Nonzero Momentum Requires Long-Range Entanglement

Lei Gioia[1,2] and Chong Wang[1]

[1]*Perimeter Institute for Theoretical Physics, Waterloo, Ontario N2L 2Y5, Canada*
[2]*Department of Physics and Astronomy, University of Waterloo, Waterloo, Ontario N2L 3G1, Canada*

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We show that a quantum state in a lattice spin (boson) system must be long-range entangled if it has nonzero lattice momentum, i.e., if it is an eigenstate of the translation symmetry with eigenvalue $e^{iP} \neq 1$. Equivalently, any state that can be connected with a nonzero momentum state through a finite-depth local unitary transformation must also be long-range entangled. The statement can also be generalized to fermion systems. Some nontrivial consequences follow immediately from our theorem: (i) Several different types of Lieb-Schultz-Mattis-Oshikawa-Hastings theorems, including a previously unknown version involving only a discrete $\mathbb{Z}_n$ symmetry, can be derived in a simple manner from our result; (ii) a gapped topological order (in space dimension $d > 1$) must *weakly* break translation symmetry if one of its ground states on torus has nontrivial momentum—this generalizes the familiar physics of Tao-Thouless; (iii) our result provides further evidence of the "smoothness" assumption widely used in the classification of crystalline symmetry-protected topological phases.

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I. INTRODUCTION

The ubiquitous appearance of translation symmetry in physical systems signals the importance of having a complete picture of the complex role it may play. In particular, although the ground state energy (associated with time-translation symmetry) of a many-body quantum system or a quantum field theory is frequently studied, the ground state *momentum* (associated with space-translation symmetry) is rarely discussed. Rather, in most cases, one focuses on the momentum difference between excited states and the ground state. In this work, we reveal a connection between the momentum and the entanglement structure of a quantum state, in the context of lattice spin (boson) systems.

Theorem 1.—If a quantum state $|\Psi\rangle$ in a lattice spin (boson) system is an eigenstate of the lattice translation operator $T: |\Psi\rangle \to e^{iP}|\Psi\rangle$ with a nontrivial momentum $e^{iP} \neq 1$, then $|\Psi\rangle$ must be long-range entangled; namely, $|\Psi\rangle$ cannot be transformed to an unentangled product state $|000\ldots\rangle$ through an adiabatic evolution or a finite-depth quantum circuit (local unitary).

The intuition behind this statement follows from the sharp difference between translation T and an ordinary on-site symmetry G that is defined as a tensor product of operators acting on each lattice site [such as the electromagnetic U(1)]. A product state may recreate any total symmetry charge Q under G by simply assigning individual local Hilbert space states to carry some charge Q_α such that $Q = \sum_\alpha Q_\alpha$. However, in the case of non-on-site translation symmetry, all translation-symmetric product states, which take the form $|\alpha\rangle^{\otimes L}$, can carry only trivial charge (lattice momentum). This suggests that nontrivial momentum is an inherently nonlocal quantity that cannot be reproduced without faraway regions still retaining some entanglement knowledge of each other; i.e., the state must be long-range entangled.

In condensed matter physics, we are often interested in ground states of translational-invariant local Hamiltonians. If the ground state is short-range entangled (SRE) [1] in the sense that it is connected to a product state through a finite-depth (FD) quantum circuit, then we expect the ground state to be unique, with a finite gap separating it from the excited states. In contrast, for long-range entangled (LRE) [1] ground states, we expect certain "exotic" features: Possible options include spontaneous symmetry-breaking cat states (e.g., Greenberger-Horne-Zeilinger-like states), topological orders (e.g., fractional quantum Hall states), and gapless states (e.g., metallic or quantum critical states). Theorem 1 provides us an opportunity to explore the interplay between translation symmetry and the above modern notions. An immediate corollary is as follows.

Corollary 1.1.—If a nonzero momentum state $|\Psi\rangle$ is realized as a ground state of a local spin Hamiltonian, then the ground state cannot be simultaneously unique and gapped. Possible options include (i) a gapless spectrum, (ii) intrinsic topological order, and (iii) spontaneous translation symmetry breaking.

In fact, we show in Sec. III B that option (ii) is a special subset of option (iii) through the mechanism of "weak symmetry breaking" [2].

Our result is reminiscent of the celebrated Lieb-Schultz-Mattis-Oshikawa-Hastings (LSMOH) theorems [3–5], which state that, in systems with charge U(1) and translation symmetries, a ground state with fractional U(1) charge filling (per unit cell) cannot be SRE. In our case, the nontrivial lattice momentum $e^{iP}\neq 1$ plays a very similar role as the fractional charge density in LSMOH. In fact, as we discuss in Sec. III A, our theorem can be viewed as a more basic version of LSMOH that involves only translation symmetry, from which the standard LSMOH can be easily derived. As a by-product, we also discover a previously unknown version of LSMOH constraint that involves an on-site $\mathbb{Z}_n$ symmetry and lattice translations.

The rest of this paper is structured as follows: In Sec. II, we provide a proof of Theorem 1 via a quantum circuit approach and generalize it to fermion systems. Three consequences of Theorem 1 are discussed in Sec. III: In Sec. III A, we discuss several LSMOH-type theorems; in Sec. III B, we show that a gapped topological order must *weakly* break translation symmetry if one of its ground states on torus has nonzero momentum—this is a generalization of the Tao-Thouless physics in the fractional quantum Hall effect [6,7]; in Sec. III C, we discuss the implication of Theorem 1 for the classification of crystalline symmetry-protected topological phases. We end with some discussions in Sec. IV.

II. PROOF

In this section, we prove that SRE states necessarily possess trivial momentum, conversely implying that all nontrivial momentum ground states must be LRE. The approach that we take utilizes the quantum circuit formalism, which is equivalent to the usual adiabatic Hamiltonian evolution formulation [1,8] but conceptually cleaner. In particular, we harness the causal structure of quantum circuits, which allows us to "cut and paste" existing circuits to create useful new ones.

We shall first prove Theorem 1 in one space dimension, from which the higher-dimensional version follows immediately.

A. Proof in 1D

First, let us specify our setup more carefully. We consider a spin (boson) system with a local tensor product Hilbert space $\mathcal{H}=\otimes_i \mathcal{H}_i$ where $\mathcal{H}_i$ is the local Hilbert space at unit cell i. The system is put on a periodic ring with L unit cells so $i\in\{1,2\ldots L\}$. In each unit cell, the Hilbert space $\mathcal{H}_i$ is q dimensional (q does not depend on i), with a basis labeled by $\{|a_i\rangle_i\}$ ($a_i\in\{0,1\ldots q-1\}$). The translation symmetry is implemented by a unitary operator that is uniquely defined through its action on the tensor product basis

$$\begin{aligned}T\colon\ &|a_1\rangle_1\otimes|a_2\rangle_2\otimes\cdots|a_{L-1}\rangle_{L-1}\otimes|a_L\rangle_L\\ &\to|a_L\rangle_1\otimes|a_1\rangle_2\otimes\cdots|a_{L-2}\rangle_{L-1}\otimes|a_{L-1}\rangle_L.\end{aligned}\quad(1)$$

Under this definition of translation symmetry (which is the usual definition), we have [9] $T^L=1$, and any translational-symmetric product state $|\varphi\rangle^{\otimes L}$ has trivial lattice momentum $e^{iP}=1$.

Now consider a SRE state $|\Psi_{P(L)}\rangle$ with momentum $P(L)$. By SRE, we mean that there is a quantum circuit U with depth $\xi\ll L$ that sends $|\Psi_{P(L)}\rangle$ to the product state $|\mathbf{0}\rangle\equiv|0\rangle^{\otimes L}$ (we do not assume U to commute with translation). The depth ξ is roughly the correlation length of $|\Psi_{P(L)}\rangle$. Our task is to prove that $P(L)=0 \bmod 2\pi$ as long as $\xi\ll L$. Notice that this statement is, in fact, stronger than that for a FD circuit, which requires $\xi\sim O(1)$ as $L\to\infty$. For example, our result holds even if $\xi\sim\mathrm{PolyLog}(L)$, which is relevant if we want the quantum circuit to simulate an adiabatic evolution more accurately [10]. Our result is also applicable if the existence of U requires extra ancilla degrees of freedom (d.o.f.) that enlarges the on-site Hilbert space to $\tilde{\mathcal{H}}_i$ with dimension $\tilde{q}>q$ (for example, see Ref. [11]), since ancilla d.o.f. by definition come in product states and, therefore, cannot change the momentum.

