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Self-Interacting Random Walks: Aging, Exploration, and First-Passage Times

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Self-interacting random walks are endowed with long-range memory effects that emerge from the interaction of the random walker at time t with the territory that it has visited at earlier times $t' < t$. This class of non-Markovian random walks has applications in a broad range of examples, from insects to living cells, where a random walker locally modifies its environment—leaving behind footprints along its path and, in turn, responding to its own footprints. Because of their inherent non-Markovian nature, the exploration properties of self-interacting random walks have remained elusive. Here, we show that long-range memory effects can have deep consequences on the dynamics of generic self-interacting random walks; they can induce aging and nontrivial persistence and transience exponents, which we determine quantitatively, in both infinite and confined geometries. Based on this analysis, we quantify the search kinetics of self-interacting random walkers and show that the distribution of the first-passage time to a target site in a confined domain takes universal scaling forms in the large-domain size limit, which we characterize quantitatively. We argue that memory abilities induced by attractive self-interactions provide a decisive advantage for local space exploration, while repulsive self-interactions can significantly accelerate the global exploration of large domains.

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I. INTRODUCTION

Random-walk theory provides a natural framework to model transport processes at all scales. Beyond the historical examples provided by particle transport in simple fluids at the molecular and supramolecular scales [1–3], it has also been used more recently to powerfully describe the dynamics of more complex, passive or active, larger-scale systems—ranging from polymers, molecular motors, or self-propelled colloids to cells or animals, whose dynamics take place in potentially complex environments [4–10]. In the latter case, the coupling of the internal degrees of freedom of the random walker to those of the environment generically leads to complex correlations and requires a non-Markovian description of the evolution over time of the position $X(t)$ of the random walker. Taking into account such memory effects remains a theoretical challenge even if several examples of model systems have been analyzed [11–18].

In this paper, we focus on a broad class of such non-Markovian random walkers, where memory effects emerge from the interaction of the random walker at time t with the territory that it has visited at earlier times $t' < t$ [19–27]. This class of self-interacting random walks has clear applications in a broad range of examples where a random walker locally modifies its environment—leaving behind footprints along its path and, in turn, responding to its own footprints [28,29]. Such behaviors have been reported for ants depositing pheromones along their path [30] or larger territorial animals [31]. More recently, self-interactions have been identified quantitatively in the case of living cells [32,33]. It was found *in vitro* in one- and two-dimensional settings that different cell types can chemically modify and durably remodel the extracellular matrix. In turn, it was shown that these long-lived cellular footprints deeply impact cell trajectories, with cells preferentially remaining within the previously explored territory and thus effectively attracted by their footprints. This provides a prototypical example of attractive self-interacting random walk, for which memory effects are demonstrated experimentally to have striking consequences on space exploration, such as aging and subdiffusion.

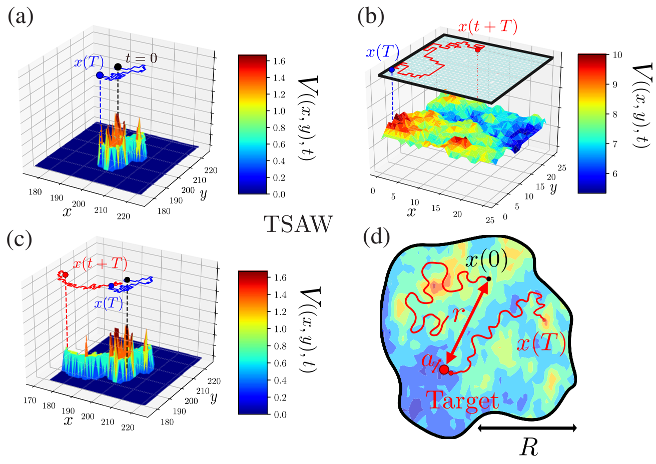

FIG. 1. Aging and first-passage times for self-interacting random walks in infinite space and in confined domains. Examples of $2d$ self-interacting trajectories (TSAW model, see text): In panel (a), the random path from $t = 0$ to T generates the local energy landscape $V((x, y), t)$ [proportional to the total number of visits to (x, y) until t], plotted along the vertical axis z. This dynamics can lead to long-range memory effects and aging at all timescales. The trajectory after T plotted in panel (c) explicitly depends on the full territory visited by the walker until T. In confined domains (b), the statistics of visits to a given site is radically modified by confinement. The dynamics of self-interacting random walks is thus geometry dependent. In panel (d), we aim at quantifying space exploration and search kinetics of confined self-interacting random walks.

More precisely, self-interacting random walks can be defined as nearest-neighbor random walks on a d-dimensional lattice, for which the probability to jump to a neighboring site i at time t is proportional to a weight function $w(n_i)$ that depends on the number of previous visits n_i of the random walker to site i up to time t (see Fig. 1). Writing $w(n) = e^{-V(n)}$, the process has the following clear interpretation: Upon visiting site i, the random walker deposits a signal that, in turn, modifies the local energy landscape V experienced by the walker. Note that, in contrast to autochemotactic or autophoretic systems [34,35], the deposited signal is assumed to be static and permanent, but not diffusive, which, as we argue below, leads to long-lived memory effects. To cover a broad spectrum of possible behaviors, we consider both attractive (V decreasing) and repulsive (V increasing) self-interacting random walks, with effective potentials ranging from linear [$V(n) = \beta n$] to bounded [$V(n) = \beta H(n)$], where $H(n)$ denotes the Heaviside function [22,24–27,36].

Despite their relevance in various contexts, the properties of self-interacting random walks remain poorly understood, even if significant results have been obtained in the mathematical [22,24,27,37] and physical communities [19–21,23,25,26,28,29,38–41]. This stems from the strongly non-Markovian nature of self-interacting random walks, whose dynamics depends on the set of number of visits (or local times) $\{n_i\}_{i\in\mathbb{Z}^d}$ at all sites i of the lattice at time t, and therefore on the full trajectory $\{X(t')\}_{t'\leq t}$ of the random walker up to time t. This dependence leads to memory effects at all timescales, which can have important consequences depending on the potential $V(n)$ and space dimension d, such as anomalous diffusion—defined as the anomalous scaling of the mean squared displacement (MSD):

$$\langle X^2(t)\rangle \underset{t\to\infty}{\propto} t^{2/d_w}, \tag{1}$$

with the walk dimension $d_w \neq 2$, or aging—which can be defined as the dependence of increments

$$\Delta^2(T,t) \equiv \langle (X(T+t) - X(T))^2\rangle \equiv 2D(T,t)t^{2/dw} \tag{2}$$

on the observation time T, where $D(T,t)$ is the effective time-dependent diffusion coefficient. As we recapitulate below, the analytical determination of d_w remains a theoretical challenge; so far, it has been obtained analytically or numerically for different examples of $V(n)$ and d, but even its numerical determination remains debated for attractive linear $V(n)$ for $d = 2$. In turn, the aging properties of $\Delta^2(T,t)$ have not been studied until recently [40], and they will be analyzed in this paper.

