Characterizing Symmetry-Protected Thermal Equilibrium by Work Extraction

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The second law of thermodynamics states that work cannot be extracted from thermal equilibrium, whose quantum formulation is known as complete passivity; a state is called completely passive if work cannot be extracted from any number of copies of the state by any unitary operations. It has been established that a quantum state is completely passive if and only if it is a Gibbs ensemble. In physically plausible setups, however, the class of possible operations is often restricted by fundamental constraints such as symmetries imposed on the system. In the present work, we investigate the concept of complete passivity under symmetry constraints. Specifically, we prove that a quantum state is completely passive under a symmetry constraint described by a connected compact Lie group, if and only if it is a generalized Gibbs ensemble including conserved charges associated with the symmetry. Remarkably, our result applies to noncommutative symmetry such as SU(2) symmetry, suggesting an unconventional extension of the notion of generalized Gibbs ensemble. Furthermore, we consider the setup where a quantum work storage is explicitly included, and prove that the characterization of complete passivity remains unchanged. Our result extends the notion of thermal equilibrium to systems protected by symmetries, and would lead to flexible design principles of quantum heat engines and batteries. Moreover, our approach serves as a foundation of the resource theory of thermodynamics in the presence of symmetries.

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I. INTRODUCTION

The second law of thermodynamics, also known as Kelvin's principle, dictates that a positive amount of work can never be extracted by any cyclic operation from a single heat bath at a uniform temperature [1]. Ever since its establishment in the 19th century, the second law has served as the most fundamental constraint on our capability of energy harvesting, which prohibits the perpetual motion of the second kind. In recent years, the frontier of thermodynamics is extended to the quantum regime due to the development of quantum technologies. Quantum heat engines have been experimentally realized by quantum technologies such as ion traps [2,3], superconducting qubits [4–6], and NMR [7,8], which have opened new opportunities of power generation by utilizing quantum effects.

On the theory side, quantum information theory sheds new light on quantum thermodynamics. In particular, an information-theoretic framework called resource theory has attracted much attention [9–19], which identifies work with resources and thermal equilibrium states with resource-free states. From this perspective, a concept called *passive* state plays a key role [20,21], from which positive work cannot be extracted by any unitary operation. It is known, however, that one can extract a positive amount of work from multiple copies of a certain passive state, and thus the concept of passivity is not sufficient to characterize thermal equilibrium from which energy harvesting should be strictly prohibited. The full characterization of thermal equilibrium is given by *complete passivity*: A positive amount of work cannot be extracted from any number of copies of a completely passive state. It is known that a state is completely passive if and only if it is a Gibbs ensemble, which suggests that complete passivity provides a physically meaningful, as well as information-theoretically accurate, definition of thermal equilibrium.

In the above approach to characterize thermal equilibrium, a central assumption is that all unitary operations are allowed for work extraction. In real physical situations, on the other hand, several constraints are often imposed on possible unitary operations, which often make the class of physically plausible unitary operations strictly smaller than all unitary operations. Among such constraints, we here focus on the symmetry of quantum systems. There are various kinds of symmetry and the corresponding conservation laws [22], such as U(1) symmetry and particle number conservation, SU(2) symmetry and spin (magnetization) conservation, and $\mathbb{Z}_2$ symmetry and parity conservation.

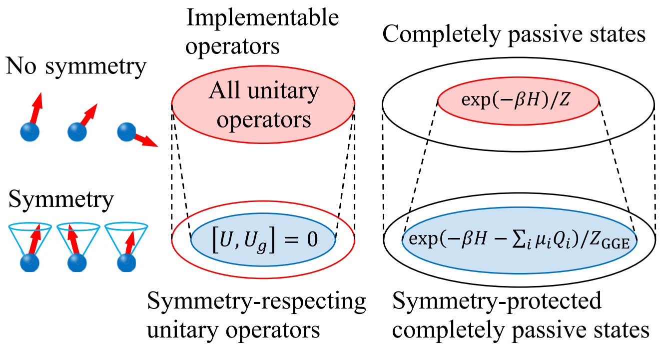

FIG. 1. Symmetry constraints on unitary operations and completely passive states. When the class of allowed operations is restricted, the class of quantum states from which positive work cannot be extracted is expanded. This leads to the question of identifying the states that behave as effective thermal equilibrium under symmetry constraints. In this paper, we completely identify the class of such effective thermal equilibrium states, which we name as symmetry-protected thermal equilibrium states, by proving that those states are always given by GGEs of the form Eq. (1), including the case that the conserved charges are noncommutative.

Once the class of possible unitary operations is restricted by such symmetries, thermal equilibrium states are no longer necessarily Gibbs ensembles. In fact, there are some non-Gibbs ensembles from which one cannot extract positive work by symmetry-respecting unitaries. In other words, a broader class of states looks like effective thermal equilibrium, as long as the symmetry is respected (see Fig. 1). We call this extended notion of thermal equilibrium *symmetry-protected* thermal equilibrium. In terms of resource theory [23], the above observation implies that the class of free states is expanded if the class of free operations is restricted. Therefore, the conventional Gibbs ensemble would be insufficient to represent all symmetry-protected thermal equilibrium states. Then, a natural question raised is, what are concrete expressions of symmetry-protected thermal equilibrium states? More specifically, how should the notion of complete passivity be extended if only symmetry-respecting unitaries are considered?

In this paper, we answer this question by proving that a quantum state is completely passive under a symmetry constraint, if and only if it is a generalized Gibbs ensemble (GGE) that involves conserved charges associated with the symmetry. This result is applicable to noncommutative symmetry such as SU(2) symmetry, which leads to an unconventional extension of the notion of GGE.

More explicitly, consider a continuous symmetry described by a connected compact Lie group, and let Q_i's be its conserved charges such as the particle number operator or the spin operators. Then, what we prove is that any completely passive state is given in the form

$$\rho_{\mathrm{GGE}} \coloneqq \frac{1}{Z_{\mathrm{GGE}}} \exp\left(-\beta H - \sum_i \mu_i Q_i\right), \tag{1}$$

where $Z_{\mathrm{GGE}} \coloneqq \mathrm{tr}[\exp(-\beta H - \sum_i \mu_i Q_i)]$ is the generalized partition function, $\beta \geq 0$ is the inverse temperature, and μ_i's are generalized "chemical potentials." In the commutative U(1) case, the GGE is reduced to the conventional grand canonical ensemble. Furthermore, we consider the setup where a quantum storage that stores work is explicitly introduced [24,25], and prove that the above characterization of symmetry-protected complete passivity remains unchanged from the setup where work is treated as a classical variable.

Our result establishes that any state other than the GGE is not completely passive and thus cannot be regarded as thermal equilibrium in terms of work extraction. From the experimental point of view, this would lead to a more flexible design principle of quantum heat engines [2–8] and quantum batteries [26–29]. From the theoretical point of view, our result would serve as a foundation of the resource theory of thermodynamics under symmetry constraints, as our result specifies the free states of such a resource theory.

In the context of thermalization, the GGE has been investigated as a state describing equilibration in integrable systems [30–33]. Most of the previous works consider the case where conserved charges are commutative with each other (but see also Refs. [34–36]). A noncommutative extension of the GGE has been proposed by Yunger Halpern *et al.* in Ref. [37], and our result supports that the expression proposed by them is a proper form of the noncommutative GGE. We emphasize, however, that our setup is different from the setup of Ref. [37]; in the present paper, we consider the purely energetic work extraction (instead of the chemical work extraction) under a symmetry constraint that is imposed only on the system (instead of including charge storages). Therefore, we adopt a tighter constraint on the operations than Ref. [37]. Our setup is physically plausible given that heat engines and external systems are often very different (e.g., matter and light) [2–8], where it would be natural to suppose that symmetries are imposed only on the system of interest.

This paper is organized as follows. In Sec. II, we give the definition of symmetry-protected complete passivity and show our main theorem, stating that only GGEs are completely passive states under symmetry constraints. Since the proof of this theorem is highly involved, we leave the full description of the proof to Supplemental Material [38]. Instead, we illustrate the physical implications of the theorem by some examples. In Sec. III, we consider a setup including a quantum work storage. In Sec. IV, we discuss the relation between the present study and other relevant previous studies. In the Appendix A, we show the condition for symmetry-protected passivity. In Appendix B, we describe the full proof of the main theorem in Sec. II only for a simplest nontrivial example of a dimer model. In Appendix C, we deal with the cases of some finite-group symmetries and time-reversal symmetry, where only conventional Gibbs states are symmetry-protected completely passive states.

II. SETUP AND THE MAIN THEOREM

In this section, we discuss the setup and the main result of this paper. In Sec. II A, we describe our definition of passivity and complete passivity under symmetry constraints. In Sec. II B, we present our main theorem, stating that any completely passive state under a symmetry constraint is a GGE. In Sec. II C, we illustrate the setup and the theorem by some examples.

A. Complete passivity under symmetry constraints

As a preliminary, we first formalize ordinary passivity without symmetry constraints [20,21]. Let H be the initial and final Hamiltonians of the system, which should be the same because we consider cyclic processes. The time evolution of the system is represented by a unitary operator U. Note that U is an arbitrary unitary operator and is not necessarily given by $\exp(-\mathrm{i}tH)$, because the Hamiltonian can be time dependent during the operation. Then, the average work extracted from the system is defined by

$$W(\rho, U) \coloneqq \mathrm{tr}(\rho H) - \mathrm{tr}(U\rho U^\dagger H). \tag{2}$$

Now, a state ρ is called passive if $W(\rho, U) \leq 0$ holds for all U. It is proved [20,21] that a state is passive, if and only if the state ρ is diagonal in the energy eigenbasis as $\rho = \sum_j p_j |E_j\rangle\langle E_j|$, and the probabilities $\{p_j\}$ and the energy eigenvalues $\{E_j\}$ satisfy $p_1 \geq p_2 \geq \ldots$ and $E_1 \leq E_2 \leq \ldots$.

Even if a state ρ is passive, there remains the possibility to extract positive work from multiple copies of ρ. In such a case, ρ cannot be regarded as truly thermal equilibrium, because one must not extract positive work from any number of copies of an equilibrium state. We therefore define ρ as completely passive, if $\rho^{\otimes N}$ is passive for all $N \in \mathbb{N}$. It has been proved in Refs. [20,21] that ρ is completely passive if and only if it is the Gibbs ensemble $\rho = e^{-\beta H}/Z$ for some $\beta \geq 0$.

