Adjoint Kirchhoff's Law and General Symmetry Implications for All Thermal Emitters

Cheng Guo,[1,*] Bo Zhao,[2] and Shanhui Fan[2,†]

[1]*Department of Applied Physics, Stanford University, Stanford, California 94305, USA*
[2]*Ginzton Laboratory and Department of Electrical Engineering, Stanford University, Stanford, California 94305, USA*

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We study the relation between angular spectral absorptivity and emissivity for any thermal emitter, which consists of any linear media that can be dispersive, inhomogeneous, bianisotropic, or nonreciprocal. First, we establish an adjoint Kirchhoff's law for mutually adjoint emitters. This law is based on generalized reciprocity and is a natural generalization of conventional Kirchhoff's law for reciprocal emitters. Using this law, we derive all the relations between absorptivity and emissivity for an arbitrary thermal emitter We reveal that such relations are determined by the symmetries of the system, which are characterized by a Shubnikov point group. We classify all thermal emitters based on their symmetries using the known list of all three-dimensional Shubnikov point groups. Each class possesses its own set of laws that relates the absorptivity and emissivity. We numerically verify our theory for all three types of Shubnikov point groups: Gray groups, colorless groups, and black and white groups. We also verify the theory for both planar and nonplanar structures with single or multiple diffraction channels. Our theory provides a theoretical foundation for further exploration of thermal radiation in general media.

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I. INTRODUCTION

Thermal radiation is a fundamental aspect of nature [1–7]. The control of thermal radiation has important implications for a wide range of applications such as energy harvesting [8–13], far-field radiation control [14–21], near-field thermal management [22–36], radiative cooling [37–40], imaging [41], and heat-assisted magnetic recording [42]. Any thermal emitter is fundamentally characterized by two key quantities: the angular spectral absorptivity $\alpha(\omega, -\hat{n}, \hat{p})$ and the angular spectral emissivity $e(\omega, \hat{n}', \hat{p}')$. $\alpha(\omega, -\hat{n}, \hat{p})$ represents the absorption coefficient for incident light at frequency ω and direction $-\hat{n}$ with a complex polarization vector $\hat{p}$. $e(\omega, \hat{n}', \hat{p}')$ measures the spectral emission power per unit area at the frequency ω into the direction $\hat{n}'$ with a polarization $\hat{p}'$, normalized against a blackbody at the same temperature as the emitter.

Most existing studies of thermal radiation are restricted to reciprocal thermal emitters [4]. Reciprocal thermal emitters are made of materials that satisfy Lorentz reciprocity. Reciprocity imposes fundamental constraints on their properties. In particular, it imposes a direct relation between the angular spectral absorptivity and emissivity [1,43]:

$$\alpha(\omega, -\hat{n}, \hat{p}) = e(\omega, \hat{n}, \hat{p}^*), \tag{1}$$

where $\hat{p}^*$ is the complex conjugation of $\hat{p}$. Equation (1) is Kirchhoff's law [44]. Besides its fundamental significance [1], Kirchhoff's law also has great practical importance [2–5,43,45]. It provides a useful guideline for designing reciprocal thermal emitters: to attain the desired emissivity, one only needs to design the absorptivity.

Despite its wide applicability, Kirchhoff's law does not hold for all thermal emitters. It is not required by the second law of thermodynamics, but by Lorentz reciprocity. Not all thermal emitters are reciprocal. Nonreciprocal thermal emitters that break Kirchhoff's law have been constructed using magneto-optical materials [46–53] and magnetic Weyl semimetals [54–57].

The capability of breaking Kirchhoff's law has fundamental significance. For example, solar energy harvesting requires an efficient solar absorber. However, by Kirchhoff's law, if the absorber is reciprocal, it must also radiate efficiently back to the sun. Radiation back to the sun represents an intrinsic loss mechanism. Consequently, reciprocal systems can not reach the ultimate efficiency limit for solar energy harvesting, known as the Landsberg limit [58]. Instead, to reach the Landsberg limit, one must use nonreciprocal systems [59–61]. It has been shown that the

*guocheng@stanford.edu
†shanhui@stanford.edu

Landsberg limit can be reached using nonreciprocal thermal emitters that efficiently absorb the incident sunlight but emit in a direction different from the incident direction so that the emitted power can be reharvested [62]. This example highlights the significant opportunities enabled by nonreciprocal thermal emitters. On the other hand, it also points to the need for a better understanding of the relation between angular spectral absorptivity and emissivity, which are no longer equal.

The development of nonreciprocal thermal emitters, therefore, requires a deeper understanding and proper generalization of conventional Kirchhoff's law. We need a theory that answers the following fundamental question: What is the relation, if any, between $\alpha(\omega, -\hat{n}, \hat{p})$ and $e(\omega, \hat{n}', \hat{p}')$ for *any* thermal emitter? The desired answer should be a natural generalization of Kirchhoff's law, and should reduce to Kirchhoff's law when restricted to reciprocal emitters. Similar to the role of Kirchhoff's law in reciprocal thermal radiation, such a theory would play a foundational role in the theoretical understanding of nonreciprocal thermal radiation. It would also provide a practical guideline for the design of nonreciprocal thermal emitters.

Several recent works have obtained results related to this problem [63,64]. In particular, Ref. [63] showed that for any thermal emitter, for every input mode with an absorptivity α, there exists an output mode of which the emissivity is equal to α. However, Ref. [63] does not in general provide a direct relation between $\alpha(\omega, -\hat{n}, \hat{p})$ and $e(\omega, \hat{n}', \hat{p}')$.

In this paper, we provide a generalization of Kirchhoff's law to all thermal emitters. The central idea of our work is this: the relation between $\alpha(\omega, -\hat{n}, \hat{p})$ and $e(\omega, \hat{n}', \hat{p}')$ for any thermal emitter is determined by its symmetry. This viewpoint allows us to illustrate the symmetry origin of conventional Kirchhoff's law and provide its broadest generalization based on symmetry. There is a symmetry that underlies reciprocity: any reciprocal system is invariant under an adjoint transformation $\mathcal{T}$. (A precise definition of $\mathcal{T}$ is provided in Sec. II C.) Kirchhoff's law is a direct consequence of this symmetry $\mathcal{T}$. A nonreciprocal emitter does not have $\mathcal{T}$ as a symmetry, but it may have other symmetries that can strongly constrain absorptivity and emissivity. These symmetries include geometric symmetries and compound symmetries. Geometric symmetries include the usual rotations, reflections, and improper rotations. Compound symmetries are not purely geometric; instead, they are the combination of geometric symmetries and $\mathcal{T}$. Both types of symmetries are common in many practical nonreciprocal thermal emitters. As a simple example, consider a nonreciprocal thermal emitter as shown in Fig. 1 as first proposed in Ref. [46], which consists of an n-InAs grating atop a uniform metal layer subjected to an external magnetic field B. This structure has both geometric and compound symmetries. The geometric symmetry is $\sigma_v(yz)$. The compound symmetries are $\mathcal{T}\sigma_v(xz)$ and $\mathcal{T}C_2(z)$. The set of all the geometric and compound symmetries determines the relations between absorptivity and emissivity for this structure.

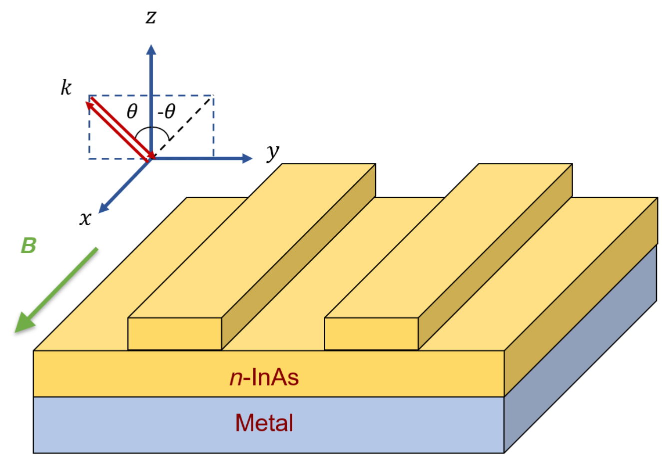

FIG. 1. A nonreciprocal thermal emitter as first proposed in Ref. [46]. It has a geometric symmetry $\sigma_v(yz)$ and compound symmetries $\mathcal{T}\sigma_v(xz)$ and $\mathcal{T}C_2(z)$. Here $\sigma_v(yz)$ is the mirror symmetry with respect to the yz plane, $\sigma_v(xz)$ is the mirror operation with respect to the xz plane, $C_2(z)$ is the twofold rotation around the z axis, and $\mathcal{T}$ is equivalent to reversing the magnetic field direction in this particular system.