A central question that arises in random-walk theory is the quantification of space exploration by a random walker [42–45]. Beyond the MSD and increments of the position, which provide a first quantification of the dynamics of spreading in space, several observables have been proposed to quantify space exploration. Among these observables, the first-passage time (FPT) and its distribution have proved to play a key role [17,42,43,46–51]. Indeed, beyond being a prominent technical tool of random-walk theory that gives access to various observables, it quantifies the kinetics of general target search problems at all timescales and, as such, has a broad range of applications—from diffusion-limited reactions to animal foraging behavior.

In infinite space, the first-passage statistics to a target follows two very distinct behaviors depending on the so-called type of random walk [2]. In the compact or recurrent case, the survival probability $S(t)$, i.e., the probability that the target has not been found until time t, typically vanishes at long timescales as

$$S(t) \underset{t\to\infty}{\propto} t^{-\theta}, \tag{3}$$

where θ is the persistence exponent, which has been the focus of numerous studies [52]. In the noncompact or transient case, the survival probability admits a nonzero large time limit, which defines the hitting probability Π [2,42] according to $S \underset{t\to\infty}{\longrightarrow} 1 - \Pi$. In turn, the hitting probability is expected to decrease with the distance r from the starting position of the random walk and the target radius a according to

$$\Pi \underset{r\to\infty}{\propto} (a/r)^{\psi}. \quad (4)$$

The corresponding transience exponent ψ was recently introduced in Refs. [50,53], and it parallels the persistence exponent of recurrent processes. In spite of their pivotal role in quantifying first-passage properties of random walks, analytically determining the exponents θ, ψ for general non-Markovian aging processes remains a theoretical challenge [52,54]. In particular, they remain unknown analytically for most examples of self-interacting processes, with the exception of Ref. [40]; we will analyze them numerically and analytically in this paper.

In the case of geometrically confined spaces, which is relevant to most practical situations where the space accessible to the random walker is ultimately bounded, space exploration is known to be radically different. In particular, a target is eventually found with a probability of 1 for both compact and noncompact processes, and the broad tails of the FPT distribution are generally suppressed [47,48,50,55]. FPT statistics in confinement has been the subject of intense study over the last decade, and results have been obtained for general scale-invariant Markovian processes [47,55] or Gaussian non-Markovian processes [17]. More recently, a universal scaling form of the FPT distribution was derived in the limit of large confining volume for a class of scale-invariant non-Markovian processes that display power-law aging [50]; importantly, in this case, the scaling of the FPT distribution in confinement was found to be fully determined asymptotically by the exponents d_w, θ, ψ (in addition to the space dimension d), which are all defined in infinite space and independent of the geometric confinement.

In this context, quantifying space exploration of self-interacting random walks in confined geometry brings a new conceptual challenge. Indeed, qualitatively, it is expected that geometric confinement will modify the statistics of the numbers of visits $\{n_i\}$ at the sites i of the confined domain over time and thus the effective potential $V(n_i)$ experienced by the walker, thereby impacting the very dynamics of the process, as compared to that in infinite space. This can be illustrated by the simple example of the normal random walk for $d = 3$: In infinite space, the mean number of visits $\langle n_i \rangle$ to any site i converges to a finite value for $t \to \infty$, whereas it diverges as $\langle n_i \rangle \propto t$ in a confined domain. In the case of self-interacting random walks, the kinetics of space exploration thus directly feeds back to the dynamics of the process in a geometry-dependent manner, which suggests that key intrinsic features of the dynamics, such as increments [quantified by $D(T,t)$ and d_w], persistence, and transience exponents θ, ψ could in fact be different in confined and infinite geometries. This, in particular, makes earlier approaches to determine FPT statistics inapplicable and calls for new theoretical tools to quantify the space exploration of confined self-interacting random walks, which is at the core of this paper.

Our findings can be summarized as follows. We show that universal scaling forms of the FPT distributions of general self-interacting random walks in confinement can be derived in the large volume limit by generalizing the approach introduced in Ref. [50]. Because of the intrinsic aging properties of self-interacting random walks, different cases emerge depending on the preparation protocol. For "fresh" initial conditions, for which the random walker starts the search for a target in a domain that has never been explored, we find that the exponents d_w, θ, ψ that determine the FPT distribution are generally identical to those defined in infinite space. In other words, the FPT distribution in confinement can be asymptotically predicted from the knowledge of the process in infinite space only. This is quite remarkable because, as we show, geometric confinement ultimately deeply modifies the dynamics of the process and can even change the corresponding exponents $d_{w,c}, \theta_c, \psi_c$ defined in confinement. In contrast, for aged initial conditions, for which the random walker has been extensively wandering in the domain before the search starts, the exponents that determine the FPT distribution are those defined in confinement $d_{w,c}, \theta_c, \psi_c$ and can thus be different from the classical infinite space exponents d_w, θ, ψ. In that case, the process in confinement must therefore be characterized to determine the FPT distribution. In all cases, scaling functions are not universal and are process dependent. This analysis is made possible by a systematic quantitative characterization of the aging properties [quantified by $D(T,t)$ and d_w] and exponents θ or ψ of self-interacting random walks in both confined and infinite geometries, which highlights the impact of geometric confinement on their dynamics. Finally, this paper proposes a unified, quantitative analysis of aging, exploration, and FPT statistics of self-interacting random walks in confined and infinite geometries.