In the foregoing conventional definition of passivity, any unitary operators are allowed to be implemented for work extraction. In order to describe symmetry constraints, we restrict the class of possible unitary operations in the following manner. Consider a group G that describes a symmetry, and fix a unitary representation of G. Let U_g be the unitary labeled by $g \in G$, which should satisfy $U_g U_{g'} = U_{gg'}$ for any $g, g' \in G$. We do not assume that the unitary representation is irreducible, but technically, we assume that the representation is faithful. (If a unitary representation is not faithful, the structure of G is not fully represented by the representation. Physically, therefore, we can suppose that a unitary representation is faithful without loss of generality.)

When G is a Lie group (a smooth continuous group), we can introduce the generators of the symmetry operators, which is the representation of the basis of the Lie algebra. Physically, those generators are conserved charges $\{Q_i\}_{i=1}^n$, which are Hermitian operators linearly independent of each other. If G is connected and compact, they can generate all the symmetry operators as $U_g = \exp(\mathrm{i}\sum_{i=1}^n \alpha_i Q_i)$ with some $\alpha_1, \ldots, \alpha_n \in \mathbb{R}$ determined by g. In the following, we assume that G is a connected compact Lie group unless stated otherwise.

Now we say that a unitary U respects the symmetry, if it commutes with all symmetry operators, that is, $[U, U_g] = 0$ holds for all $g \in G$. Or equivalently, U commutes with all conserved charges, that is, $[U, Q_i] = 0$ holds for all i. Moreover, we suppose that the Hamiltonian H also respects the symmetry: $[H, U_g] = 0$ for all $g \in G$, or equivalently $[H, Q_i] = 0$ for all i. We then define ρ as symmetry-protected passive if $W(\rho, U) \leq 0$ for all symmetry-respecting unitaries U, where $W(\rho, U)$ is defined by Eq. (2).

It is more nontrivial to properly define complete passivity under symmetry constraints, because we need to specify the physically feasible class of symmetry operations on the multiple copies $\rho^{\otimes N}$. For this purpose, we here adopt the *global* symmetry, which collectively acts on all the copies. That is, we consider the unitary representation of the form $U_g^{\otimes N}$, by which all the copies are independently operated with the same symmetry operation U_g (see also Fig. 2). Then, we say that an operator U acting on N copies respects the global symmetry, if it satisfies $[U, U_g^{\otimes N}] = 0$ for all $g \in G$. We can also introduce the total charges,

$$Q_i^{(N)} \coloneqq \sum_{k=1}^N I^{\otimes k-1} \otimes Q_i \otimes I^{\otimes N-k}, \tag{3}$$

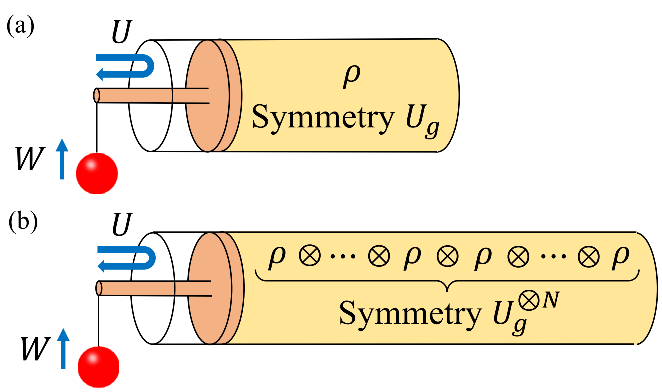

FIG. 2. Schematics of passivity and complete passivity under symmetry constraints. The work W is extracted by the unitary operation U. (a) A state ρ is called symmetry-protected passive if one cannot extract positive work from a single state by any symmetry-respecting unitary U, which commutes with all U_g. (b) A state ρ is called symmetry-protected completely passive if one cannot extract positive work from any number of copies of it by any symmetry-respecting unitary U, which commutes with all $U_g^{\otimes N}$ for all $N \in \mathbb{N}$. Here, the symmetry is supposed to be global; i.e., the symmetry operations collectively act on all the copies.

with I being the identity operator. By using this notation, we can say that U respects the global symmetry if the total charges are conserved: for all i,

$$[U, Q_i^{(N)}] = 0. \tag{4}$$

Meanwhile, the work extracted from the N copies is defined by

$$W^{(N)}(\rho^{\otimes N}, U) = \mathrm{tr}(\rho^{\otimes N} H^{(N)}) - \mathrm{tr}(U\rho^{\otimes N} U^\dagger H^{(N)}), \tag{5}$$

where

$$H^{(N)} \coloneqq \sum_{k=1}^{N} I^{\otimes k-1} \otimes H \otimes I^{\otimes N-k} \tag{6}$$

is the total Hamiltonian without interaction (as is the case for ordinary complete passivity). We also suppose that the Hamiltonian H respects the symmetry for individual copies; i.e., $[H, U_g] = 0$ holds for all $g \in G$. Finally, ρ is called symmetry-protected completely passive if $W^{(N)}(\rho^{\otimes N}, U) \le 0$ holds for all $N \in \mathbb{N}$ and for all symmetry-respecting unitaries U satisfying Eq. (4).

B. Main theorem

We now state our main theorem of this paper, which gives the complete characterization of symmetry-protected complete passivity. Note that the characterization of symmetry-protected (not complete) passivity is given in Appendix A.

We first exclude the trivial situation where the Hamiltonian H is of the form of $\alpha_0 I + \sum_{i=1}^n \alpha_i Q_i$ with some $\alpha_0, \alpha_1, \ldots, \alpha_n \in \mathbb{R}$. This is because in such a case, the energy is conserved under any symmetry-respecting unitary and thus all states are trivially completely passive. We also suppose that ρ is positive definite. Then, the main theorem is stated as follows.

Theorem 1.—Let G be a connected compact Lie group, $\{U_g\}_{g\in G}$ be its (faithful) unitary representation, and $\{Q_i\}$ be the corresponding conserved charges. A state ρ is symmetry-protected completely passive with respect to a symmetry-respecting Hamiltonian H, if and only if ρ is given by the GGE Eq. (1) with some $\beta \ge 0$ and $\mu_i \in \mathbb{R}$.

The mathematically rigorous proof of this theorem is presented in Supplemental Material [38] (Theorem S3). In particular, the proof of the *only if* part is quite complicated and requires advanced tools from mathematical theory of Lie groups. However, we describe the proof for a special example in Appendix B.

At this stage, we only mention the proof of the *if* part, which is much easier than the *only if* part. That is, we here show that the GGE Eq. (1) is symmetry-protected completely passive. To see this, we remark that for given β, μ_i, the GGE Eq. (1) can be seen as the Gibbs ensemble of the "Hamiltonian" $H' \coloneqq H + \beta^{-1}\sum_i \mu_i Q_i$ and is completely passive with respect to H'. If unitary U respects the symmetry, it does not change the expectation value of the second term of H', and thus the extracted work defined by H' and H are the same. This implies that the GGE Eq. (1) is symmetry-protected completely passive with respect to H. Note that here we did not use the assumption that H also respects the symmetry. The above argument is essentially the same as a part of Ref. [37], while in our setup this kind of argument cannot be applied to the converse part (i.e., the main part of this paper).

Meanwhile, we can determine the parameters β and $\{\mu_i\}$ in the GGE Eq. (1) in the following manner. First, we consider the Hilbert-Schmidt inner product and orthonormalize the conserved charges $I, H, Q_1, \ldots, Q_n$ by the Gram-Schmidt orthonormalization into $I/\sqrt{d}, Q_0', Q_1', \ldots, Q_n'$ that satisfy $\mathrm{tr}(Q_i') = 0$ and $\mathrm{tr}(Q_i' Q_j') = \delta_{ij}$ for $i, j = 0, 1, \ldots, n$, where d is the dimension of the Hilbert space of the system. Then, the GGE is written as $\rho_{\mathrm{GGE}} = \exp(-\sum_{i=0}^n \mu_i' Q_i')/Z_{\mathrm{GGE}}'$ with $\mu_0', \mu_1', \ldots, \mu_n' \in \mathbb{R}$ and the normalization constant Z_{GGE}'. Here, $\mu_0', \mu_1', \ldots, \mu_n'$ can be regarded as the coefficients of the orthonormalized basis $\{Q_i'\}$ in $-\log(\rho_{\mathrm{GGE}})$, and are given by $\mu_i' = -\mathrm{tr}[\log(\rho_{\mathrm{GGE}}) Q_i']$. By expressing Q_i' by a linear combination of I, H, and Q_i, we obtain β and $\{\mu_i\}$.

C. Examples

We show some illustrative examples. In the case of $G = \mathrm{U}(1)$, there is a single charge Q. It often describes the particle number N, where Eq. (4) means the conservation of total particle number [see Fig. 3(a)]. We note that the particle number of an individual ρ is not necessarily conserved, but that of the multiple copies is globally conserved. In this case, Theorem 1 states that a state is symmetry-protected completely passive, if and only if it is the grand canonical ensemble,

$$\rho_{\mathrm{GGE}} = \frac{1}{Z_{\mathrm{GGE}}} \exp(-\beta H - \mu N), \tag{7}$$

with inverse temperature $\beta \ge 0$ and chemical potential $\mu \in \mathbb{R}$.

In the case of $G = \mathrm{SU}(2)$, there are three charges, Q^x, Q^y, Q^z. They often describe the spin operators in the x, y, z directions, where Eq. (4) means the total spin (magnetization) conservation in all the directions. We write $\boldsymbol{Q} \coloneqq (Q^x, Q^y, Q^z)$.

In the following, let us elucidate the SU(2) case by considering a "dimer" model [see Fig. 3(b)]. Suppose that the system consists of two spin-$1/2$ systems with the spin operators $\boldsymbol{s}_1, \boldsymbol{s}_2$. The unitary representation of SU(2) on this system is generated by $\boldsymbol{Q} \coloneqq \boldsymbol{s}_1 \otimes I + I \otimes \boldsymbol{s}_2$, which consists of two irreducible sectors with the total spin 0 and 1.

