad (A32)$$

where f_α is defined in Appendix Sec. A 1. Because of convexity or concavity of f_α and their multiplicativity, by Lemma 11, putting A = SCat and $B = \mathrm{G}$, we obtain

$$f_\alpha(\rho_S^0 \otimes \rho_{Cat}^0) \geq f_\alpha(\rho_{SCat}^F), \quad \text{for } \alpha > 1,$$
$$f_\alpha(\rho_S^0 \otimes \rho_{Cat}^0) \leq f_\alpha(\rho_{SCat}^F), \quad \text{for } \alpha \in (0,1). \qquad (A33)$$

This implies, by the definition of Tsallis entropy T_α [Eq. (A6)], that for all $\alpha > 0$, $\alpha \neq 1$ we have

$$T_\alpha(\rho_S^0 \otimes \rho_{Cat}^0) \leq T_\alpha(\rho_{SCat}^F). \qquad (A34)$$

We now use $\|\rho_{SCat}^F - \rho_S^F \otimes \rho_{Cat}^0\|_1 \leq \epsilon_{emb}$ and the continuity Lemma 21 to find for $\alpha > 0$,

$$T_\alpha(\rho_{SCat}^F) \leq T_\alpha(\rho_S^F \otimes \rho_{Cat}^0) + \eta_\alpha \qquad (A35)$$

for all η_α satisfying

$$\eta_\alpha \geq 6D\left(\frac{\epsilon_{emb}}{D}\right)^\alpha \quad \text{for } \alpha \in (0, 1/2], \qquad (A36)$$

$$\eta_\alpha \geq -32D\sqrt{\frac{\epsilon_{emb}}{D}} \ln \sqrt{\frac{\epsilon_{emb}}{D}}$$
$$\text{for } \epsilon_{emb} \leq \frac{1}{32D^2}, \qquad \alpha \in \left(\frac{1}{2}, 2\right), \qquad (A37)$$

$$\eta_\alpha \geq 6\sqrt{D\epsilon_{emb}} \quad \text{for } \alpha \in [2, \infty), \qquad (A38)$$

where $D = d_S D_{Cat}$. We rewrite the above equation back in terms of functions f_α, which gives

$$f_\alpha(\rho_{SCat}^F) \geq f_\alpha(\rho_S^F \otimes \rho_{Cat}^0) - (\alpha - 1)\eta_\alpha \quad \text{for } \alpha > 1,$$
$$f_\alpha(\rho_{SCat}^F) \leq f_\alpha(\rho_S^F \otimes \rho_{Cat}^0) - (\alpha - 1)\eta_\alpha \quad \text{for } \alpha \in (0,1). \qquad (A39)$$

Then, by using Eq. (A33) followed by the multiplicativity of the f_α's, we obtain from the above equations

$$f_\alpha(\rho_S^0) \geq f_\alpha(\rho_S^F) - \frac{(\alpha - 1)}{f_\alpha(\rho_{Cat}^0)}\eta_\alpha \quad \text{for } \alpha > 1, \qquad (A40)$$

$$f_\alpha(\rho_S^0) \leq f_\alpha(\rho_S^F) - \frac{(\alpha - 1)}{f_\alpha(\rho_{Cat}^0)}\eta_\alpha \quad \text{for } \alpha \in (0,1). \qquad (A41)$$

Finally, using $f_\alpha(p) \geq d^{1-\alpha}$ for $\alpha > 1$ and $f_\alpha(p) \geq 1$ for $\alpha \in (0,1)$ [these inequalities follow from setting $r = 1$, $p = \alpha$, and $r = \alpha$, $p = 1$, respectively, in Eq. (A268) in Lemma (29)] rewriting back in terms of T_α's we obtain

$$T_\alpha(\rho_S^0) \leq T_\alpha(\rho_S^F) + \eta_\alpha D_{Cat}^{\alpha-1} \quad \text{for } \alpha \geq 1, \qquad (A42)$$

$$T_\alpha(\rho_S^0) \leq T_\alpha(\rho_S^F) + \eta_\alpha \quad \text{for } \alpha \in (0,1). \qquad (A43)$$

Here we include the case $\alpha = 1$, which is obtained by taking the limit $\alpha \to 1$ [95]. We can somewhat crudely unify this equation into

$$T_\alpha(\rho_S^0) \leq T_\alpha(\rho_S^F) + \eta_\alpha D_{Cat}^\alpha \quad \text{for } \alpha > 0. \qquad (A44)$$

Furthermore, Eq. (A33) implies that for $\alpha > 1$,

$$S_\alpha(\rho_S^0 \otimes \rho_{Cat}^0) \leq S_\alpha(\rho_{SCat}^F), \qquad (A45)$$

and by taking limit $\alpha \to \infty$, we get

$$S_\infty(\rho_S^0 \otimes \rho_{Cat}^0) \leq S_\infty(\rho_{SCat}^F), \qquad (A46)$$

which by Lemma 21 and additivity of S_∞ gives

$$S_\infty(\rho_S^0) \leq S_\infty(\rho_S^F) + \eta_\infty, \qquad (A47)$$

where

$$\eta_\infty = D_{Cat}\epsilon_{emb}. \qquad (A48)$$

Let us now define as in Proposition 19

$$\sigma_S^F(\epsilon) = \begin{cases} \mathbb{1}_S/d_S & \text{when } \|\rho_S^F - \mathbb{1}_S/d_S\|_1 < \epsilon, \\ (1-\epsilon)\rho_S^F + \epsilon \mathbb{1}_S/d_S & \text{when } \|\rho_S^F - \mathbb{1}_S/d_S\|_1 \geq \epsilon. \end{cases} \qquad (A49)$$

Equations (A44) and (A47) by using Proposition 19 lead to the following conclusion: For

$$\tilde{\epsilon}_T(\alpha) = \begin{cases} (16\eta_\alpha D_{Cat}^\alpha d_S^{\alpha-1})^{\frac{1}{3}} & \text{for } \alpha \geq 1, \\ (16\eta_\alpha D_{Cat}^\alpha d_S^{\alpha-1}\alpha^{-1})^{\frac{1}{3}} & \text{for } \alpha \in (0,1), \end{cases} \qquad (A50)$$

$$\epsilon_\infty(\alpha) = 4\sqrt{\frac{\ln d_S}{\alpha} + \eta_\infty} \quad \text{for } \alpha > 1, \qquad (A51)$$

$$\epsilon_0(\alpha) = \left(1 - \frac{1}{d_S}\right)^{\frac{1}{2\alpha}} \quad \text{for } \alpha \in (0,1), \qquad (A52)$$

we have

$$T_\alpha(\rho_S^0) \leq T_\alpha(\sigma_S^F(\tilde{\epsilon}_T(\alpha)))$$
$$- \min\{D_{Cat}^\alpha \eta_\alpha, T_\alpha(\mathbb{1}/d_S) - T_\alpha(\rho_S^0)\} \quad \text{for } \alpha > 0, \qquad (A53)$$

$$S_\alpha(\rho_S^0) \leq S_\alpha(\sigma_S^F(\epsilon_\infty(\alpha))) - \min\{D_{Cat}^\alpha \eta_\alpha, \ln d_S - S_1(\rho_S^0)\}$$
$$\text{for } \alpha > 1, \qquad (A54)$$

$$S_\alpha(\rho_S^0) \le S_\alpha(\sigma_S^F(\epsilon_0(\alpha))) \tag{A55}$$

$$-\frac{1}{2}\ln\left(\frac{d_S}{d_S-1}\right) \quad \text{for } \alpha \in (0,1), \tag{A56}$$

from which we achieve

$$T_\alpha(\rho_S^0) < T_\alpha(\sigma_S^F(\tilde{\epsilon}_T(\alpha))) \quad \text{for } \alpha > 0, \tag{A57}$$

$$S_\alpha(\rho_S^0) < S_\alpha(\sigma_S^F(\epsilon_\infty(\alpha))) \quad \text{for } \alpha > 1, \tag{A58}$$

$$S_\alpha(\rho_S^0) < S_\alpha(\sigma_S^F(\epsilon_0(\alpha))) \quad \text{for } \alpha \in (0,1). \tag{A59}$$

Let us now insert explicitly the η's from Eqs. (A36)–(A38) and Eq. (A48) into Eqs. (A50)–(A52). For

$$\epsilon_{\text{emb}} \le \frac{1}{32D^2}, \tag{A60}$$

we achieve the upper bounds

$$\tilde{\epsilon}_T(\alpha) \le \left(96D\frac{\epsilon_{\text{emb}}^\alpha}{\alpha}\right)^{\frac{1}{3}} =: \bar{\epsilon}_{T\,\min}(\alpha) \quad \text{for } \alpha \in (0, 1/2], \tag{A61}$$

$$\left(-1024D^2\sqrt{\frac{\epsilon_{\text{emb}}}{D}}\ln\sqrt{\frac{\epsilon_{\text{emb}}}{D}}\right)^{\frac{1}{3}} =: \bar{\epsilon}_{T\text{mid}} \quad \text{for } \alpha \in (1/2, 2], \tag{A62}$$

$$(96\sqrt{D\epsilon_{\text{emb}}}D^\alpha)^{\frac{1}{3}} =: \bar{\epsilon}_{T\,\max}(\alpha) \quad \text{for } \alpha \in (2,\infty), \tag{A63}$$

$$\epsilon_\infty(\alpha) \le 4\sqrt{\frac{\ln d_S}{\alpha} + D\epsilon_{\text{emb}}} =: \bar{\epsilon}_\infty(\alpha) \quad \text{for } \alpha \in [1,\infty), \tag{A64}$$

$$\epsilon_0(\alpha) \le \left(\frac{d_S-1}{d_S}\right)^{\frac{1}{2\alpha}} =: \bar{\epsilon}_0(\alpha) \quad \text{for } \alpha \in (0,1), \tag{A65}$$

where $D = d_S D_{\text{Cat}}$. We now divide the set $(0,\infty)$ into five subintervals (some of which may be empty). For $\alpha_{\min} \in (0,1)$, $\alpha_{\max} \in [2,\infty)$, we have $(0,\infty) = (0,\alpha_{\min}] \cup (\alpha_{\min}, 1/2] \cup (1/2, 2] \cup [2, \alpha_{\max}] \cup (\alpha_{\max}, \infty)$. For each of these intervals, we compute upper bounds on our epsilons. Specifically, from Eqs. (A61), (A64), and (A65), we observe that

$$\epsilon_0(\alpha) \le \bar{\epsilon}_0(\alpha_{\min}) \quad \forall\, \alpha \in (0,\alpha_{\min}), \quad \forall\, \alpha_{\min} \in (0,1), \tag{A66}$$

$$\tilde{\epsilon}_T(\alpha) \le \begin{cases} \bar{\epsilon}_{T\,\min}(\alpha_{\min}) & \forall\, \alpha \in (\alpha_{\min}, 1/2],\ \forall\, \alpha_{\min} \in (0,1/2], \\ \bar{\epsilon}_{T\text{mid}} & \forall\, \alpha \in (1/2, 2], \\ \bar{\epsilon}_{T\,\max}(\alpha_{\max}) & \forall\, \alpha \in [2,\alpha_{\max}],\ \forall\, \alpha_{\max} \in [2,\infty), \end{cases} \tag{A67}$$

$$\epsilon_\infty(\alpha) \le \bar{\epsilon}_\infty(\alpha_{\max}) \quad \forall\, \alpha \in (\alpha_{\max},\infty), \quad \forall\, \alpha_{\max} \in [1,\infty). \tag{A68}$$

Now we define ϵ_{res} as any value satisfying

$$\epsilon_{\text{res}}(\alpha_{\min},\alpha_{\max}) \ge \max\{\varepsilon_{\min}(\alpha_{\min}), \varepsilon_{\max}(\alpha_{\max}), \bar{\epsilon}_{T\text{mid}}\}, \tag{A69}$$

where

$$\varepsilon_{\min}(\alpha_{\min}) = \max\{\bar{\epsilon}_{T\,\min}(\alpha_{\min}), \bar{\epsilon}_0(\alpha_{\min})\}, \tag{A70}$$

$$\varepsilon_{\max}(\alpha_{\max}) = \max\{\bar{\epsilon}_{T\,\max}(\alpha_{\max}), \bar{\epsilon}_\infty(\alpha_{\max})\}. \tag{A71}$$

Thus, using Lemma 20, we have

$$S_\alpha(\rho_S^0) < S_\alpha(\sigma_S^F(\bar{\epsilon}_0(\alpha_{\min}))) < S_\alpha(\sigma_S^F(\epsilon_{\text{res}}(\alpha_{\min},\alpha_{\max}))) \quad \forall\, \alpha \in (0,\alpha_{\min}), \tag{A72}$$

$$T_\alpha(\rho_S^0) < T_\alpha(\sigma_S^F(\bar{\epsilon}_{T\,\min}(\alpha_{\min}))) < T_\alpha(\sigma_S^F(\epsilon_{\text{res}}(\alpha_{\min},\alpha_{\max}))) \quad \forall\, \alpha \in (\alpha_{\min}, 1/2), \tag{A73}$$

$$T_\alpha(\rho_S^0) < T_\alpha(\sigma_S^F(\bar{\epsilon}_{T\text{mid}})) < T_\alpha(\sigma_S^F(\epsilon_{\text{res}}(\alpha_{\min},\alpha_{\max}))) \quad \forall\, \alpha \in [1/2, 2], \tag{A74}$$

$$T_\alpha(\rho_S^0) < T_\alpha(\sigma_S^F(\bar{\epsilon}_{T\max}(\alpha_{\max}))) < T_\alpha(\sigma_S^F(\epsilon_{\text{res}}(\alpha_{\min},\alpha_{\max}))) \quad \forall\, \alpha \in [2, \alpha_{\max}), \tag{A75}$$

$$S_\alpha(\rho_S^0) < S_\alpha(\sigma_S^F(\bar{\epsilon}_{\infty}(\alpha_{\max}))) < S_\alpha(\sigma_S^F(\epsilon_{\text{res}}(\alpha_{\min},\alpha_{\max}))) \quad \forall\, \alpha \in [\alpha_{\max}, \infty) \tag{A76}$$

holds for all $\alpha_{\min} \in (0,1)$, $\alpha_{\max} \in (2,\infty)$ [96].