Our main results are twofold. First, we establish an adjoint Kirchhoff's law that relates the angular spectral absorptivity and emissivity, respectively, of a pair of mutually adjoint emitters. (Mutually adjoint emitters are related by the adjoint transformation $\mathcal{T}$; a precise definition is provided in Sec. II C.) This law directly results from generalized reciprocity; hence it is a natural extension of conventional Kirchhoff's law. Second, based on adjoint Kirchhoff's law, we establish the general relations between angular spectral absorptivity and emissivity for an arbitrary emitter based on its symmetry. For a finite object, the set of all the relevant symmetries forms a Shubnikov point group [65–68]. There is a complete list of all the three-dimensional Shubnikov point groups, which provides a complete classification of any finite linear thermal emitters. Each class possesses its own set of laws that constrain absorptivity and emissivity. These Shubnikov point groups can be further categorized into three types: gray groups, colorless groups, and black and white groups. Emitters in a gray group are reciprocal, for which the usual Kirchhoff's law holds. Emitters in a colorless group are nonreciprocal, and there is no relation between any pair of $\alpha(\omega, -\hat{n}, \hat{p})$ and $e(\omega', \hat{n}', \hat{p}')$. Emitters in a black and white group are also nonreciprocal; however, there is a set of modified Kirchhoff's law specific to each group that relates particular pairs of $\alpha(\omega, -\hat{n}, \hat{p})$ and $e(\omega', \hat{n}', \hat{p}')$.

We note that the Shubnikov point group is widely used in crystallography and commonly associated with three-dimensionally periodic systems in the solid-state physics literature. However, as a mathematical concept, such a group is not restricted to 3D periodic systems. The thermal emitters considered here are not three-dimensionally periodic systems. But we show that their symmetry properties, in general, can be described using the mathematical construction of the Shubnikov point group.

The rest of the paper is organized as follows. Section II provides the general theory. Section III provides numerical verification of our theory. We conclude in Sec. IV. Detailed mathematical proofs are found in the Appendixes A–E.

II. THEORY

A. Geometry and conventions

To start, we define the geometry and provide the conventions. As shown schematically in Fig. 2, we consider an object of arbitrary shape made of a linear local inhomogeneous dispersive bianisotropic medium. It can be described by a 6×6 constitutive matrix $C(\omega,\mathbf{r})$:

$$\begin{pmatrix}\mathbf{D}\\ \mathbf{B}\end{pmatrix}=C(\omega,\mathbf{r})\begin{pmatrix}\mathbf{E}\\ \mathbf{H}\end{pmatrix}=\begin{pmatrix}\varepsilon(\omega,\mathbf{r}) & \zeta(\omega,\mathbf{r})\\ \eta(\omega,\mathbf{r}) & \mu(\omega,\mathbf{r})\end{pmatrix}\begin{pmatrix}\mathbf{E}\\ \mathbf{H}\end{pmatrix}, \qquad (2)$$

where ε, μ, ζ, η are 3×3 matrices of electric permittivity, magnetic permeability, electric-magnetic coupling strength, and magnetoelectric coupling strength, respectively. ω and $\mathbf{r}$ denote the angular frequencies and the spatial coordinates, respectively. We assume that the object is surrounded by vacuum as described by $\varepsilon=\varepsilon_0, \mu=\mu_0, \zeta=0, \eta=0$.

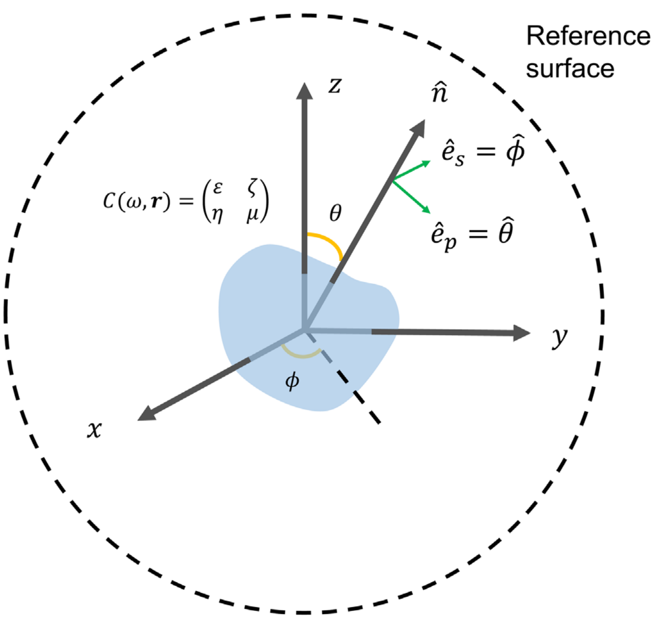

FIG. 2. Scheme. A thermal emitter of arbitrary shape is made of an inhomogeneous dispersive bianisotropic medium described by a constitutive matrix $C(\omega,r)$. (x,y,z) denote the Cartesian coordinates. (θ,ϕ) denote the polar and azimuthal angle of an outgoing direction $\hat{n}$. $\hat{e}_s$ and $\hat{e}_p$ denote the linear polarization basis. A reference surface enclosing the emitter is chosen to specify the phase of plane-wave basis.

Following Ref. [69], we choose the plane wave basis for both incoming and outgoing waves in free space outside the emitter. In particular, $|\omega,\hat{n},\hat{p}\rangle$ denotes an outgoing plane wave at a frequency ω propagating along a direction $\hat{n}$ with electric field along a complex polarization vector $\hat{p}$, and $|\omega,-\hat{n},\hat{p}\rangle$ denotes the corresponding incoming plane wave propagating along the direction $-\hat{n}$. To fix the phase convention, we choose a reference surface enclosing the emitter and specify the phase of each plane wave at the corresponding intersection point on the reference surface. We choose the linear polarization basis $|\sigma\rangle\equiv|\hat{e}_\sigma\rangle$, where $\sigma=p,s$ denote the p and s polarizations, respectively. By definition, the complex conjugate of a linear polarization is itself: $|\sigma^*\rangle=|\sigma\rangle$. A general polarization state is given by $|\hat{p}\rangle=\sum_\sigma c_\sigma|\sigma\rangle$, where c_σ are complex coefficients. The complex conjugate of $|\hat{p}\rangle$ is defined as $|\hat{p}^*\rangle=\sum_\sigma c_\sigma^*|\sigma\rangle$. For the coordinate system as defined in Fig. 2, we define the polarization vectors $\hat{e}_p=\hat{\theta}$ and $\hat{e}_s=\hat{\phi}$ for both the outgoing and incoming waves. Note $\hat{e}_p$ and $\hat{e}_s$ depend on $\hat{n}$. For an outgoing wave propagating along $\hat{n}$, $(\hat{e}_p,\hat{e}_s,\hat{n})$ forms a right triad. For an incoming wave propagating along $-\hat{n}$, $(\hat{e}_p,\hat{e}_s,-\hat{n})$ forms a left triad. Every basis state is normalized such that it carries unit intensity.

B. Scattering matrix, angular spectral absorptivity, and emissivity

The scattering property of a linear object is characterized by its scattering matrix S, where the matrix element $\langle\omega,\hat{n}',\hat{p}'|S|\omega,-\hat{n},\hat{p}\rangle$ denotes the scattering amplitude from an incoming plane wave $|\omega,-\hat{n},\hat{p}\rangle$ to an outgoing plane wave $|\omega,\hat{n}',\hat{p}'\rangle$.

The second law of thermodynamics establishes the following relations between the scattering matrix and angular spectral absorptivity and emissivity:

$$\alpha(\omega,-\hat{n},\hat{p})=1-\langle\omega,-\hat{n},\hat{p}|S^\dagger S|\omega,-\hat{n},\hat{p}\rangle, \qquad (3)$$

$$e(\omega,\hat{n},\hat{p})=1-\langle\omega,\hat{n},\hat{p}|SS^\dagger|\omega,\hat{n},\hat{p}\rangle. \qquad (4)$$

A detailed proof of the above relations can be found in Ref. [63]. But as a simple check, if S is unitary, both absorptivity and emissivity as calculated in Eqs. (3) and (4) vanish as expected.

From these relations Eqs. (3) and (4), we can readily derive the following integrated radiation law: for any linear object, the sums of the absorptivity and emissivity over all directions and polarizations are equal:

$$\sum_\sigma\int d\hat{n}\alpha(\omega,-\hat{n},\sigma)=\sum_\sigma\int d\hat{n}e(\omega,\hat{n},\sigma). \qquad (5)$$

The proof of Eq. (5) is provided in Appendix A.