The paper is organized as follows. First, we briefly define the main classes of attractive and repelling self-interacting walks and recall their walk dimension d_w when it is known. In particular, we provide a general criterion for attractive walks that leads to bounded exploration ($d_w = \infty$) or to the regime of strong attraction ($\theta = \infty$, to be defined below); in this case, the FPT problem in confinement is trivially equivalent in the large volume limit to the problem in infinite space. Second, we quantitatively characterize the increments and their aging behavior, as well as the persistence and transience exponents θ, ψ in both confined and infinite geometries. Third, based on this analysis, we derive the asymptotic FPT distribution in confinement for both compact and noncompact self-interacting random walks for fresh initial conditions. Last, we discuss the impact of aging on FPT statistics by analyzing the case of aged initial conditions, which allows us to assess the impact of memory effects on target search kinetics of self-interacting random walks.

II. DEFINITIONS AND MAIN CLASSES OF SELF-INTERACTING RANDOM WALKS

As stated above, self-interacting random walks can be defined as nearest-neighbor random walks on a d-dimensional lattice, for which the probability to jump to a neighboring site i at time t is proportional to a weight function $w(n_i)$ that depends on the number of previous visits n_i of the random walker to site i up to time t. Denoting $w(n) = e^{-V(n)}$, the process has the following clear interpretation: Upon visiting site i, the random walker deposits a signal that in turn modifies the local energy landscape V experienced by the walker. Different classes of random walks are obtained, depending on the choice of weight function $w(n)$; below, we note the main known results concerning the MSD of these processes.

A. True self-avoiding walk (TSAW): $w(n) \propto e^{-\beta n}$

In the TSAW model [19,22,56,57], the effective potential $V(n)$ depends linearly on the local time n. For $\beta < 0$, the interaction is attractive, and it leads (almost certainly), as we show below, to the complete trapping of the random walker on a finite set of sites for all d, and thus formally to $d_w = \infty$:

$$\langle X(t)^2 \rangle \underset{t\to\infty}{\rightarrow} C(\beta), \tag{5}$$

where $C(\beta)$ is a d-dependent constant. For $\beta > 0$, the interaction is repulsive, and the random walker qualitatively avoids its own path. It has been shown that this leads to the following scaling of the MSD for $t \to \infty$ [19,38,56,57]:

(i) $d = 1$: $\langle X(t)^2 \rangle \propto t^{(4/3)}$, $d_w = (3/2)$,
(ii) $d = 2$: $\langle X(t)^2 \rangle \propto t \ln(t)^{\alpha}$, $d_w = 2$, $\alpha \approx 0.5$,
(iii) $d = 3$: $\langle X(t)^2 \rangle \propto t$, $d_w = 2$.

Note that the scaling of the MSD is thus anomalous (superdiffusive) for $d \leq 2$ because of self-repulsion, while it is diffusive for $d > 2$.

B. Subexponential self-repelling walk (SESRW): $w(n) \propto e^{-\beta n^k}$

The SESRW model [21,22,58] extends the TSAW to effective potentials $V(n)$ that depend sublinearly on the local time n: $V(n) = \beta n^k$, with $0 < k < 1$. Similarly to the TSAW, in the attractive case ($\beta < 0$), the random walker is (almost certainly) trapped for all d, and thus $d_w = \infty$. For $\beta > 0$, the effect of self-avoidance is clearly weaker than for the TSAW; however, it has been shown to still lead to superdiffusion for $d = 1$ [21,58]. This can be summarized as follows:

(i) $d = 1$: $\langle X(t)^2 \rangle \propto t^{(2(1+k)/(2+k))}$, $d_w = (2+k)/(1+k)$,
(ii) $d = 2$: $\langle X(t)^2 \rangle \propto t \ln(t)^{\alpha_k}$, $d_w = 2$, $\alpha_k \geq 0$,
(iii) $d = 3$: $\langle X(t)^2 \rangle \propto t$, $d_w = 2$.

C. Self-attractive random walk (SATW): $w(n) \propto e^{-\beta H(n)}$

In the SATW model [23–27,59,60], the effect of the self-interaction is assumed to saturate with the number of visits, so the effective potential $V(n)$ is bounded for $n \to \infty$. For the sake of simplicity, it is assumed in the SATW model that $V(n) = \beta H(n)$, with $H(0) = 0$ and $H(n \geq 1) = 1$. Note that the SATW can thus be seen as the $k \to 0$ limit of the SESRW defined above. For $\beta > 0$, self-avoidance is insufficient to modify the scaling of the MSD, which remains diffusive for all d:

$$\langle X(t)^2 \rangle \propto t, \qquad d_w = 2. \tag{6}$$

In the attractive case $\beta < 0$, the random walker is never trapped. For $d = 1$, the MSD satisfies [23,59]

$$\langle X(t)^2 \rangle \propto t, \qquad d_w = 2, \tag{7}$$

while for $d = 3$, different behaviors emerge depending on the value of the parameter β:

(i) $|\beta| < |\beta_c|$: $\langle X(t)^2 \rangle \propto t$, $\quad d_w = 2$,
(ii) $|\beta| > |\beta_c|$: $\langle X(t)^2 \rangle \propto t^{1/2}$, $\quad d_w = 4$.

For $d = 2$, the scaling of the MSD is still debated [25,26]; while the existence of a subdiffusive regime with $d_w = 3$ is consistently observed numerically, the existence of a transition for a critical value $\beta'_c \neq 0$ to a diffusive regime with $d_w = 2$ for $|\beta| < |\beta'_c|$ has been proposed, but it was later questioned in Ref. [25].

III. ATTRACTIVE SELF-INTERACTING RANDOM WALKS: TRAPPING AND SUBDIFFUSION

Qualitatively, attractive self-interacting random walks are attracted by their own path. Strikingly, this can lead to the full trapping of the walker within a finite set of sites in the $t \to \infty$ limit and, therefore, to a bounded MSD. This effect was demonstrated mathematically [27,61] for one-dimensional, attractive self-interacting random walks and later generalized to arbitrary d [36,62–64]. More precisely, these results state that if $\sum_{n=1}^{\infty} w(n)^{-1} = \infty$, the random walker is free and will visit infinitely many sites of the lattice [note that the limit case $w(n)^{-1} \propto 1/n$ must be discussed independently]. Conversely, for $\sum_{n=1}^{\infty} w(n)^{-1} < \infty$, the random walker visits only a finite set of sites, and the MSD is bounded (see Supplemental Material in Ref. [65]). This immediately yields that attractive TSAW and SESRW lead to the full trapping of the random walker and to a bounded MSD for all d (see Fig. 2). Among the classes of attractive self-interacting random walks introduced above, the only case that leads to a nontrivial exploration of space is the SATW, for which the MSD diverges for $t \to \infty$ (see Fig. 2; an additional example is provided in Ref. [65] for completeness). Despite the diverging MSD, the effect of attractive self-interactions can still have important consequences on the dynamics of