Thus, for any particular choice of $\alpha_{\min} \in (0,1)$ and $\alpha_{\max} \in (2,\infty)$, $\epsilon_{\text{res}}(\alpha_{\min},\alpha_{\max})$ is such that the Klimesh conditions are satisfied, so that for any ρ_S^0 there exists catalyst $\tilde{\rho}_{\text{Cat}}$ such that

$$\rho_S^0 \otimes \tilde{\rho}_{\text{Cat}} \succ \sigma_S^F(\epsilon_{\text{res}}) \otimes \tilde{\rho}_{\text{Cat}}. \tag{A77}$$

Our next aim is to find an explicit expression for $\epsilon_{\text{res}}(\alpha_{\min},\alpha_{\max})$ with the aim of choosing the parameters $\alpha_{\min}$, $\alpha_{\max}$, so that $\epsilon_{\text{res}}(\alpha_{\min},\alpha_{\max})$ is not too large. In Lemma 22, we show that the ϵ_{res} given in the statement of the theorem upper bounds $\epsilon_{\text{res}}(\alpha_{\min},\alpha_{\max})$ for some $\alpha_{\min}$ and $\alpha_{\max}$. This finalizes the proof. ■

To see how to write Theorem 5 in the form of Theorem 1, see Corollary 33 in the Supplemental Material [33].

3. Introduction to the quasi-ideal clock and proof of Theorem 2

In the following subsection, we start with a brief overview of the properties of the quasi-ideal clock. These are necessary for the proof of Theorem 2, which is in Appendix Sec. A 4 a.

a. Brief overview of the quasi-ideal clock

In this section, we recall the clock construction from Ref. [67] which is subsequently used to prove Proposition 4, which in turn leads to the proof of Theorem 2.

The time-independent total Hamiltonian over system $\rho_A \otimes \rho_{Cl}$ is

$$\hat{H}_{ACl} = \hat{H}_A \otimes \mathbb{1}_{Cl} + \hat{H}_A^{\text{int}} \otimes \hat{V}_d + \mathbb{1}_A \otimes \hat{H}_{Cl}, \tag{A78}$$

where $\hat{H}_A$ is the system Hamiltonian which commutes with the target unitary U_A^{target}. The term $\hat{H}_A^{\text{int}}$ encodes the target unitary via the relation $U_A^{\text{target}} = e^{-i\hat{H}_A^{\text{int}}}$ with

$$H_A^{\text{int}} = \sum_{n=1}^{d_A} \Omega_n |n\rangle\langle n|. \tag{A79}$$

The clock's free Hamiltonian $\hat{H}_{Cl}$ is a truncated harmonic oscillator Hamiltonian. Namely, $\hat{H}_{Cl} = \sum_{n=0}^{d-1} \omega n |n\rangle\langle n|$. The free evolution of any initial clock state under this Hamiltonian has a period of $T_0 = 2\pi/\omega$, specifically, $e^{-iT_0\hat{H}_{Cl}}\rho_{Cl}e^{iT_0\hat{H}_{Cl}} = \rho_{Cl}$ for all ρ_{Cl}. The clock interaction term $\hat{V}_d$ takes the form

$$\hat{V}_d = \frac{d}{T_0}\sum_{k=0}^{d-1} V_d(k)|\theta_k\rangle\langle\theta_k|, \tag{A80}$$

where the basis $\{|\theta_k\rangle\}_{k=0}^{d-1}$ is the Fourier transform of the energy eigenbasis $\{|n\rangle\}_{n=0}^{d-1}$. The function $V_d : \mathbb{R} \mapsto \mathbb{R}$ will be called potential and is defined by

$$V_d(y) = \frac{2\pi}{d} V_0\left(\frac{2\pi}{d}(y - y_0)\right), \tag{A81}$$

where V_0 is an infinitely differentiable periodic function of period 2π centered on 0 (so that V_d has period d and is centered on y_0). A lot of results hold for this general form of potential. To obtain all the results, we need a specialized form of potential given by

$$V_0(x) = A_c \cos^{2n}\left(\frac{x}{2}\right) \quad \text{with} \quad A_c = \frac{2^{2n}}{2\pi\binom{2n}{n}}, \tag{A82}$$

and where n is later taken to be a suitable function of the clock dimension (specifically, later we take $n \sim d^{1/4}$). Here, A_c is a normalization constant so that $\int_{-\pi}^{\pi} V_0(x)dx = 1$. It is important that V_0 has exponentially decaying tails

$$\tilde{\epsilon}_V \coloneqq \int_{-2\pi(1-\delta_V)}^{-2\pi\delta_V} V_0(x)dx \le \frac{1}{\delta_V} e^{-\delta_V^2 n} \quad \text{for } \delta_V \in (0,\pi). \tag{A83}$$

The bound in Ref. [67] is tighter and does not diverge as $\delta_V \to 0^+$, but the present one is just enough, as we care just about scaling for the proof anyway (see Lemma 36).

Recall that for the quasi-ideal clock, the initial state is pure $\rho_{Cl} = |\Psi_{\text{nor}}(k_0)\rangle\langle\Psi_{\text{nor}}(k_0)|$, where

$$|\Psi_{\text{nor}}(k_0)\rangle = \sum_{k\in\mathcal{S}_d(k_0)} \psi(k_0;k)|\theta_k\rangle, \tag{A84}$$

$$\psi(k_0;x) = A e^{-\frac{\pi}{\sigma^2}(x-k_0)^2} e^{i2\pi n_0(x-k_0)/d}, \qquad x \in \mathbb{R} \tag{A85}$$

with $\sigma \in (0,d)$, $n_0 \in (0, d-1)$, $k_0 \in \mathbb{R}$, $A \in \mathbb{R}^+$, and $\mathcal{S}_d(k_0)$ is the set of d integers closest to k_0 defined as

$$\mathcal{S}_d(k_0) = \left\{k : k \in \mathbb{Z} \text{ and } -\frac{d}{2} \le k_0 - k < \frac{d}{2}\right\}. \tag{A86}$$

Note that for k larger than $d-1$ or smaller than 0, we define θ_k as $\theta_{k \bmod d}$. The quantity A is defined so that the state is normalized, namely,

$$A = A(\sigma; k_0) = \frac{1}{\sqrt{\sum_{k\in\mathcal{S}_d(k_0)} e^{-\frac{2\pi}{\sigma^2}(k-k_0)^2}}} = \mathcal{O}\left(\left(\frac{2}{\sigma^2}\right)^{\frac{1}{4}}\right). \tag{A87}$$

The parameter n_0 is approximately the mean energy of the clock, and for good clock performance, it should be not too close to 0 or to d. We later set it to $(d-1)/2$ as suggested in Ref. [67]; see Definition 1.

4. Small error on the clock

Here we prove a proposition that is crucial to prove Theorem 2. The proposition states that for the clock described above, the state of the clock acquires a small error.

Proposition 4. Consider the quasi-ideal clock described above. Consider times t_1, t_2 satisfying $0 < t_1 < t_2 < T_0$. Then for the potential V_d determined by Eqs. (A81) and (A82) with $y_0 = (t_1 - t_2)d/T_0$ and $n = \lceil d^{1/4} \rceil$, we have

$$\|\rho^F_{\mathrm{Cl}}(t) - \rho^0_{\mathrm{Cl}}(t)\|_1 \leq \frac{1}{t_2 - t_1}\mathrm{poly}(d)e^{-c_2 d^{1/4}}, \tag{A88}$$

where $c_2 = \min\{\frac{1}{64\pi}, (t_2 - t_1)^2/(32T_0^2)\}$.

Remark 5.—The unbounded from above factor $1/(t_2 - t_1)$ is not necessary, and in the original paper [67] it did not appear. Here it is the price for a simpler proof of potential concentration properties.

Proof.—Let us set arbitrary times t_1 and t_2. We want to show that we can choose the potential in the clock described in Appendix Sec. A 3 a so that, apart from times near the boundaries (i.e., those not satisfying $0 < t_1 < t_2 < T_0$), it will be close to the free evolution of the clock state. In other words, the evolution may be different in the "interaction zones." The potential has been already determined with two free parameters: the peak position y_0 and n determining the concentration of the potential around the peak. As we argue later, the clock state (we call it pointer) will approximately travel around the circle with linear speed d/T_0, so that to times t_1 and t_2 there correspond positions $y_1 = t_1 d/T_0$ and $y_2 = t_2 d/T_0$ (see Fig. 5). Since the interaction can take place on the interval where the potential is non-negligible, we aim to have the potential concentrated in the area between y_1 and y_2. To this end, we choose the peak of the potential to be in the middle between y_1 and y_2:

$$y_0 = \frac{y_1 + y_2}{2} = \frac{t_2 + t_2}{2}\frac{d}{T_0}. \tag{A89}$$

The concentration parameter n is chosen later. With such a potential, we want to estimate the following quantity:

$$\|\rho^F_{\mathrm{Cl}}(t) - \rho^0_{\mathrm{Cl}}(t)\|_1. \tag{A90}$$

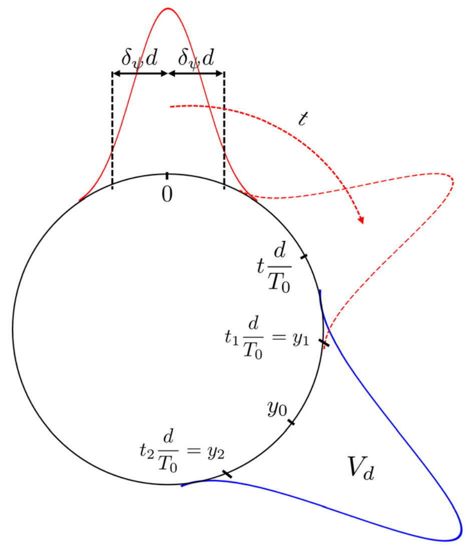

FIG. 5. Dynamics of the clock. The circumference of the circle is d. The red profile represents the amplitudes of the clock state (called pointer) with the weight concentrated within $\pm\delta_\psi d$ from the center. It moves around approximately with speed d/T_0. The potential V_d has peak at y_0. The positions y_1 and y_2 are determined by the times t_1 and t_2 and denote places which the peak of the clock state will reach at times t_1 and t_2.