We consider the Hamiltonian of the system with the isotropic Heisenberg-type interaction between the two

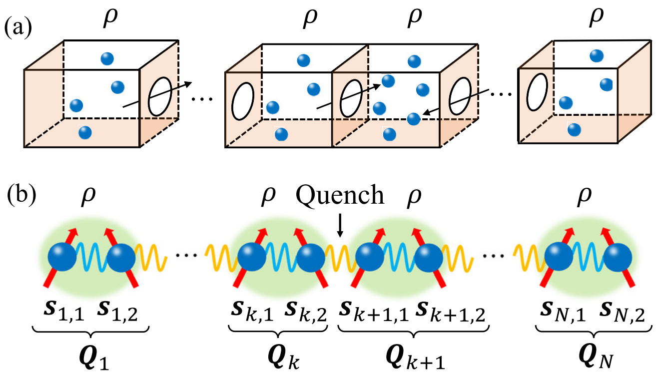

FIG. 3. Toy examples of multiple copies of the system under global symmetry constraints. (a) U(1) symmetry and the particle number conservation. The total particle number should be conserved, while individual copies can exchange particles with each other. (b) SU(2) symmetry and the spin conservation. Each copy of the system is represented by a "dimer" consisting of two spin-1/2 systems. We consider the situation where the total spin is conserved in all the spatial directions, while each spin is not necessarily conserved. Such symmetry-respecting interaction between the dimers can be induced, for example, by quenching the second term on the Hamiltonian Eq. (8).

spins, $H = \boldsymbol{s}_1 \cdot \boldsymbol{s}_2$. It is straightforward to check that H commutes with all the components of $\boldsymbol{Q}$, implying the total spin conservation. We consider N copies of this system (i.e., $2N$ spin-1/2 systems). Let $\boldsymbol{s}_{1,k}, \boldsymbol{s}_{2,k}$ be the spin operators of the kth copy. The total Hamiltonian is given by $H^{(N)} = \sum_{k=1}^{N} \boldsymbol{s}_{1,k} \cdot \boldsymbol{s}_{2,k}$, which describes a trivial sequence of the dimers without interaction.

A simple example of symmetry-respecting unitaries is given by $U = \exp(-\mathrm{i}H')$ with H' being the 1D XXX Hamiltonian. That is, H' is obtained by quenching the interaction between the dimers:

$$H' = H^{(N)} + \sum_{k=1}^{N-1} \boldsymbol{s}_{2,k} \cdot \boldsymbol{s}_{1,k+1}. \quad (8)$$

It is again straightforward to check that H' conserves the total spin of N copies.

In this dimer example, Theorem 1 states that a state is symmetry-protected completely passive, if and only if it is the GGE,

$$\rho_{\mathrm{GGE}} = \frac{1}{Z_{\mathrm{GGE}}} \exp\left(-\beta H - \sum_{\alpha=x,y,z} \mu_\alpha Q^\alpha\right), \quad (9)$$

with inverse temperature $\beta \geq 0$ and generalized chemical potentials $\mu_x, \mu_y, \mu_z \in \mathbb{R}$. We give a full proof of Theorem 1 for the case of this dimer model in Appendix B.

We note that the conserved charges already satisfy the orthogonal relation, and thus we only need to normalize the charges. Then, β and μ_α's are simply given by $\beta = -\mathrm{tr}[\log(\rho_{\mathrm{GGE}})H]/\mathrm{tr}(H^2) = -4\mathrm{tr}[\log(\rho_{\mathrm{GGE}})H]/3$, $\mu_\alpha = -\mathrm{tr}[\log(\rho_{\mathrm{GGE}})Q^\alpha]/\mathrm{tr}[(Q^\alpha)^2] = -\mathrm{tr}[\log(\rho_{\mathrm{GGE}})Q^\alpha]/2$.

Finally, we remark on the ways of connecting dimers by interaction Hamiltonians. Since we consider the total magnetization as the conserved charge in the above example, it is conserved in both the cases where we connect dimers in a 1D geometry by Eq. (8) and we connect them in a 2D geometry. On the other hand, only in the 1D case the spin chain has many local conserved charges other than the total magnetization, which characterizes the integrability of the 1D XXX model. Since those conserved charges play a crucial role in isolated quantum systems [32,33,39], it would be interesting to investigate the role of the charges in our setup. We consider, however, that it is not straightforward to apply those charges to the setup like Fig. 3(b), because those charges (except for the total magnetization) cannot be written as the sum of operators each of which acts on a single dimer, while in our setup we treat conserved charges in the form of Eq. (3). It is an important future issue to generalize our framework to more general conserved charges and to clarify the relation between complete passivity and integrability.

III. ROLE OF WORK STORAGE

Thus far, we have considered the setup where the work is defined as the difference of the average energies of the system before and after a unitary operation. On the other hand, there is another setup that reflects the first law of thermodynamics more explicitly [9–11,24,25,40–42], where a quantum work storage is introduced in addition to the system of interest, and unitary operations on the total system should conserve the total energy. In this section, we consider this setup with the fully quantum treatment of the work storage.

In Sec. III A, we describe our setup of the work storage. In Sec. III B, in the absence of symmetry constraints, we show the relationship between the maximal works with and without the work storage, which is of separate interest. In Sec. III C, we prove that even in the presence of the work storage, a state is completely passive under a symmetry constraint if and only if it is a GGE, independently of the initial state of the work storage.

A. Setup

We introduce a work storage attached to the system of interest in line with Refs. [24,25,40–42]. The work storage is a continuous system described by position and momentum, and its Hamiltonian is given by the position operator x.

We impose the following two conditions on implementable unitary operator V that acts on the composite system including the work storage: (I) V conserves the total energy, i.e.,

$$[V, H \otimes I + I \otimes x] = 0, \quad (10)$$

and (II) V is invariant under energy translation of the work storage, i.e.,

$$[V, I \otimes p] = 0, \tag{11}$$

where p is the momentum operator (the generator of energy translation) of the work storage. Note that the canonical commutation relation is given by $[x, p] = \mathrm{i}$.

We also suppose that there is no correlation between the system and the work storage in the initial state. Then, we define the extracted work as the difference of the average energies of the work storage before and after an operation V:

$$W^{\mathrm{WS}}(\rho, \rho_{\mathrm{W}}, V) \\ \coloneqq \mathrm{tr}[V(\rho \otimes \rho_{\mathrm{W}})V^{\dagger}(I \otimes x)] - \mathrm{tr}[(\rho \otimes \rho_{\mathrm{W}})(I \otimes x)], \tag{12}$$

where ρ and ρ_{W} are the initial states of the system and the work storage, respectively. From condition (II), the extracted work is invariant under energy translation of the initial state of the work storage.

We note that the average work extraction adopted here is different from a protocol investigated in Ref. [9], which can be referred to as almost deterministic work extraction. We also note that we adopted strict energy conservation Eq. (10) as in Refs. [24,40], but not the average energy conservation in the sense of Ref. [25].

B. Without symmetry constraints

We consider work extraction in the case without symmetry constraints. We clarify the relation between the maximal extracted works with and without work storage. Moreover, we show that in the presence of the work storage, only Gibbs states are completely passive states.

First, we consider the correspondence between the implementable unitary operations in the setups with and without the work storage. For a unitary operator U on the system of interest, we can construct a unitary operator that acts on the composite system including the work storage and satisfies conditions (I) and (II) by the following Kitaev construction [43]:

$$\mathcal{C}(U) = \int_{-\infty}^{\infty} dq e^{\mathrm{i}qH} U e^{-\mathrm{i}qH} \otimes |q\rangle\langle q|, \tag{13}$$

where $|q\rangle$ represents the momentum eigenstate of the work storage with eigenvalue q. This unitary operator is equivalent to the one that appears in Refs. [40,41]. It is shown in Ref. [40] that all the unitary operators that satisfy conditions (I) and (II) are represented by $\mathcal{C}(U)$ with some unitary operator U. In Ref. [43], the operator $\mathcal{C}(U)$ is introduced as an operator that simulates U by giving the same action on ρ if ρ is symmetry respecting. However, we note that if the state ρ is not symmetry respecting, the action of $\mathcal{C}(U)$ on the system of interest is not necessarily the same as that of U, and moreover, the extracted work from $\rho \otimes \rho_{\mathrm{W}}$ by the action of $\mathcal{C}(U)$ is not necessarily the same as that from ρ by the action of U.

By using the Kitaev construction Eq. (13), we derive the relation between the maximal extracted works with and without the work storage. The maximal extracted work from a state ρ in the setups with and without the work storage are respectively defined as

$$W^{\mathrm{WS}}_{\max}(\rho, \rho_{\mathrm{W}}) \coloneqq \max_{V} W^{\mathrm{WS}}(\rho, \rho_{\mathrm{W}}, V), \tag{14}$$

$$W_{\max}(\rho) \coloneqq \max_{U} W(\rho, U), \tag{15}$$

where V ranges over all the unitaries on the composite system that satisfy conditions (I) and (II), and U ranges over all the unitaries on the system of interest (see Fig. 4). Since any V can be represented as $\mathcal{C}(U)$ for some U, $W^{\mathrm{WS}}_{\max}(\rho, \rho_{\mathrm{W}})$ is also written as

$$W^{\mathrm{WS}}_{\max}(\rho, \rho_{\mathrm{W}}) = \max_{U} W^{\mathrm{WS}}[\rho, \rho_{\mathrm{W}}, \mathcal{C}(U)]. \tag{16}$$

From Eq. (13), the extracted work is given as

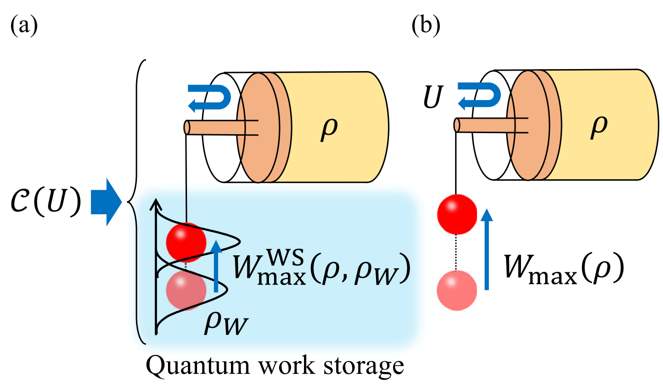

FIG. 4. Work extraction in the setups (a) with and (b) without the work storage. $\mathcal{C}(U)$ is a unitary operator acting on the composite system of the system and the work storage, which is defined by a unitary operator U acting only on the system through the Kitaev construction Eq. (13). $\mathcal{C}(U)$ is constructed such that it satisfies energy conservation of the composite system and energy translation invariance of the work storage. Moreover, $\mathcal{C}(U)$ gives a one-to-one correspondence between the possible operations in the setups with and without the work storage. As shown in Eqs. (19) and (20), the maximal extracted work with the work storage is no greater than that without the work storage. If U satisfies $[U^{\dagger}HU, H] = 0$, the extracted work from ρ without the work storage is the same as that from ρ with the work storage by unitary operation $\mathcal{C}(U)$ independently of the work storage ρ_{W}, which is stated in Proposition 2.