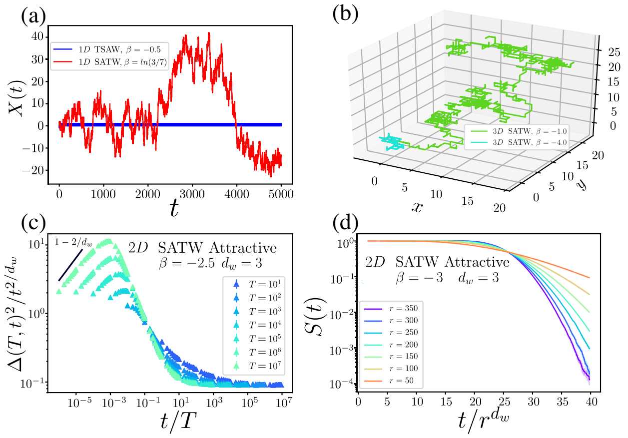

FIG. 2. General properties of attractive self-interacting random walks. (a) Example of a trapped trajectory performed by a $1d$ attractive TSAW random walker (blue), compared to a diffusive trajectory of a $1d$ attractive SATW random walker (red). (b) For $d > 1$, the SATW is subdiffusive for $|\beta| > |\beta_c|$ (blue sample trajectory) and diffusive for $|\beta| < |\beta_c|$ (green sample trajectory). Here, $d = 3$, and thus $|\beta_c| \neq 0$ [25]. (c) Aging of the increments for the subdiffusive SATW ($d_w = 3$ for $d = 2$) normalized by the expected subdiffusive scaling at long times. Each curve corresponds to a fixed value of T. Note that the increments are diffusive for $t \ll T$. (d) In the subdiffusive regime, the SATW performs an extremely compact exploration of space: The survival probability $S(t)$ decays faster than any power law ($\theta = \infty$).

space exploration; in particular, for $d = 2$ and $d = 3$ (for $|\beta| > |\beta_c|$), the process is subdiffusive [25], and we find that the survival probability $S(t)$ in infinite space decays faster than any power law; thus, $\theta = \infty$ (see Fig. 2 and Ref. [65]). In this case, determining the FPT distribution starting at a distance r from the target $F(t, r, R)$ in confined domains of volume $V \propto R^d$ is straightforward in the large volume limit because all moments of $F(t, r, R)$ have a finite limit, so

$$F(t, r, R) \underset{t\to\infty}{\sim} -\frac{dS}{dt}. \tag{8}$$

Thus, $F(t, r, R)$ is asymptotically independent of R. Defining the rescaled variable $\eta = t/r^{d_w}$, a scaling argument finally indicates that its asymptotic distribution can be simply written as

$$\bar{F}(\eta, r, R) = h(\eta), \tag{9}$$

where h is an undetermined scaling function. In the rest of this paper, we thus focus on diffusive attractive and repulsive self-interacting random walks, for which determining the FPT distribution $F(t, r, R)$ in confined domains is nontrivial.

IV. IMPACT OF CONFINEMENT ON INCREMENTS θ, ψ

In this section, we quantitatively characterize the exploration properties of diffusive attractive self-interacting random walks and repulsive self-interacting random walks. We focus on the following observables: increments, and survival probability characterized by θ (for compact processes) and ψ (for noncompact processes) in both infinite and confined geometries. We show numerically and provide heuristic arguments to justify that geometric confinement can deeply and nonlocally modify the dynamics of the process, beyond imposing locally reflecting boundary conditions. As can be expected, it is useful to separately analyze compact and noncompact processes. While this property is known to impact many properties of random walks, it is expected to play a prominent role in the case of self-interacting random walks, whose dynamics is controlled by the number of visits n at each site.

A. Compact (recurrent) processes

The compact case is exemplified by the $1d$ (repulsive) TSAW, the $1d$ (repulsive) SESRW, and the $1d$ (attractive or repulsive) SATW (additional examples are provided in Ref. [65] for completeness). In the compact case, the mean number of visits $\langle n_i \rangle$ to a given site diverges with time T by definition, even in infinite space. The local energy landscape $V(n_i)$ experienced by the random walker therefore depends on the observation time T at all timescales. We argue below that this leads to aging of the increments in infinite space at all timescales, i.e., a dependence on T of the effective diffusion coefficient $D(T, t)$ defined in Eq. (2) for all T. In a confined domain, the dynamics of the random walk starting typically from the bulk is not modified by confinement up to an observation time $T \sim R^{d_w}$, where R is the typical linear size of the domain; in this regime, we therefore expect the increments to be identical in both confined and infinite geometries (note that, for the same reason, in confined domains the analysis of increments is restricted to $t \ll R^{d_w}$). For $T \gtrsim R^{d_w}$, confinement modifies the statistics of visits to a given site; however, the number of visits to a given site still diverges with time T, even if the explicit dependence on T is different in confinement and in infinite space. Aging of the increments is thus expected in confinement as well.

To make this analysis quantitative, it is useful to write $V(n_i)$ as a Taylor series:

$$V(n_i) = V(\bar{n}) + \sum_{p\geq 1} V^{(p)}(\bar{n}) \frac{(\delta n_i)^p}{p!}, \tag{10}$$