To this end, we evaluate the fidelity and use $\|\rho - \sigma\| \leq 2\sqrt{1 - F(\rho,\sigma)^2}$. In Ref. [67] (see the proof of Lemma 10.0.3, page 192) after a bit of algebra, the following is obtained:

$$\rho_{\mathrm{Cl}}(t) = \sum_{n=1}^{d_{\mathrm{A}}} \rho_{n,n}(0)|\bar{\Phi}_n(t)\rangle\langle\bar{\Phi}_n(t)|_{\mathrm{Cl}}, \tag{A91}$$

where $\{\rho_{n,n}(0)\}_n$ are the eigenvalues of the initial system state ρ_{A}, and thus also constitute a set of normalized probabilities. $|\bar{\Phi}_n(t)\rangle_{\mathrm{Cl}}$ is defined by

$$|\bar{\Phi}_n(t)\rangle_{\mathrm{Cl}} = \hat{\Gamma}_n(t)|\Psi_{\mathrm{nor}}(k_0)\rangle_{\mathrm{Cl}}, \tag{A92}$$

$$\hat{\Gamma}(t,\Omega_n) = e^{-it(\Omega_n\hat{V}_d + \hat{H}_{\mathrm{Cl}})}, \tag{A93}$$

where $\{\Omega_n \in [-\pi,\pi]\}_{n=1}^{d_{\mathrm{A}}}$ is a set of phases which determine the target unitary one wishes to apply [see Eq. (A79)]. Using $F(\rho,\psi) = \langle\psi|\rho|\psi\rangle$, we get

$$F(\rho^0_{\mathrm{Cl}}(t), \rho_{\mathrm{Cl}}(t))$$
$$= \sum_{n=1}^{d_{\mathrm{Cl}}} \rho_{n,n}(0)|\langle\Psi_{\mathrm{nor}}(k_0)|e^{it\hat{H}_{\mathrm{Cl}}}\hat{\Gamma}(t,\Omega_n)|\Psi_{\mathrm{nor}}(k_0)\rangle|^2 \tag{A94}$$

$$\geq \min_{\Omega\in[-\pi,\pi]} |\langle\Psi_{\mathrm{nor}}(k_0)|e^{it\hat{H}_{\mathrm{Cl}}}\hat{\Gamma}(t,\Omega)|\Psi_{\mathrm{nor}}(k_0)\rangle|^2. \tag{A95}$$

We thus aim to show that the following states

$$e^{-it\hat{H}_{\mathrm{Cl}}}|\Psi_{\mathrm{nor}}(k_0)\rangle \quad \text{and} \quad \hat{\Gamma}(t,\Omega)|\Psi_{\mathrm{nor}}(k_0)\rangle \tag{A96}$$

have overlap close to 1, irrespective of phase Ω. To this end, we use the core theorems in Ref, [67] (Theorem VIII.1 on page 19 and Theorem IX.1 on page 35). They say that

(i) under evolution $e^{it\hat{H}_{\mathrm{Cl}}}$ the state $|\Psi_{\mathrm{nor}}(k_0)\rangle$ up to a small correction evolves in a trivial way; namely, its peak undergoes translation

(ii) under evolution $\Gamma(\Omega,t)$ the above translation occurs too, but in addition, the kth amplitude of the state acquires phase equal to the potential integrated over the interval that k traveled

More specifically, for $n_0=(d-1)/2$ (cf. Definition 2 in Ref. [67]), which means that n_0, which has the interpretation of the average energy, is not too close to 0 or to the maximal energy, we have

$$e^{-it\hat{H}_{\mathrm{Cl}}}|\Psi_{\mathrm{nor}}(k_0)\rangle \tag{A97}$$

$$=|\Psi_{\mathrm{nor}}(k_0+td/T_0)\rangle+|\varepsilon_c\rangle \tag{A98}$$

$$=\sum_{k\in\mathcal{S}_d(k_0+td/T_0)}\psi(k_0+td/T_0;k)|\theta_k\rangle+|\varepsilon_c\rangle, \tag{A99}$$

$$\Gamma(t,\Omega)|\Psi_{\mathrm{nor}}(k_0)\rangle \tag{A100}$$

$$=|\bar{\Psi}_{\mathrm{nor}}(k_0+td/T_0,td/T_0)\rangle+|\varepsilon_\nu\rangle \tag{A101}$$

$$=\sum_{k\in\mathcal{S}_d(k_0+td/T_0)}e^{-i\phi_k(t)}\psi(k_0+td/T_0;k)|\theta_k\rangle+|\varepsilon_\nu\rangle, \tag{A102}$$

where the phase acquired by the kth amplitude is given by

$$\phi_k(t)=\Omega\int_{k-td/T_0}^{k}dyV_d(y). \tag{A103}$$

Now, for $\sigma=\sqrt{d}$ and for $n=\frac{1}{2}\lceil d^{1/4}\rceil$ in the potential form of Eq. (A82) we have (see Lemma 37)

$$\||\varepsilon_\nu\rangle\|_2=:\varepsilon_\nu\lesssim t\mathrm{poly}(d)e^{-\frac{1}{16\pi}d^{1/4}},$$
$$\||\varepsilon_c\rangle\|_2=:\varepsilon_c=\mathcal{O}(\mathrm{poly}(d_{\mathrm{Cl}})e^{-\frac{\pi}{4}d_{\mathrm{Cl}}}). \tag{A104}$$

Actually, only the estimate on ε_ν depends on the potential form. The bound for ε_c holds for arbitrary periodic V_0.

Now, since for normalized $|\psi\rangle$, $|\phi\rangle$ and $|x\rangle$, $|y\rangle$ such that $\||x\rangle\|, \||y\rangle\|\le 1$, we have $|(\langle\psi|+\langle x|)(|\phi\rangle+|y\rangle)|^2\ge|\langle\psi|\phi\rangle|^2-3\||x\rangle\|-3\||y\rangle\|$, we obtain from Eq. (A94)

$$F(\rho^0_{\mathrm{Cl}}(t),\rho_{\mathrm{Cl}}(t))$$
$$\ge\min_{\Omega\in[-\pi,\pi]}|\langle\Psi_{\mathrm{nor}}(k_0+td/T_0)|\bar{\Psi}_{\mathrm{nor}}(k_0+td/T_0,td/T_0)\rangle|^2 \tag{A105}$$

$$-3\varepsilon_\nu-3\varepsilon_c. \tag{A106}$$

We thus have the situation that $\Psi_{\mathrm{nor}}(k_0+td/T_0)$ and $\bar{\Psi}_{\mathrm{nor}}(k_0+td/T_0)$ have the peak moving around with speed d/T_0, while $\bar{\Psi}_{\mathrm{nor}}$ in addition acquires phase. We now write explicitly the above inner product

$$\Delta(\Omega)\coloneqq\langle\Psi_{\mathrm{nor}}(k_0+td/T_0)|\bar{\Psi}_{\mathrm{nor}}(k_0+td/T_0,td/T_0)\rangle \tag{A107}$$

$$=\sum_{k\in\mathcal{S}_d(k_0+td/T_0)}e^{-i\Omega\int_{k-td/T_0}^{k}dyV_d(y)}|\psi_{\mathrm{nor}}(k_0+td/T_0;k)|^2, \tag{A108}$$

and the goal is to show that it is close to 1 independent of Ω for all times before t_1 and after t_2.

The idea to prove this (along the lines of Ref. [67]) is the following (see also Fig. 5). Let us denote the position of the peak of the pointer by $k_0(t)=k_0+td/T_0$. We set the initial pointer's peak to be at 12 o'clock, i.e., $k_0=0$. First, since Gaussians have rapidly decaying tails, we have (see Lemma 35)

$$\varepsilon_{\mathrm{LR}}\coloneqq\sum_{k:|k-k_0|\ge\delta_\psi d}|\psi_{\mathrm{nor}}(k_0;k)|^2\le\mathrm{poly}(d)e^{-\delta_\psi{}^2 d} \tag{A109}$$

for $\delta_\psi>0$. We can restrict the sum in Eq. (A107) and leave only $k's$ lying within the interval $k_0(t)\pm\delta_\psi d$. Since the pointer's peak travels with speed d/T_0, i.e., $k_0(t)=td/T_0$, for times "before the interaction," i.e., $t\le t_1$, the k's will be to the left of $y_1+\delta_\psi d$, and for times "after the interaction," i.e., $t\ge t_2$, they will be to the right of $y_2-\delta_\psi d$.

The potential is strongly peaked around y_0 which sits between those two positions. Thus, for times $t\le t_1$ the $k's$ are traveling within the tail of the potential, while for times $t\ge t_2$, all the k's have passed the "body" of the potential. Thus, for times $t\le t_1$, by virtue of Eq. (A103), the acquired phase for all those k's will be close to zero (less than $\Omega\tilde{\epsilon}_V$, where ϵ_V is the area of the tail), while for times $t\ge t_2$, the phase will be close to Ω [larger than $\Omega(1-\tilde{\epsilon}_V)$, since the total integral of the potential is 1].

We now write it rigorously. First of all, we cut the tails of the pointer. We split $\Delta(\Omega)$ as

$$\Delta(\Omega)=\Delta_C(\Omega)+\Delta_{\mathrm{LR}}(\Omega) \tag{A110}$$

with

$$\Delta_C(\Omega) \tag{A111}$$

$$\coloneqq\sum_{|k-td/T_0|\le\delta_\psi d}e^{-i\Omega\int_{k-td/T_0}^{k}dyV_d(y)}|\psi_{\mathrm{nor}}(td/T_0;k)|^2, \tag{A112}$$

$$\Delta_{\mathrm{LR}}(\Omega) \tag{A113}$$

$$:= \sum_{\delta_\psi d<|k-td/T_0|\le d/2} e^{-i\Omega\int_{k-td/T_0}^{k} dy V_d(y)} |\psi_{\mathrm{nor}}(td/T_0;k)|^2. \tag{A114}$$

We then have

$$|\Delta_{\mathrm{LR}}(\Omega)| \tag{A115}$$

$$\le \sum_{k:|k-k_0|\ge\delta_\psi d} |\psi_{\mathrm{nor}}(k_0;k)|^2 = \varepsilon_{\mathrm{LR}} \le \mathrm{poly}(d) e^{-\delta_\psi{}^2 d}, \tag{A116}$$

where the tail bound is, for completeness, given in Lemma 35. By the tail estimate (A115), for $\varepsilon_{\mathrm{LR}} \le 1$ we then get

$$|\Delta(\Omega)|^2 \ge |\Delta_C(\Omega)|^2 - 2\varepsilon_{\mathrm{LR}}, \tag{A117}$$

so that it is enough to show that $\Delta_C(\Omega)$ is close to 1 irrespective of Ω.

We now bound the phase $\phi_k(t)$ for our restricted set of k's.

Times before interaction: Consider time $t \le t_1$, and as said we are restricting to k such that $|k - td/T_0| \le \delta_\psi d$. This implies

$$k \le td/T_0 + \delta_\psi d \le t_1 d/T_0 + \delta_\psi d = y_1 + \delta_\psi d,$$
$$k - td/T_0 \ge -\delta_\psi d, \tag{A118}$$

which gives (see Fig. 6)

$$\phi_k(t) = \Omega \int_{k-td/T_0}^{k} V_d(y)dy \le \Omega \int_{-\delta_\psi d}^{y_1+\delta_\psi d} V_d(y)dy \tag{A119}$$

$$\le \Omega \int_{y_0+\delta_V d-d}^{y_0-\delta_V d} V_d(y)dy = \Omega \int_{-2\pi(1-\delta_V)}^{-2\pi\delta_V} V_0(x)dx = \tilde{\epsilon}_V, \tag{A120}$$

where δ_V is determined by $\delta_V d = y_0 - y_1 - \delta_\psi d$. Denoting $t_2 - t_1 = \Delta t$, we have $\delta_V + \delta_\psi = \Delta t/T_0$, and we may choose $\delta_V = \delta_\psi = \Delta t/(4T_0)$. We know from Eq. (A83) that $\tilde{\epsilon}_V$ decays exponentially in the concentration parameter n of the potential.