C. Generalized reciprocity

At the heart of Kirchhoff's law is reciprocity [70]. Here we formulate generalized reciprocity for arbitrary objects. These relations reduce to ordinary reciprocity as a special case. The study of generalized reciprocity is important for the understanding of our theory of adjoint Kirchhoff's law.

First, we define mutually adjoint systems [69,71–73]. Our choice of the term "adjoint" is in agreement with Refs. [69,72]. For an "original" system defined by a constitutive matrix,

$$C(\omega,\boldsymbol{r}) = \begin{pmatrix} \varepsilon(\omega,\mathbf{r}) & \zeta(\omega,\mathbf{r}) \\ \eta(\omega,\mathbf{r}) & \mu(\omega,\mathbf{r}) \end{pmatrix}, \tag{6}$$

its adjoint system is defined by

$$\tilde{C}(\omega,\boldsymbol{r}) = \begin{pmatrix} \varepsilon^T(\omega,\mathbf{r}) & -\eta^T(\omega,\mathbf{r}) \\ -\zeta^T(\omega,\mathbf{r}) & \mu^T(\omega,\mathbf{r}) \end{pmatrix}. \tag{7}$$

We define the transformation $C(\omega,\boldsymbol{r}) \to \tilde{C}(\omega,\boldsymbol{r})$ as the adjoint transformation $\mathcal{T}$. $\mathcal{T}$ is involutory; i.e., the adjoint of the adjoint system is the original system. Thus, it defines a mutual relation. A self-adjoint system, i.e., a system whose adjoint is itself, is called "reciprocal"; e.g., vacuum is reciprocal.

We make three further remarks about $\mathcal{T}$. First, $\mathcal{T}$ is the symmetry operation that underlies reciprocity: a system is reciprocal if and only if it is invariant under $\mathcal{T}$. Second, compared to geometric transformation, $\mathcal{T}$ may seem unfamiliar to readers, but it is well defined and entirely conceivable. It may even be physically realizable for some systems. For example, for magneto-optical materials, $\mathcal{T}$ is equivalent to reversing the magnetic field direction; for moving media, $\mathcal{T}$ is equivalent to reversing the moving direction [74]. Nonetheless, the physical realizability of $\mathcal{T}$ is irrelevant for our purpose. Third, $\mathcal{T}$ is *not* the time-reversal transformation. Our entire theory does not involve any time-reversal transformation.

From Maxwell's equations, one can show that the dyadic Green's functions of mutually adjoint systems are the transpose of each other [69,75]. Moreover, for a medium that is lossy, lossless, or with gain, its adjoint is also lossy, lossless, or with gain, respectively, due to the relation between the relevant parts of the constitutive matrices of mutually adjoint media. (Detailed proof can be found in Ref. [69], pp. 107–108.)

It is natural to define the scattering matrix $\tilde{S}$ for the adjoint system with the same basis states and the same reference surface as S for the original system. Then the following generalized reciprocity relation holds between $\tilde{S}$ and S:

$$\langle \omega, \hat{n}', \sigma' | S | \omega, -\hat{n}, \sigma \rangle = \langle \omega, \hat{n}, \sigma | \tilde{S} | \omega, -\hat{n}', \sigma' \rangle, \tag{8}$$

where $\sigma = p,\ s$ and $\sigma' = p,\ s$ label the basis linear polarizations. Equation (8) holds because the dyadic Green's functions of mutually adjoint systems are mutually transpose. The original proof of Eq. (8) can be found in Ref. [69]; a shorter proof is provided in Appendix B.

From the above relation for the linear polarization basis, we can readily derive the following generalized reciprocity relation for arbitrary polarizations $\hat{p}'$ and $\hat{p}$:

$$\langle \omega, \hat{n}', \hat{p}' | S | \omega, -\hat{n}, \hat{p} \rangle = \langle \omega, \hat{n}, \hat{p}^* | \tilde{S} | \omega, -\hat{n}', \hat{p}'^* \rangle. \tag{9}$$

The proof of Eq. (9) is provided in Appendix C.

D. Adjoint Kirchhoff's law

Now we are ready to state our first main result.

Adjoint Kirchoff's law.—For mutually adjoint emitters,

$$\alpha(\omega, -\hat{n}, \hat{p}) = \tilde{e}(\omega, \hat{n}, \hat{p}^*), \quad e(\omega, \hat{n}, \hat{p}) = \tilde{\alpha}(\omega, -\hat{n}, \hat{p}^*), \tag{10}$$

where $\alpha(\tilde{\alpha}), e(\tilde{e})$ are the angular spectral absorptivity and emissivity for the original (adjoint) system, respectively.

The proof of Eq. (10) is provided in Appendix D. For self-adjoint (reciprocal) systems, the tildes in Eq. (10) can be dropped; thus adjoint Kirchhoff's law reduces to conventional Kirchhoff's law [Eq. (1)].

E. Relevant transformations, symmetry, and group

Adjoint Kirchhoff's law relates the angular spectral absorptivity and emissivity of mutually adjoint emitters. However, one is usually more interested in the relation between the angular spectral absorptivity and emissivity for a *single* emitter. We answer this question in the remaining part of this section. We will see that adjoint Kirchhoff's law plays a key role in connecting the angular spectral absorptivity and emissivity for a single emitter.

It is the intrinsic symmetry of an emitter that determines the relation between its angular spectral absorptivity and emissivity. In general, symmetry is invariance under transformation and is mathematically described by groups. Therefore, we must first identify the relevant transformation, symmetry, and group.

There are two types of relevant transformations. The first type is the usual geometric transformation including rotation, reflection, and improper rotation. The second type, called compound transformations, are geometric transformations combined with an adjoint transformation $\mathcal{T}$, where $\mathcal{T}$ by definition transforms a system to its adjoint system. $\mathcal{T}$ has two important properties: (1) $\mathcal{T}$ commutes with all geometric transformations and (2) $\mathcal{T}^2 = E$, where E is the identity transformation.

A symmetry is a transformation that leaves the system invariant. For any finite system, the set of all the geometric and compound symmetries forms a group G, which is mathematically termed a Shubnikov point group [65,66,68]. We denote $G = \{A, \mathcal{T}B\}$, where the two subsets A and $\mathcal{T}B$ contain geometric symmetries and compound symmetries, respectively. A and B are sets of geometric transformations.

There is a complete classification of all Shubnikov point groups in three dimensions. Any thermal emitter belongs to one and only one Shubnikov point group. Thus, we have a complete classification of all linear thermal emitters based on their symmetry.

From the general mathematical theory, for any Shubnikov point group $G = \{A, \mathcal{T}B\}$, A is nonempty since it contains E, while B can be empty or nonempty. Moreover, if B is nonempty, it must have the same cardinality (size) as A. Accordingly, all Shubnikov point groups can be classified into three types [68]:

(i) gray groups: $B = A$. In particular, $\mathcal{T} = \mathcal{T}E \in G$;
(ii) colorless groups: $B = \varnothing$;
(iii) black and white groups: $B \neq \varnothing$ and $B \cap A = \varnothing$. In this case, $\{A, B\}$ forms an ordinary point group G', and A forms a subgroup of G' with index 2.

An emitter is reciprocal if it belongs to a gray group. An emitter is nonreciprocal if it belongs to either a colorless group or a black and white group.

F. Relation of angular spectral absorptivity and emissivity for a single emitter

Now we are ready to state our second main result.

Relation of angular spectral absorptivity and emissivity for a single emitter.—For an arbitrary linear emitter with a Shubnikov point group $G = \{A, \mathcal{T}B\}$,

(1) $\forall\, a \in A$,

$$\begin{aligned} \alpha(\omega, -\hat{n}, \hat{p}) &= \alpha(\omega, -\hat{n}', \hat{p}'), \\ e(\omega, \hat{n}, \hat{p}) &= e(\omega, \hat{n}', \hat{p}'), \end{aligned} \tag{11}$$

where a transforms $\hat{n}, \hat{p}$ into $\hat{n}', \hat{p}'$.

(2) $\forall\, b \in B$,

$$\begin{aligned} \alpha(\omega, -\hat{n}, \hat{p}) &= e(\omega, \hat{n}', \hat{p}'^*), \\ e(\omega, \hat{n}, \hat{p}) &= \alpha(\omega, -\hat{n}', \hat{p}'^*), \end{aligned} \tag{12}$$

where b transforms $\hat{n}, \hat{p}$ into $\hat{n}', \hat{p}'$.