$$\begin{aligned} &W^{\mathrm{WS}}[\rho, \rho_{\mathrm{W}}, \mathcal{C}(U)] \\ &\quad = \int_{-\infty}^{\infty} dq \langle q|\rho_{\mathrm{W}}|q\rangle W(e^{-\mathrm{i}qH}\rho e^{\mathrm{i}qH}, U) \\ &\quad = W[\mathcal{D}_{\rho_{\mathrm{W}}}(\rho), U], \end{aligned} \tag{17}$$

where $\mathcal{D}_{\rho_{\mathrm{W}}}$ is defined as

$$\mathcal{D}_{\rho_{\mathrm{W}}}(\rho) \coloneqq \int_{-\infty}^{\infty} dq \langle q|\rho_{\mathrm{W}}|q\rangle e^{-\mathrm{i}qH}\rho e^{\mathrm{i}qH}. \tag{18}$$

From Eqs. (16) and (17), we obtain

$$W_{\max}^{\mathrm{WS}}(\rho, \rho_{\mathrm{W}}) = W_{\max}[\mathcal{D}_{\rho_{\mathrm{W}}}(\rho)]. \tag{19}$$

We next prove that this value is no greater than the maximal extracted work from ρ in the setup without the work storage:

$$W_{\max}[\mathcal{D}_{\rho_{\mathrm{W}}}(\rho)] \le W_{\max}(\rho). \tag{20}$$

This inequality is proved as follows:

$$\begin{aligned} &W_{\max}[\mathcal{D}_{\rho_{\mathrm{W}}}(\rho)] \\ &\quad = W_{\max}\left(\int_{-\infty}^{\infty} dq \langle q|\rho_{\mathrm{W}}|q\rangle e^{-\mathrm{i}qH}\rho e^{\mathrm{i}qH}\right) \\ &\quad \le \int_{-\infty}^{\infty} dq \langle q|\rho_{\mathrm{W}}|q\rangle W_{\max}(e^{-\mathrm{i}qH}\rho e^{\mathrm{i}qH}) \\ &\quad = \int_{-\infty}^{\infty} dq \langle q|\rho_{\mathrm{W}}|q\rangle W_{\max}(\rho) \\ &\quad = W_{\max}(\rho), \end{aligned} \tag{21}$$

where we used the concavity of $W_{\max}(\rho)$ to obtain the third line. Here, the concavity of $W_{\max}(\rho)$ is shown as follows. Take arbitrary $s \in [0,1]$ and arbitrary states ρ_1 and ρ_2. We take one of the unitary operators U_0 that extract the maximal extracted work from $\rho = s\rho_1 + (1-s)\rho_2$. Then,

$$\begin{aligned} W_{\max}(\rho) &= \mathrm{tr}[(\rho - U_0\rho U_0^\dagger)H] \\ &= s\mathrm{tr}[(\rho_1 - U_0\rho_1 U_0^\dagger)H] \\ &\quad + (1-s)\mathrm{tr}[(\rho_2 - U_0\rho_2 U_0^\dagger)H] \\ &\le sW_{\max}(\rho_1) + (1-s)W_{\max}(\rho_2), \end{aligned} \tag{22}$$

which implies the concavity of $W_{\max}(\rho)$.

We consider two extreme examples of $\mathcal{D}_{\rho_{\mathrm{W}}}$. When ρ_{W} is a momentum eigenstate, $\mathcal{D}_{\rho_{\mathrm{W}}}$ is the identical mapping. In this case, Eq. (19) states that $W_{\max}^{\mathrm{WS}}(\rho, \rho_{\mathrm{W}}) = W_{\max}(\rho)$, i.e., the extracted works are the same in the setups with and without work storage, which is consistent with the result in Ref. [24]. On the other hand, when ρ_{W} is a position eigenstate, $\mathcal{D}_{\rho_{\mathrm{W}}}$ is the dephasing mapping Δ defined as $\Delta(\rho) \coloneqq \sum_E \Pi_E \rho \Pi_E$, where Π_E is the projection operator onto the energy eigenspace of E. In this case, Eq. (19) states that $W_{\max}^{\mathrm{WS}}(\rho, \rho_{\mathrm{W}}) = W_{\max}[\Delta(\rho)]$. This means that we cannot extract work from coherence, which is reminiscent of a phenomenon called work locking [9]. It is shown, however, in Refs. [9,40] that if we have infinitely many copies of the state, we can again extract work from coherence.

We now prove that only Gibbs states are completely passive states in the presence of the work storage, which is of separate interest. For that purpose, we prepare the following two propositions.

The first proposition states that in the setup without work storage, we can extract positive work from multiple copies of any state other than the Gibbs state, even if we further impose constraints $[U^\dagger H U, H] = 0$ on the possible operations U. This is a stronger statement than the conventional characterization of complete passivity [20,21].

Proposition 1.—Let ρ be a state such that $W(\rho^{\otimes N}, U) \le 0$ holds for any $N \in \mathbb{N}$ and any unitary operator U acting on $\rho^{\otimes N}$ satisfying $[U^\dagger H^{(N)} U, H^{(N)}] = 0$, where $H^{(N)}$ is defined by Eq. (6). Then, ρ is the Gibbs ensemble at positive temperature.

The proof is the simplest case of that of Theorem 1 without symmetry constraints (see also Appendix B).

Proof.—We define a sequence of unitary operators $\{U_m\}_{m\in\mathbb{N}}$ that satisfies $[U_m^\dagger H^{(2m+1)} U_m, H^{(2m+1)}] = 0$, and consider the extracted work from $\rho^{\otimes 2m+1}$ by U_m. Since the Hamiltonian is not trivial, there exist energy eigenstates $|E_{k_0}\rangle, |E_{k_1}\rangle$ with different eigenvalues $E_{k_0} < E_{k_1}$. For any l, we define $R_{ij} \coloneqq \frac{1}{2}[I - (-1)^i T][|E_l\rangle\langle E_l| \otimes (I - |E_l\rangle\langle E_l|)][I - (-1)^j T]$ with the swapping operator T between two copies of the system. We consider unitary operator $U_m \coloneqq I - \sum_{i,j\in\{0,1\}} (-1)^{i-j} R_{ij}^{\otimes m} \otimes |E_{k_i}\rangle\langle E_{k_j}|$, which satisfies $[U_m^\dagger H^{(2m+1)} U_m, H^{(2m+1)}] = 0$ and

$$\begin{aligned} &H^{(2m+1)} - U_m^\dagger H^{(2m+1)} U_m \\ &\quad = (E_{k_1} - E_{k_0})(R_{11}^{\otimes m} \otimes |E_{k_1}\rangle\langle E_{k_1}| - R_{00}^{\otimes m} \otimes |E_{k_0}\rangle\langle E_{k_0}|). \end{aligned} \tag{23}$$

Therefore, the extracted work is given by

$$\begin{aligned} W(\rho^{\otimes 2m+1}, U_m) &= \mathrm{tr}[\rho^{\otimes 2m+1}(H^{(2m+1)} - U_m^\dagger H^{(2m+1)} U_m)] \\ &= (E_{k_1} - E_{k_0})[\mathrm{tr}(\rho^{\otimes 2} R_{11})^m \langle E_{k_1}|\rho|E_{k_1}\rangle \\ &\quad - \mathrm{tr}(\rho^{\otimes 2} R_{00})^m \langle E_{k_0}|\rho|E_{k_0}\rangle] \\ &= a\left[1 - b\left(\frac{\mathrm{tr}(\rho^{\otimes 2} R_{00})}{\mathrm{tr}(\rho^{\otimes 2} R_{11})}\right)^m\right] \\ &= a[1 - b(1 + c\|[\rho, |E_l\rangle\langle E_l|]\|_{\mathrm{HS}}^2)^{-m}], \end{aligned} \tag{24}$$

where $a \coloneqq (E_{k_1} - E_{k_0})[\mathrm{tr}(\rho^{\otimes 2}R_{11})]^m \langle E_{k_1}|\rho|E_{k_1}\rangle > 0$, $b \coloneqq \langle E_{k_0}|\rho|E_{k_0}\rangle / \langle E_{k_1}|\rho|E_{k_1}\rangle > 0$, $c \coloneqq [\mathrm{tr}(\rho^{\otimes 2}R_{00})]^{-1} > 0$, $\|\cdot\|_{\mathrm{HS}}$ is the Hilbert-Schmidt norm, and we used

$$\mathrm{tr}(\rho^{\otimes 2}R_{11}) - \mathrm{tr}(\rho^{\otimes 2}R_{00}) = \|[\rho, |E_l\rangle\langle E_l|]\|_{\mathrm{HS}}^2. \quad (25)$$

From Eq. (24), if $W(\rho^{\otimes 2m+1}, U_m) \leq 0$ holds for all m, then $\|[\rho, |E_l\rangle\langle E_l|]\|_{\mathrm{HS}} = 0$; i.e., $[\rho, |E_l\rangle\langle E_l|] = 0$ must be satisfied. Therefore, ρ can be written as $\rho = \sum_l p_l |E_l\rangle\langle E_l|$ with $p_l \in (0,1)$. It is proved in Ref. [44] that we can extract positive work from multiple copies of such a state by a unitary operator U that satisfies $[U^\dagger H^{(N)} U, H^{(N)}] = 0$ for some $N \in \mathbb{N}$, unless the state is the Gibbs ensemble at positive temperature. ■

The second proposition states that if a unitary operator U satisfies $[U^\dagger H U, H] = 0$, we can extract the same amount of work for both the cases where U is implemented without the work storage and $\mathcal{C}(U)$ is implemented with the work storage, independently of its initial state.

Proposition 2.—If U satisfies $[U^\dagger H U, H] = 0$, the extracted work from a state ρ of the system of interest by the action of U without the work storage is the same as that from the state ρ with the work storage by the action of $\mathcal{C}(U)$ acting on the composite system with any state ρ_{W} of the work storage.

It is proved in Ref. [41] that if a work-extracting unitary operator U only permutes energy eigenstates, the extracted work without the work storage equals the corresponding extracted work with the work storage. Since we find that a unitary operator U satisfies $[U^\dagger H U, H] = 0$ if and only if U only permutes energy eigenstates, we can prove Proposition 2 However, we can also prove it without using the fact that U only permutes energy eigenstates, as shown in the direct proof below.