Times after interaction: Now we consider $t \ge t_2$. Similar to before, the condition $|k - td/T_0| \le \delta_\psi$ implies

$$k \ge td/T_0 - \delta_\psi d \ge t_2 d/T_0 - \delta_\psi d = y_2 - \delta_\psi d,$$
$$k - td/T_0 \le \delta_\psi d, \tag{A121}$$

hence,

$$\phi_k(t) = \Omega \int_{k-td/T_0}^{k} V_d(y)dy \ge \Omega \int_{\delta_\psi d}^{y_2-\delta_\psi d} V_d(y)dy \tag{A122}$$

$$\ge \Omega \int_{y_0-\delta_V d}^{y_0+\delta_V d} V_d(y)dy = \Omega \int_{-2\pi\delta_V}^{2\pi\delta_V} V_0(x)dx \tag{A123}$$

$$= \Omega(1 - \tilde{\epsilon}_V). \tag{A124}$$

Altogether, for times $t \le t_1$ and $t \ge t_2$, respectively, we obtain

$$|\phi_k - 0| \le \Omega\tilde{\epsilon}_V, \qquad |\phi_k - \Omega| \le \Omega\tilde{\epsilon}_V. \tag{A125}$$

We can now come back to the estimation of Δ_C. Denoting

$$\mathrm{A}_k = |\psi_{\mathrm{nor}}(td/T_0;k)|^2, \tag{A126}$$

we have

$$\Delta_C = \sum_{|k-td/T_0|\le\delta_\psi d} e^{-i\phi_k} \mathrm{A}_k. \tag{A127}$$

Because of normalization and the tail estimate of Eq. (A115), we have

$$1 - \varepsilon_{\mathrm{LR}} \le \sum_{|k-td/T_0|\le\delta_\psi d} \mathrm{A}_k \le 1. \tag{A128}$$

Thus, we have an expression where the A_k's are almost normalized, and the ϕ_k's almost equal to each other; hence, the expression must be close to 1. Indeed, Lemma 34 implies that

$$|\Delta_C(\Omega)| \ge 1 - \varepsilon_{\mathrm{LR}} - \pi\tilde{\epsilon}_V, \tag{A129}$$

irrespective of whether we are before or after the interaction (i.e., whether $t \le t_1$ or $t \ge t_2$) because we use only the fact that the phase is approximately equal, which happens in both cases as in Eq. (A125).

We can now come back to the fidelity. From estimates (A105), (A117), and (A129), we have

$$F(\rho^0_{\mathrm{Cl}}(t), \rho_{\mathrm{Cl}}(t)) \ge \min_{\Omega\in[-\pi,\pi]} |\Delta(\Omega)|^2 - 6\varepsilon_\nu \tag{A130}$$

$$\ge \min_{\Omega\in[-\pi,\pi]} |\Delta_C(\Omega)|^2 - 2\varepsilon_{\mathrm{LR}} - 6\varepsilon_\nu \tag{A131}$$

$$\ge 1 - \varepsilon_{\mathrm{LR}} - \pi\tilde{\epsilon}_V - 2\varepsilon_{\mathrm{LR}} - 3\varepsilon_\nu - 3\varepsilon_\nu. \tag{A132}$$

We now use the exponential bounds on all the ϵ's. Recall that $\varepsilon_{\mathrm{LR}}$ and $\tilde{\epsilon}_V$ are tails of the pointer shape and the potential, and that ε_ν, ε_c describe deviations of the pointer evolution from the simple picture of movement and phase

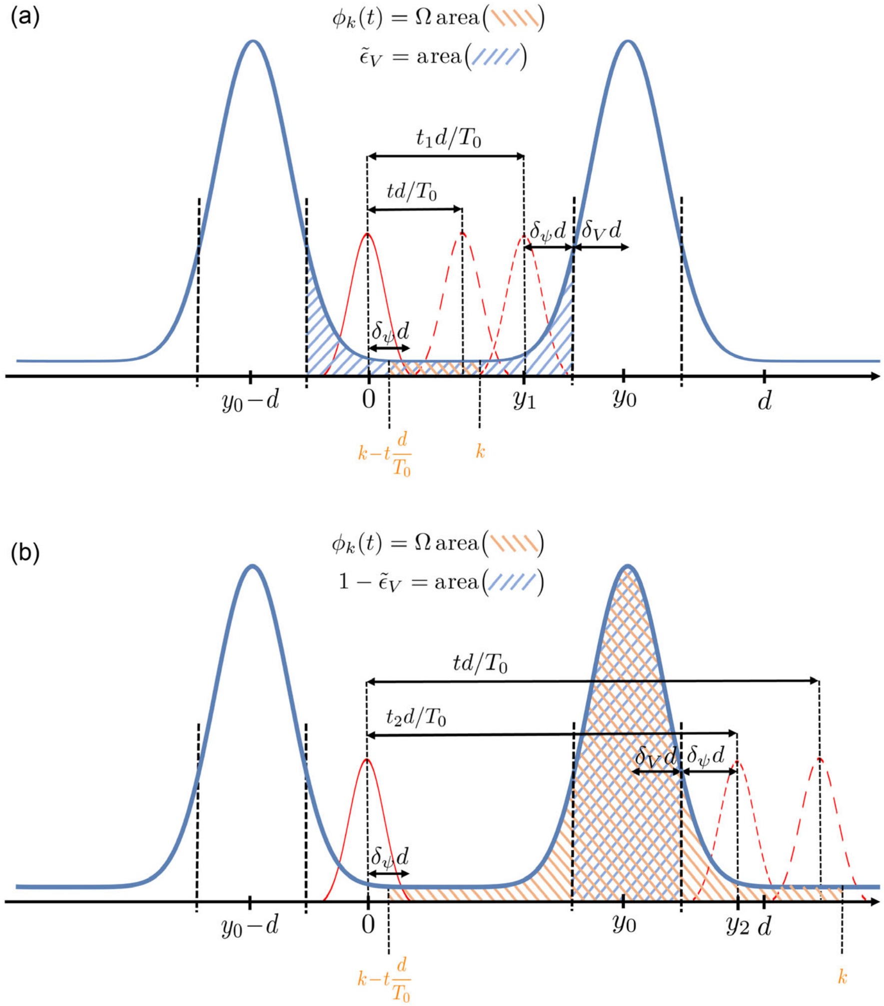

FIG. 6. Acquiring phase by the clock states $|\theta_k\rangle$. The quantities area ($\backslash\backslash\backslash\backslash$) and area ($////$) are the areas of the orange and blue regions, respectively. (a) Times $t \leq t_1$. For such times, the peak of the pointer will travel with speed d/T_0 up to y_1, so that the body of the pointer (i.e., the part within $\pm\delta_\psi d$ from the peak) will always be within the blue area. We consider arbitrary k within the body of the pointer at time t (the left dashed pointer) and its past position $k - td/T_0$. The phase $\phi_k(t)$ acquired by $|\theta_k\rangle$ is proportional to the yellow area, which is contained in the blue one, which in turn is the tail of the potential and therefore small. (b) Times $t \geq t_2$. In this case, the pointer is initially before the blue area (the body of the potential) and ends up after it. Thus, any k from the body of the pointer at time t had to travel through the blue area from its past position $k - td/t_0$. The acquired phase $\phi_k(t)$ is proportional to the yellow area. The latter for any k is larger than the blue one (body of the potential) and therefore the phase is close to Ω.

acquisition. Here we write the bounds for those quantities given by Eqs. (A104), (A115), and (A83),

$$\begin{aligned} \varepsilon_\nu &\lesssim t\mathrm{poly}(d) e^{-\frac{1}{32\pi} d^{1/4}}, \\ \varepsilon_c &\lesssim \mathrm{poly}(d_{\mathrm{Cl}}) e^{-\frac{\pi}{4} d_{\mathrm{Cl}}}, \\ \varepsilon_{\mathrm{LR}} &\leq \mathrm{poly}(d) e^{-\delta_\psi{}^2 d}, \\ \tilde{\epsilon}_V &\leq \frac{1}{\delta_V} e^{-\delta_V{}^2 n}, \end{aligned} \tag{A133}$$

where we choose $\delta_\psi = \delta_V = (t_2 - t_1)/(4T_0)$, and to get the estimate for ε_ν, we choose the potential concentration parameter n to be $n = \lceil d^{1/4} \rceil$; hence, we also have

$$\tilde{\epsilon}_V \leq \frac{1}{\delta_V} e^{-\delta_V{}^2 d^{1/4}}. \tag{A134}$$

We thus have

$$F(\rho^0_{\mathrm{Cl}}(t), \rho_{\mathrm{Cl}}(t)) \lesssim 1 - \frac{1}{t_2 - t_1} \mathrm{poly}(d) e^{-c_1 d^{1/4}}, \tag{A135}$$

where $c_1 = \min\{\frac{1}{32\pi}, (t_2 - t_1)^2/(16T_0^2)\}$. We now use

$$\|\rho - \sigma\|_1 \leq 2\sqrt{1 - F(\rho, \sigma)^2}, \tag{A136}$$

so that $F \geq 1 - \epsilon$ implies $\|\rho - \sigma\|_1 \leq 2\sqrt{2}\sqrt{\epsilon}$. Using this, we obtain that for times t satisfying $t \leq t_1$ or $t \geq t_2$,

$$\|\rho^0_{\mathrm{Cl}}(t),\rho_{\mathrm{Cl}}(t)\| \lesssim \frac{1}{t_2-t_1}\mathrm{poly}(d)e^{-c_2 d^{1/4}}, \quad (A137)$$

where $c_2=\min\{\frac{1}{64\pi},(t_2-t_1)^2/(32T_0^2)\}$. (We can put t_2-t_1 instead of $\sqrt{t_2-t_1}$ in front of poly, as the differences are bounded from above by T_0, so for scaling, the small values of the differences are relevant, so this rough estimate is legitimate.) ■

a. Proof of Theorem 2

We start with a definition and proposition whose usefulness is soon apparent in the proof of Theorem 2 below.

Definition 6. (Autonomous control device error). Let $\rho_{\mathrm{A}}^{\mathrm{target}}(t)$ denote the idealized or targeted control of system A, namely,

$$\rho_{\mathrm{A}}^{\mathrm{target}}(t)=\begin{cases}\rho^0_{\mathrm{A}} & \text{for } t\in[0,t_1],\\ U_{\mathrm{A}}^{\mathrm{target}}\rho^0_{\mathrm{A}}U_{\mathrm{A}}^{\mathrm{target}\dagger} & \text{for } t\in[t_2,T_0],\end{cases} \quad (A138)$$

where we associate the time interval $[t_1,t_2]$ with the time in which the CPTP (completely positive and trace preserving) map is being implemented in the ideal case. Furthermore, let $\rho^F_{\mathrm{ACl}}(t)$ denote the autonomous evolution of A and the control system (the clock cl),

$$\rho^F_{\mathrm{ACl}}(t)=e^{-it\hat{H}_{\mathrm{ACl}}}(\rho_{\mathrm{A}}\otimes\rho_{\mathrm{Cl}})e^{it\hat{H}_{\mathrm{ACl}}}. \quad (A139)$$

Let $\varepsilon_{\mathrm{A}}(t,d_{\mathrm{Cl}},d_{\mathrm{A}})$ and $\varepsilon_{\mathrm{Cl}}(t,d_{\mathrm{Cl}})$ be defined by the relations

$$\|\rho^F_{\mathrm{A}}(t)-\rho^{\mathrm{target}}_{\mathrm{A}}(t)\|_1\le\varepsilon_{\mathrm{A}}(t,d_{\mathrm{Cl}},d_{\mathrm{A}}), \quad (A140)$$

$$\|\rho^F_{\mathrm{Cl}}(t)-\rho^0_{\mathrm{Cl}}(t)\|_1\le\varepsilon_{\mathrm{Cl}}(t,d_{\mathrm{Cl}}), \quad (A141)$$

where $\rho^0_{\mathrm{Cl}}(t)$ is the free evolution of the clock,

$$\rho^0_{\mathrm{Cl}}(t)\coloneqq e^{-it\hat{H}_{\mathrm{Cl}}}\rho_{\mathrm{Cl}}e^{it\hat{H}_{\mathrm{Cl}}}. \quad (A142)$$

Proposition 7. There exists a clock state ρ_{Cl} and time-independent Hamiltonian called the quasi-ideal clock [67] such that for all $t\in[0,t_1]\cup[t_2,T_0]$ and for all fixed $0<t_1<t_2<T_0$, the error terms ε_{A}, $\varepsilon_{\mathrm{Cl}}$ are given by

$$\varepsilon_{\mathrm{A}}(t,d_{\mathrm{Cl}},d_{\mathrm{A}})=\sqrt{d_{\mathrm{A}}\mathrm{tr}[\rho^2_{\mathrm{A}}(0)]}\varepsilon(d_{\mathrm{Cl}}), \quad (A143)$$

$$\varepsilon_{\mathrm{Cl}}(t,d_{\mathrm{Cl}})=\varepsilon(d_{\mathrm{Cl}}), \quad (A144)$$

where $\varepsilon(d_{\mathrm{Cl}})$ is independent of the system A parameters and is of order

$$\varepsilon(d_{\mathrm{Cl}})=\mathcal{O}(\mathrm{poly}(d_{\mathrm{Cl}})\exp[-cd_{\mathrm{Cl}}^{1/4}]), \quad \text{as } d_{\mathrm{Cl}}\to\infty, \quad (A145)$$

where the constant $c>0$ depends on t_1, t_2, and $\mathrm{poly}(d_{\mathrm{Cl}})$ is a polynomial in d_{Cl}.