In words, (1) for a pair of directions and polarizations related by a geometric symmetry, the (angular spectral) absorptivities are equal, and emissivities are equal, and (2) for a pair of directions and polarizations related by a compound symmetry, the (angular spectral) absorptivity of one equals the emissivity of the other.

The proof of Eqs. (11) and (12) is provided in Appendix E.

Here we highlight three points. First, since G contains all the geometric and compound symmetries of the system, the above theorem gives all the relations about its angular spectral absorptivity and emissivity that can be stated from geometric symmetry and generalized reciprocity. Second, to determine all the *independent* relations, it is unnecessary to enumerate all the elements in G. Instead, it suffices to consider a set of generators of G. Last, only a compound symmetry containing $\mathcal{T}$ can relate the absorptivity and emissivity via adjoint Kirchhoff's law. This highlights the essential role of generalized reciprocity in connecting absorptivity and emissivity, which echoes the essence of conventional Kirchhoff's law.

These relations have different consequences for the three types of Shubnikov point groups.

(i) Gray groups: Since $\mathcal{T} \in G$, conventional Kirchhoff's law holds. This applies to reciprocal objects.
(ii) Colorless groups: Since G contains no compound symmetry, no relation exists between any pair of $\alpha(\omega, -\hat{n}, \hat{p})$ and $e(\omega, \hat{n}', \hat{p}')$.
(iii) Black and white groups: Since G contains compound symmetries, there is a set of modified Kirchhoff's law that relates specific pairs of $\alpha(\omega, -\hat{n}, \hat{p})$ and $e(\omega, \hat{n}', \hat{p}')$.

III. NUMERICAL VERIFICATION

Now we verify our theory using two sets of numerical examples. In the first set of examples, we consider a planar slab structure made of a general bianisotropic medium. In the second set of examples, we consider a multilayer grating structure made of magneto-optical and dielectric materials. The grating structure is chosen such that there exist multiple propagating diffraction channels. The planar structures and gratings possess translational symmetry; hence the integrals over all modes are reduced to the sum over the relevant diffraction orders. Using these examples, we verify our theory for both planar and nonplanar structures with single or multiple diffraction channels.

A. Bianisotropic planar slab

First, we consider a planar slab made of a general bianisotropic medium as shown in Fig. 3. We choose the slab thickness $d = 10\ \mu$m, and the wavelength $\lambda = 20\ \mu$m.

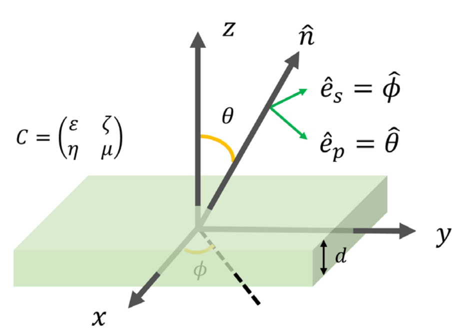

FIG. 3. The geometry for the examples in Figs. 4–6. The structure is a planar slab made of a general bianisotropic medium with a constitutive matrix C. Different C's are chosen in Figs. 4–6. The slab thickness $d = 10\ \mu$m. The light wavelength $\lambda = 20\ \mu$m. The coordinate system is the same as shown in Fig. 2.

We adopt the same coordinate system as shown in Fig. 2. For an outgoing direction $\hat{n} = (\theta, \phi)$, there is a corresponding incoming direction $-\hat{n} = (\pi - \theta, \pi + \phi)$, where $0 \leq \theta \leq \pi$ and $0 \leq \phi < 2\pi$ denote the polar and azimuthal angle of $\hat{n}$, respectively. We calculate the transmission and reflection coefficients of the structure using the transfer matrix method, then deduce the angular spectral absorptivity and emissivity from Eqs. (3) and (4), which results in

$$\begin{aligned}\alpha(\omega, -\hat{n}, \sigma) \equiv \alpha(\omega, \pi-\theta, \pi+\phi, \sigma) &= 1 - R_{s\sigma}(\omega, \pi-\theta, \pi+\phi) \\ &\quad - R_{p\sigma}(\omega, \pi-\theta, \pi+\phi) \\ &\quad - T_{s\sigma}(\omega, \pi-\theta, \pi+\phi) \\ &\quad - T_{p\sigma}(\omega, \pi-\theta, \pi+\phi),\end{aligned} \tag{13}$$

$$\begin{aligned}e(\omega, \hat{n}, \sigma) &\equiv e(\omega, \theta, \phi, \sigma) \\ &= 1 - R_{\sigma s}(\omega, \pi-\theta, \phi) - R_{\sigma p}(\omega, \pi-\theta, \phi) \\ &\quad - T_{\sigma s}(\omega, \theta, \phi) - T_{\sigma p}(\omega, \theta, \phi).\end{aligned} \tag{14}$$

Hereafter, $\sigma = s, p$. Here R and T are the power reflectance and transmittance, respectively. The first and second subscripts of R and T denote the polarization of outgoing and incoming waves, respectively.

We demonstrate three examples, one for each type of Shubnikov point groups.

1. Example 1. Gray group

In the first example, we consider a general reciprocal lossy bianisotropic medium with constitutive tensors

$$\varepsilon = \varepsilon_0 \begin{pmatrix} 0.52+0.05i & 0.91 & 1.70 \\ 0.91 & 1.59+0.08i & 1.37 \\ 1.70 & 1.37 & 2.00+0.04i \end{pmatrix}, \tag{15}$$

$$\mu = \mu_0 \begin{pmatrix} 0.09+0.03i & 1.11 & 1.77 \\ 1.11 & 0.53+0.05i & 1.01 \\ 1.77 & 1.01 & 0.75+0.03i \end{pmatrix}, \tag{16}$$

$$\zeta = -\eta^T = \sqrt{\varepsilon_0 \mu_0} \begin{pmatrix} 0.06i & 1.38i & 1.99i \\ 1.74i & 1.09i & 1.85i \\ 1.10i & 0.97i & 0.80i \end{pmatrix}. \tag{17}$$

This medium is self-adjoint, i.e., reciprocal. In addition, the constitutive tensors above are chosen such that $\mathcal{T}$ is the only symmetry of such a system. The Shubnikov point group is $G_1 = \{E, \mathcal{T}\}$. From our theory, we expect $\alpha(\omega, -\hat{n}, \sigma) = e(\omega, \hat{n}, \sigma)$ to be the only constraint.

Figure 4 shows the calculated angular spectral emissivity and absorptivity $e(\omega, \hat{n}, \sigma) = e(\omega, \theta, \phi, \sigma)$ and $\alpha(\omega, -\hat{n}, \sigma) = \alpha(\omega, \pi-\theta, \pi+\phi, \sigma)$. Indeed, we see that $\alpha(\omega, -\hat{n}, \sigma) = e(\omega, \hat{n}, \sigma)$, and there are no other constraints.

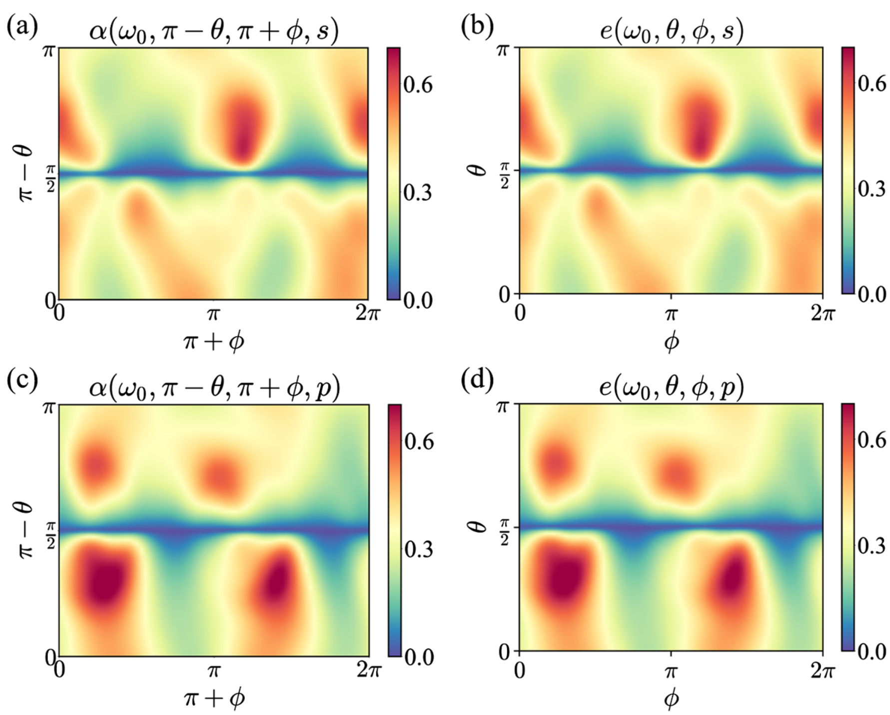

FIG. 4. Example 1. Gray group. The constitutive relation is given in Eqs. (15)–(17). (a),(c) Angular spectral absorptivity for s and p polarizations, respectively, along the incoming direction $-\hat{n} = (\pi - \theta, \pi + \phi)$. (b),(d) Angular spectral emissivity for s and p polarizations, respectively, along the outgoing direction $\hat{n} = (\theta, \phi)$.