Proof.—Since U satisfies $[U^\dagger H U, H] = 0$, we obtain for any $q \in \mathbb{R}$,

$$\begin{aligned} W(e^{-\mathrm{i}qH}\rho e^{\mathrm{i}qH}, U) &= \mathrm{tr}[\rho e^{\mathrm{i}qH}(H - U^\dagger H U)e^{-\mathrm{i}qH}] \\ &= \mathrm{tr}[\rho(H - U^\dagger H U)] \\ &= W(\rho, U). \end{aligned} \quad (26)$$

Therefore, for any ρ_{W}, we get

$$\begin{aligned} W^{\mathrm{WS}}[\rho, \rho_{\mathrm{W}}, \mathcal{C}(U)] &= \int_{-\infty}^{\infty} dq \langle q|\rho_{\mathrm{W}}|q\rangle W(e^{-\mathrm{i}qH}\rho e^{\mathrm{i}qH}, U) \\ &= \int_{-\infty}^{\infty} dq \langle q|\rho_{\mathrm{W}}|q\rangle W(\rho, U) \\ &= W(\rho, U). \end{aligned} \quad (27)$$

■

From Propositions 1 and 2, if a state is completely passive in the setup with the work storage, the state is the Gibbs ensemble independently of the initial state of the work storage. The converse is obvious from Eqs. (19) and (20). Therefore, we finally obtain the following proposition.

Proposition 3.—For any initial state ρ_{W} of the work storage, a state ρ of the system of interest is completely passive in the setup with the work storage, if and only if the state ρ is the Gibbs ensemble at positive temperature.

We here remark on the relation between the above proposition and some known results. In fact, essentially the same result has been proved in Ref. [9] by allowing for the introduction of any number of auxiliary heat baths, while in our setup we do not allow for it. Reference [45] also addresses a similar problem without allowing for auxiliary heat baths but by assuming that the work storage is initially in the uniform superposition of energy eigenstates, while in our setup we consider an arbitrary initial state of the work storage.

C. With symmetry constraints

We consider symmetry-protected completely passive states in the setup with the work storage and show that only GGEs are symmetry-protected completely passive. Since we consider the work extraction that is purely defined by the energy, we introduce the work storage that only stores the energy but does not store other conserved charges associated with symmetry constraints, in contrast to Ref. [37]. For example, we can imagine the situation where the system consists of atoms with the conserved particle number, which is coupled to light in a cavity as an external system. In such a situation, it is natural to impose U(1) symmetry only on the system of interest, instead of the total system.

We consider a symmetry-respecting unitary V in the setup with the work storage, which is supposed to satisfy not only conditions (I) and (II) mentioned in Sec. III A, but also the following: (III) V respects the symmetry of the system of interest, i.e.,

$$[V, U_g \otimes I] = 0. \quad (28)$$

This reflects the fact that the work storage does not store the conserved charges. We note that this condition implies that charges are locally conserved only in the system of interest, and is tighter than the condition used in Ref. [37], which only requires global charge conservation in the total system including the storage. Then, we define that a state is symmetry-protected completely passive, if we cannot extract positive work from any number of copies of the state by any unitary V that satisfies conditions (I), (II), and (III).

The condition for symmetry-protected complete passivity in the setup with the work storage is now stated as the following theorem.

Theorem 2.—For any initial state of the work storage, a state of the system of interest is symmetry-protected completely passive in the setup with the work storage, if and only if the state is the GGE.

We can prove Theorem 2 in the following manner. From Eqs. (19) and (20), it is obvious that we cannot extract positive work from multiple copies of the GGE with the work storage under symmetry constraints. Then, we only need to show that we can extract positive work from multiple copies of any other state than the GGE by some unitary that satisfies conditions (I), (II), and (III). In the proof of Theorem 1 (see Proposition S8 of Supplemental Material [38]), we construct a unitary operator U that extracts positive work from any state other than the GGE and satisfies $[U^\dagger H^{(N)} U, H^{(N)}] = 0$, which is the generalization of Proposition 1 to the setup under symmetry constraints. From Proposition 2, in the setup with the work storage, we can extract the same amount of work by implementing $\mathcal{C}(U)$ for any initial state of the work storage. We can also check that if U is symmetry respecting, $\mathcal{C}(U)$ satisfies condition (III). Therefore, if a state is not symmetry-protected completely passive in the setup without the work storage, then the state is not symmetry-protected completely passive in the setup with the work storage independently of the state of the work storage.

IV. DISCUSSION

In this paper, we have provided the characterization of complete passivity for systems under symmetry constraints, which is referred to as symmetry-protected thermal equilibrium. We proved that a state is symmetry-protected completely passive if and only if it is a GGE of the form Eq. (1), which is the main result of this paper (Theorem 1). Remarkably, our result applies to noncommutative symmetries, as illustrated by the dimer model with SU(2) symmetry discussed in Sec. II C. While we leave the full proof of Theorem 1 to Supplemental Material [38], that for the special case of the dimer model is provided in Appendix B. In Appendix C, we also show that under a certain class of finite group symmetry constraints, only Gibbs states are completely passive.

Moreover, we proved that the same characterization of symmetry-protected complete passivity holds true, by explicitly including the work storage as a quantum system (Theorem 2). As a by-product (Proposition 3), we proved that, in a stronger form than the known results in literature, only Gibbs ensembles are completely passive without symmetry constraints in the presence of the work storage. We note that the energy levels of the work storage introduced in our setup are unbounded from below, but we expect that we can extend our argument for the work storage bounded from below by following the idea of Ref. [24].

We here discuss the relationship between the present work and other approaches to symmetries and GGEs. Let us first clarify the difference between our study and a previous study by Yunger Halpern *et al.* [37], where noncommutative GGE Eq. (1) also appears. In our study, a symmetry constraint is imposed solely on the system of interest, while in Ref. [37] it is imposed on the entire system including external charge storages. That is, the charges of the system are solely conserved in our setup, while the charges can be transferred to the storages in their setup. It should also be emphasized that our definition of work is given purely by the energy (i.e., the Hamiltonian), while they adopted a generalized notion called chemical work. Related to this point, a characteristic of our approach to symmetry-protected (complete) passivity lies in the fact that its definition (Sec. II A) itself does not involve the parameters μ_i of the GGE Eq. (1). Therefore, our setup is different from theirs, and our result complements their result by providing a further support that Eq. (1) is a proper form of the GGE including the noncommutative cases.

In more detail, the results of Theorem 2 and Ref. [37] are summarized in Table I, where the classes of completely passive states are classified by the charge conservation constraints and the definitions of work. The charge conservation constraint is tighter in an upper row, because local charge conservation in the system of interest implies global charge conservation in the total system including the storage (given that the charges are conserved in the storage). The definitions of extracted work in the left- and right-hand columns are, respectively, the change in the expectation value of the energy $\langle H \rangle$ and that of a linear combination of the energy and the charges $\langle H + \sum_i \xi_i Q_i \rangle$. Correspondingly to these definitions of the work, allowed unitary operators commute with the total Hamiltonian H^{tot} of the system and the storage in the left-hand column, while they commute with the linear combination $H^{\mathrm{tot}} + \sum_i \xi_i Q_i^{\mathrm{tot}}$ of the total Hamiltonian and the total charges in the right-hand column. The result of Theorem 2 is shown in (a1) in Table I, where completely passive states are defined based on the extraction of $\langle H \rangle$ under local charge conservation. On the other hand, the result of Ref. [37] is shown in (b2) in Table I, where completely passive states are defined based on the extraction of $\langle H + \sum_i \xi_i Q_i \rangle$ under global charge conservation. We note that, in contrast to Theorem 1, Theorem 2 includes the storage, allowing us to fairly compare our work and Ref. [37].

The class of completely passive states in our work is strictly larger than that in Ref. [37]. This reflects the fact that the class of operations is strictly smaller in our work than in Ref. [37]. For the purpose of comparison, we consider the setup of (a2) in Table I. The class of states in (a2) is strictly larger than that in (b2) in Table I, because μ_i's in (a2) can be chosen independently of ξ_i's in the definition of work, while ξ_i's in (b2) must be the same as those in the definition of work. We note that the class of completely passive states in (a2) is the same as that in (a1), because the expectation values of charges $\langle Q_i \rangle$ are invariant, and the change in $\langle H + \sum_i \xi_i Q_i \rangle$ is the same as that in $\langle H \rangle$. We also note that as for (b1), the class of completely passive states is the Gibbs ensemble, which is obtained by

TABLE I. The classes of completely passive states classified by charge conservation constraints and the definitions of work. In the left-hand column, the total energy H^{tot} is conserved, while in the right-hand column, a linear combination $H^{\mathrm{tot}} + \sum_i \xi_i Q_i^{\mathrm{tot}}$ of the total energy and the total charges is conserved. In Theorem 2, the charges are locally conserved in the system of interest, leading to the form of GGE shown in (a1). In Ref. [37], the charges are globally conserved in the total system including the storage, leading to the form of GGE shown in (b2). Note that the class of operations is strictly smaller than that in the lower row. In particular, the classes of states shown in (a1) and (a2) are strictly smaller than those in (b1) and (b2), respectively. We can obtain the result for (c2) by changing the definition of the Hamiltonian H to $H + \sum_i \xi_i Q_i$ in (c1).

		Definition of work			
		Change in $\langle H \rangle$		Change in $\langle H + \sum_i \xi_i Q_i \rangle$	
Charge conservation	Local (Theorem 2)	(a1)	$(1/Z)\exp(-\beta H - \sum_i \mu_i Q_i)$ μ_i's are freely chosen	(a2)	$(1/Z)\exp(-\beta H - \sum_i \mu_i Q_i)$ μ_i's are freely chosen
	Global (Ref. [37])	(b1)	$(1/Z)\exp(-\beta H)$	(b2)	$(1/Z)\exp[-\beta(H + \sum_i \xi_i Q_i)]$
	None	(c1)	$(1/Z)\exp(-\beta H)$	(c2)	$(1/Z)\exp[-\beta(H + \sum_i \xi_i Q_i)]$

substituting $\xi_i = 0$ into (b2). In addition, the class of allowed operations in (b1) is strictly smaller than that in (c1) in Table I, because only operations satisfying global charge conservation are allowed in (b1).

In Refs. [46,47], other types of passivity in the presence of conserved charges are defined and investigated. In these studies, however, (complete) passivity is defined with a focus on extracting charges themselves instead of the energy (see also Ref. [45]), which is the opposite case to our setup of purely extracting energy.