Proof.—This proposition is a direct consequence of Proposition 4 and the results in Ref. [67]. Proposition 4 proves estimate (A144) with the constant in the exponent given by c_2, i.e., the constant used in estimate Eq. (A88) in Proposition 4. In Ref. [67] [see section Examples: 2) System error faster than power-law decay, page 47], the following estimate is proven:

$$\frac{\|\rho_{\mathrm{A}}(t)-\sigma_{\mathrm{A}}(t)\|_1}{\sqrt{d_{\mathrm{A}}\mathrm{tr}[\rho^2_{\mathrm{A}}(0)]}}=\mathcal{O}(t\mathrm{poly}(d_{\mathrm{Cl}})e^{-2c_0d_{\mathrm{Cl}}^{1/4}\sqrt{\ln d_{\mathrm{Cl}}}}) \quad (A146)$$

for all $t\in[0,t_1]\cup[t_2,T_0]$ and for all fixed $0<t_1<t_2<T_0$. Here we define $2c_0\coloneqq(\pi/4)\alpha_0^2\chi_2^2$, where α_0, χ are constants defined in Ref. [67]. So this proves Eq. (A143) with constant c_0. Taking $c=\min\{2c_0,c_2\}$ finalizes the proof. ■

This proposition is a generalization of the results from Ref. [67]. Specifically, these results are proven for the special case in which $t=T_0$ in Eq. (A144).

We are now ready to provide the proof of Theorem 2 located in the main text. We precede the proof with a short overview.

Overview of the proof of Theorem 2.—The aim of the theorem is to show that in our autonomous setup, the final state of the system, catalyst, and clock is close to product. Indeed, now that the catalyst and clock play the role of the total catalyst state, in order to prevent embezzling, we have to make sure that the total catalyst will not be polluted too much. Of course, the final state on the system is to be close to the required state. Proposition 4 implies that on the system and catalyst we get a state close to the required state, and we have a small error on the clock.

To pass from this outcome to what we want, we note that the initial clock state is pure. Thus, having a small error on the clock means also that the total state stays approximately product between the system-catalyst state and the clock. In the proof, we express this in terms of fidelity.

We now present the full proof.

Proof.—We start by demonstrating part 1 of the theorem. Define

$$U_{\mathrm{SCatG}}(t)=\begin{cases}\mathbb{1}_{\mathrm{SCatG}} & \text{if } t\in[0,t_1],\\ U'_{\mathrm{SCatG}} & \text{if } t\in[t_2,T_0],\end{cases} \quad (A147)$$

where U'_{SCatG} satisfies

$$\mathrm{tr}_{\mathrm{G}}[U'_{\mathrm{SCatG}}(\rho^0_{\mathrm{S}}\otimes\rho^0_{\mathrm{Cat}}\otimes\tilde{\tau}_{\mathrm{G}})U'^{\dagger}_{\mathrm{SCatG}}]=\rho^1_{\mathrm{S}}\otimes\rho^0_{\mathrm{Cat}}. \quad (A148)$$

Define

$$\sigma_{\mathrm{SCatG}}(t)\coloneqq U'_{\mathrm{SCatG}}(t)(\rho^0_{\mathrm{S}}\otimes\rho^0_{\mathrm{Cat}}\otimes\tilde{\tau}_{\mathrm{G}})U'^{\dagger}_{\mathrm{SCatG}}(t). \quad (A149)$$

It follows by the definition of t-CNO (Definition 1 and Proposition 2) that for every pair ρ^0_{S}, ρ^1_{S} for which there exists a t-CNO from ρ^0_{S} to ρ^1_{S}, there exists a unitary U_{SCatG}

satisfying the above criteria. Since the catalyst $\rho^0_{\rm Cat}$ is arbitrary, this is true if and only if Eq. (15) holds. Therefore, $\sigma_{\rm S}(t)$ in Eq. (A149) fulfills part 1 of the theorem.

We now proceed with proving part 2 of the theorem. Recalling Definition 6 and Proposition 7, and using the identifications $\mathrm{A} = \mathrm{SCatG}$, $U^{\rm target}_{\rm A} = U'_{\rm SCatG}$, for every unitary $U_{\rm SCatG}$ above, there exists an interaction term $\hat{I}_{\rm SCatClG}$ such that using the quasi-ideal clock we have

$$\|\rho^F_{\rm SCatG}(t) - \sigma_{\rm SCatG}(t)\|_1 \leq \varepsilon_{\rm A} \tag{A150}$$

$$= \sqrt{d_{\rm S} d_{\rm Cat} d_{\rm G}}\sqrt{{\rm tr}[(\rho^0_{\rm S}\otimes\rho^0_{\rm Cat}\otimes\tilde{\tau}_{\rm G})^2]}\varepsilon_{\rm Cl}(d_{\rm Cl})$$
$$\leq \sqrt{d_{\rm S} d_{\rm Cat}}\varepsilon_{\rm Cl}(d_{\rm Cl}), \tag{A151}$$

$$\|\rho^F_{\rm Cl}(t) - \rho^0_{\rm Cl}(t)\|_1 \leq \varepsilon_{\rm Cl}(d_{\rm Cl}), \tag{A152}$$

where in the last line of Eq. (A150) we take into account that $\tilde{\tau}_{\rm G} = \mathbb{1}/d_{\rm G}$. Recall that an expression for $\varepsilon(d_{\rm Cl})$ is given by Eq. (A145). We now apply Proposition 14 with the identifications

$$\rho^F_{\rm SCatG}(t) =: \rho_{\rm A}, \quad \rho^F_{\rm Cl}(t) =: \rho_{\rm A}, \quad \rho^F_{\rm SCatGCl}(t) =: \rho_{\rm AB}, \tag{A153}$$

$$\sigma_{\rm SCatG}(t) =: \sigma_{\rm A}, \qquad \rho^0_{\rm Cl}(t) =: \sigma_B, \tag{A154}$$

$$\sigma_{\rm SCatG}(t)\otimes\rho^0_{\rm Cl}(t) =: \sigma_{\rm AB}, \tag{A155}$$

hence,

$$\epsilon_1 = \varepsilon_{\rm A}, \qquad \epsilon_2 = \varepsilon_{\rm Cl}, \qquad \epsilon_3 = 0 \tag{A156}$$

[ε_3 vanishes because $\rho^0_{\rm Cl}(t)$ is a pure state] to achieve

$$\|\rho^F_{\rm SCatGCl}(t) - \sigma_{\rm SCatG}(t)\otimes\rho^0_{\rm Cl}(t)\|_1 \leq 2\sqrt{\varepsilon_{\rm Cl}} + \varepsilon_{\rm A} \tag{A157}$$

for all $t\in[0,t_1]\cup[t_2,T_0]$, and where $\varepsilon_{\rm A}$, $\varepsilon_{\rm Cl}$ are given in Eqs. (A150) and (A152). Applying the data processing inequality, we find

$$\|\rho^F_{\rm SCatCl}(t) - \sigma_{\rm SCat}(t)\otimes\rho^0_{\rm Cl}(t)\|_1 \leq 2\sqrt{\varepsilon_{\rm Cl}} + \varepsilon_{\rm A} \tag{A158}$$

for all $t\in[0,t_1]\cup[t_2,T_0]$. Using the triangle inequality, we have

$$\|\rho^F_{\rm SCatCl}(t) - \rho^F_{\rm S}(t)\otimes\rho^0_{\rm Cat}\otimes\rho^0_{\rm Cl}(t)\|_1 \tag{A159}$$

$$\leq \|\rho^F_{\rm SCatCl}(t) - \sigma_{\rm SCat}(t)\otimes\rho^0_{\rm Cl}(t)\|_1 \tag{A160}$$

$$+\|\sigma_{\rm SCat}(t)\otimes\rho^0_{\rm Cl}(t) - \rho^F_{\rm S}(t)\otimes\rho^0_{\rm Cat}\otimes\rho^0_{\rm Cl}(t)\|_1 \tag{A161}$$

$$\leq 2\sqrt{\varepsilon_{\rm Cl}(t)} + \varepsilon_{\rm A}(t) + \|\sigma_{\rm SCat}(t) - \rho^F_{\rm S}(t)\otimes\rho^0_{\rm Cat}\|_1. \tag{A162}$$

Now we note that by definition, it follows that $\sigma_{\rm SCat}(t) = \sigma_{\rm S}(t)\otimes\rho^0_{\rm Cat}$ for all $t\in[0,t_1]\cup[t_2,T_0]$. Plugging into Eq. (A162), we achieve

$$\|\rho^F_{\rm SCatCl}(t) - \rho^F_{\rm S}(t)\otimes\rho^0_{\rm Cat}\otimes\rho^0_{\rm Cl}(t)\|_1 \tag{A163}$$

$$\leq 2\sqrt{\varepsilon_{\rm Cl}} + \varepsilon_{\rm A} + \|\sigma_{\rm S}(t) - \rho^F_{\rm S}(t)\|_1 \tag{A164}$$

$$\leq 2\sqrt{\varepsilon_{\rm Cl}} + 2\varepsilon_{\rm A} \tag{A165}$$

for all $t\in[0,t_1]\cup[t_2,T_0]$ and where in the last line, we use Eq. (A150) after applying the data processing inequality to it. Without loss of generality, assume that $\varepsilon_{\rm Cl}\leq 2$ (if this does not hold, then the following bound holds anyway since the trace distance between any two states is upper bounded by 2), so that $\varepsilon_{\rm Cl}\leq\sqrt{2\varepsilon_{\rm Cl}}$ and using Eq. (A150) we achieve

$$\|\rho^F_{\rm SCatCl}(t) - \rho^F_{\rm S}(t)\otimes\rho^0_{\rm Cat}\otimes\rho^0_{\rm Cl}(t)\|_1 \tag{A166}$$

$$\leq 2\sqrt{\varepsilon_{\rm Cl}} + 2\sqrt{d_{\rm S} d_{\rm Cat}}\varepsilon_{\rm Cl} \tag{A167}$$

$$\leq (2 + 2\sqrt{2}\sqrt{d_{\rm S} d_{\rm Cat}})\sqrt{\varepsilon_{\rm Cl}} \tag{A168}$$

$$\leq (2 + 3\sqrt{d_{\rm S} d_{\rm Cat}})\sqrt{\varepsilon_{\rm Cl}} \tag{A169}$$

$$= \epsilon_{\rm emb} \tag{A170}$$

for all $t\in[0,t_1]\cup[t_2,T_0]$. Now, recalling that $\varepsilon_{\rm Cl}$ is independent of $d_{\rm S}$, $d_{\rm Cat}$, $d_{\rm G}$, and only a function of $d_{\rm Cl}, t_1, t_2, T_0$, we obtain estimate Eq. (17) of part 2 of the theorem. Next, the formula (A145) from Proposition 7 gives the estimate (18) concluding the proof of Theorem 2. ∎

5. Proof of Theorem 3

In this section, we prove Theorem 3 in the main text.

Proof of Theorem 3.—The proof is divided into two parts labeled A and B.

Part A consists of proving that the theorem statements 1 and 2 hold under a different set from those of the theorem. Namely, that 1 and 2 hold when the following two conditions both simultaneously hold:

(a) The final joint system-catalyst-bath state is very close to that of the target joint system-catalyst-bath state.
(b) The final clock state is very close to its free state. The proof of part A uses basic relationships between quantum states (such as trace distance and fidelity), but it does not take into account any dynamical properties.