The proof is split into two steps, where in step 1 we first prove that the momentum is trivial for all $L=mn$, where $m,n\in\mathbb{Z}^+$ are mutually coprime satisfying $m,n\gg\xi$. In step 2, we use the results of step 1 to show that this may be extended to all other lengths.

Step 1.—A key ingredient of the proof is to recognize that the entanglement structure of the SRE state $|\Psi_{P(L)}\rangle$ on system size $L=mn$, where $m,n\in\mathbb{Z}^+$ and $n\gg\xi$, is adiabatically connected to that of m identical unentangled length-n SRE systems. The existence of such an adiabatic deformation, which is of a similar flavor to those presented in Refs. [12,13], is due to the finite correlation length of SRE systems and is explained in the following paragraph.

Take the SRE state $|\Psi_{P(L)}\rangle$ placed on a periodic chain of length $L=mn$ with $m,n\in\mathbb{Z}^+$ and $n\gg\xi$. Let us try to decouple this system at some point (say, between site i and $i+1$) via an adiabatic evolution, creating an "open" chain. To show that such a decoupling cut exists, we use the fact that SRE states always have a FD quantum circuit U that sends the ground state to the $|\mathbf{0}\rangle\equiv|0\rangle^{\otimes L}$ product state [see Fig. 1(a)]. The appropriate cut is then created by modifying this circuit to form a new light-cone-like FD quantum

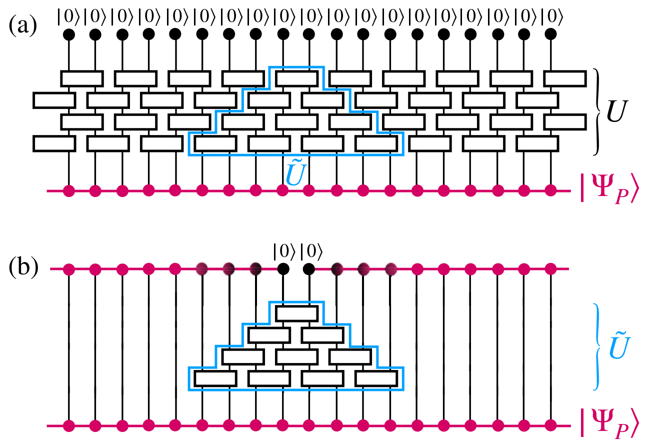

FIG. 1. Depiction of finite-depth quantum circuits applied on $|\Psi_P\rangle$. Here, qudits are depicted as solid circles, while unitaries are depicted as rectangles. (a) A SRE state $|\Psi_P\rangle$ is always connected to the $|\mathbf{0}\rangle$ trivial state via a FD quantum circuit U. From U, a light-cone-like "adiabatic cut" $\tilde{U}$ can be created (framed in blue). (b) $\tilde{U}$ connects $|\Psi_P\rangle$ to a state that is completely decoupled across the cut.

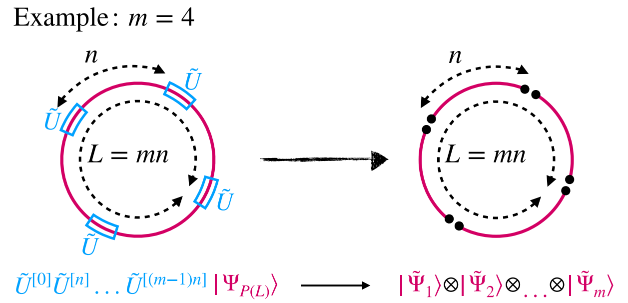

FIG. 2. Illustration of the adiabatic cutting procedure on a periodic length $L = mn$ chain. Here, we take the $m = 4$ example to demonstrate how four identical cuts, applied by $\tilde{U}$ (blue rectangle) at every nth link, on a length $L = 4n$ state $|\Psi_{P(L)}\rangle$ (purple circle) produce four decoupled length-n SRE states.

circuit $\tilde{U}$ with all unitaries outside the "light cone," i.e., those that do not affect the transformation that sends the two sites i and $i+1$ to $|0\rangle$, set to identity [see Fig. 1(b)]. Such a modified circuit spans approximately ξ qudits on either side of the cut and by construction takes the two sites on either side of the cut to $|0\rangle$, thus completely removing any entanglements across the link [14]. Let us concretely take $\tilde{U}^{[0]}$ to denote the appropriate light-cone cut between the last and first qudits (recall that we are on a ring) and define the shifted adiabatic cut between the $x-1$ and xth qudits to be $\tilde{U}^{[x]} \equiv T^x \tilde{U}^{[0]} T^{-x}$. If the ground state is translation symmetric, we have $\tilde{U}^{[x]}|\Psi_{P(L)}\rangle = e^{-ixP(L)} T^x \tilde{U}^{[0]}|\Psi_{P(L)}\rangle$, so we see that $\tilde{U}^{[x]}$ performs the same cut (up to a phase factor) at any link. By construction, this means that the local density matrices of a region surrounding the cut obey $\rho_{lr} = \rho_l \otimes |00\rangle\langle 00| \otimes \rho_r$, where the left (right) region to the cut is denoted l (r), which, in turn, implies that the operation $\tilde{U}^{[x]}$ disentangles the system along that cut.

The cutting procedure may be simultaneously applied to two separate links, as long as they are separated by a distance much greater than the correlation length. With this in mind, let us identically apply the cut on an $L = mn$ length system with a cut after every nth qudit, as depicted in Fig. 2, via the FD quantum circuit $\tilde{U}^{[0]}\tilde{U}^{[n]}\ldots\tilde{U}^{[(m-1)n]}$. Since the adiabatic deformation fully disentangles the system across the cuts, the resulting state should take the form $|\tilde{\Psi}_1\rangle \otimes |\tilde{\Psi}_2\rangle \otimes \cdots |\tilde{\Psi}_m\rangle$, where each $|\tilde{\Psi}_i\rangle$ is an n-block SRE state.

Now let us examine the symmetries of this resultant system. The original $\mathbb{Z}_{mn}$ translation symmetry, generated by operator T, of the original system is broken by the adiabatic deformation. However, the $\mathbb{Z}_m$ translation symmetry subgroup, generated by operator T^n, is preserved, since by construction identical cuts occur at every nth junction. This immediately implies that all the n-block states are identical $|\tilde{\Psi}_i\rangle = |\tilde{\Psi}\rangle$ and the total state after the cut is simply $|\tilde{\Psi}\rangle^{\otimes m}$. Thus, we know that the original $\mathbb{Z}_m$ quantum number is the same as the final one, which must be trivial, since we are dealing with an n-block product state $|\tilde{\Psi}\rangle^{\otimes m}$. This implies

$$nP(L) = 0 \mod 2\pi, \tag{2}$$

$\forall\, L = mn$ with $m, n \in \mathbb{Z}^+$ and $n \gg \xi$.

Using this relation on a general system length $L = p_1^{q_1} p_2^{q_2} \ldots p_d^{q_d}$ (here, we are using prime factorization notation), we arrive at the condition

$$P(L) = 0 \mod \frac{2\pi}{p_1^{r_1} p_2^{r_2} \ldots p_d^{r_d}}, \tag{3}$$

$\forall\, r_i \in \{1, \ldots, q_i\}$ such that $p_1^{r_1} p_2^{r_2} \ldots p_d^{r_d} \gg \xi$. If L factorizes into at least two mutually coprime numbers m, n with $m, n \gg \xi$, then these conditions can be satisfied only if

$$P(L) = 0 \mod 2\pi, \tag{4}$$

which is satisfied for almost all large enough L.