where $\bar{n}$ denotes the number of visits to a given site averaged over a spatial scale $l \ll T^{1/d_w}$ and $n_i = \bar{n} + \delta n_i$. The very definition of the dynamics of self-interacting random walks shows that it depends only on the spatial fluctuations of V; increments are thus independent of the site-independent contribution $V(\bar{n})$ for $t \lesssim l^{d_w} \ll T$. In the case of the SATW and the SESRW, one has $V^{(p)}(\bar{n}) \propto \bar{n}^{k-p}$ by definition (we note that, for the SESRW, $V \propto n^k$, where $k \to 0$ yields the SATW); in addition, a mean-field argument (see Ref. [65]) yields the scaling $\delta n_i \propto \bar{n}^{(1-k)/2}$. This shows that in the regime $1 \ll t \ll T$, all site-dependent terms $V^{(p)}(\bar{n})(\delta n_i)^p/p!$ for $p \geq 1$ appearing in Eq. (10) vanish in the limit $T \to \infty$ in both confined and infinite geometries because $\bar{n} \to \infty$. Self-interactions are thus eventually negligible in this limit: The SATW and the SESRW are equivalent to a simple random walk, and one identically has $\Delta^2(T,t) \sim t$ in both confined and infinite geometries. In the case of the TSAW, one has $V'(\bar{n}) = \beta$ and $V^{(p)}(\bar{n}) = 0$ for $p > 1$, independently of $\bar{n}$. Using, in addition, the fact that the spatial fluctuations δn_i reach a steady state in the limit $\bar{n} \to \infty$ (see Ref. [39] and Ref. [65]), this shows that, in the regime $1 \ll t \ll T$, the dynamics of increments is identical in both confined and infinite cases; it can be shown to satisfy $\Delta^2(T,t) \propto t^{2/d_w}$. Last, in the regime $1 \ll T \ll t$, one recovers the scaling of the MSD in all cases: $\Delta^2(T,t) \propto t^{2/d_w}$.

These results can be recapitulated for all examples by the following scaling forms, which are identical in confined and infinite geometries:

$$
\begin{aligned}
&1 \ll t \ll T\colon\ \Delta^2(T,t) \sim 2D_<(t)t^{2/d_w},\\
&1 \ll T \ll t\colon\ \Delta^2(T,t) \sim 2D_> t^{2/d_w},
\end{aligned}
\tag{11}
$$

where the constant $D_>$ and function $D_<(t)$ are process dependent. Numerical simulations confirm this analysis in all examples of compact self-interacting random walks (see Fig. 3): Increments display aging (as seen by a dependence of Δ^2 on the observation time T), and their dynamics is found to be the same in infinite space and in confined domains in both regimes $t, T \ll R^{d_w}$ and $T \gtrsim R^{d_w}, t \ll R^{d_w}$.

In contrast to the dynamics of increments, we now argue that the persistence exponent θ can be modified by confinement. Following Ref. [66], we introduce here the aged persistence exponent θ_c in confinement, which can be defined by

$$
S(t|T) \propto t^{-\theta_c}
\tag{12}
$$

for $T \gg R^{d_w}$ and $1 \ll t \ll R^{d_w}$, where $S(t|T)$ denotes the (survival) probability that the random walker has not reached the target between T and $T + t$. It is known that θ depends on the dynamics of increments at all timescales [52] and not only on their long-time asymptotics. The exponents θ and θ_c can thus be different, as was found earlier in Ref. [66] for models of fluctuating interfaces,

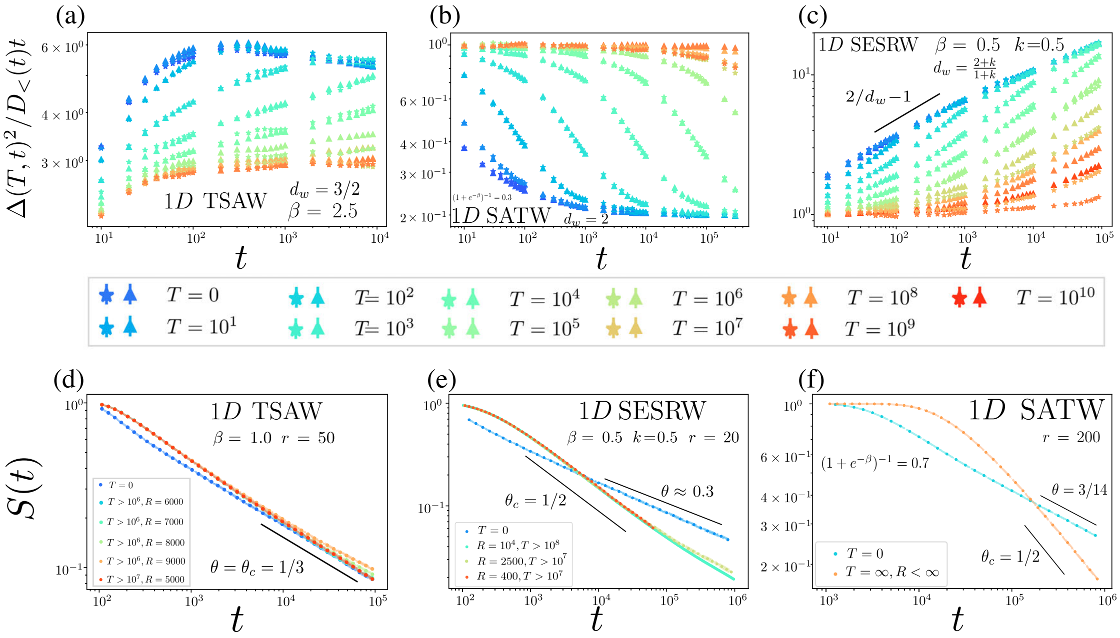

FIG. 3. Aging and first-passage properties for compact self-interacting random walks in infinite space and in confined geometries. (a) Aging of the increments for the $1d$ TSAW, (b) the $1d$ SATW, and (c) the $1d$ SESRW in both infinite space and confined domains (increments are normalized by the expected scaling for $T \gg t$). The dynamics is found to be identical in infinite space (triangle) and in confined domains (stars) in both regimes $t, T \ll R^{d_w}$ and $T \gtrsim R^{d_w}, t \ll R^{d_w}$. This is shown by the collapse of triangles and stars for each color, which overlap. The TSAW is superdiffusive at all timescales and displays aging; the SATW is diffusive at all timesscales and displays aging. In contrast, the SESRW is diffusive for $T \gg t$, but superdiffusive for $t \gg T$ with $d_w = (2+k)/(k+1)$. Persistence exponents in infinite space (θ) and in confined domains (θ_c) for (d) the $1d$ TSAW, (e) the $1d$ SESRW, and (f) the $1d$ SATW. The persistence exponent is modified by confinement for the SATW and the SESRW models but is numerically found to be unchanged for the TSAW.