Part B consists of proving that the conditions in the theorem from which 1 and 2 should follow, do indeed imply conditions a and b from part A. The proof uses the dynamical properties of the states.

a. Part A of the proof of Theorem 3

We start with a comment on notation and a few immediate consequences. We denote $U^{\mathrm{target}}_{\mathrm{SCatG}}(t) = e^{-i\theta(t)\hat{H}^{\mathrm{int}}_{\mathrm{SCatG}}}$ from the main text by $U^{\mathrm{target}(\epsilon_H)}_{\mathrm{SCatG}}(t) = e^{-i\theta(t)\hat{H}^{\mathrm{int}}_{\mathrm{SCatG}}}$ here to remind ourselves that $\hat{H}^{\mathrm{int}}_{\mathrm{SCatG}}$ induces a small error ϵ_H onto the final catalyst and system state [see Eqs. (21) and (23)]. We also denote $U^{\mathrm{target}(0)}_{\mathrm{SCatG}}(t) \coloneqq e^{-i\theta(t)\hat{I}^{\mathrm{int}}_{\mathrm{SCatG}}}$, since $\hat{I}^{\mathrm{int}}_{\mathrm{SCatG}}$ corresponds to the case of no error, i.e., $\epsilon_H = 0$ [see Eq. (22)]. Accordingly, we denote

$$\rho^{\mathrm{target}(0)}_{\mathrm{SCatG}}(t) \tag{A171}$$

$$\coloneqq U^{\mathrm{target}(0)}_{\mathrm{SCatG}}(t)[\rho^0_{\mathrm{S}}(t)\otimes\rho^0_{\mathrm{Cat}}(t)\otimes\tau_{\mathrm{G}}]U^{\mathrm{target}(0)\dagger}_{\mathrm{SCatG}}(t), \tag{A172}$$

$$\rho^{\mathrm{target}(\epsilon_H)}_{\mathrm{SCatG}}(t) \tag{A173}$$

$$\coloneqq U^{\mathrm{target}(\epsilon_H)}_{\mathrm{SCatG}}(t)[\rho^0_{\mathrm{S}}(t)\otimes\rho^0_{\mathrm{Cat}}(t)\otimes\tau_{\mathrm{G}}]U^{\mathrm{target}(\epsilon_H)\dagger}_{\mathrm{SCatG}}(t). \tag{A174}$$

Recall that

$$\theta(t) = \begin{cases} 0 & \text{if } t\in[0,t_1], \\ 1 & \text{if } t\in[t_2,t_3]. \end{cases} \tag{A175}$$

Therefore, similar to Eqs. (27) and (28), we have for $t\in[0,t_1]$,

$$\rho^{\mathrm{target}(0)}_{\mathrm{SCat}}(t) = \rho^0_{\mathrm{S}}(t)\otimes\rho^0_{\mathrm{Cat}}(t), \tag{A176}$$

while for $t\in[t_2,t_3]$,

$$\rho^{\mathrm{target}(0)}_{\mathrm{SCat}}(t) = e^{-it(\hat{H}_{\mathrm{S}}+\hat{H}_{\mathrm{Cat}})}\rho^1_{\mathrm{S}}\otimes\rho^0_{\mathrm{Cat}}e^{it(\hat{H}_{\mathrm{S}}+\hat{H}_{\mathrm{Cat}})} \tag{A177}$$

$$= \rho^1_{\mathrm{S}}(t)\otimes\rho^0_{\mathrm{Cat}}(t). \tag{A178}$$

Hence, Eqs. (A176) and (A177) together imply

$$\rho^{\mathrm{target}(0)}_{\mathrm{SCat}}(t) = \rho^{\mathrm{target}(0)}_{\mathrm{S}}(t)\otimes\rho^0_{\mathrm{Cat}}(t) \tag{A179}$$

for $t\in[0,t_1]\cup[t_2,t_3]$.

Part A consists of proving that the following holds. Let $\varepsilon_{\mathrm{SCatG}}(\epsilon_H;t) > 0$ and $\varepsilon_{\mathrm{Cl}}(t) > 0$ satisfy

$$\|\rho^F_{\mathrm{SCatG}}(t) - \rho^{\mathrm{target}(\epsilon_H)}_{\mathrm{SCatG}}(t)\|_1 \le \varepsilon_{\mathrm{SCatG}}(\epsilon_H;t), \tag{A180}$$

$$\|\rho^F_{\mathrm{Cl}}(t) - \rho^0_{\mathrm{Cl}}(t)\|_1 \le \varepsilon_{\mathrm{Cl}}(t). \tag{A181}$$

It follows that

(1) The deviation from the idealized dynamics is bounded by

$$\|\rho^F_{\mathrm{SCatCl}}(t) - \rho^F_{\mathrm{S}}(t)\otimes\rho^0_{\mathrm{Cat}}(t)\otimes\rho^0_{\mathrm{Cl}}(t)\|_1 \tag{A182}$$

$$\le 2\varepsilon_{\mathrm{SCatG}}(\epsilon_H;t) + 2\sqrt{\varepsilon_{\mathrm{Cl}}(t)} + 2\epsilon_H\theta(t). \tag{A183}$$

(2) The final state $\rho^F_{\mathrm{S}}(t)$ is

$$\|\rho^F_{\mathrm{S}}(t) - \rho^{\mathrm{target}(0)}_{\mathrm{S}}(t)\|_1 \le \varepsilon_{\mathrm{SCatG}}(\epsilon_H;t) + \epsilon_H\theta(t) \tag{A184}$$

close to the one which can be reached via t-CTO: For all $t\in[0,t_1]\cup[t_2,t_3]$, the transition

$$\rho^0_{\mathrm{S}}\otimes\rho^0_{\mathrm{Cat}}\otimes\rho^0_{\mathrm{Cl}} \text{ to } \rho^{\mathrm{target}(0)}_{\mathrm{S}}(t)\otimes\rho^0_{\mathrm{Cat}}(t)\otimes\rho^0_{\mathrm{Cl}}(t) \tag{A185}$$

is possible via a TO, i.e., ρ^0_{S} to $\rho^{\mathrm{target}(0)}_{\mathrm{S}}$ via a t-CTO.

We begin with proving item 2. To prove the Eqs. (A184) and (A185), we start by extending the definitions of $\rho^{\mathrm{target}(\epsilon_H)}_{\mathrm{SCatG}}(t)$ and $\rho^{\mathrm{target}(0)}_{\mathrm{SCatG}}(t)$ in Eqs. (A176) and (A177) to include the clock system:

$$\rho^{\mathrm{target}(\epsilon_H)}_{\mathrm{SCatGCl}}(t) = \rho^{\mathrm{target}(\epsilon_H)}_{\mathrm{SCatG}}(t)\otimes\rho^0_{\mathrm{Cl}}(t), \tag{A186}$$

$$\rho^{\mathrm{target}(0)}_{\mathrm{SCatGCl}}(t) = \rho^{\mathrm{target}(0)}_{\mathrm{SCatG}}(t)\otimes\rho^0_{\mathrm{Cl}}(t), \tag{A187}$$

where $\rho^0_{\mathrm{Cl}}(t)$ is the free evolution of the clock defined in Eq. (6). Therefore, from Eq. (A179), it follows that the reduced state after tracing out the Gibbs state on G is

$$\rho^{\mathrm{target}(0)}_{\mathrm{SCatCl}}(t) = \rho^{\mathrm{target}(0)}_{\mathrm{S}}(t)\otimes\rho^0_{\mathrm{Cat}}(t)\otimes\rho^0_{\mathrm{Cl}}(t) \tag{A188}$$

for $t\in[0,t_1]\cup[t_2,t_3]$. Thus, taking into account property Eq. (20), it follows by definition of CTOs and t-CTOs that a transition from ρ^0_{S} to $\rho^{\mathrm{target}(0)}_{\mathrm{S}}(t)$ is possible via a t-CTO. Finally, applying the data processing inequality to Eq. (A180), we achieve

$$\|\rho^F_{\mathrm{S}}(t) - \rho^{\mathrm{target}(\epsilon_H)}_{\mathrm{S}}(t)\|_1 \le \varepsilon_{\mathrm{SCatG}}(\epsilon_H;t), \tag{A189}$$

while applying the date processing inequality to Eqs. (A172) and (A174), we find $\|\rho^{\mathrm{target}(0)}_{\mathrm{S}}(t) - \rho^{\mathrm{target}(\epsilon_H)}_{\mathrm{S}}(t)\|_1 = 0$ for $t\in[0,t_1]$, while from Eqs. (21)–(23), we see $\|\rho^{\mathrm{target}(0)}_{\mathrm{S}}(t) - \rho^{\mathrm{target}(\epsilon_H)}_{\mathrm{S}}(t)\|_1 \le \epsilon_H$ for $t\in[t_2,t_3]$. Hence, combining both equations, we have $\|\rho^{\mathrm{target}(0)}_{\mathrm{S}}(t) - \rho^{\mathrm{target}(\epsilon_H)}_{\mathrm{S}}(t)\|_1 \le \epsilon_H\theta(t)$ for $t\in[0,t_1]\cup[t_2,t_3]$. Then, Eq. (A184) in item 2 above follows from the triangle inequality:

$$\|\rho^F_{\mathrm{S}}(t) - \rho^{\mathrm{target}(0)}_{\mathrm{S}}(t)\|_1 \le \|\rho^F_{\mathrm{S}}(t) - \rho^{\mathrm{target}(\epsilon_H)}_{\mathrm{S}}(t)\|_1 \tag{A190}$$

$$+\|\rho^{\mathrm{target}(\epsilon_H)}_{\mathrm{S}}(t) - \rho^{\mathrm{target}(0)}_{\mathrm{S}}(t)\|_1 \le \varepsilon_{\mathrm{SCatG}}(\epsilon_H;t) \tag{A191}$$

$$+\epsilon_H\theta(t). \tag{A192}$$

We now prove the above item 1 [i.e., the estimate (A183)]. We begin by using the triangle inequality followed by the identity $\rho_{\mathrm{SCat}}^{\mathrm{target}(0)}(t)=\rho_{\mathrm{S}}^{\mathrm{target}(0)}(t)\otimes\rho_{\mathrm{Cat}}^{0}(t)$, which follows from Eq. (A188),

$$\|\rho_{\mathrm{SCatCl}}^{F}(t)-\rho_{\mathrm{S}}^{F}(t)\otimes\rho_{\mathrm{Cat}}^{0}(t)\otimes\rho_{\mathrm{Cl}}^{0}(t)\|_1 \tag{A193}$$

$$=\|\rho_{\mathrm{SCatCl}}^{F}(t)-\rho_{\mathrm{SCat}}^{\mathrm{target}(\epsilon_H)}(t)\otimes\rho_{\mathrm{Cl}}^{0}(t) \tag{A194}$$

$$+\rho_{\mathrm{SCat}}^{\mathrm{target}(\epsilon_H)}(t)\otimes\rho_{\mathrm{Cl}}^{0}(t)-\rho_{\mathrm{S}}^{F}(t)\otimes\rho_{\mathrm{Cat}}^{0}(t)\otimes\rho_{\mathrm{Cl}}^{0}(t)\|_1 \tag{A195}$$

$$\le\|\rho_{\mathrm{SCatCl}}^{F}(t)-\rho_{\mathrm{SCat}}^{\mathrm{target}(\epsilon_H)}(t)\otimes\rho_{\mathrm{Cl}}^{0}(t)\|_1 \tag{A196}$$

$$+\|\rho_{\mathrm{SCat}}^{\mathrm{target}(\epsilon_H)}(t)\otimes\rho_{\mathrm{Cl}}^{0}(t)-\rho_{\mathrm{SCat}}^{\mathrm{target}(0)}(t)\otimes\rho_{\mathrm{Cl}}^{0}(t)\|_1 \tag{A197}$$

$$+\|\rho_{\mathrm{SCat}}^{\mathrm{target}(0)}(t)\otimes\rho_{\mathrm{Cl}}^{0}(t)-\rho_{\mathrm{S}}^{F}(t)\otimes\rho_{\mathrm{Cat}}^{0}(t)\otimes\rho_{\mathrm{Cl}}^{0}(t)\|_1 \tag{A198}$$

$$=\|\rho_{\mathrm{SCatCl}}^{F}(t)-\rho_{\mathrm{SCat}}^{\mathrm{target}(\epsilon_H)}(t)\otimes\rho_{\mathrm{Cl}}^{0}(t)\|_1 \tag{A199}$$

$$+\|\rho_{\mathrm{SCat}}^{\mathrm{target}(\epsilon_H)}(t)-\rho_{\mathrm{SCat}}^{\mathrm{target}(0)}(t)\|_1 \tag{A200}$$

$$+\|\rho_{\mathrm{S}}^{\mathrm{target}(0)}\otimes\rho_{\mathrm{Cat}}^{0}(t)-\rho_{\mathrm{S}}^{F}(t)\otimes\rho_{\mathrm{Cat}}^{0}(t)\|_1$$

$$\le\|\rho_{\mathrm{SCatCl}}^{F}(t)-\rho_{\mathrm{SCat}}^{\mathrm{target}(\epsilon_H)}(t)\otimes\rho_{\mathrm{Cl}}^{0}(t)\|_1 \tag{A201}$$