Step 2.—There is a sparse set of cases for which step 1 does not enforce trivial momentum, the most notable case being when $\tilde{L} = p^q$ with p prime and $q \in \mathbb{Z}^+$. Factorizations such as $\tilde{L} = p_1^{q_1} p_2$ are also not covered if $p_1^{q_1} \gg \xi$.

To show that these cases also possess trivial momentum, once again take a SRE state $|\Psi_{P(L)}\rangle$ on a general length-L system with momentum $P(L)$ mod 2π. By the definition of a SRE state, there exists a FD quantum circuit V_L such that $|\Psi_{P(L)}\rangle = V_L|\mathbf{0}\rangle$. This circuit obeys $TV_L T^\dagger|\mathbf{0}\rangle = e^{iP(L)} V_L|\mathbf{0}\rangle$, meaning that it boosts the trivial momentum of the $|\mathbf{0}\rangle$ state by $P(L)$ mod 2π. Consider the composition of a circuit

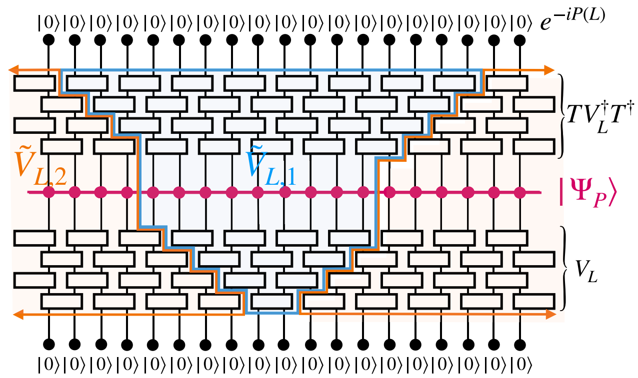

FIG. 3. Illustration of splitting $TV_L^\dagger T^\dagger V_L = \tilde{V}_{L,1}\tilde{V}_{L,2}$ with $\tilde{V}_{L,1}\tilde{V}_{L,2}|\mathbf{0}\rangle = e^{-iP(L)}|\mathbf{0}\rangle$. Here, we take a snapshot of the circuit to focus on $\tilde{V}_{L,1}$ (framed in blue); however, the support of $\tilde{V}_{L,1}$ (in the depicted example, 16 qudits) is actually much smaller than the system length. Recall that the circuit is periodic such that the orange arrows, corresponding to components of $\tilde{V}_{L,2}$ (framed in orange), eventually connect on the far side of the ring.

$$(TV_L^\dagger T^\dagger)V_L|\mathbf{0}\rangle = e^{-iP(L)}|\mathbf{0}\rangle. \tag{5}$$

As may be understood via the causality structure, the phase $e^{-iP(L)}$ comes piecewise from light-cone circuits. Let us understand this in detail: Split $\tilde{V}_L \equiv TV_L^\dagger T^\dagger V_L$ into a light-cone circuit $\tilde{V}_{L,1}$ and reverse light-cone circuit $\tilde{V}_{L,2}$ such that $\tilde{V}_L = \tilde{V}_{L,1}\tilde{V}_{L,2}$, as depicted in Fig. 3. The causal structure of the light cone guarantees that a gate U_1 in $\tilde{V}_{L,1}$ and a gate U_2 in $\tilde{V}_{L,1}$ must commute if U_2 appears in a layer after U_1, which then allows for the decomposition $\tilde{V}_L = \tilde{V}_{L,1}\tilde{V}_{L,2}$. Although the exact form of this decomposition is quite malleable, for concreteness let us define $\tilde{V}_{L,1}$ to be constructed causally such that the first (lowest) layer consists of a single two-qudit gate (as seen in Fig. 3). $\tilde{V}_{L,1}$ has support over qudits in the range $[L-\eta, L]$, where by the SRE nature $\eta \ll L$. Because of Eq. (5), we see that

$$\tilde{V}_{L,2}|\mathbf{0}\rangle = |0\ldots0\rangle^{[1,L-\eta-1]} \otimes |\alpha\rangle^{[L-\eta,L]} \tag{6}$$

for some $|\alpha\rangle$. By construction, we have

$$\tilde{V}_{L,1}|\alpha\rangle = e^{-iP(L)}|0\ldots0\rangle^{[L-\eta,L]}, \tag{7}$$

such that we satisfy Eq. (5).

Now, we extend the circuit V_L from length L to nL for some $n \in \mathbb{Z}^+$, where n, L are coprime and $\gg \xi$, and denote this extended circuit V_{nL}. To do this, we simply unstitch the circuit V_L at some link and reconnect the ends of n consecutive copies of this unstitched V_L circuit to create a FD quantum circuit V_{nL}. Let us see what happens to $\tilde{V}_{nL} \equiv TV_{nL}^\dagger T^\dagger V_{nL}$ by once again splitting the circuit into two $\tilde{V}_{nL} = \tilde{V}_{nL,1}\tilde{V}_{nL,2}$, where $\tilde{V}_{nL,k} = \prod_{j=0}^{n-1} T^{jL}\tilde{V}_{L,k}(T^\dagger)^{jL}$ with $k \in \{1,2\}$. By construction and due to the SRE nature of state construction

$$\tilde{V}_{nL,2}|\mathbf{0}\rangle^{\otimes n} = (|0\ldots0\rangle^{[1,L-\eta-1]} \otimes |\alpha\rangle^{[L-\eta,L]})^{\otimes n}. \tag{8}$$

However, by Eq. (7), we have

$$\tilde{V}_{nL,1}\tilde{V}_{nL,2}|\mathbf{0}\rangle^{\otimes n} = e^{-inP(L)}|\mathbf{0}\rangle^{\otimes n}, \tag{9}$$

so this implies

$$TV_{nL}T^\dagger|\mathbf{0}\rangle^{\otimes n} = e^{inP(L)}V_{nL}|\mathbf{0}\rangle^{\otimes n}, \tag{10}$$

which means that V_{nL} boosts the momentum of $|\mathbf{0}\rangle$ on a length-nL system to a state with momentum $P(nL) = nP(L) \bmod 2\pi$. In step 1, we show that $P(nL) = 0 \bmod 2\pi$, so this implies $nP(L) = 0 \bmod 2\pi$. Since this holds for two mutually coprime values of n, one concludes that 1D SRE translation-symmetric states have $P(L) = 0 \bmod 2\pi$ for all $L \gg \xi$.

B. Higher-dimensional extension

Our result can be extended to higher dimensions. Consider a d-dimensional lattice system and a state $|\Psi\rangle$ that has nontrivial momentum P along, say, the $\hat{x}$ direction. We can view the state as a 1D state along the $\hat{x}$ axis, with an enlarged Hilbert space per unit cell (generally exponentially large in $\prod_i L_i$ with i denoting the transverse directions). A finite-depth quantum circuit of the d-dimensional system also is a finite-depth quantum circuit when viewed as a 1D circuit along the $\hat{x}$ direction (a proof and a somewhat subtle example are presented in Appendix A; the converse is not true, but that does not concern us here). This immediately implies that a SRE state on the d-dimensional system must also be SRE when viewed as a 1D state along $\hat{x}$. What we prove in Sec. II A thus implies that the nontrivial momentum state $|\Psi\rangle$ must be long-range entangled. In particular, imposing locality in the transverse directions only further restricts possible FD circuit and certainly does not lead to possibilities beyond the 1D proof. This completes the proof of Theorem 1. ■

C. Fermion systems

It is not difficult to generalize our Theorem 1 to a fermionic system. The only subtlety is that the usual definition of translation symmetry in fermion systems has an extra $\mathbb{Z}_2$ sign structure compared to the naive implementation in Eq. (1). Instead of specifying the sign structure in the tensor product basis as in Eq. (1), it is more convenient to define the translation operator through $Tc_{i,\alpha}T^{-1} = c_{i+1,\alpha}$, where $c_{i,\alpha}$ is a fermion operator in unit cell i with some internal index α and $c_{L+1,\alpha} = c_{1,\alpha}$. This operator relation, together with $T|\mathbf{0}\rangle = |\mathbf{0}\rangle$ for the fermion vacuum, uniquely determines the action of T on any state.