because θ involves the dynamics of increments at all timescales T, t, while the definition of θ_c only involves the timescales $t \ll R^{d_w}$ and $T \gg R^{d_w}$. This is straightforwardly confirmed in the case of the $1d$ SATW. It is clear that for $T \gg R^{d_w}$, the confined SATW is equivalent to a simple random walk [in this regime, all sites have been visited, and $V(n_i) = \beta$ for all sites], so $\theta_c = 1/2$; in contrast, it was shown recently that in infinite space, one has $\theta = e^{-\beta}/2$ [40]. In the case of the SESRW, the above analysis (see Fig. 3 and Ref. [65]) shows that in the regime $T \gg 1$, the process is also equivalent to a simple random walk, so $\theta_c = 1/2$; in contrast, we numerically find $\theta \neq \theta_c$. Note, however, that it is found numerically that $\theta_c \approx \theta \approx 1 - 1/d_w$ for the TSAW for all $\beta > 0$. Note that this value of θ_c is the one expected for processes with stationary increments [50,52]. The fact that this relation holds for the TSAW, which has nonstationary increments, might be related to the fact that the fluctuations of the number of visits to a given site reach a steady state for this process (see Refs. [39] and [65]).

B. Noncompact (transient) processes

The noncompact case is exemplified by the $3d$ (repulsive) TSAW, the $3d$ (repulsive) SESRW, and the $3d$ (diffusive attractive or repulsive) SATW (additional examples are provided in Ref. [65] for completeness). In the noncompact case, in infinite space, a random walker visits only a fraction of sites and ultimately only makes, on average, a finite number of visits to a given site i. The local energy landscape $V(n_i)$ therefore reaches a stationary state at large observation time T, so aging, if any, is expected to be transient: $D(T, t)$ is asymptotically independent of T for $T \gg 1$. This is indeed observed numerically in all examples of noncompact self-interacting random walks: Increments display weak aging at short timescales t and cross over to diffusive increments with numerically close diffusion coefficients at larger t for all observation times T [see Figs. 4(a)–4(c)]. The effect of self-interaction is thus moderate for noncompact self-interacting random walks, which are all eventually diffusive. This can be heuristically justified as follows: At time t, the typical volume covered scales as t^{d/d_w}, while the number of visited sites scales as t; thus, the local fraction of sites where the local energy landscape is nonzero eventually vanishes for $t \to \infty$ as t^{1-d/d_w}. Self-interactions are thus negligible in the large time limit for noncompact processes, which are diffusive in this limit (note, however, that the diffusion coefficient is nontrivial and depends on the small t dynamics).

The case of confined geometries is radically different for noncompact processes because confinement leads to a divergence of the number of visits to a given site and thus has important consequences at timescales $T \gtrsim R^{d_w}$. In this regime, the above reasoning developed after Eq. (10) for compact processes in fact also applies to confined noncompact processes because the locally averaged number of visits $\bar{n}$ diverges in both cases. In particular, this similarly yields that in the regime $1 \ll t \ll T$, both the noncompact SATW and the noncompact SESRW are equivalent to a simple random walk, so $\Delta^2(T, t) \sim t$. In the case of the confined noncompact TSAW, one finds numerically (see also Refs. [19,37] and [65] for a heuristic argument) that the spatial fluctuations δn_i reach a steady state in the limit $T \to \infty$. Together with Eq. (10), this allows us to conclude that, in this limit, increments are similar (scaling wise) to the infinite space case; thus, they are diffusive and independent of T (see Ref. [65]).

Finally, for all confined, noncompact, self-interacting random walks, these results can be recapitulated as follows for $R \gg 1$:

$$\begin{aligned} T \lesssim R^{d_w}:\ & \Delta^2(T, t) \underset{T\gg1}{\sim} 2D_<(t)t,\ D_<(t) \underset{t\gg1}{\to} D_<, \\ T \gtrsim R^{d_w}:\ & \Delta^2(T, t) \underset{t\gg1}{\sim} 2D_>(T)t,\ D_>(T) \underset{T\gg1}{\to} D_>, \end{aligned} \tag{13}$$

where $D_<, D_>$ are constants. The first regime $T \lesssim R^{d_w}$ is the same in confined and infinite geometries, while the second regime $T \gtrsim R^{d_w}$ is controlled by geometric confinement. Numerical simulations confirm this analysis in all examples of noncompact self-interacting random walks [see Fig. 4(d)–4(f)]: Increments, even if they are always asymptotically diffusive, are found numerically, in all examples, to be quantitatively different for confined and nonconfined, noncompact self-interacting random walks.

Last, for the sake of completeness, we note that, similarly to the persistence exponent in the compact case [see Eq. (12)], the aged hitting probability Π_c can be defined in confinement according to $S(t|T) \sim 1 - \Pi_c$ for $T \gg R^{d_w}$ and $a^{d_w} \ll t \ll R^{d_w}$. In turn, the aged transience exponent ψ_c can be defined in confinement according to

$$\Pi_c \underset{a\to0}{\propto} \left(\frac{a}{r}\right)^{\psi_c}. \tag{14}$$

While, in principle, ψ_c can be different from its infinite space counterpart ψ, our above analysis shows that all examples of noncompact self-interacting processes are diffusive and independent of T for $t \gg 1$ in the limit $T \to \infty$ in both confined and nonconfined cases; this suggests that $\psi = \psi_c = 1$, which is consistent with our numerical simulations [see Figs. 4(h) and 4(i)]. For completeness, in Ref. [65], we provide an example of a non-Markovian random walk with $\psi_c \neq \psi$.