Next, let us discuss some possible applications of our study. Since symmetry constraints restrict the class of operations, the amount of work that one can extract from a particular state is reduced by symmetry constraints as shown in Eq. (A9). This observation has also been made in the context of thermalization of many-body systems by quench [48,49] and by Bragg scattering [50]. While this implies that the presence of symmetry can be a disadvantage for heat engines, we would emphasize that it can be an advantage at the same time: symmetry constraints expand the class of thermally stable states. If one attempts to construct a heat engine operating under symmetry constraints, the equilibrium state of the heat bath can be chosen to be a GGE, which cannot be thermally stable without symmetry constraints. This would lead to a more flexible design principle of quantum heat engines than the conventional approach where the equilibrium state must be the Gibbs ensemble [2–8].

Meanwhile, quantum batteries proposed in Ref. [26] make use of the fact that one can extract work from multiple copies of a state if it is passive but not completely passive. Correspondingly, our result provides a designing principle of symmetry-protected quantum batteries, which make use of symmetry-protected passive states. They offer a wider choices of states than the conventional quantum batteries, and thus may have the potential to store more work. In fact, we have explicitly constructed the protocol by which one can extract work from multiple copies of a state that is symmetry-protected passive but not symmetry-protected completely passive (see Appendix B for the dimer model and Supplemental Material for the general case [38]). This result suggests a concrete method to design symmetry-protected quantum batteries.

Our work characterizes the noncommutative GGE Eq. (1) in a different setup from Refs. [37,46,47], and suggests that this form of GGE would be useful in the context of thermalization. For example, in symmetric systems, the noncommutative GGE can be used in the investigation of the eigenstate thermalization hypothesis (ETH) [51], which states that even a single energy eigenstate presents thermalization properties. One of the future research directions would be to compare the equilibrated value of an observable for a single energy eigenstate and the predicted value from the GGE of the form Eq. (1) in a system that has noncommutative symmetry such as SU(2). As a relevant study, Ref. [52] investigates the relation between the ETH and passivity by considering work extraction from a single energy eigenstate. This study numerically shows that it is impossible to extract work from any eigenstates in a nonintegrable system and from most eigenstates in an integrable system by simple unitary operations such as quench protocols. Combining this study and the present work may lead to a future study of work extraction from an eigenstate under symmetry constraints.

Adding a locality constraint to the symmetry constraints in our work is another future perspective. About work extraction under a locality constraint, Refs. [53,54] show that the difference between the maximal extracted work by local and global operations is related to entanglement. These studies and our work motivate us to study how entanglement is related to the difference between extracted work by local and global operations under symmetry constraints, as our Theorem 3 identifies the maximal extracted work under symmetry constraints from arbitrary systems. This direction of study may reveal a new insight into the relation between entanglement and symmetry.

We finally note that the resource theory of asymmetry adopts a broader class of free operations [55,56] than our setup. The resource theory that adopts our smaller class of free operations would also be useful in quantum thermodynamics in the presence of conserved charges, because our setup requires the conservation of the expectation values of the charges, but their setup does not. For example, in their setup the operation that increases the particle number independently of the initial state is allowed, while in our setup such operation is not allowed. From a general perspective of resource theories, we can say that our result has determined the class of free states of the resource theory of thermodynamics with conserved charges, and thus would serve as a foundation of a new class of resource theories in the presence of symmetries.

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APPENDIX A: SYMMETRY-PROTECTED PASSIVITY

In this Appendix, we reveal the characterization of symmetry-protected (not complete) passivity. Symmetry-protected passive states are defined as the states from which no positive amount of work can be extracted by any symmetry-respecting unitary operations [see also Fig. 2(a)]. We prove that a state is symmetry-protected passive, if and only if every sector of the symmetrized density operator of the state is passive with respect to the corresponding Hamiltonian.

First, we specify the form of symmetry-respecting operators. We follow the method in Ref. [57] and make use of the decomposition of the Hilbert space induced by a representation of a group:

$$\mathcal{H} = \bigoplus_{\lambda\in\Lambda} \mathcal{R}_\lambda \otimes \mathcal{M}_\lambda, \tag{A1}$$

where Λ is the set of the labels of inequivalent irreducible representations that appear in a given representation, $\mathcal{R}_\lambda$ is a space carrying an irreducible representation, and $\mathcal{M}_\lambda$ is a space carrying a trivial representation. We note that a representation is called irreducible if it cannot be seen as the composition of simpler representations, or equivalently, it does not have any invariant subspace. Correspondingly, the symmetry representation U_g can be decomposed in the following form [see Fig. 5(a)]:

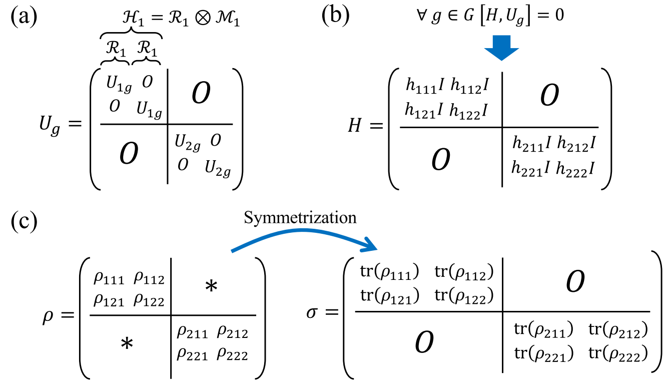

FIG. 5. (a) Schematic of the irreducible decomposition of a unitary representation. (b) The corresponding form of a symmetry-respecting Hamiltonian. (c) The effective density operator corresponding to the irreducible representation.

$$U_g = \bigoplus_{\lambda\in\Lambda} U_{\lambda g} \otimes I_{\mathcal{M}_\lambda}, \tag{A2}$$

where $\{U_{\lambda g}\}_{g\in G}$ is an irreducible representation of G acting on $\mathcal{R}_\lambda$, and $I_{\mathcal{M}_\lambda}$ is the identity operator on $\mathcal{M}_\lambda$. For example, SU(2) symmetry representation on the system composed of two spin-1/2 systems can be decomposed into spin-0 representation on the singlet space and spin-1 representation on the triplet space. Under the decomposition Eq. (A2), Schur's lemma (e.g., Proposition 4.8 of Ref. [58]) states that every symmetry-respecting Hamiltonian H and symmetry-respecting unitary operator U can be written as

$$H = \bigoplus_{\lambda\in\Lambda} I_{\mathcal{R}_\lambda} \otimes H_\lambda, \qquad U = \bigoplus_{\lambda\in\Lambda} I_{\mathcal{R}_\lambda} \otimes U_\lambda, \tag{A3}$$

with some Hermitian operator H_λ and unitary operator U_λ on $\mathcal{M}_\lambda$, where $I_{\mathcal{R}_\lambda}$ is the identity operator on $\mathcal{R}_\lambda$ [see Fig. 5(b)].

In order to describe the characterization of symmetry-protected passivity, we introduce the following symmetrized state:

$$\sigma := \int_G dg U_g \rho U_g^\dagger, \tag{A4}$$

where dg is the group-invariant (Haar) measure over G. This symmetrizing mapping is studied in the resource theory of asymmetry [57]. Since σ is symmetry respecting, σ can be written as

$$\sigma = \bigoplus_{\lambda\in\Lambda} I_{\mathcal{R}_\lambda} \otimes \sigma_\lambda, \tag{A5}$$

with some Hermitian operator σ_λ on $\mathcal{M}_\lambda$ [see Fig. 5(c)].

Now, symmetry-protected passivity of ρ is equivalent to passivity of all σ_λ's. This can be formally stated as the following theorem.

Theorem 3.—Let G be a group and $\{U_g\}_{g\in G}$ be its unitary representation. A state ρ is symmetry-protected passive with respect to a symmetry-respecting Hamiltonian H, if and only if σ_λ defined by Eq. (A5) is passive with respect to H_λ for all $\lambda \in \Lambda$.

Proof.—First, for any symmetry-respecting unitary operator U, we show that the extracted work from ρ and σ by U are the same. From Eq. (A4),

$$\begin{aligned} W(\sigma, U) &= \mathrm{tr}[\sigma(H - U^\dagger H U)] \\ &= \mathrm{tr}\left[\left(\int_G dg U_g \rho U_g^\dagger\right)(H - U^\dagger H U)\right] \\ &= \int_G dg \mathrm{tr}[U_g \rho U_g^\dagger (H - U^\dagger H U)] \\ &= \int_G dg \mathrm{tr}[\rho U_g^\dagger (H - U^\dagger H U) U_g] \\ &= \int_G dg \mathrm{tr}[\rho(H - U^\dagger H U)] \\ &= \mathrm{tr}[\rho(H - U^\dagger H U)] \\ &= W(\rho, U). \end{aligned} \tag{A6}$$

We next show that the extracted work from σ can be written by the extracted work from σ_λ. From Eqs. (A3) and (A5),

$$\begin{aligned} W(\sigma, U) &= \mathrm{tr}[\sigma(H - U^\dagger H U)] \\ &= \mathrm{tr}\{(\bigoplus_{\lambda\in\Lambda} I_{\mathcal{R}_\lambda} \otimes \sigma_\lambda)[\bigoplus_{\lambda\in\Lambda} I_{\mathcal{R}_\lambda} \otimes (H_\lambda - U_\lambda^\dagger H_\lambda U_\lambda)]\} \\ &= \mathrm{tr}[\bigoplus_{\lambda\in\Lambda} I_{\mathcal{R}_\lambda} \otimes \sigma_\lambda (H_\lambda - U_\lambda^\dagger H_\lambda U_\lambda)] \\ &= \sum_{\lambda\in\Lambda} \mathrm{tr}[I_{\mathcal{R}_\lambda} \otimes \sigma_\lambda (H_\lambda - U_\lambda^\dagger H_\lambda U_\lambda)] \\ &= \sum_{\lambda\in\Lambda} \mathrm{tr}(I_{\mathcal{R}_\lambda}) \mathrm{tr}[\sigma_\lambda (H_\lambda - U_\lambda^\dagger H_\lambda U_\lambda)] \\ &= \sum_{\lambda\in\Lambda} r_\lambda W(\sigma_\lambda, U_\lambda), \end{aligned} \tag{A7}$$

where r_λ is the dimension of $\mathcal{R}_\lambda$. By comparing Eqs. (A6) and (A7), the extracted work from ρ is written as

$$W(\rho, U) = \sum_{\lambda\in\Lambda} r_\lambda W(\sigma_\lambda, U_\lambda). \tag{A8}$$

Therefore, the maximal extracted work from ρ under the symmetry constraint is given by

$$W_{\max,G}(\rho) = \sum_{\lambda\in\Lambda} r_\lambda W_{\max}(\sigma_\lambda), \tag{A9}$$

where $W_{\max}(\sigma_\lambda)$ is the maximal extracted work from σ_λ under no symmetry constraints and $W_{\max,G}(\rho)$ is the maximal extracted work from ρ under the symmetry constraint. This shows that ρ is symmetry-protected passive with respect to H, if and only if σ_λ is passive in the ordinary sense with respect to H_λ. ∎

Finally, we remark on the setup with the work storage discussed in Sec. III. From Eq. (17), we can prove that ρ is symmetry-protected passive with the initial state ρ_{W} of the work storage, if and only if $\mathcal{D}_{\rho_{\mathrm{W}}}(\rho)$ is symmetry-protected passive in the setup without the work storage. In order to prove this, we can prove (Lemma S12 of Supplemental Material [38]) that the Kitaev construction Eq. (13) gives a one-to-one correspondence between symmetry-respecting unitaries in the setups with and without the work storage. Therefore, the proof goes in the same way as that in Sec. III B, and even under symmetry constraints, the maximal extracted work from ρ with the work storage ρ_{W} equals the maximal extracted work from $\mathcal{D}_{\rho_{\mathrm{W}}}(\rho)$ without the work storage.