Now consider a product state $|\varphi\rangle^{\otimes L}$; it is easy to verify that the momentum is $e^{iP} = 1$ for odd L and $e^{iP} = \pm 1$ for even L, where the sign is the fermion parity on each site $\langle\varphi|(-1)^{\sum_\alpha c_\alpha^\dagger c_\alpha}|\varphi\rangle$. We can then go through the proof in Sec. II, but now with fermion parity-preserving FD quantum circuits, and conclude the following.

Theorem 2.—Any short-range entangled translation eigenstate $|\Psi\rangle$ in a lattice fermion system must have momentum (say, in the x direction) $e^{iP_x} = 1$ if L_x is odd and $e^{iP_x} = \pm 1$ if L_x is even. States violating this condition must, in turn, be long-range entangled.

The details of the proof are presented in Appendix B.

Using the same proof technique, we can extend the above result further in various directions. We mention two such extensions without going into the details: (i) For L_x even, the option of $e^{iP_x} = -1$ is possible for a SRE state only if V/L_x is odd ($V = L_x L_y \ldots$ being the volume); (ii) if the total fermion parity is odd in a system with even V, then any translation eigenstate must be LRE.

III. CONSEQUENCES

One of the beauties of Theorem 1 lies in the nontrivial consequences that easily follow. For this section, it is useful to introduce an alternative, but equivalent, formulation of Theorem 1.

Theorem 1 (equivalent).—If there exists a finite-depth local unitary that boosts a state's momentum to a different value (mod 2π), then the state is necessarily long-range entangled.

The equivalence of this new formulation with the one introduced in Sec. I can be understood as follows: If all translation-symmetric SRE states possess trivial momentum, then nontrivial momentum states must be LRE. Thus, if there exists a finite-depth local unitary that can boost a state's momentum to a different value, then at least one of either the original or final state possesses nontrivial momentum and must be LRE. The other state is connected to the LRE state via a finite-depth local unitary and, thus, must also be LRE. The converse follows by contradiction: Assume there exists a SRE state that has nontrivial momentum. Such a state (by definition of SRE) is connected via a FD local unitary to the translation-symmetric direct-product state $|\alpha\rangle^{\otimes L}$, which, in turn, has trivial momentum. Since there now exists a FD local unitary that boosts the momentum to a different value, this implies that the original state is LRE, which leads to the contradiction.

This equivalent formulation allows for a direct test for long-range entanglement that we demonstrate on known and previously unknown Lieb-Schultz-Mattis (LSM) theories and topological orders. In the following discussions, we mostly focus on spin (boson) systems for simplicity, but the results can be generalized quite readily to fermion systems as well.

A. LSMOH constraints

The original LSM theory [3] along with the extensions by Oshikawa [4] and Hastings [5], collectively referred to as LSMOH, and their descendants are powerful tools for understanding the low-energy nature of lattice systems. In one of its most potent forms, the theorem states that systems with U(1) and translational symmetry that have noncommensurate U(1) charge filling must be "exotic," meaning that they cannot be SRE states. Since the conception of the original LSM theory, the field has flourished rapidly with many extensions that impose similar simple constraints based on symmetry [15–21] and connections to various fields of physics such as symmetry-protected topological (SPT) order and 't Hooft anomaly in quantum field theory [22–27]. These sort of constraints also have immediate experimental consequences, as they provide general constraints in determining candidate materials of exotic states such as quantum spin liquids [28]. Thus, unsurprisingly, there is a lot of interest in generating more LSMOH-like theorems that provide simple rules to find exotic states. In the following section, we provide simple proofs of some known and previously unknown LSMOH theorems.

The first example we consider is the aforementioned noncommensurate 1D $U(1) \times T$ LSM (T denotes the translation symmetry). In this case, there exists a local unitary momentum boost that is the large gauge transformation $U = e^{i(2\pi/L)\sum x\hat{n}_x}$, where $\hat{n}_x$ is the local number operator at x. Notice that this transformation is an on-site phase transformation and, thus, a FD quantum circuit of depth 1. The commutation relation with translation is $TUT^\dagger = e^{i2\pi(\hat{N}/L)}U$ ($\hat{N}$ being the total charge), which means that, for noncommensurate filling $(\hat{N}/L) \notin \mathbb{Z}$, we may always boost the momentum by a nontrivial value $2\pi(\hat{N}/L)$. Via the equivalent formulation of Theorem 1, this immediately implies that noncommensurate filling leads to a LRE state. This observation may be summarized as follows.

Corollary 2.1 [U(1) × *T LSM*].—A 1D translation and U(1) symmetric state that possesses noncommensurate U(1) charge filling must be long-range entangled.

The standard LSM theorem follows from this statement, since we may now apply it to a *ground* state of a 1D translation and U(1) symmetric local spin Hamiltonian to show that the state must be either gapless or a spontaneously symmetry-broken cat state. Notice that, strictly speaking, the statement we prove differs slightly from the standard LSM theorem, in that we do not directly prove the vanishing of the energy gap. Rather, we show that any simultaneous eigenstate of translation and $\hat{N}$ such that $\langle\hat{N}\rangle/L \notin \mathbb{Z}$ must be LRE. In principle, we do not even need to assume the parent Hamiltonian to be translation or U(1) symmetric, just that the state itself be translation and U(1) symmetric. In fact, the statement encompasses all states, not just the ground state, which is perhaps unsurprising, since LRE is fundamentally a property of a state and not the Hamiltonian.

The higher-dimensional $\mathrm{U}(1)\times T$ LSMOH theorem may be proved following the same logic if $\langle\hat{N}\rangle/L_i\notin\mathbb{Z}$ for some direction i (similar to what is done in Ref. [4]). For generic values of L_i, the above condition may not hold, and more elaborate arguments are needed (for example, see Ref. [29]), which we do not discuss here.

Our proof of the LSM theorem has an appealing feature compared to the classic proof [3]: We do not need to show that the state $|\Omega'\rangle = U|\Omega\rangle$ has excitation energy $\sim O(1/L)$ (relative to the ground state $|\Omega\rangle$); rather, it suffices for us to show that $|\Omega'\rangle$ has a different lattice momentum compared to $|\Omega\rangle$, which is enough to establish the LRE nature of $|\Omega\rangle$. Next, we use this simplifying feature to generalize the $\mathrm{U}(1)\times T$ LSM theorem to a new constraint involving only discrete $\mathbb{Z}_n\times T$ symmetries.

Let us consider a spin chain (1D) with translation symmetry and an on-site $\mathbb{Z}_n$ symmetry generated by $Z\equiv\otimes_i Z_i$ ($Z_i^n=1$). We consider the case when the system size $L=nM$ for some $M\in\mathbb{N}$ and study simultaneous eigenstates of the translation and $\mathbb{Z}_n$ symmetries. If such a state is an unentangled product state $\otimes_i|\varphi\rangle_i$, then by definition $Z=1$ when acting on this state; namely, the state carries trivial $\mathbb{Z}_n$ charge. This turns out to be true for any symmetric SRE state, which we now prove. Suppose a translation eigenstate $|\Psi\rangle$ has $Z|\Psi_P\rangle = e^{i2\pi Q/n}|\Psi\rangle$ for some $Q\neq 0$ (mod n). We can construct a local unitary which is a $\mathbb{Z}_n$ analog of the large gauge transform

$$U=\otimes_i Z_i^i, \tag{11}$$

where i is the unit cell index. For system size $L=nM$, one can verify that $TUT^{-1}U^\dagger = Z^\dagger$. This means that the momentum of the twisted state $U|\Psi\rangle$ differs from that of the untwisted $|\Psi\rangle$ by $\langle\Psi|Z^\dagger|\Psi\rangle = e^{-i2\pi Q/n}\neq 1$. By the equivalent form of Theorem 1, $|\Psi\rangle$ must be LRE. We, therefore, have the following.