2. Example 2. Colorless group

In the second example, we consider a general nonreciprocal lossy bianisotropic medium with constitutive tensors:

$$\varepsilon=\varepsilon_0\begin{pmatrix}0.06+0.10i & 0.36+0.43i & 1.01-0.48i\\ 0.36-0.43i & 1.14+0.03i & 0.65+0.06i\\ 1.01+0.48i & 0.65-0.06i & 1.50+0.04i\end{pmatrix}, \tag{18}$$

$$\mu=\mu_0\begin{pmatrix}1.41+0.01i & 0.82+0.45i & 0.70-0.08i\\ 0.82-0.45i & 1.73+0.08i & 0.54-0.21i\\ 0.70+0.08i & 0.54+0.21i & 0.59+0.054i\end{pmatrix}, \tag{19}$$

$$\zeta=\eta^{\dagger}=\sqrt{\varepsilon_0\mu_0}\begin{pmatrix}0.50+0.18i & 0.47+0.32i & 0.98+0.12i\\ 0.79+0.82i & 0.84+0.91i & 0.29+0.55i\\ 0.41+0.24i & 0.52+1.00i & 0.70+0.32i\end{pmatrix}. \tag{20}$$

Here we have assumed that the material loss is nongyrotropic, which is reasonable for many gyrotropic media 54,55]]. Such a system has neither geometric nor compound symmetry. The Shubnikov point group is $G_2=\{E\}$. Therefore, we expect that no relation exists between any pair of $\alpha(\omega,-\hat{n},\sigma)$ and $\alpha(\omega,\hat{n}',\sigma)$, $e(\omega,-\hat{n},\sigma)$ and $e(\omega,\hat{n}',\sigma)$, or $\alpha(\omega,-\hat{n},\sigma)$ and $e(\omega,\hat{n}',\sigma)$.

Figure 5 shows the calculated angular spectral emissivity and absorptivity $e(\omega,\hat{n},\sigma)=e(\omega,\theta,\phi,\sigma)$ and $\alpha(\omega,-\hat{n},\sigma)=\alpha(\omega,\pi-\theta,\pi+\phi,\sigma)$. Indeed, we see no relation exists.

3. Example 3. Black and white group

In the third example, we consider a specific nonreciprocal lossy medium with the following constitutive tensors:

$$\varepsilon=\varepsilon_0\begin{pmatrix}4.0+0.16i & 1.5i & 0\\ -1.5i & 4.0+0.16i & 2.0i\\ 0 & -2.0i & 4.0+0.16i\end{pmatrix},$$
$$\mu=\mu_0,\qquad \zeta=0,\quad \eta=0, \tag{21}$$

which characterize a uniaxial magneto-optical material with the magnetization $\hat{m}$ along the $0.8\hat{x}+0.6\hat{z}$ direction.

Such a system has both geometric and compound symmetries. The geometric symmetry is the inversion symmetry I. (We note that $\hat{m}$ is a pseudo-vector, and hence is invariant under inversion.) The compound symmetries include $\mathcal{T}\sigma_v(xz)$ and $\mathcal{T}C_2(y)$, where $\sigma_v(xz)$ is the mirror operation with respect to the xz plane and $C_2(y)$ is the twofold rotation around the y axis. (These symmetries are readily identified because $\mathcal{T}$ reverses $\hat{m}$.) The Shubnikov point group is $G_3=\{E,I,\mathcal{T}\sigma_v(xz),\mathcal{T}C_2(y)\}$.

From our theory, we determine all the relations from G_3. I requires

$$\alpha(\omega,\pi-\theta,\pi+\phi,\sigma)=\alpha(\omega,\theta,\phi,\sigma), \tag{22}$$

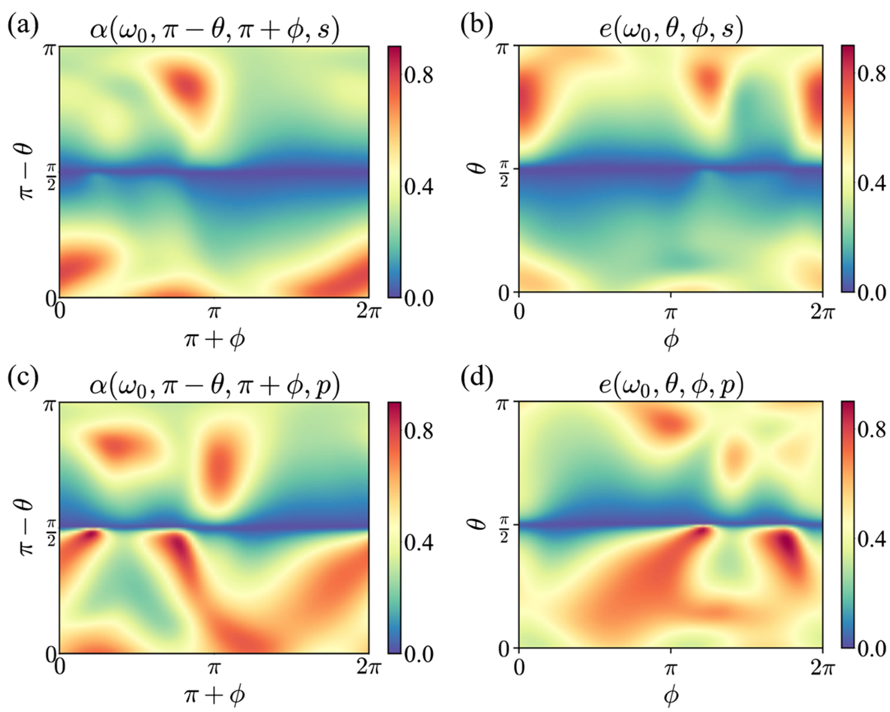

FIG. 5. Example 2. Colorless group. The constitutive relation is given in Eqs. (18)–(20). (a),(c) Angular spectral absorptivity for s and p polarizations, respectively, along the incoming direction $-\hat{n}=(\pi-\theta,\pi+\phi)$. (b),(d) Angular spectral emissivity for s and p polarizations, respectively, along the outgoing direction $\hat{n}=(\theta,\phi)$.

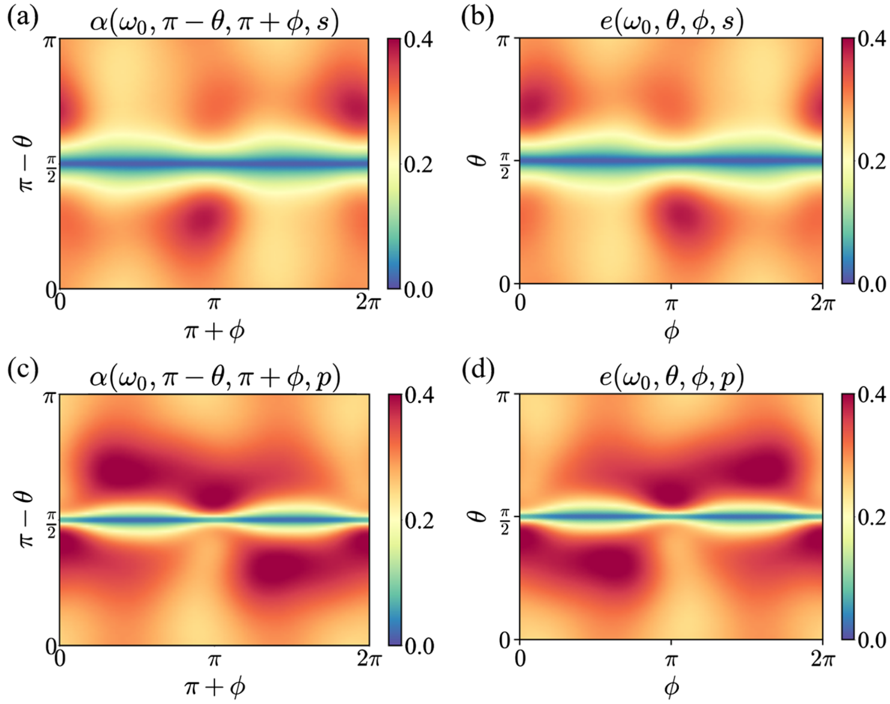

FIG. 6. Example 3. Black and white group. The constitutive relation is given in Eq. (21). (a),(c) Angular spectral absorptivity for s and p polarizations, respectively, along the incoming direction $-\hat{n} = (\pi - \theta, \pi + \phi)$. (b),(d) Angular spectral emissivity for s and p polarizations, respectively, along the outgoing direction $\hat{n} = (\theta, \phi)$.