$$+\epsilon_H\theta(t)+\|\rho_{\mathrm{S}}^{\mathrm{target}(0)}-\rho_{\mathrm{S}}^{F}(t)\|_1 \tag{A202}$$

$$\le\|\rho_{\mathrm{SCatCl}}^{F}(t)-\rho_{\mathrm{SCat}}^{\mathrm{target}(\epsilon_H)}(t)\otimes\rho_{\mathrm{Cl}}^{0}(t)\|_1 \tag{A203}$$

$$+2\epsilon_H\theta(t)+\varepsilon_{\mathrm{SCatG}}(\epsilon_H;t), \tag{A204}$$

where we apply the data processing inequality to Eq. (A180) and use the resultant equation in the last line. Now we make the following identifications, noting that $\rho_{\mathrm{Cl}}^{0}(t)$ all $t\in\mathbb{R}$ is pure by assumption of the theorem,

$$\rho_{\mathrm{SCatG}}^{F}(t)=:\rho_{\mathrm{A}},\quad \rho_{\mathrm{Cl}}^{F}(t)=:\rho_{\mathrm{B}},\quad \rho_{\mathrm{SCatGCl}}^{F}(t)=:\rho_{\mathrm{AB}}, \tag{A205}$$

$$\rho_{\mathrm{SCatG}}^{\mathrm{target}(\epsilon_H)}(t)=:\sigma_{\mathrm{A}},\qquad \rho_{\mathrm{Cl}}^{0}(t)=:\sigma_{\mathrm{B}}, \tag{A206}$$

$$\rho_{\mathrm{SCatG}}^{\mathrm{target}(\epsilon_H)}(t)\otimes\rho_{\mathrm{Cl}}^{0}(t)=:\sigma_{\mathrm{AB}}, \tag{A207}$$

and apply Proposition 14 with use of Eqs. (A180) and (A181) arriving at

$$\epsilon_1=\varepsilon_{\mathrm{SCatG}}(\epsilon_H;t),\qquad \epsilon_2=\varepsilon_{\mathrm{Cl}}(t),\qquad \epsilon_3=0, \tag{A208}$$

and thus,

$$\|\rho_{\mathrm{SCatGCl}}^{F}(t)-\rho_{\mathrm{SCatG}}^{\mathrm{target}(\epsilon_H)}(t)\otimes\rho_{\mathrm{Cl}}^{0}(t)\|_1 \tag{A209}$$

$$\le 2\sqrt{\varepsilon_{\mathrm{Cl}}(t)}+\varepsilon_{\mathrm{SCatG}}(\epsilon_H;t). \tag{A210}$$

Applying the data processing inequality to the above equation, followed by substituting into Eq. (A204), gives

$$\|\rho_{\mathrm{SCatCl}}^{F}(t)-\rho_{\mathrm{S}}^{F}(t)\otimes\rho_{\mathrm{Cat}}^{0}(t)\otimes\rho_{\mathrm{Cl}}^{0}(t)\|_1 \tag{A211}$$

$$\le 2\sqrt{\varepsilon_{\mathrm{Cl}}(t)}+\varepsilon_{\mathrm{SCatG}}(\epsilon_H;t)+\varepsilon_{\mathrm{SCatG}}(\epsilon_H;t)+2\epsilon_H\theta(t)$$
$$=2\varepsilon_{\mathrm{SCatG}}(\epsilon_H;t)+2\sqrt{\varepsilon_{\mathrm{Cl}}(t)}+2\epsilon_H\theta(t). \tag{A212}$$

b. Part B of the proof of Theorem 3

We now set out to prove the second part, which consists of deriving expressions for $\varepsilon_{\mathrm{SCatG}}(\epsilon_H;t)$ and $\varepsilon_{\mathrm{Cl}}(t)$ such that the contents of items 1 and 2 above are consistent with the claims in 1 and 2 of the theorem.

To start with, since $\hat{H}_{\mathrm{S}}+\hat{H}_{\mathrm{Cat}}+\hat{H}_{\mathrm{G}}$ and $\hat{H}_{\mathrm{SCatG}}^{\mathrm{int}}$ commute, they share a common eigenbasis which we denote $\{|E_j\rangle\}_j$. We can write the interaction term in terms of this basis as follows: $\hat{H}_{\mathrm{SCatG}}^{\mathrm{int}}=\sum_{j=1}^{d_{\mathrm{S}}d_{\mathrm{Cat}}d_{\mathrm{G}}}\Omega_j|E_j\rangle\langle E_j|$ with eigenvalues Ω_j in the range $\Omega_j\in[-\pi,\pi]$ since $\|\hat{H}_{\mathrm{SCatG}}^{\mathrm{int}}\|_\infty\le\pi$. We can also expand the state $\rho_{\mathrm{S}}^{0}\otimes\rho_{\mathrm{Cat}}^{0}\otimes\tau_{\mathrm{G}}$ in the energy eigenbasis $\{|E_j\rangle\}_j$. Doing so allows one to simplify the expression for $\rho_{\mathrm{SCatGCl}}^{F}(t)$. We find

$$\rho_{\mathrm{SCatGCl}}^{F}(t)=e^{-it(\hat{H}_{\mathrm{S}}+\hat{H}_{\mathrm{Cat}}+\hat{H}_{\mathrm{G}}+\hat{H}_{\mathrm{SCatG}}^{\mathrm{int}}\otimes\hat{H}_{\mathrm{Cl}}^{\mathrm{int}}+\hat{H}_{\mathrm{Cl}})}\rho_{\mathrm{S}}^{0}\otimes\rho_{\mathrm{Cat}}^{0}\otimes\tau_{\mathrm{G}}\otimes|\rho_{\mathrm{Cl}}^{0}\rangle\langle\rho_{\mathrm{Cl}}^{0}|e^{it(\hat{H}_{\mathrm{S}}+\hat{H}_{\mathrm{Cat}}+\hat{H}_{\mathrm{G}}+\hat{H}_{\mathrm{SCatG}}^{\mathrm{int}}\otimes\hat{H}_{\mathrm{Cl}}^{\mathrm{int}}+\hat{H}_{\mathrm{Cl}})} \tag{A213}$$

$$=\sum_{j,j'=1}^{d_{\mathrm{S}}d_{\mathrm{Cat}}d_{\mathrm{G}}}e^{-it(\hat{H}_{\mathrm{S}}+\hat{H}_{\mathrm{Cat}}+\hat{H}_{\mathrm{G}}+\Omega_j\hat{H}_{\mathrm{Cl}}^{\mathrm{int}}+\hat{H}_{\mathrm{Cl}})}\rho_{\mathrm{SCatG},j,j'}^{0}|E_j\rangle\langle E_{j'}|\otimes|\rho_{\mathrm{Cl}}^{0}\rangle\langle\rho_{\mathrm{Cl}}^{0}|e^{it(\hat{H}_{\mathrm{S}}+\hat{H}_{\mathrm{Cat}}+\hat{H}_{\mathrm{G}}+\Omega_{j'}\hat{H}_{\mathrm{Cl}}^{\mathrm{int}}+\hat{H}_{\mathrm{Cl}})} \tag{A214}$$

$$=\sum_{j,j'=1}^{d_{\mathrm{S}}d_{\mathrm{Cat}}d_{\mathrm{G}}}\rho_{\mathrm{SCatG},j,j'}^{0}(t)|E_j\rangle\langle E_{j'}|\otimes|\rho_{\mathrm{Cl},j}^{0}(t)\rangle\langle\rho_{\mathrm{Cl},j'}^{0}(t)|, \tag{A215}$$

where

$$\sum_{j,j'=1}^{d_S d_{Cat} d_G} \rho^0_{SCatG,j,j'}(t)|E_j\rangle\langle E_{j'}| = \rho_S(t)\otimes\rho^0_{Cat}(t)\otimes\tau_G, \quad (A216)$$

$$|\rho^0_{Cl,j}(t)\rangle = e^{-it(\Omega_j \hat{H}^{int}_{Cl}+\hat{H}_{Cl})}|\rho^0_{Cl}\rangle. \quad (A217)$$

We thus have by taking partial traces

$$\rho^F_{SCatG}(t) \quad (A218)$$

$$= \sum_{j,j'=1}^{d_S d_{Cat} d_G} \rho^0_{SCatG,j,j'}(t)|E_j\rangle\langle E_{j'}|\langle\rho^0_{Cl,j'}(t)|\rho^0_{Cl,j}(t)\rangle, \quad (A219)$$

$$\rho^F_{Cl}(t) = \sum_{j=1}^{d_S d_{Cat} d_G} \rho^0_{SCatG,j,j}|\rho^0_{Cl,j}(t)\rangle\langle\rho^0_{Cl,j}(t)|. \quad (A220)$$

Similarly,

$$\rho^{target(\epsilon_H)}_{SCatG}(t) \quad (A221)$$

$$= U^{target(\epsilon_H)}_{SCatG}(t)[\rho^0_S(t)\otimes\rho^0_{Cat}(t)\otimes]U^{target(\epsilon_H)\dagger}_{SCatG}(t) \quad (A222)$$

$$= e^{-it\theta(t)\hat{H}^{int}_{SCatG}} \quad (A223)$$

$$\left(\sum_{j,j'=1}^{d_S d_{Cat} d_G} \rho^0_{SCatG,j,j'}(t)|E_j\rangle\langle E_{j'}|\right)e^{it\theta(t)\hat{H}^{int}_{SCatG}} \quad (A224)$$

$$= \sum_{j,j'=1}^{d_S d_{Cat} d_G} \rho^0_{SCatG,j,j'}(t)e^{-it(\Omega_j-\Omega_{j'})\theta(t)}|E_j\rangle\langle E_{j'}|. \quad (A225)$$

Noting that the Frobenious norm $\|\cdot\|_F$ upper bounds the trace distance by the inequality $\|\cdot\|_F \geq \|\cdot\|_1/\sqrt{d}$ for a d-dimensional space, we find

$$\|\rho^F_{SCatG}(t) - \rho^{target(\epsilon_H)}_{SCatG}(t)\|_1 \leq \sqrt{d_S d_{Cat} d_G}\|\rho^F_{SCatG}(t) - \rho^{target(\epsilon_H)}_{SCatG}(t)\|_F \quad (A226)$$

$$= \sqrt{d_S d_{Cat} d_G}\sqrt{\sum_{j,j'=1}^{d_S d_{Cat} d_G} |\rho^0_{SCatG,j,j'}(t)|^2 |e^{-it(\Omega_j-\Omega_{j'})\theta(t)} - \langle\rho^0_{Cl,j'}(t)|\rho^0_{Cl,j}(t)\rangle|^2} \quad (A227)$$

$$\leq \sqrt{d_S d_{Cat} d_G}\sqrt{\sum_{j,j'=1}^{d_S d_{Cat} d_G} |\rho^0_{SCatG,j,j'}(t)|^2 (\max_{m,n}|e^{-it(\Omega_m-\Omega_n)\theta(t)} - \langle\rho^0_{Cl,n}(t)|\rho^0_{Cl,m}(t)\rangle|^2)} \quad (A228)$$

$$\leq \sqrt{d_S d_{Cat} d_G}\sqrt{\mathrm{tr}[\rho^0_S(t)^2\otimes\rho^0_{Cat}(t)^2\otimes\tau^2_G](\max_{m,n}|e^{-it(\Omega_m-\Omega_n)\theta(t)} - \langle\rho^0_{Cl,n}(t)|\rho^0_{Cl,m}(t)\rangle|^2)} \quad (A229)$$

$$= \sqrt{d_S \mathrm{tr}[\rho^{0\,2}_S]d_{Cat}\mathrm{tr}[\rho^0_{Cat}{}^2]d_G\mathrm{tr}[\tau_G{}^2]}\max_{m,n}|e^{-it(\Omega_m-\Omega_n)\theta(t)} - \langle\rho^0_{Cl,n}(t)|\rho^0_{Cl,m}(t)\rangle| \quad (A230)$$

$$\leq \sqrt{d_S \mathrm{tr}[\rho^{0\,2}_S]d_{Cat}\mathrm{tr}[\rho^0_{Cat}{}^2]d_G\mathrm{tr}[\tau_G{}^2]}\max_{x,y\in[-\pi,\pi]}|1 - \langle\rho^0_{Cl}|\hat{\Gamma}^\dagger(x,t)\hat{\Gamma}(y,t)|\rho^0_{Cl}\rangle| \quad (A231)$$

$$= A\max_{x,y\in[-\pi,\pi]}|1-\Delta(t;x,y)|, \quad (A232)$$

where we denote

$$\Delta(t;x,y) \coloneqq \langle\rho^0_{Cl}|\Gamma^\dagger(x,t)\Gamma(y,t)|\rho^0_{Cl}\rangle, \quad (A233)$$

$$A \coloneqq \sqrt{d_S \mathrm{tr}[\rho^{0\,2}_S]d_{Cat}\mathrm{tr}[\rho^0_{Cat}{}^2]d_G\mathrm{tr}[\tau_G{}^2]}. \quad (A234)$$