Corollary 2.2 ($\mathbb{Z}_n\times T$ LSM).—A 1D translation and $\mathbb{Z}_n$ symmetric ground state that possesses nontrivial $\mathbb{Z}_n$ charge on system lengths $L=nM$ for some $M\in\mathbb{N}$ cannot be short-range entangled and, thus, is either a gapless or spontaneously symmetry-broken cat state.

The above statement also generalizes to higher dimensions if $L_i=nM$ for some direction i. For systems with U(1) symmetry, we can choose to consider a Z_L subgroup of the U(1), and the above $\mathbb{Z}_n\times T$ LSM theorem leads to the familiar $\mathrm{U}(1)\times T$ LSMOH theorem.

The two LSM-type theorems discussed so far, together with our Theorem 1, can all be viewed as "filling-type" LSM theorems, in the sense that these theorems constrain a symmetric many-body state $|\Psi\rangle$ to be LRE when $|\Psi\rangle$ carries certain nontrivial quantum numbers, such as lattice momentum $e^{iP}\neq 1$, total U(1) charge $Q\notin L\mathbb{Z}$, or total $\mathbb{Z}_n$ charge $Q\notin L\mathbb{Z}/n\mathbb{Z}$.

There is another type of LSM theorems that involve projective symmetry representations in the on-site Hilbert space, the most familiar example being the spin-1/2 chain with SO(3) symmetry. Our Theorem 1 can also be used to understand some (but possibly not all) of the projective symmetry types of LSM. Here, we discuss one illuminating example with on-site $\mathbb{Z}_2\times\mathbb{Z}_2$ symmetry in one dimension [30–32], such that the generators of the two $\mathbb{Z}_2$ group anticommutes when acting on the local Hilbert space: $X_iZ_i=-Z_iX_i$ (this can simply be represented by the Pauli matrices σ_x and σ_z). Now set the length $L=2N$ with odd N, and consider the three local unitaries $U_x=(\mathbb{1}\otimes\sigma_x)^{\otimes N}$, $U_z=(\mathbb{1}\otimes\sigma_z)^{\otimes N}$, and $U_{xz}=(\sigma_x\otimes\sigma_z)^{\otimes N}$. One can verify the commutation relations $TU_xT^\dagger=(-1)^{Q_x}U_x$, $TU_zT^\dagger=(-1)^{Q_z}U_z$, and $TU_{xz}T^\dagger=(-1)^{N+Q_x+Q_z}U_{xz}$. These commutation relations imply that the momentum of any symmetric state $|\Psi\rangle$ is boosted by $\Delta P=\pi$ by at least one of the three unitaries; therefore, $|\Psi\rangle$ must be LRE by Theorem 1.

B. Topological orders: Weak charge density wave

We now consider an intrinsic (bosonic) topological order on a d-dimensional torus. By definition, there are multiple degenerate ground states, separated from the excitation continuum by a finite energy gap. If one of the ground states $|\Psi_a\rangle$ has a nontrivial momentum, say, along the $\hat{x}$ direction, then according to Theorem 1 this state should be LRE even when viewed as a one-dimensional system in the $\hat{x}$ direction (with the other dimensions $y,z\ldots$ viewed as internal indices). Since there is no intrinsic topological order in one dimension, the only mechanism for the LRE ground state is spontaneous symmetry breaking. The lattice translation symmetry is the only relevant symmetry here—all the other symmetries can be explicitly broken without affecting the LRE nature of $|\Psi_a\rangle$, since the state still has nontrivial momentum. Therefore, $|\Psi_a\rangle$ must be a cat state that spontaneously breaks the $\hat{x}$-translation symmetry [33], also known as a charge density wave (CDW) state [34]. Furthermore, any other ground state $|\Psi_{b\neq a}\rangle$ can be obtained from $|\Psi_a\rangle$ by a unitary operator U_{ba} that is nonlocal in the directions transverse to $\hat{x}$ but crucially is local in $\hat{x}$—for example, in two dimensions U_{ba} corresponds to moving an anyon around the transverse cycle. By Theorem 1, we then conclude that $|\Psi_b\rangle$ is also a CDW in $\hat{x}$.

Perhaps the most familiar example of the above statement is the fractional quantum Hall effect. It is known that the $1/k$ Laughlin state on the torus is adiabatically connected to a quasi-one-dimensional CDW state in the Landau gauge, also known as the Tao-Thouless state [6,7]. For example, for $k=2$ the Tao-Thouless state with momentum $P=\pi n$, in the Landau orbit occupation number basis, reads

$$|101010\ldots\rangle + e^{i\pi n}|010101\ldots\rangle. \quad (12)$$

The CDW nature of the ground states is perfectly compatible with the topological order being a symmetric state, since there is no *local* CDW order parameter with a nonzero expectation value. The CDW order parameter in this case is nonlocal in the directions transverse to $\hat{x}$. For example, in two dimensions the CDW order parameter is defined on a large loop that wraps around the cycle transverse to $\hat{x}$. This phenomenon is dubbed weak symmetry breaking in Ref. [2]. The weak spontaneous symmetry breaking requires a certain degeneracy for the ground state. This degeneracy is naturally accommodated by the ground state manifold of the topological order. For example, for the above Tao-Thouless state at $k = 2$ the CDW order requires a twofold ground state degeneracy, which is nothing but the two degenerate Laughlin states on torus.

The above results can be summarized as follows.

Corollary 2.3.—If a ground state of a gapped topological order on a d-dimensional torus ($d > 1$) has a nontrivial momentum in $\hat{x}$, then any ground state of this topological order must *weakly* break translation symmetry in $\hat{x}$.

A further example of these results, alongside the effects of anyon condensation, applied upon the $\mathbb{Z}_2$ topologically ordered toric code is demonstrated in Appendix C. The above result also implies the following constraint on possible momentum carried by a topologically ordered ground state.

Corollary 2.4.—If a gapped topological order has q degenerate ground states on torus, then the momentum of any ground state in any direction is quantized:

$$P_i^{(a)} = 2\pi N_i^{(a)}/q, \quad (13)$$

where $N_i^{(a)}$ is an integer depending on the ground state (labeled by a) and direction i.

This is simply because, for other values of the momentum, the ground state degeneracy required by the spontaneous translation-symmetry-breaking order is larger than the ground state degeneracy from the topological order, which results in an inconsistency. An immediate consequence of the above corollary is that invertible topological orders (higher-dimensional states that are LRE by our definition but have only a unique gapped ground state on closed manifolds), such as the chiral E_8 state [2], cannot have nontrivial momentum on a closed manifold since $q = 1$.

The above statement immediately implies that the momenta of topological ordered ground states are robust under adiabatic deformations, as long as the gap remains open and translation symmetries remain unbroken. For the Tao-Thouless states, this conclusion can also be drawn from the LSM theorem if the U(1) symmetry is unbroken. Our result implies that the momenta of Laughlin-Tao-Thouless states are robustly quantized even if the U(1) symmetry is explicitly broken.

C. Crystalline symmetry-protected topological phases

There is growing interest and successes in understanding the SPT phases associated with crystalline symmetries [12,13,35–38]. When the protecting symmetry involves lattice translation, a crucial "smoothness" assumption [13,35] is used. Essentially, one assumes that for such SPT phases the inter-unit-cell entanglement can be adiabatically removed, possibly with the help of additional ancilla degrees of freedom. This allows one to formally "gauge" the translation symmetry [35] and build crystalline topological phases out of lower-dimensional states [13,37,38].

Our result, namely, Theorem 1, serves as a nontrivial check on the smoothness assumption in the following sense. If there were SRE states with nontrivial lattice momenta, such states would have irremovable inter-unit-cell entanglement, since unentangled states cannot have nontrivial momentum. Equivalently, the correlation length ξ cannot be tuned to be smaller than the unit cell size a. In fact, if such states exist, they would by definition be nontrivial SPT states protected solely by translation symmetry—such SPT states would be beyond all the recent classifications.

We note that our result is a necessary condition, but not a proof, for the smoothness assumption, as there may be other ways to violate the assumption without involving a ground state momentum. It will be interesting to see if the arguments used in this work can be extended to fully justify the smoothness assumption.