$$e(\omega, \theta, \phi, \sigma) = e(\omega, \pi - \theta, \pi + \phi, \sigma). \tag{23}$$

$\mathcal{T}\sigma_v(xz)$ requires

$$\alpha(\omega, \pi - \theta, \pi + \phi, \sigma) = e(\omega, \theta, -\phi, \sigma). \tag{24}$$

$\mathcal{T}C_2(y)$ requires

$$\alpha(\omega, \pi - \theta, \pi + \phi, \sigma) = e(\omega, \pi - \theta, \pi - \phi, \sigma). \tag{25}$$

We note that these relations Eqs. (22)–(25) are not independent. For example, Eq. (25) can be derived from Eqs. (22)–(24), because $\mathcal{T}C_2(y) = I\mathcal{T}\sigma_v(xz)$.

Figure 6 shows the calculated angular spectral emissivity and absorptivity $e(\omega, \hat{n}, \sigma) = e(\omega, \theta, \phi, \sigma)$ and $\alpha(\omega, -\hat{n}, \sigma) = \alpha(\omega, \pi - \theta, \pi + \phi, \sigma)$. Indeed, we verify all the relations Eqs. (22)–(25) hold.

B. Nonreciprocal diffraction grating

Next, we consider a class of more complicated nonplanar structures: multilayer diffraction gratings. As shown in Fig. 7, the structures consist of a top grating, a middle dielectric layer, and a bottom substrate. We choose the operation wavelength $\lambda = 15$ μm. The grating and the substrate have a permittivity tensor,

$$\varepsilon_{mo} = \begin{pmatrix} -76.507 + 0.014i & 0 & 0 \\ 0 & -76.507 + 0.014i & -26.618i \\ 0 & 26.618i & -76.507 + 0.014i \end{pmatrix}, \tag{26}$$

describing a magnetic Weyl semimetal with the Weyl node separation $2b = 4.8$ nm^{-1} along the x direction. (See Ref. [55] for more details.) The dielectric material has a permittivity $\varepsilon = 10.254 + 0.052i$, close to that of SiC or Si at $\lambda = 15$ μm. For comparison, we consider two structures, as shown in Figs. 7(a) and 7(d). Both the gratings have a period $\Lambda = 20$ μm, and have two parallel ridges of different widths in a unit cell. The centers of neighboring ridges are separated by Λ_1 and Λ_2, respectively. The former grating has $\Lambda_1 = \Lambda_2 = \Lambda/2 = 10$ μm, whereas the latter grating has $\Lambda_1 = 12$ μm, $\Lambda_2 = 8$ μm. The other structure parameters are identical and provided in the caption of Fig. 7. These two structures have different symmetries. The former has both the geometric symmetry $\sigma_v(yz)$ and the

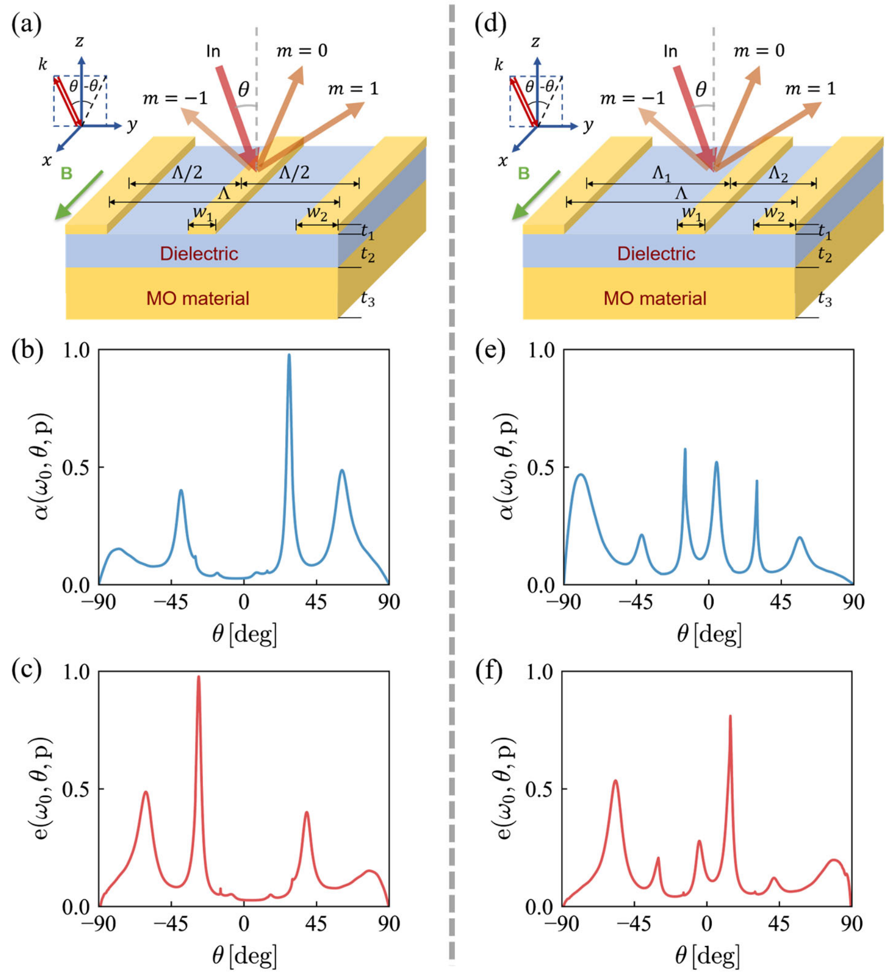

FIG. 7. (a)–(c) A nonreciprocal diffraction grating with a higher symmetry. (a) The structure consists of a top grating of thickness $t_1 = 0.68\ \mu$m, a middle dielectric of thickness $t_2 = 1.10\ \mu$m, and a bottom substrate of thickness $t_3 = 50\ \mu$m. The grating and the substrate are made of a magneto-optical material subjected to a magnetic field B along the x direction. Its permittivity tensor is given in Eq. (26). The dielectric has a permittivity $\varepsilon = 10.254 + 0.052i$. The grating has a periodicity $\Lambda = 20\ \mu$m along the y direction, and has two parallel ridges in a unit cell with widths $w_1 = 3.085\ \mu$m and $w_2 = 4.928\ \mu$m, respectively. The centers of two neighboring ridges are separated by $\Lambda_1 = 10\ \mu$m and $\Lambda_2 = 10\ \mu$m. The incident light is p polarized with a wavelength $\lambda = 15\ \mu$m and incident angle θ. There exist multiple propagating diffraction orders $\{m\}$ that depend on θ. (See Table I.) (b) Angular spectral absorptivity along the incoming direction $-\hat{n}$ denoted by θ. (c) Angular spectral emissivity along the outgoing direction $\hat{n}$ also denoted by θ. (d)–(f) A nonreciprocal diffraction grating with a lower symmetry. (d) The structure is identical to that in (a), except $\Lambda_1 = 12\ \mu$m and $\Lambda_2 = 8\ \mu$m. (e),(f) Angular spectral absorptivity and emissivity as functions of θ.

compound symmetries $\mathcal{T}\sigma_v(xz)$ and $\mathcal{T}C_2(z)$; the Shubnikov point group is $G_4 = \{E, \sigma_v(yz), \mathcal{T}\sigma_v(xz), \mathcal{T}C_2(z)\}$. The latter has only the mirror symmetry $\sigma_v(yz)$; the Shubnikov point group is $G_5 = \{E, \sigma_v(yz)\}$.

We consider p-polarized light incident in the yz plane with an incident angle θ. Since $\lambda < \Lambda$, there exist multiple propagating diffraction orders. When θ varies, the propagating diffraction orders $\{m\}$ vary accordingly (see Table I). Despite such a complicated diffraction scenario, our theory provides a simple prediction: $\alpha(\omega, \theta, p) = e(\omega, -\theta, p)$ for all θ for the former structure, while no relation exists for the latter structure. Here $\omega = 2\pi c/\lambda$. Our examples are inspired by the structure in Fig. 1 first proposed in Ref. [46]. For readers' convenience of comparison, we follow Ref. [46] and use θ to denote the incoming $-\hat{n}$ and outgoing $\hat{n}$ direction. Such a notation differs from that in the previous examples where (θ, ϕ) denotes $\hat{n}$ and $(\pi - \theta, \pi + \phi)$ denotes $-\hat{n}$.