APPENDIX B: PROOF OF THEOREM 1 FOR THE DIMER MODEL

In this Appendix, we present a full proof of Theorem 1 in the special case of the dimer model introduced in Sec. II C, as a simplest nontrivial example that has noncommutative symmetry. See Supplemental Material [38] for the complete proof for the general case.

In the proof of the *only if* part of Theorem 1, we consider work extraction by a series of symmetry-respecting unitary operators. We prove that if we cannot extract positive work from multiple copies of a state by any of those operations, then the state is a GGE at positive temperature. This is a generalization of the proof of Proposition 1 in Sec. III B.

We use the same notations as in Sec. II C for the dimer model. Specifically, the Hamiltonian is given by $H = \boldsymbol{s}_1 \cdot \boldsymbol{s}_2$, and we denote the total spin operator in the α direction of the dimer by $Q^\alpha \coloneqq s_1^\alpha \otimes I + I \otimes s_2^\alpha$ for $\alpha = x, y, z$. Let ρ be the initial state of the dimer.

Proof.—To prove Theorem 1 for the dimer setup, we consider the following three steps.

Step 1: (Proposition S3 of Supplemental Material [38]) First, we prove that if a state ρ is completely passive, then $\rho^{\otimes 2}$ commutes with the spin inner product $\boldsymbol{Q} \cdot \boldsymbol{Q} \coloneqq \sum_{\alpha=x,y,z} Q^\alpha \otimes Q^\alpha$. For that purpose, we construct a series of unitary operators such that if positive work cannot be extracted from any number of multiple copies of a state ρ by the operations, then $\rho^{\otimes 2}$ commutes with $\boldsymbol{Q} \cdot \boldsymbol{Q}$.

Let the spectral decomposition of $\boldsymbol{Q} \cdot \boldsymbol{Q}$ be written as $\boldsymbol{Q} \cdot \boldsymbol{Q} = \sum_\omega \omega P_\omega$, where ω is an eigenvalue and P_ω is the projection operator onto the eigenspace of ω. We take arbitrary P_ω and consider unitary operators $U_m \coloneqq I - \sum_{i,j\in\{0,1\}} (-1)^{i-j} R_{ij}^{\otimes m} \otimes |\Psi_i\rangle\langle\Psi_j|$ acting on $4m+3$ copies of the system for $m \in \mathbb{N}$, where $R_{ij} \coloneqq \frac{1}{2}[I - (-1)^i T][P_\omega \otimes (I - P_\omega)][I - (-1)^j T]$ with the swapping operator T of the states of the two pairs of dimers, and

$|\Psi_0\rangle := |s\rangle|s\rangle|s\rangle$, $|\Psi_1\rangle := (1/\sqrt{6})\sum_{i,j,k\in\{1,2,3\}}\epsilon_{ijk}|t_i\rangle|t_j\rangle|t_k\rangle$ with the singlet state $|s\rangle$, the triplet states $|t_1\rangle, |t_2\rangle, |t_3\rangle$ of a dimer, and the Levi-Civita symbol ϵ_{ijk}. U_m is symmetry respecting because P_ω and T commute with $Q^{\alpha(2)}$ and $Q^{\alpha(4)}$, respectively, and $|\Psi_i\rangle$ is an eigenstate of $Q^{\alpha(3)}$ with eigenvalue 0 for $\alpha = x, y, z$, where $Q^{\alpha(N)}$ is defined by Eq. (3).

We calculate the extracted work $W(\rho^{\otimes 4m+3}, U_m)$ from $\rho^{\otimes 4m+3}$ by U_m. Since R_{ij} commutes with $H^{(4)}$ defined by Eq. (6) and $|\Psi_i\rangle$ satisfies $H^{(3)}|\Psi_0\rangle = -\frac{9}{4}|\Psi_0\rangle$, $H^{(3)}|\Psi_1\rangle = \frac{3}{4}|\Psi_1\rangle$, we obtain

$$H^{(4m+3)} - U_m^\dagger H^{(4m+3)} U_m = 3(R_{11}^{\otimes m} \otimes |\Psi_1\rangle\langle\Psi_1| - R_{00}^{\otimes m} \otimes |\Psi_0\rangle\langle\Psi_0|). \quad \text{(B1)}$$

Therefore, $W(\rho^{\otimes 4m+3}, U_m)$ is given by

$$\begin{aligned} W(\rho^{\otimes 4m+3}, U_m) &= \mathrm{tr}[\rho^{\otimes 4m+3}(H^{(4m+3)} - U_m^\dagger H^{(4m+3)} U_m)] \\ &= 3[\mathrm{tr}(\rho^{\otimes 4}R_{11})^m \langle\Psi_1|\rho^{\otimes 3}|\Psi_1\rangle \\ &\quad -\mathrm{tr}(\rho^{\otimes 4}R_{00})^m \langle\Psi_0|\rho^{\otimes 3}|\Psi_0\rangle] \\ &= a\left[1 - b\left(\frac{\mathrm{tr}(\rho^{\otimes 4}R_{00})}{\mathrm{tr}(\rho^{\otimes 4}R_{11})}\right)^m\right] \\ &= a[1 - b(1 + c\|[\rho^{\otimes 2}, P_\omega]\|_{\mathrm{HS}}^2)^{-m}], \end{aligned} \quad \text{(B2)}$$

where $a := 3[\mathrm{tr}(\rho^{\otimes 4}R_{11})]^m \langle\Psi_1|\rho^{\otimes 3}|\Psi_1\rangle > 0$, $b := \langle\Psi_0|\rho^{\otimes 3}|\Psi_0\rangle / \langle\Psi_1|\rho^{\otimes 3}|\Psi_1\rangle > 0$, $c := [\mathrm{tr}(\rho^{\otimes 4}R_{00})]^{-1} > 0$, and we used

$$\mathrm{tr}(\rho^{\otimes 4}R_{11}) - \mathrm{tr}(\rho^{\otimes 4}R_{00}) = \|[\rho^{\otimes 2}, P_\omega]\|_{\mathrm{HS}}^2. \quad \text{(B3)}$$

If $W(\rho^{\otimes 4m+3}, U_m) \leq 0$ holds for all m, then $\|[\rho^{\otimes 2}, P_\omega]\|_{\mathrm{HS}} = 0$; i.e., $[\rho^{\otimes 2}, P_\omega] = 0$ must be satisfied. Since this holds for all P_ω, $\rho^{\otimes 2}$ commutes with $\boldsymbol{Q}\cdot\boldsymbol{Q}$. Note that $b \geq 1$ is shown later, implying that $\|[\rho^{\otimes 2}, P_\omega]\|_{\mathrm{HS}} = 0$ is sufficient for $W(\rho^{\otimes 4m+3}, U_m) \leq 0$.

Step 2: (Proposition S5 of Supplemental Material [38]) Next, we prove that if $\rho^{\otimes 2}$ commutes with $\boldsymbol{Q}\cdot\boldsymbol{Q}$, then ρ can be written as the product of a symmetry-respecting operator and the exponential of a linear combination of the conserved charges $\{Q^\alpha\}_{\alpha=x,y,z}$. We define $\xi := -\log(\rho)$, $\mathcal{P}(\xi) := \sum_\alpha \frac{1}{2}\mathrm{tr}(\xi Q^\alpha)Q^\alpha$ and $\eta := \xi - \mathcal{P}(\xi)$. $\mathcal{P}(\xi)$ can be seen as the projection of ξ onto the linear subspace spanned by $\{Q^\alpha\}$ in terms of the Hilbert-Schmidt inner product in the operator space. ξ and η satisfy the following relation for $\alpha = x, y, z$:

$$\begin{aligned} &\mathrm{tr}_{\mathcal{H}_2}\{(I\otimes Q^\alpha)[\xi\otimes I + I\otimes\xi, \boldsymbol{Q}\cdot\boldsymbol{Q}]\} \\ &= \sum_\beta \mathrm{tr}_{\mathcal{H}_2}([\xi, Q^\beta]\otimes Q^\alpha Q^\beta + Q^\beta\otimes Q^\alpha[\xi, Q^\beta]) \\ &= \sum_\beta\{\mathrm{tr}(Q^\alpha Q^\beta)[\xi, Q^\beta] - \mathrm{tr}(\xi[Q^\alpha, Q^\beta])Q^\beta\} \\ &= \sum_\beta\left[2\delta_{\alpha\beta}[\xi, Q^\beta] - \mathrm{tr}\left(\xi\sum_\gamma\epsilon_{\alpha\beta\gamma}Q^\gamma\right)Q^\beta\right] \\ &= 2[\xi, Q^\alpha] - \sum_\gamma\left[\mathrm{tr}(\xi Q^\gamma)\sum_\beta\epsilon_{\gamma\alpha\beta}Q^\beta\right] \\ &= 2[\xi, Q^\alpha] - \sum_\gamma\mathrm{tr}(\xi Q^\gamma)[Q^\gamma, Q^\alpha] \\ &= 2\left[\xi - \sum_\gamma\frac{1}{2}\mathrm{tr}(\xi Q^\gamma)Q^\gamma, Q^\alpha\right] \\ &= 2[\xi - \mathcal{P}(\xi), Q^\alpha] \\ &= 2[\eta, Q^\alpha], \end{aligned} \quad \text{(B4)}$$

where $\mathcal{H}_2$ is the Hilbert space of the second dimer and $\epsilon_{\alpha\beta\gamma}$ is the Levi-Civita symbol with $\epsilon_{xyz} = 1$. If $\rho^{\otimes 2}$ commutes with $\boldsymbol{Q}\cdot\boldsymbol{Q}$, $\xi^{(2)} = -\log(\rho^{\otimes 2})$ also commutes with $\boldsymbol{Q}\cdot\boldsymbol{Q}$. Then from Eq. (B4), we get $[\eta, Q^\alpha] = 0$; i.e., η is symmetry respecting. Therefore, ρ can be written as $\rho = \exp(-\xi) = \exp(-\eta)\exp(-\sum_{\alpha=x,y,z}\mu_\alpha Q^\alpha)$, where $\mu_\alpha := \frac{1}{2}\mathrm{tr}(\xi Q^\alpha)$.