IV. DISCUSSIONS

In this paper, we have shown that a quantum many-body state with nontrivial lattice momentum is necessarily long-range entangled, hence establishing a simple yet intriguing connection between two extremely familiar concepts in physics: translation symmetry and quantum entanglement. Many directions can be further explored, which we briefly comment on in the remainder of this section.

One important aspect that we have so far skipped over is that LSM theory is, in fact, intimately connected to quantum anomalies [22–27]. This is natural, since they both provide UV conditions that constrain the low-energy behaviors. For the "projective symmetry" type of LSM theorems, this connection has been precisely established, and it is known that such LSM constraints correspond to certain discrete (quantized) 't Hooft anomalies. For the "partial filling" type of LSM such as the familiar $\mathrm{U}(1) \times T$ constraint, however, the connection has been discussed [39–42] but has yet to be fully developed. As we discuss in Sec. III A, our main result (Theorem 1) can be viewed as a partial filling type of LSM that involves only translation symmetry. It is, therefore, natural to ask whether Theorem 1 can be understood from an anomaly perspective. To achieve this goal, it is clear that the standard quantized 't Hooft anomaly is insufficient [a point which is also emphasized in Ref. [40] for the $\mathrm{U}(1)\times T$ LSM]—for example, the toric code discussed in Appendix C has no 't Hooft anomaly, since one can condense the e particle to obtain a trivial symmetric state. One would, therefore, need to expand the notion of anomaly to accommodate the partial filling type of LSM constraints including the one discussed in this work, possibly along the line of the "unquantized anomaly" discussed in Ref. [42]. We leave this aspect to future work.

Another powerful consequence of the traditional $\mathrm{U}(1)\times T$ LSM theorem is on the stability of the LRE ground states (with partial charge filling) under symmetric perturbations: Assuming the charge compressibility is finite (could be zero), then a small perturbation will not change the charge filling discontinuously, so the system remains LRE under small symmetric perturbations (unless the perturbation leads to spontaneous symmetry breaking like the BCS attraction). It is natural to ask whether the other partial filling types of LSM theorems can serve similar purposes. In fact, Ref. [42] discusses precisely this point under the notion of unquantized anomaly. The unquantized anomalies are very similar to Theorems 1 and 2 and Corollary 2.2, except that the key quantity is not the discrete charges (lattice momentum or $\mathbb{Z}_n$ charges) on a specific system size L, but the charge densities (momentum density or $\mathbb{Z}_n$ charge density). Such discrete charge densities cannot be defined for a fixed L but may be defined for a sequence of systems with $L\to\infty$. Reference [42] argues that, in the context of Weyl and Dirac semimetals, as long as these discrete charge densities are well behaved in the $L\to\infty$ limit, the unquantized anomalies protect the LRE nature of the states under symmetric perturbations. Our work here can be viewed as a rigorous justification of the unquantized anomalies in Ref. [42] on fixed system sizes.

Assuming a well-behaving momentum density in the thermodynamic limit, we can also apply our results to a Fermi liquid with a generic Fermi surface shape, such that the ground state from the filled Fermi sea has a nonvanishing momentum density (this requires breaking of time-reversal, inversion, and reflection symmetries). This can be viewed as a nonperturbative explanation for the stability of such a low-symmetry Fermi surface, even in the absence of the charge U(1) symmetry. (Recall that perturbatively the stability comes from the fact that the Cooper pairing terms no longer connect opposite points on the Fermi surface.)

Another question one may ask is whether a broader group of non-on-site symmetries obey similar charge and entanglement restrictions. It is easy to see that exactly the same constraint holds for glide reflections and screw rotations, since when the system is viewed as 1D there is no difference between glide reflection, screw rotation, and translation. It is also easy to see that the constraint does *not* hold for point group symmetries (rotations and reflections), because such symmetries are on site at some points in space (the fixed points of point groups). It is, therefore, important that translation symmetry is *everywhere* non-on-site. The question becomes even more intriguing if we consider more general unitary operators (such as quantum cellular automata [43]).

There are many more natural avenues for further exploration. The interplay between the nonlocal nature of translation symmetry with crystalline symmetry anomalies is not yet well understood and requires more concrete mathematical grounding such as a rigorous proof of when the smoothness condition is valid. Relatedly, it remains to be determined whether translation symmetry may be truly treated as an on-site symmetry and gauged or whether its nonlocality and nontrivial momentum may hinder or require modifications to the usual gauging process. Implications of our results on the "emergibility" of phases may also provide fruitful insights to achievable and unachievable states on the lattice [27,44]. Our work has shown without a doubt that translation symmetry is many faceted and plays a crucial role in the entanglement properties of crystalline materials.

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APPENDIX A: HIGHER-DIMENSIONAL FD QUANTUM CIRCUIT

Here, we show that a higher-dimensional ($d>1$) FD quantum circuit viewed in 1D (say, along $\hat{x}$), where each enlarged unit cell Hilbert space is now exponential in the transverse dimension $\prod_i L_i$, is also a FD quantum circuit.

To see this, let us decompose the higher-dimensional FD quantum circuit U into two sets of unitaries via "zigzag" cuts following light-cone pathways along $\hat{x}$. We depict an example of such a cut applied to a 2D FD quantum circuit U in Fig. 4. The two sets of unitaries consist of self-commuting "extended light-cone" unitaries $\{V_i\}$ and a set of self-commuting "extended reverse light-cone" unitaries $\{W_i\}$, such that $U=\prod_i W_i\prod_j V_j$. Because of the finite correlation length ξ in SRE systems and correlations

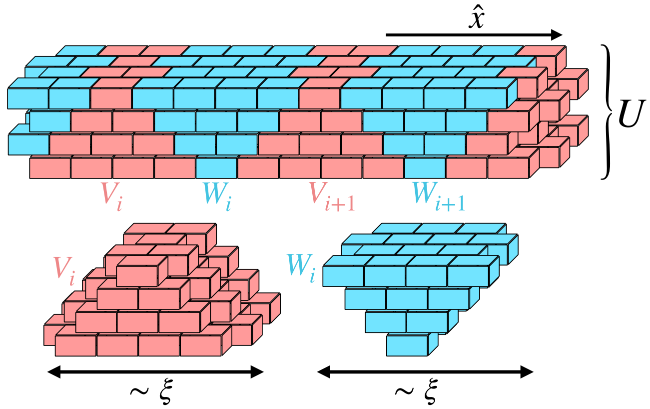

FIG. 4. A sample 2D FD quantum circuit U decomposed along $\hat{x}$ into extended light-cone unitaries $\{V_i\}$ (shaded red) and extended reverse light-cone unitaries $\{W_i\}$ (shaded blue). The exact position to begin the light-cone cut is variable, although here we do so symmetrically.

necessarily arise from the light-cone structure, each unitary component spans approximately $\xi \ll L$ unit cells in $\hat{x}$. This decomposition forms a 1D FD quantum circuit with two layers ($\{V_i\}$ and $\{W_i\}$).

Thus, we show that higher-dimensional SRE states remain SRE when viewed in 1D.

We now discuss a somewhat subtle example to further illustrate the point [45]. Consider a $(2+1)d$ system of fermions with global symmetry $\mathbb{Z}_2 \times \mathbb{Z}_2^f$, where $\mathbb{Z}_2^f$ is the fermion parity conservation. Essentially, we have two flavors of fermions, one that transforms trivially under the global $\mathbb{Z}_2$ and another that transforms with a minus sign. Now put the $\mathbb{Z}_2$-even fermion in a $p+ip$ superconductor and the $\mathbb{Z}_2$-odd fermion in a $p-ip$ superconductor. It seems natural to consider this state SRE, since the state can be trivialized by breaking the $\mathbb{Z}_2$ symmetry (which can be seen, for example, by examining the edge states). However, it turns out that, when put on torus, the state is strictly SRE only if the two fermions ($\mathbb{Z}_2$ even and odd) have the same boundary conditions in space. If the two fermions have opposite boundary conditions—say, one with periodic and the other with antiperiodic boundary conditions in $\hat{y}$—then, when viewed as a one-dimensional system along $\hat{x}$, the system forms a Kitaev chain with unpaired Majorana zero modes at the ends. Crucially, a Kitaev chain does not require any global symmetry (besides the $\mathbb{Z}_2^f$ which is anyway unbreakable) and, therefore, cannot be adiabatically connected to a trivial state. Such an "invertible" topological state is considered LRE in the definition adopted in this work. So, by simply twisting the boundary condition in $\hat{y}$ direction, we convert a SRE state to a LRE one.