We use the COMSOL Multiphysics® software [76] to calculate the scattering matrix S of the structure as a function of θ, then determine $\alpha(\omega, \theta, p)$ and $e(\omega, \theta, p)$ using Eqs. (3) and (4). Figures 7(b) and 7(c) and Figs. 7(e) and 7(f) plot the results for the two structures, respectively. Indeed, we verify all the theoretical predictions hold. In particular, we see a clear relation between the absorptivity and emissivity in the example of Fig. 7(a), since the two processes can be related by the compound symmetry $\mathcal{T}\sigma_v(xz)$ or $\mathcal{T}C_2(z)$. In contrast, no such relation exists

TABLE I. Incident angle dependence of diffraction orders for the structure in Fig. 7.

Incident angle θ	Diffraction orders $\{m\}$
$(-90°, -30°]$	$\{0, 1, 2\}$
$(-30°, -14.5°]$	$\{0, 1\}$
$(-14.5°, 14.5°]$	$\{-1, 0, 1\}$
$(14.5°, 30°]$	$\{-1, 0\}$
$(30°, 90°)$	$\{-2, -1, 0\}$

for the example in Fig. 7(d), since the structure no longer supports such compound symmetries.

IV. FINAL REMARK AND CONCLUSION

Before concluding, we make several final remarks.

First, we have focused on the relations between angular spectral absorptivity and angular spectral emissivity, two of the key surface radiative properties of thermal emitters [4]. Similar relations can be established for other surface radiative properties such as bidirectional spectral reflectivity and bidirectional spectral transmissivity [4], and for absorption and emission cross sections of antennas [69]. For all these extensions, the concepts of adjoint transformation and compound symmetry will again play an essential role.

Second, we have focused on the adjoint transformation associated with reciprocity. Such a transformation is appropriate for typical thermal emitters since it preserves the dissipative characteristics pointwise [69]. In addition to this adjoint transformation, there are other internal transformations and internal symmetries, such as those associated with time reversal or energy conservation [77]. However, the transformations associated with time reversal or energy conservation transform a lossy medium into a medium with gain, and therefore are less relevant for the discussions of typical thermal emitters that are lossy.

Third, our theory provides a useful guideline for the practical design of nonreciprocal thermal emitters. As we pointed out, there exist two types of nonreciprocal thermal emitters, black and white and colorless. An emitter of black and white type violates the conventional Kirchhoff's law, but there is a specific correlation between its emissivity and absorptivity. An emitter of colorless type has no direct relation between its emissivity and absorptivity. Either type may be preferable depending on application scenarios. In the design of nonreciprocal thermal emitters for a specific application, one should first decide a preferable type, then choose the corresponding symmetry group, and finally carry out a detailed design while respecting the symmetry. Thus, the symmetry principle can be used to regulate the design process and ensures the desired relation between absorptivity and emissivity. As an example, we have applied the symmetry analysis in the design of a semi-transparent nonreciprocal thermal emitter [13].

In conclusion, we study the general relations between angular spectral absorptivity and emissivity for an arbitrary thermal emitter that can be dispersive, inhomogeneous, bianisotropic, and nonreciprocal. We establish an adjoint Kirchhoff's law for mutually adjoint emitters based on generalized reciprocity. From this law, we reveal the intrinsic Shubnikov point group symmetries of any single object that determine the relations between its angular spectral absorptivity and emissivity. We provide a classification of thermal emitters based on their symmetry using the complete list of three-dimensional Shubnikov point groups. We numerically verify the theory for all three types of Shubnikov point groups: gray groups, colorless groups, and black and white groups. We also verify the theory for both planar and nonplanar structures with single or multiple diffraction channels. Our general theory provides a theoretical foundation for further exploration of thermal radiation in general media.

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APPENDIX A: PROOF OF THE INTEGRATED RADIATION LAW [EQ. (5)]

Here we prove Eq. (5). Note

$$\begin{aligned}\sum_\sigma \int d\hat{n}\alpha(\omega, -\hat{n}, \sigma) &= \sum_\sigma \int d\hat{n}(1 - \langle \omega, -\hat{n}, \sigma | S^\dagger S | \omega, -\hat{n}, \sigma\rangle) \\ &= \mathrm{Tr}(I - S^\dagger S),\end{aligned} \tag{A1}$$

$$\begin{aligned}\sum_\sigma \int d\hat{n}e(\omega, \hat{n}, \sigma) &= \sum_\sigma \int d\hat{n}(1 - \langle \omega, \hat{n}, \sigma | S S^\dagger | \omega, \hat{n}, \sigma\rangle) \\ &= \mathrm{Tr}(I - S S^\dagger),\end{aligned} \tag{A2}$$

where $\mathrm{Tr}(\cdot)$ is the trace of a linear operator, which is independent of the choice of basis. Since $\mathrm{Tr}(S^\dagger S) = \mathrm{Tr}(S S^\dagger)$, Eqs. (A1) and Eq. (A2) are equal, thus Eq. (5) holds. This completes the proof.

APPENDIX B: PROOF OF GENERALIZED RECIPROCITY FOR LINEAR POLARIZATION [EQ. (8)]

Here we prove Eq. (8). See Ref. [69], pp. 102–106, for the original proof, and Ref. [77] for additional details.

First, we recall the definition of scattering matrices [78,79]. As shown in Fig. 2, we consider a general sourceless linear time-invariant system characterized by $C(\omega, \boldsymbol{r})$ within a volume V enclosed by a surface ∂V. The system is connected to its exterior by P incoming and P outgoing modes. For simplicity, we consider the case where P is finite. (One can take the continuum limit to extend the results to incorporate plane waves in all directions. See Ref. [69] for more details.) The modes are chosen to have real transverse fields $\{\boldsymbol{e}_i(\boldsymbol{r}^t), \boldsymbol{h}_i(\boldsymbol{r}^t)\}$, $1 \le i \le P$, which satisfy the orthonormal conditions [79]:

$$\oint_{\partial V} \mathbf{d}\boldsymbol{S} \cdot \boldsymbol{e}_i(\boldsymbol{r}^t) \times \boldsymbol{h}_j(\boldsymbol{r}^t) = \oint_{\partial V} \mathbf{d}\boldsymbol{S} \cdot \boldsymbol{e}_j(\boldsymbol{r}^t) \times \boldsymbol{h}_i(\boldsymbol{r}^t) = -2\delta_{ij}, \qquad 1 \le i, j \le P. \tag{B1}$$

Then the transverse fields of light outside ∂V can be expressed as

$$\boldsymbol{E}^t(\boldsymbol{r}^t, z) = \sum_{i=1}^{P} (a_i e^{-i\beta_i z} + b_i e^{i\beta_i z}) \boldsymbol{e}_i(\boldsymbol{r}^t),$$
$$\boldsymbol{H}^t(\boldsymbol{r}^t, z) = \sum_{i=1}^{P} (a_i e^{-i\beta_i z} - b_i e^{i\beta_i z}) \boldsymbol{h}_i(\boldsymbol{r}^t). \tag{B2}$$

Here the local coordinates $\boldsymbol{r}^t = (x, y)$ are tangential to ∂V and z is along the outgoing direction. We set $z = 0$ at ∂V. Thus the incoming and outgoing waves can be represented by complex vectors:

$$\boldsymbol{a} = [a_1, \ldots, a_P]^T, \qquad \boldsymbol{b} = [b_1, \ldots, b_P]^T, \tag{B3}$$

where a_i and b_i are the complex coefficients of the ith incoming and outgoing modes, respectively. There is a linear relation between $\boldsymbol{a}$ and $\boldsymbol{b}$:

$$\boldsymbol{b} = S\boldsymbol{a}, \tag{B4}$$

S is a matrix of size $P \times P$, called the *scattering matrix*. Its element S_{ij} gives the scattering amplitude from the jth basis mode into the ith basis mode.

From the original system as described by $C(\omega, \boldsymbol{r})$, we can define its adjoint system characterized by $\tilde{C}(\omega, \boldsymbol{r})$, as defined by Eq. (7), within the same volume V enclosed by the same surface ∂V. The ports are identical to those of the original system. Thus, we can choose the same orthonormal basis and define the scattering matrix for the adjoint system:

$$\tilde{\boldsymbol{b}} = \tilde{S}\tilde{\boldsymbol{a}}, \tag{B5}$$

where $\tilde{\boldsymbol{a}}, \tilde{\boldsymbol{b}}, \tilde{S}$ are the incoming amplitudes, outgoing amplitudes, and scattering matrices of the adjoint system, respectively.