To summarize this section, we have showed quantitatively that geometric confinement can deeply and nonlocally modify the dynamics of self-interacting random walks, beyond imposing locally reflecting boundary conditions. In the compact case, increments remain unchanged (in the regime $t \ll R^{d_w}$) in confined and unconfined geometries, but the persistence exponent can be modified. In the noncompact case, increments remain asymptotically

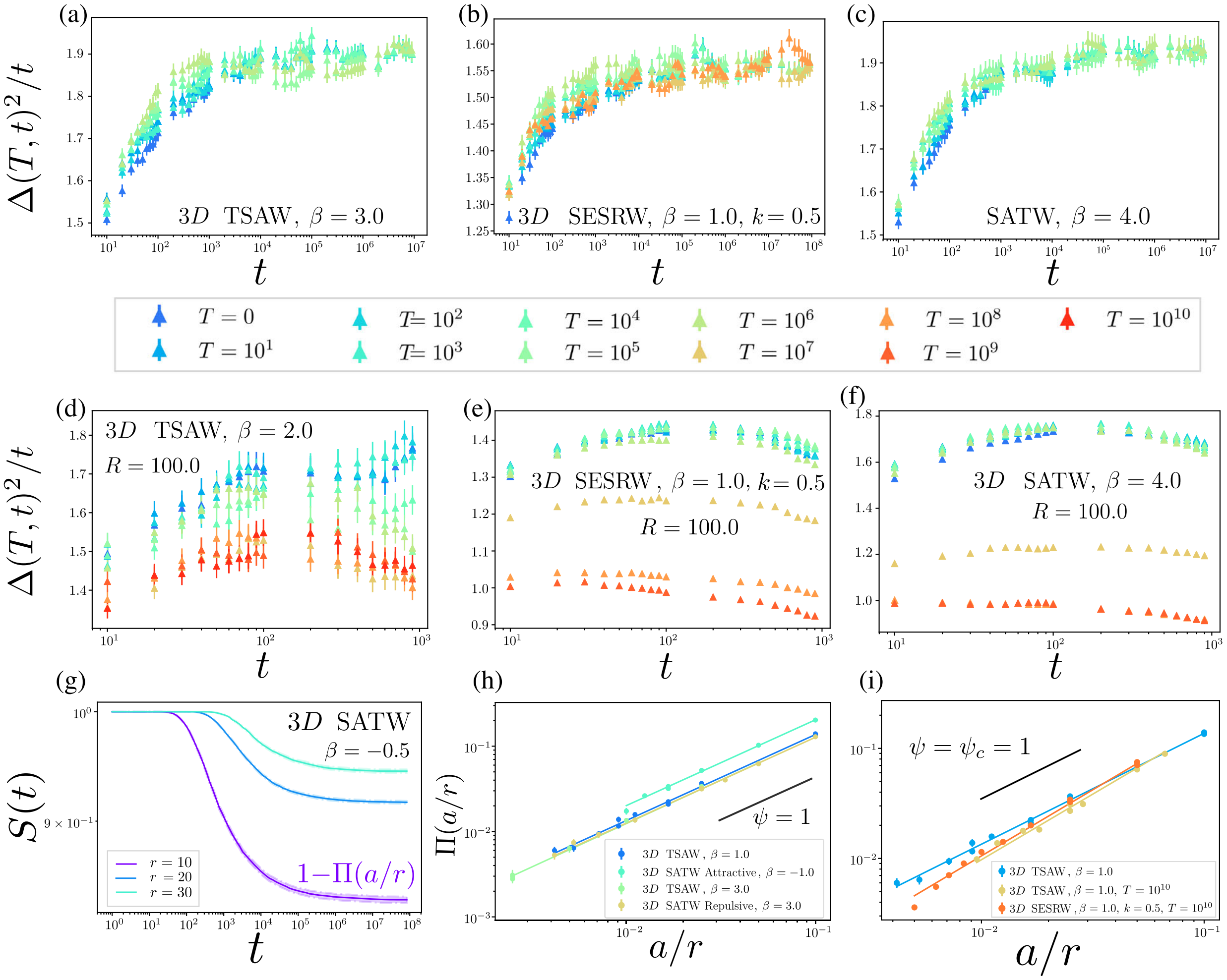

FIG. 4. Aging and first-passage properties for noncompact self-interacting random walks in infinite space and in confined geometries. (a) Aging of the increments for the $3d$ TSAW, (b) the $3d$ SESRW, and (c) the $3d$ SATW in infinite space (increments are normalized by the expected diffusive scaling at long times). Note that the increments are stationary at timescales $T \gg 1$ as shown by the collapse of all curves. In contrast, in confined geometries, aging occurs at longer timescales, $\gtrsim R^d$, for (d) the TSAW, (e) the SESRW, and (f) the SATW (increments are normalized by the expected diffusive scaling at long times). (g) For noncompact random walks (here, the $3d$ diffusive SATW), the survival probability tends, for $t \to \infty$ in infinite space, to a nonzero value $1-\Pi$, which defines the hitting probability that depends on the initial distance to the target r and the target radius a. Hitting probability and transience exponent in infinite space (h) and in confined domains (i) [in this case, the hitting probability is defined in the regime $R^{d_w} \gg t \gg 1$ according to Eq. (14)]. Numerical simulations (symbols) and power-law fits (solid lines). Our numerical results indicate $\psi = \psi_c = 1$ for the TSAW, the SESRW, and the SATW, in agreement with the asymptotic diffusive behavior of noncompact self-interacting random walks.

diffusive in both cases, but their dynamics is quantitatively modified by geometric confinement; in turn, it is found that the transience exponent is unchanged.

V. FPT DISTRIBUTION IN CONFINED DOMAINS

The above analysis of increments and exponents θ and ψ shows that these observables can be impacted by confinement (even if we found $\psi = \psi_c$ for most examples). Turning to the analysis of FPT properties of self-interacting random walks in confinement, one therefore needs to develop a new methodology. Indeed, so far, available methods to determine FPT statistics in confinement [50] rely implicitly on the hypothesis that increments and exponents θ and ψ, which are the key quantities defining the universality classes of FPT statistics in confinement, are not modified by confinement. Below, we extend the method developed originally in Ref. [50] to the case of self-interacting random walks by explicitly taking into account the impact of confinement on the dynamics. In this section, we consider the case of "fresh" initial conditions: At $t = 0$, the random walker, confined in a domain of volume $V = R^d$ with reflecting walls, starts at a distance r from the target of radius a, and the number of visits to all sites i of the domain is set to $n_i = 0$. As stated in the Introduction, we focus on diffusive attractive and repulsive self-interacting random walks and consider separately the cases of compact processes [for which the survival probability $S(t)$ has a power-law decay in infinite space] and noncompact processes; the case of marginal exploration ($2d$ processes with $d_w = 2$) is discussed in Ref. [65].