Step 3: (Proposition S4 of Supplemental Material [38]) Finally, we combine the results in steps 1 and 2, consider work extraction again, and prove Theorem 1. Suppose that ρ is symmetry-protected completely passive. From steps 1 and 2, ρ can be written as $\rho = \exp(-\eta)\exp(-\sum_{\alpha=x,y,z}\mu_\alpha Q^\alpha)$ with some symmetry-respecting operator η and $\mu_\alpha \in \mathbb{R}$. Since the total spin symmetry operators irreducibly act on the singlet space and the triplet space, Schur's lemma implies that symmetry-respecting η can be written as $\eta = c_s|s\rangle\langle s| + c_t\sum_{i=1}^3|t_i\rangle\langle t_i|$ with some $c_s, c_t \in \mathbb{R}$. Since $H = -\frac{3}{4}|s\rangle\langle s| + \frac{1}{4}\sum_{i=1}^3|t_i\rangle\langle t_i|$, η can be written as $\eta = \mu I + \beta H$ with some $\mu, \beta \in \mathbb{R}$. Note that such a simple relation between η and H is specific to this dimer model that has only two energy levels. Then ρ can be written as $\rho = \exp(-\mu I - \beta H)\times\exp(-\sum_{\alpha=x,y,z}\mu_\alpha Q^\alpha) = \exp(-\beta H - \sum_{\alpha=x,y,z}\mu_\alpha Q^\alpha)/\exp(\mu)$. From the normalization condition, $\exp(\mu) = \mathrm{tr}[\exp(-\beta H - \sum_{\alpha=x,y,z}\mu_\alpha Q^\alpha)] = Z_{\mathrm{GGE}}$ and we obtain $\rho = \exp(-\beta H - \sum_{\alpha=x,y,z}\mu_\alpha Q^\alpha)/Z_{\mathrm{GGE}}$. In order to prove that $\beta \geq 0$, we consider the case where $m = 0$ in Eq. (B2). In this case, the extracted work is given by $W(\rho^{\otimes 3}, U_0) = a(1-b) = a[1 - \exp(3\beta)]$. Since ρ is symmetry-protected completely passive, $W(\rho^{\otimes 3}, U_0) \leq 0$, and thus we get $\beta \geq 0$ (i.e., $b \geq 1$). ∎

In the proof for the general case (see Supplemental Material [38]), we decompose a connected compact Lie group into a compact Abelian Lie group and a semisimple Lie group by Levi decomposition (Theorem 4.29 of Ref. [58]). When we deal with the semisimple Lie group, we use a Casimir operator (Lemma 3.3.7 of Ref. [59]) as a generalization of the spin inner product $\boldsymbol{Q}\cdot\boldsymbol{Q}$ along with a generalized version of totally antisymmetric states $|\Psi_i\rangle$. We also use the fact that every conserved charge Q associated with a semisimple Lie group symmetry satisfies $\mathrm{tr}(Q)=0$. When we deal with the Abelian Lie group, we use the notion of virtual temperature introduced in Ref. [44] under symmetry constraints.

APPENDIX C: FINITE-GROUP SYMMETRY AND TIME-REVERSAL SYMMETRY

We consider the case where symmetry constraints on the operations are described by some finite groups. Specifically, we prove that if the symmetry group is a finite cyclic group or a dihedral group, every symmetry-protected completely passive state is just a conventional Gibbs ensemble. We note that in a one-dimensional lattice with the periodic boundary condition, spatial translation generates a finite cyclic group, and the combination of spatial translation and inversion generates a dihedral group (see Fig. 6). In addition, we investigate the case of time-reversal symmetry without spin degrees of freedom, which is an antiunitary symmetry (class AI). In this case, every symmetry-protected completely passive state is again a conventional Gibbs ensemble.

First, we consider the case of finite-group symmetry. As for passivity, the same argument as in Appendix A can be applied, where the symmetrizing mapping Eq. (A4) can be replaced with $(1/|G|)\sum_{g\in G}U_g\rho U_g^\dagger$, with $|G|$ being the order of G. Then, ρ is symmetry-protected passive if and only if all σ_λ's are passive.

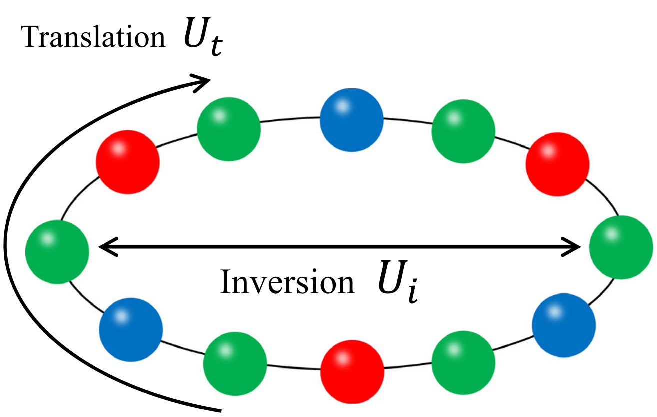

FIG. 6. An example of dihedral group symmetry in a one-dimensional lattice with the periodic boundary condition. The system is invariant under spatial translation and inversion; these two operations generate a dihedral group.

As for complete passivity, we restrict ourselves to the cases of finite cyclic group symmetry and dihedral group symmetry, and prove that only Gibbs ensembles at positive temperature are symmetry-protected completely passive. The proof is parallel to that in Appendix B except for step 2.

The reason why the same argument as step 2 cannot be applied is that for finite groups, there does not exist an explicit counterpart of Casimir operators in Lie groups. Instead, we can show (Proposition S9 in Supplemental Material [38]) that if ρ is symmetry-protected completely passive, then ρ is symmetry-respecting; i.e., $[\rho, U_g]=0$ for all $g\in G$. In the case of a finite cyclic group, the proof of this statement is straightforward. Since U_g's are symmetry respecting, by a similar argument as step 1 of Appendix B, every symmetry-protected completely passive state commutes with all U_g's. On the other hand, in the case of a dihedral group, the proof is more complicated due to its noncommutativity. We can construct symmetry-respecting operators with the projection operators onto the eigenspaces of a symmetry operator U_t, where t is an element of a dihedral group of order $2n$ and satisfies $t^n=1$ (see Supplemental Material [38] for details).

We note that for the case of general finite groups, the characterization of completely passive states is an open problem, while we conjecture that only Gibbs ensembles are symmetry-protected completely passive as in the foregoing cases.

Next, we consider the case of time-reversal symmetry without spin degrees of freedom. In this case, time-reversal operator $\mathcal{T}$ is represented by the complex conjugation operator with respect to some basis of the Hilbert space of the system. We can prove (Theorem S2 of Supplemental Material [38]) that a state ρ is passive under the time-reversal symmetry, if and only if the time-reversal symmetrized state $\sigma \coloneqq (\rho+\mathcal{T}\rho\mathcal{T}^{-1})/2$ is passive in the ordinary sense.

We illustrate time-reversal symmetry-protected passivity by the following translation-symmetric five-site spinless fermion system with the periodic boundary condition. Let the Hamiltonian of the system be $H=\sum_{j=1}^2\sum_{i=1}^5[-t_j(c_i^\dagger c_{i+j}+c_{i+j}^\dagger c_i)+V_j n_i n_{i+j}]$, where c_i and n_i are the annihilation and the number operators at site i, and $t_j, V_j>0$ are parameters for intersite hopping and Coulomb repulsion. The ground state of this system is denoted as $|\phi_0\rangle$, and the first excited states are denoted as $|\phi_{\pm1}\rangle$ when $V_1>V_2>5t_1>15t_2$, where $|\phi_k\rangle \coloneqq \sum_{i=1}^5\exp[\mathrm{i}(2\pi/5)ik]c_i^\dagger|0\rangle/\sqrt{5}$ for $k=0,\pm1,\pm2$ with $|0\rangle$ being the vacuum. Then, one can see that $|\psi\rangle\coloneqq(\sqrt{3}|\phi_0\rangle+\mathrm{i}|\phi_1\rangle)/2$ is passive under the time-reversal symmetry constraint, but not without the constraint. In fact, we can extract work by a unitary operator that transforms $|\psi\rangle$ into $|\phi_0\rangle$, while this operator does not respect the time-reversal symmetry.

The outline of the proof of the above statement is as follows. For any symmetry-respecting unitary U, the extracted work from ρ and σ by U are the same, which implies that the maximal extracted work from ρ and σ under the time-reversal symmetry are the same. Since both of σ and H are symmetry respecting, σ can be converted to an ordinary passive state by a symmetry-respecting unitary operation. This implies that the maximal extracted work from σ with and without the time-reversal symmetry are the same. By combining these two relations, the maximal extracted work from ρ under the time-reversal symmetry and that from σ without the time-reversal symmetry are the same. Therefore, ρ is symmetry-protected passive if and only if σ is passive.

We can also prove (Theorem S5 of Supplemental Material [38]) that only Gibbs ensembles are completely passive under the time-reversal symmetry. It is obvious that Gibbs ensembles are symmetry-protected completely passive, and therefore we only need to prove the converse. Since the Hamiltonian is symmetry respecting, all the projection operators onto the energy eigenspaces are symmetry respecting. In the same way as step 1 of Appendix B, every symmetry-protected completely passive state commutes with all symmetry-respecting operators, and thus it commutes with all the projection operators onto the energy eigenspaces. This implies that the density operator of the state is diagonal in the energy eigenbasis. The rest of the proof can be constructed by a standard technique considering virtual temperatures [44].

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