Our objective is to establish the relation between S and $\tilde{S}$. This is achieved by using the generalized reciprocity theorem [71]: if a current density $\boldsymbol{J}(\boldsymbol{r})$ at frequency ω produces fields $\boldsymbol{E}(\boldsymbol{r})$ and $\boldsymbol{H}(\boldsymbol{r})$ in the *original* system, and another $\tilde{\boldsymbol{J}}(\boldsymbol{r})$ at ω produces $\tilde{\boldsymbol{E}}(\boldsymbol{r})$ and $\tilde{\boldsymbol{H}}(\boldsymbol{r})$ in its *adjoint* system, then

$$\oint_{\partial V} (\boldsymbol{E} \times \tilde{\boldsymbol{H}} - \tilde{\boldsymbol{E}} \times \boldsymbol{H}) \cdot \mathbf{d}\boldsymbol{S} = \int_V (\tilde{\boldsymbol{E}} \cdot \boldsymbol{J} - \boldsymbol{E} \cdot \tilde{\boldsymbol{J}}) dV. \tag{B6}$$

If we assume that there are no sources within V, Eq. (B6) becomes

$$\oint_{\partial V} (\boldsymbol{E} \times \tilde{\boldsymbol{H}} - \tilde{\boldsymbol{E}} \times \boldsymbol{H}) \cdot \mathbf{d}\boldsymbol{S} = 0. \tag{B7}$$

We express the fields at the surface ∂V in terms of the incoming and outgoing amplitudes using the orthonormal basis, then perform the integration over the cross sections. Using mode orthonormality [Eq. (B1)], Eq. (B7) becomes

$$\sum_{i=1}^{P} [(a_i + b_i)(\tilde{a}_i - \tilde{b}_i) - (\tilde{a}_i + \tilde{b}_i)(a_i - b_i)] = 0, \tag{B8}$$

which can be simplified as

$$\sum_{i=1}^{P} (b_i \tilde{a}_i - a_i \tilde{b}_i) = \boldsymbol{b}^T \tilde{\boldsymbol{a}} - \boldsymbol{a}^T \tilde{\boldsymbol{b}} = 0. \tag{B9}$$

Substituting Eqs. (B4) and (B5) into Eq. (B9), we obtain

$$\boldsymbol{a}^T (S^T - \tilde{S}) \tilde{\boldsymbol{a}} = 0. \tag{B10}$$

Since Eq. (B10) holds for any $\boldsymbol{a}$ and $\tilde{\boldsymbol{a}}$, it requires

$$\tilde{S} = S^T. \tag{B11}$$

This is the relation between $\tilde{S}$ and S.

In the case of linear polarized plane-wave basis, Eq. (B11) becomes

$$\langle \omega, \hat{n}', \sigma' | S | \omega, -\hat{n}, \sigma \rangle = \langle \omega, \hat{n}, \sigma | \tilde{S} | \omega, -\hat{n}', \sigma' \rangle, \tag{B12}$$

where $\sigma = p, s$ and $\sigma' = p, s$ are the polarization labels. This completes the proof.

APPENDIX C: PROOF OF GENERALIZED RECIPROCITY FOR ARBITRARY POLARIZATION [EQ. (9)]

Here we prove Eq. (9). Let

$$|\hat{p}\rangle = \sum_{\sigma} c_\sigma |\sigma\rangle, \qquad |\hat{p}'\rangle = \sum_{\sigma'} d_{\sigma'} |\sigma'\rangle, \tag{C1}$$

where c_σ and $d_{\sigma'}$ are complex coefficients. By definition,

$$|\hat{p}^*\rangle = \sum_\sigma c_\sigma^* |\sigma\rangle, \qquad |\hat{p}'^*\rangle = \sum_{\sigma'} d_{\sigma'}^* |\sigma'\rangle, \tag{C2}$$

then

$$\begin{aligned}\langle\omega,\hat{n}',\hat{p}'|S|\omega,-\hat{n},\hat{p}\rangle &= \sum_\sigma\sum_{\sigma'} d_{\sigma'}^* c_\sigma \langle\omega,\hat{n}',\sigma'|S|\omega,-\hat{n},\sigma\rangle \\ &= \sum_\sigma\sum_{\sigma'} c_\sigma d_{\sigma'}^* \langle\omega,\hat{n},\sigma|\tilde{S}|\omega,-\hat{n}',\sigma'\rangle \\ &= \langle\omega,\hat{n},\hat{p}^*|\tilde{S}|\omega,-\hat{n}',\hat{p}'^*\rangle. \end{aligned} \tag{C3}$$

This completes the proof.

APPENDIX D: PROOF OF ADJOINT KIRCHOFF'S LAW [EQ. (10)]

Here we prove Eq. (10). We note

$$\begin{aligned}&\alpha(\omega,-\hat{n},\hat{p}) \\ &= 1-\langle\omega,-\hat{n},\hat{p}|S^\dagger S|\omega,-\hat{n},\hat{p}\rangle \\ &= 1-\sum_{\sigma'}\int d\hat{n}'\langle\omega,-\hat{n},\hat{p}|S^\dagger|\omega,\hat{n}',\sigma'\rangle\langle\omega,\hat{n}',\sigma'|S|\omega,-\hat{n},\hat{p}\rangle \\ &= 1-\sum_{\sigma'}\int d\hat{n}'\langle\omega,-\hat{n}',\sigma'|\tilde{S}^\dagger|\omega,\hat{n},\hat{p}^*\rangle\langle\omega,\hat{n},\hat{p}^*|\tilde{S}|\omega,-\hat{n}',\sigma'\rangle \\ &= 1-\sum_{\sigma'}\int d\hat{n}'\langle\omega,\hat{n},\hat{p}^*|\tilde{S}|\omega,-\hat{n}',\sigma'\rangle\langle\omega,-\hat{n}',\sigma'|\tilde{S}^\dagger|\omega,\hat{n},\hat{p}^*\rangle \\ &= 1-\langle\omega,\hat{n},\hat{p}^*|\tilde{S}\tilde{S}^\dagger|\omega,\hat{n},\hat{p}^*\rangle \\ &= \tilde{e}(\omega,\hat{n},\hat{p}^*). \end{aligned} \tag{D1}$$

Following a similar procedure,

$$\begin{aligned}e(\omega,\hat{n},\hat{p}) &= 1-\langle\omega,\hat{n},\hat{p}|SS^\dagger|\omega,\hat{n},\hat{p}\rangle \\ &= 1-\langle\omega,-\hat{n},\hat{p}^*|\tilde{S}^\dagger\tilde{S}|\omega,-\hat{n},\hat{p}^*\rangle \\ &= \tilde{\alpha}(\omega,-\hat{n},\hat{p}^*). \end{aligned} \tag{D2}$$

In Eq. (D1), the first and sixth equalities use Eqs. (3) and (4), the second and fifth equalities use the completeness of the plane-wave basis, and the third equality uses Eq. (9). This completes the proof.

APPENDIX E: PROOF OF RELATIONS EQS. (11) AND (12)

Here we prove Eqs. (11) and (12). Equation (11) is obvious by the definition of geometric symmetry. We only need to prove Eq. (12). Suppose $\mathcal{T}b$ is a symmetry. Since $\mathcal{T}$ commutes with any geometric transformation, $\mathcal{T}b = b\mathcal{T}$. We first apply $\mathcal{T}$ to convert the system into its adjoint system. From adjoint Kirchhoff's law,

$$\alpha(\omega,-\hat{n},\hat{p}) = \tilde{e}(\omega,\hat{n},\hat{p}^*), \qquad e(\omega,\hat{n},\hat{p}) = \tilde{\alpha}(\omega,-\hat{n},\hat{p}^*). \tag{E1}$$

We then apply b to convert the adjoint system back into the original system, since $b\mathcal{T}$ is a symmetry. Therefore,

$$\tilde{e}(\omega,\hat{n},\hat{p}^*) = e(\omega,\hat{n}',\hat{p}'^*), \quad \tilde{\alpha}(\omega,-\hat{n},\hat{p}^*) = \alpha(\omega,-\hat{n}',\hat{p}'^*), \tag{E2}$$

where b transforms $\hat{n},\hat{p}$ into $\hat{n}',\hat{p}'$. Combining Eqs. (E1) and (E2), we get Eq. (12). This completes the proof.

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