The above example, in fact, does not contradict our result in this section. What it really shows is that the $\mathbb{Z}_2$ boundary condition cannot be twisted adiabatically for this state. Namely, there is no adiabatic path (FD quantum circuit) that can change the boundary condition (along a space cycle) from $\mathbb{Z}_2$ periodic to $\mathbb{Z}_2$ antiperiodic. Indeed, the most familiar adiabatic operation that could change the boundary condition (say, in $\hat{y}$) involves creating two $\mathbb{Z}_2$ vortices, moving one in $\hat{x}$ across the entire system, and reannihilating with the other one at the end. This process requires a timescale (or circuit depth) of the order of $O(L_x)$. One may wonder if a more clever construction can bring the circuit depth down to $O(1)$, but the previous discussion on the SRE vs LRE nature shows that this is impossible [46].

APPENDIX B: PROOF FOR FERMION SYSTEMS

In this section, we carefully go through the 1D proof for fermionic systems. The key difference between fermionic and bosonic systems, as discussed in Sec. II C, is that a product state in fermion system has momentum $P = 0 \bmod \pi$ instead of mod 2π for bosons. More specifically, for fermion systems with odd system length, all product states have zero momentum, just like the bosonic case. However, for even system length, product states may have either zero or π momentum, depending on the fermion parity per site being even or odd, respectively [47]. Additionally, we note that for even system length all translation symmetric product states possess even total fermion parity.

Fermionic local unitaries are defined via fermion parity-preserving Hamiltonians [48] and, thus, can be represented only via parity-preserving FD quantum circuits. More specifically, since parity is an on-site symmetry, each unitary that makes up the parity-preserving FD quantum circuit must themselves be parity preserving. The proof in Sec. II A directly carries over for SRE fermionic systems, but we must now keep in mind the system size and total fermion parity of the system. These initial conditions lead to different possibilities dependent on the achievable translation symmetric fermionic product states, as alluded to above.

The proof for odd-length fermionic systems for both even and odd total fermion parity follows step by step with the bosonic proof. For example, in step 1, there is no trouble with "cutting" a system of length $L = mn$ into m segments of length n, since the number of segments (m) is still odd if L is odd. The resulting odd number of segments implies trivial momentum when translating by n such that $nP(L) = 0 \bmod 2\pi$, just as in the bosonic case. Similarly, in step 2, we may always glue n numbers of length L segments with n being odd; nL is still odd, so that we may then apply step 1 to arrive at the same conclusion that $P(L) = 0 \bmod 2\pi$.

The story is slightly more complicated for even-length fermionic systems. Let us first consider the even total parity case. Here, in step 1, we must be careful when we cut the $L = mn$ length system into m length-n segments. If m is even, then we have the condition $nP(L) = 0 \bmod \pi$ (note that this is π instead of 2π). If m is odd, then n must be even, and we have the condition $nP(L) = 0 \bmod 2\pi$. If L is divisible by two mutually coprime numbers $p_1 \gg \xi$ and $p_2 \gg \xi$, i.e., $L = p_1 p_2 p_3$ for some $p_3 \in \mathbb{Z}^+$, then we have two scenarios: (i) One is even, say, p_1, and one is odd, say, p_2, such that we have $p_1 P(L) = 0 \bmod \pi$, $p_2 P(L) = 0 \bmod 2\pi$ for which we conclude $P(L) = 0 \bmod \pi$; (ii) both p_1 and p_2 are odd (in this case, p_3 is even such that L is even), and then we have $p_1 P(L) = 0 \bmod \pi$ and $p_2 P(L) = 0 \bmod \pi$ such that $P(L) = 0 \bmod \pi$. So the best condition we may arrive at is $P(L) = 0 \bmod \pi$, as opposed to $P(L) = 0 \bmod 2\pi$ in the bosonic case. For length that step 1 does not cover (e.g., $L = 2p$, where p is a prime and $2 \ll \xi$), we again turn to step 2. Here, the proof for the bosonic case applies with a minor alteration that in the final step, after the gluing procedure, we can conclude via step 1 only that $nP(L) = 0 \bmod \pi$. Choosing two mutually coprime values for n, we may then conclude that $P(L) = 0 \bmod \pi$ for all L. Here, we may intuitively gain an understanding of the mod π factor from observing the translation symmetric product states with even total fermion parity on even system lengths: The fermion vacuum state $|\mathbf{0}\rangle$ possesses zero momentum, and a state with one fermion per site, say, $\prod_{i=1}^{L} c_i^\dagger |\mathbf{0}\rangle$, possesses π momentum. The two states can be related to one another via a fermionic FD quantum circuit, e.g., a layer of $|0\rangle_i \otimes |0\rangle_{i+1} \to c_i^\dagger c_{i+1}^\dagger |0\rangle_i \otimes |0\rangle_{i+1}$ operators. This indicates that for SRE states zero and π momentum may be adiabatically connected with each other, thus leading to mod π rather than mod 2π.

The story is drastically different for even-length systems with odd total fermion parity. Here, the cutting procedure in step 1 leads to a contradiction: For $L = mn$ with $m, n \gg \xi$, m and/or n must be even. Let m be even such that we may create a FD quantum circuit that can divide the system into m identical segments of length n. However, since m is even and the segments are identical, then the total fermion parity must be even. This contradicts our initial assumption, so we must conclude that the initial state cannot be SRE; i.e., a FD quantum circuit that divides the system cannot be created, since the FD quantum circuit $U: |\Psi_{P(L)}\rangle \to |\mathbf{0}\rangle$ does not exist. For lengths that step 1 does not cover, we may again apply the logic of step 2 to arrive at a contradiction: If $|\Psi_{P(L)}\rangle$ is SRE, then, via the gluing procedure, we may construct a FD quantum circuit for length nL with odd n such that the total fermion parity is still odd. n may always be chosen such that $nL = \tilde{n}\,\tilde{m}$ with $\tilde{m}$ is even and $\tilde{m} \gg \xi$, so we may again create a circuit that divides the system into $\tilde{m}$ identical segments of length $\tilde{n}$, from which we conclude that the total fermion parity is even. By contradiction, this means that all translation symmetric even-length fermionic systems with odd total fermion parity must be LRE. Intuitively, this may be understood, since there exist no even-length translation symmetric product states with odd total fermion parity.

APPENDIX C: WEAK CDW EXAMPLE—TORIC CODE

In this section, we demonstrate the effects of weak translation symmetry breaking and anyon condensation on a well-known $\mathbb{Z}_2$ topological order example: the toric code with a gauge charge at each lattice site.

Take such a modified toric code on a square lattice $L_x \times L_y$ with even L_x and odd L_y and periodic boundary conditions, with a Hamiltonian given by

$$H = J_e \sum_{+} \prod_{i\in +} \sigma_x - J_m \sum_{\square} \prod_{i\in\square} \sigma_z, \tag{C1}$$

where $J_e, J_m > 0$. Here, we choose a positive sign in front of the $+$ term instead of the usual negative sign, which corresponds to a configuration with a gauge charge ("e" anyon) at each lattice site. By construction, this system respects translation symmetry in x and y, enacted by operators T_x and T_y.

Let us define large cycle electric and magnetic charge operators $V_x = \prod_{\bar{C}_x} \sigma_x$, $V_y = \prod_{\bar{C}_y} \sigma_x$, $W_x = \prod_{C_x} \sigma_z$, and $W_y = \prod_{C_y} \sigma_z$, where $C_{x,y}$ are given by cycles along the lattice links in the x/y directions and $\bar{C}_{x,y}$ are cycles in the x/y direction that are perpendicular to the lattice links. Physically, these operators correspond to creating anyon pairs (charge e excitations for W operators and flux "m" excitations for V operators), moving one of the anyons along the relevant cycle and then reannihilating the anyons.