(Note here that since $d\mathrm{tr}[\rho^2] \geq 1$ for any d-dimensional state, we have $A \geq 1$.) Thus, $\varepsilon_{SCatG}(\epsilon_H;t)$ from Eq. (A180), we set as

$$\varepsilon_{SCatG}(\epsilon_H;t) = A\max_{x,y\in[-\pi,\pi]}|1-\Delta(t;x,y)|. \quad (A235)$$

Noting that the fidelity F between a pure state $|\rho^0_{Cl}(t)\rangle = e^{-it\hat{H}_{Cl}}|\rho^0_{Cl}\rangle$ and a state $\rho^F_{Cl}(t)$ is given by $F = \langle\rho^0_{Cl}(t)|\rho^F_{Cl}(t)|\rho^0_{Cl}(t)\rangle$ using Eq. (A220) and the usual bound for the trace distance in terms of the fidelity, we find

$$\|\rho^F_{Cl}(t) - \rho^0_{Cl}(t)\|_1 \quad (A236)$$

$$\leq 2\sqrt{1 - F(\rho_{\rm Cl}^F(t), |\rho_{\rm Cl}^0(t)\rangle)} \tag{A237}$$

$$= 2\sqrt{1 - \langle\rho_{\rm Cl}^0(t)|\rho_{\rm Cl}^F(t)|\rho_{\rm Cl}^0(t)\rangle} \tag{A238}$$

$$= 2\sqrt{1 - \sum_{j=1}^{d_{\rm S}d_{\rm Cat}d_{\rm G}} \rho^0_{{\rm SCatG},j,j}\langle\rho_{\rm Cl}^0(t)|\rho_{{\rm Cl},j}^0(t)\rangle\langle\rho_{{\rm Cl},j}^0(t)|\rho_{\rm Cl}^0(t)\rangle}$$

$$\leq 2\sqrt{1 - \min_j|\langle\rho_{\rm Cl}^0(t)|\rho_{{\rm Cl},j}^0(t)\rangle|^2} \tag{A239}$$

$$\leq 2\sqrt{1 - \min_{x\in[-\pi,\pi]}|\langle\rho_{\rm Cl}^0(t)|e^{-ix\theta(t)}\Gamma(x,t)|\rho_{\rm Cl}^0\rangle|^2} \tag{A240}$$

$$= 2\sqrt{1 - \min_{x\in[-\pi,\pi]}|\langle\rho_{\rm Cl}^0|\Gamma^\dagger(0,t)\Gamma(x,t)|\rho_{\rm Cl}^0\rangle|^2} \tag{A241}$$

$$\leq 2\max_{x,y\in[-\pi,\pi]}\sqrt{1 - |\langle\rho_{\rm Cl}^0|\Gamma^\dagger(y,t)\Gamma(x,t)|\rho_{\rm Cl}^0\rangle|^2} \tag{A242}$$

$$= 2\max_{x,y\in[-\pi,\pi]}\sqrt{1 - |\Delta(t;x,y)|^2}, \tag{A243}$$

so that we can set $\varepsilon_{\rm Cl}(t)$ from Eq. (A181) to

$$\varepsilon_{\rm Cl}(t) = 2\max_{x,y\in[-\pi,\pi]}\sqrt{1 - |\Delta(t;x,y)|^2}. \tag{A244}$$

Inserting Eqs. (A244) and (A235) into Eqs. (A183) and (A184), we conclude

$$\|\rho_{\rm SCatCl}^F(t) - \rho_{\rm S}^F(t)\otimes\rho_{\rm Cat}^0(t)\otimes\rho_{\rm Cl}^0(t)\|_1 \tag{A245}$$

$$\leq 2\sqrt{A}\max_{x,y\in[-\pi,\pi]}\sqrt{|1 - \Delta(t;x,y)|} \tag{A246}$$

$$+4\max_{x,y\in[-\pi,\pi]}\sqrt{1 - |\Delta(t;x,y)|^2} + 2\epsilon_H\theta(t), \tag{A247}$$

and

$$\|\rho_{\rm S}^F(t) - \rho_{\rm S}^{\rm target(0)}(t)\|_1 \tag{A248}$$

$$\leq \epsilon_H\theta(t) + 2\max_{x,y\in[-\pi,\pi]}\sqrt{1 - |\Delta(t;x,y)|^2}. \tag{A249}$$

Finally, to finish the proof we need to find some simplifying upper bounds to the rhs of Eqs. (A247) and (A249) to conclude the bounds stated in Theorem 3.

To this end, we apply Lemma 38, which implies, by identifying $c = \Delta(t;x,y)$,

$$1 - |\Delta(t;x,y)|^2 \leq |1 - \Delta(t;x,y)^2|, \tag{A250}$$

$$|1 - \Delta(t;x,y)| \leq |1 - \Delta(t;x,y)^2|. \tag{A251}$$

Observe that we can make the identification $c = \Delta(t;x,y)$ since $|\Delta(t;x,y)| \leq 1$ follows from unitarity and $|1-c| = |1 - \Delta(t;x,y)| \leq 1$ can be assumed without loss of generality. Indeed, if $|1 - \Delta(t;x,y)| > 1$, then the bounds would be greater than 2 [see Eqs. (A247) and (A249) and recall $A \geq 1$], hence, not relevant, since the trace norm is always no greater than 2.

We then obtain

$$\|\rho_{\rm SCatCl}^F(t) - \rho_{\rm S}^F(t)\otimes\rho_{\rm Cat}^0(t)\otimes\rho_{\rm Cl}^0(t)\|_1 \tag{A252}$$

$$\leq 2\epsilon_H\theta(t) + 6\sqrt{A\max_{x,y\in[-\pi,\pi]}|1 - \Delta^2(t;x,y)|}, \tag{A253}$$

where we use $A \geq 1$ and

$$\|\rho_{\rm S}^F(t) - \rho_{\rm S}^{\rm target(0)}(t)\|_1 \leq \epsilon_H\theta(t) \tag{A254}$$

$$+\sqrt{A}\max_{x,y\in[-\pi,\pi]}|1 - \Delta^2(t;x,y)|, \tag{A255}$$

■

where A and $\Delta(t;x,y)$ are given by Eq. (A233). Finally, since $\mathrm{tr}[\rho^2] \leq 1$ for any normalized density matrix ρ, we have

$$A \leq d_{\rm S}d_{\rm Cat}d_{\rm G}\mathrm{tr}[\tau_{\rm G}{}^2]. \tag{A256}$$

Inserting this into Eqs. (A253) and (A255), we get the thesis of Theorem 3.

6. Calculating $\Delta(t;x,y)$ for the idealized momentum clock

In the case of the idealized momentum clock, we have $\hat{H}_{\rm Cl} = \hat{p}$, $\hat{H}_{\rm Cl}^{\rm int} = g(\hat{x})$, where $\hat{x}$ and $\hat{p}$ are the position and momentum operators of a particle in one dimension satisfying the Weyl form of the canonical commutation relations $[\hat{x},\hat{p}] = i$, while g is an integrable function from the reals to the reals, normalized such that $\int_{\mathbb{R}} g(x)dx = 1$ [97]. Therefore, we find for the idealized momentum clock, for $z, y \in \mathbb{R}$,

$$\Delta(t;z,y) \tag{A257}$$

$$= \langle\rho_{\rm Cl}^0|\hat\Gamma_{\rm Cl}^\dagger(z,t)\hat\Gamma_{\rm Cl}(y,t)|\rho_{\rm Cl}^0\rangle \tag{A258}$$

$$= e^{-i(z-y)\theta(t)}\langle\rho_{\rm Cl}^0|e^{it\hat{p}+iztg(\hat{x})}e^{-it\hat{p}-iytg(\hat{x})}|\rho_{\rm Cl}^0\rangle \tag{A259}$$

$$= e^{-i(z-y)\theta(t)}\int_{\mathbb{R}} dx\langle\rho_{\rm Cl}^0|e^{it\hat{p}+iztg(\hat{x})}|x\rangle\langle x|e^{-it\hat{p}-iytg(\hat{x})}|\rho_{\rm Cl}^0\rangle. \tag{A260}$$

We can now use the relation $\hat{p} = -i(\partial/\partial x)$ and solve the first-order two-variable differential equation resulting from the Schrödinger equation for the Hamiltonian $\hat{p} + yg(\hat{x})$ and initial wave function $\langle x|\rho^0_{\mathrm{Cl}}\rangle$. Plugging the solution into the above, we arrive at

$$\Delta(t;z,y) = e^{-i(z-y)\theta(t)} \int_{\mathbb{R}} dx |\langle x|\rho^0_{\mathrm{Cl}}\rangle|^2 e^{-i(y-z)\int_x^{x+t} g(x')dx'}. \tag{A261}$$

We now choose the support of the initial wave function $\langle x|\rho^0_{\mathrm{Cl}}\rangle$ to be $x \in [x_{\psi l}, x_{\psi r}]$ and the support of $g(x)$ to be $x \in [x_{gl}, x_{gr}]$. Noting that

$$\int_x^{x+t} g(x')dx' = \begin{cases} 0 & \text{if } x+t \le x_{gl}, \\ 1 & \text{if } x \le x_{gl} \text{ and } x+t \ge x_{gr}, \end{cases} \tag{A262}$$

and taking into account the support interval of $\langle x|\rho^0_{\mathrm{Cl}}\rangle$, we conclude

$$\int_{\mathbb{R}} dx |\langle x|\rho^0_{\mathrm{Cl}}\rangle|^2 e^{-i(y-z)\int_x^{x+t} g(x')dx'} \tag{A263}$$

$$= \begin{cases} 1 & \text{if } t \le x_{gl} - x_{\psi r}, \\ e^{i(z-y)} & \text{if } t \ge x_{gr} - x_{\psi l}. \end{cases} \tag{A264}$$

Therefore, choosing $t_1 = x_{gr} - x_{\psi l}$ and $t_2 = x_{gr} - x_{\psi l}$, from Eq. (A261) we arrive at

$$\Delta(t;x,y) = 1 \quad \forall\, x,y \in [-\pi,\pi], \tag{A265}$$

as we claim in Sec. III B of the main text. Furthermore, note that the derivation holds for all $t_1 < t_2$ by appropriately choosing the parameters $x_{gr}, x_{\psi l}, x_{gr}, x_{\psi l}$.

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[42] Note that the upper bound Eq. (A218) is non-negative. Therefore, if assumption $\|p^{-1}\|_{|\alpha|}/\|p'^{-1}\|_{|\alpha|} \geq 0$ does not hold, the bound will be trivially true since $S_\alpha(p) - S_\alpha(p')$ will be negative.
[43] Here it is assumed that $g_\alpha(p, p')$ is differentiable with respect to α on the interval $(1, \alpha)$ for $\alpha \geq 1$ and $(\alpha, 1)$ for $\alpha \leq 1$. Later, we calculate explicitly its derivative, thus verifying this assumption.
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[60] For any bipartite state ρ_A, we use the notation of reduced states $\rho_A \coloneqq \mathrm{tr}_B(\rho_{AB})$, $\rho_B \coloneqq \mathrm{tr}_A(\rho_{BA})$.
[61] It is likely that the no-go theorem holds for all nontrivial U_{AG}, i.e., all cases for which there exists $t \in [0, t_1] \cup [t_2, t_3]$ such that $\rho_A^F(t) \neq \rho_A^0(t)$. However, the point of the no-go theorem is simply to show that the problem is nontrivial for most cases of interest.
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[96] Note that the range of α for which Eq. (A73) holds is the empty set if $\alpha_{\min}$ is large enough. Under such circumstances, this equation contains no information.

[97] One can also come to the same conclusions for the idealized momentum clock on a circle, rather than a line. In this case, $[\hat{x}, \hat{p}]$ still satisfy the Heisenberg form of the canonical commutation relations, but not the Weyl form. See Ref. [98] for details.

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