The degenerate ground states of such a system may be derived from the following translation symmetric topological ground state:

$$|\mathbf{0}\rangle = (1 + V_x) \prod_{+} \left(1 - \prod_{i\in +} \sigma_x\right) |\Uparrow\rangle, \tag{C2}$$

where $|\Uparrow\rangle$ is the all spin-up state and for simplicity we ignore normalization. This state is an eigenstate of V_x and W_x with eigenvalue $|v_x, w_x\rangle = |1, 1\rangle$. Because of the relations

$$W_x V_y = -V_y W_x, \qquad W_y V_x = -V_x W_y, \tag{C3}$$

$$T_x V_y |\mathbf{0}\rangle = -V_y T_x |\mathbf{0}\rangle, \qquad T_x W_y |\mathbf{0}\rangle = W_y T_x |\mathbf{0}\rangle, \tag{C4}$$

operators V_y and W_y allow us to generate the remaining ground states $\{|\mathbf{0}\rangle, V_y|\mathbf{0}\rangle = |1, -1\rangle, W_y|\mathbf{0}\rangle = |-1, 1\rangle, V_y W_y|\mathbf{0}\rangle = |-1, -1\rangle\}$, of which $V_y|\mathbf{0}\rangle$ and $V_y W_y|\mathbf{0}\rangle$ have an $\hat{x}$ momentum boost of π compared to the other two ground states. Thus, by Corollary 2.3, all ground states must weakly break translational symmetry. How can we see this more concretely?

The easiest way to see this CDW effect is to take this toric code system with $L_y = 1$. Since all topological information is contained in a single plaquette, such a

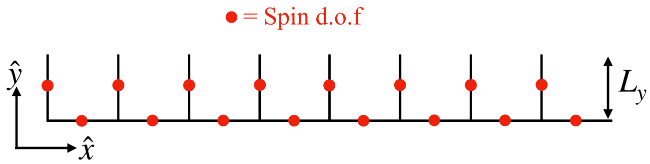

FIG. 5. The toric code system on a periodic lattice with $L_y = 1$. There are two spin degrees of freedom (d.o.f.) per unit cell in $\hat{x}$.

system is topologically no different compared to a general odd L_y system. With $L_y = 1$, the system reduces to two spins per unit cell in the $\hat{x}$ direction, which we depict in Fig. 5.

Here, the toric code Hamiltonian reduces to

$$H = J_e \sum_{i\in -} \sigma_x^{[i]} \sigma_x^{[i+1]} - J_m \sum_{i\in |} \sigma_z^{[i]} \sigma_z^{[i+1]}, \tag{C5}$$

where the i sum is over the horizontal lattice sites (denoted $-$) for the first term and over vertical lattice sites (denoted $|$) for the second term. This Hamiltonian simply describes two decoupled Ising chains, where the first term is in an antiferromagnetic state while the second term is in a ferromagnetic state. The ground states for the respective chains are $\{|\Rightarrow\Leftarrow\rangle \equiv |\rightarrow\leftarrow\rightarrow\leftarrow\cdots\rightarrow\leftarrow\rangle, |\Leftarrow\Rightarrow\rangle \equiv |\leftarrow\rightarrow\leftarrow\rightarrow\cdots\leftarrow\rightarrow\rangle\}$ and $\{|\Uparrow\rangle \equiv |\uparrow\uparrow\cdots\uparrow\rangle, |\Downarrow\rangle \equiv |\downarrow\downarrow\cdots\downarrow\rangle\}$. The four ground states of the total system are, thus, given by $\{|\Rightarrow\Leftarrow\rangle|\Uparrow\rangle, |\Rightarrow\Leftarrow\rangle|\Downarrow\rangle, |\Leftarrow\Rightarrow\rangle|\Uparrow\rangle, |\Leftarrow\Rightarrow\rangle|\Downarrow\rangle\}$. It is clear that these correspond to CDW states, since the antiferromagnetic Ising chain breaks the $\hat{x}$ directional translational $\mathbb{Z}_{L_x}$ symmetry group to $\mathbb{Z}_{L_x/2}$. Relating back to our original ground ("cat") state notation, we have

$$|\mathbf{0}\rangle = |\Rightarrow\Leftarrow\rangle|\Uparrow\rangle + |\Rightarrow\Leftarrow\rangle|\Downarrow\rangle + |\Leftarrow\Rightarrow\rangle|\Uparrow\rangle + |\Leftarrow\Rightarrow\rangle|\Downarrow\rangle, \tag{C6}$$

$$V_y|\mathbf{0}\rangle = |\Rightarrow\Leftarrow\rangle|\Uparrow\rangle + |\Rightarrow\Leftarrow\rangle|\Downarrow\rangle - |\Leftarrow\Rightarrow\rangle|\Uparrow\rangle - |\Leftarrow\Rightarrow\rangle|\Downarrow\rangle, \tag{C7}$$

$$W_y|\mathbf{0}\rangle = |\Rightarrow\Leftarrow\rangle|\Uparrow\rangle - |\Rightarrow\Leftarrow\rangle|\Downarrow\rangle + |\Leftarrow\Rightarrow\rangle|\Uparrow\rangle - |\Leftarrow\Rightarrow\rangle|\Downarrow\rangle, \tag{C8}$$

$$V_yW_y|\mathbf{0}\rangle = |\Rightarrow\Leftarrow\rangle|\Uparrow\rangle - |\Rightarrow\Leftarrow\rangle|\Downarrow\rangle - |\Leftarrow\Rightarrow\rangle|\Uparrow\rangle + |\Leftarrow\Rightarrow\rangle|\Downarrow\rangle, \tag{C9}$$

so we see that the toric code ground state indeed corresponds to weak translation symmetry breaking with nonlocal order parameter $\langle V_y\rangle$, which when viewed in 1D along $\hat{x}$ can be interpreted as a *local* order parameter.

Let us now consider the effects of anyon condensation and the phases that it may lead to. On general ground, we expect that a symmetric, confined state could arise from condensing the e anyon, since the e anyon does not carry any nontrivial projective quantum number in the toric code model. This means that, when viewed as a 1D system, the e condensation gives a transition between the CDW phase and a symmetric phase. Since the CDW phase has twofold symmetry breaking, it is natural to expect that the transition is simply of the Ising type. These phenomena can be easily demonstrated in the $L_y = 1$ limit, as we describe as follows. To drive condensation of the e anyons, we may add a $-h_e \sum_i \sigma_z$ term to Eq. (C1). For the horizontal bonds, this simply leads to a transverse-field Ising model. For $h_e < J_e$, we are still in the topological ordered state, for $h_e = J_e$, we have the Ising critical point, and, for $h_e > J_e$, we are in the disordered (trivial) state. In the $L_y = 1$ example, increasing h_e corresponds to transitioning to the $|\Uparrow\rangle$ phase for the antiferromagnetic Ising chain, which restores the $\hat{x}$ directional translation symmetry to give the trivial symmetric state.

Similarly, we may try to condense the type m anyons by adding a $-h_m \sum_i \sigma_x$ term to Eq. (C1). In this case, the anyon behaves nontrivially under either T_x or T_y, since they anticommute when acting on m. Condensing the anyon leads to symmetry breaking of either T_x or T_y (dependent on the specific energetics of the system), so we expect to transition to a true 2D cat (i.e., symmetry-broken) state. The effects of this condensation cannot be readily seen on the $L_y = 1$ lattice example, since translation in $\hat{y}$ ceases to have meaning. However, it is known that such a translation symmetry-breaking transition occurs such that the final state forms a valence bond solid [49].

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[46] If the symmetry is not $\mathbb{Z}_2$ but a continuous one such as U(1), the twist can be achieved adiabatically by slowly threading a (continuous) gauge flux. So for continuous symmetries we do not expect the SRE vs LRE nature to change under twisted boundary conditions—this is indeed compatible with the fact that $p \pm ip$ superconductors are not compatible with U(1) symmetry.

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