A. Compact (recurrent) case

In this section, we sketch the derivation of the asymptotic FPT distribution $F(t, r, R)$ for compact self-interacting random walks in the large volume limit $R \to \infty$. For compact processes, the FPT distribution is independent of the target linear size a for $r \gg a$; we focus on this regime below. Following Ref. [50], $F(t, r, R)$ can be written as a partition over trajectories that either hit the reflecting boundary before the target [with probability $\pi(r, R)$ and conditional FPT distribution to the target $F_b(t, r, R)$] or hit the target before the boundary [with probability $1 - \pi(r, R)$ and conditional FPT distribution to the target $F_t(t, r, R)$]:

$$F(t, r, R) = \pi F_b(t, r, R) + (1 - \pi) F_t(t, r, R). \tag{15}$$

Importantly, the weight $1 - \pi$ of trajectories that hit the target first can be expressed in the limit $R \to \infty$ (with r fixed) in terms of the FPT distribution in infinite space $F_\infty(t, r)$:

$$\pi(r, R) \underset{R \gg r}{\propto} \int_{R^{d_w}}^{\infty} F_\infty(t, r) dt, \tag{16}$$

which expresses the fact that most trajectories that hit the target before the boundary yield a FPT smaller than the timescale R^{d_w}. Making use of the definition of θ for processes in infinite space, we then obtain, from dimensional analysis,

$$F_\infty(t, r) \propto \frac{r^{d_w \theta}}{t^{\theta+1}} \tag{17}$$

in the regime $1 \ll t \ll R^{d_w}$, which yields, from Eq. (16),

$$\pi(r, R) \underset{R \gg r}{\propto} \left(\frac{r}{R}\right)^{d_w \theta}. \tag{18}$$

Here, we stress that the persistence exponent θ is defined in infinite space and not in confined geometry. Next, the above argument leading to Eq. (16) also implies that

$$F_t(t, r, R) \propto \Theta(t/R^{d_w}) F_\infty(t, r) \propto \Theta(t/R^{d_w}) \frac{r^{d_w \theta}}{t^{\theta+1}}, \tag{19}$$

where Θ denotes a step function with $\Theta(x \ll 1) = 1$ and $\Theta(x \gg 1) = 0$. At this stage, the conditional FPT distribution $F_b(t, r, R)$ remains to be determined. By definition, this quantity involves trajectories that interact with the domain boundary. However, our analysis above shows that the increments of compact processes are identical in confined and infinite geometries. In the limit $R \to \infty$ with r fixed, $F_b(t, r, R)$ can thus depend only on the timescales t and R^{d_w}; dimensional analysis then yields the following scaling form:

$$F_b(t, r, R) \sim g(t/R^{d_w})/t, \tag{20}$$

where g is an undetermined function that depends on the process. Finally, it is convenient to introduce the rescaled variable $\eta = t/T_{\mathrm{typ}}$, with $T_{\mathrm{typ}} = R^{d_w}$, and write, from Eqs. (15) and (18)–(20), its asymptotic distribution for $R \to \infty$ for $\eta > 0$ with r fixed:

$$\bar{F}(\eta, r, R) = \left(\frac{r}{R}\right)^{d_w \theta} h(\eta). \tag{21}$$

Thus, the corresponding survival probability reads

$$\bar{S}(\eta, r, R) = \left(\frac{r}{R}\right)^{d_w \theta} \int_\eta^\infty h(\eta') d\eta'. \tag{22}$$

Here, h is an undetermined function that depends on the process. Finally, this explicitly captures the dependence of the FPT distribution on the geometrical parameters r, R. In particular, even if this scaling form unavoidably involves an undetermined process-dependent function h, it allows for explicit, quantitative determinations of all moments of the FPT (when they exist) and, in particular, of their functional dependence on r, R, where h enters only through a numerical prefactor. For example, the mean FPT can be readily derived, and it satisfies

$$\langle T \rangle \propto R^{d_w(1-\theta)} r^{d_w \theta}. \tag{23}$$

The mean FPT thus scales nonlinearly with the confining volume $V \sim R^d$ (because one has $\theta \neq 1 - d/d_w$) for SESRW and SATW, as was found for other examples of aging processes; notably, this scaling is linear for the TSAW. Strikingly, the asymptotic form of the FPT distribution (21) is comparable to that obtained in Ref. [50] and can be determined solely from the knowledge of d_w, θ, which are defined in infinite space. This holds even if the dynamics of the process is ultimately impacted by the geometric confinement, as we have shown above. This result is, in particular, independent of the persistence exponent in confinement θ_c. Figure 5 shows an excellent quantitative agreement between numerical simulations and this analytical result. The data collapse of the properly rescaled FPT distribution shows that our approach fully captures its dependence on both r and R for all examples of compact self-interacting random walks that we have studied. Additional examples are provided in Ref. [65] and further confirm the applicability of our approach. In particular, we provide in Ref. [65] an example of a compact self-interacting random walk ($1d$ SATW) for which the

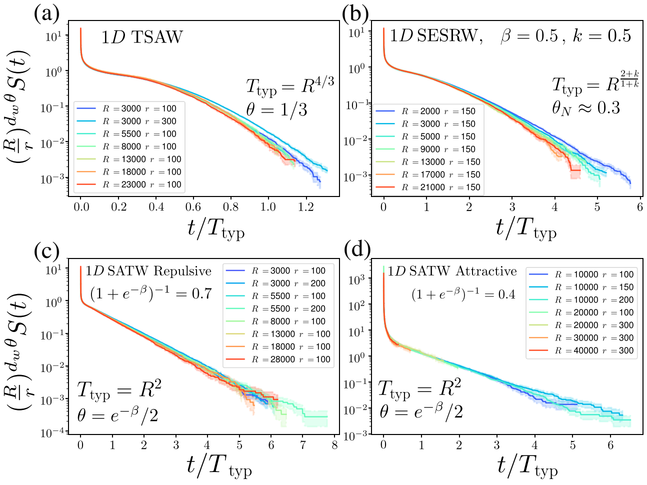

FIG. 5. Asymptotic FPT distribution of compact self-interacting random walks in confined domains for fresh initial conditions. Here, $S(t)$ is the survival probability of the random walker, whose scaling with geometrical parameters is deduced from Eq. (21). The collapse of numerical simulations after rescaling for different values of geometric parameters captures the dependence of the FPT distribution on the geometric parameters r, R (the target is a unique site of the network). Simulations are performed in $1d$ boxes of size R with reflecting boundary conditions. (a) $1D$ TSAW with $d_w = 3/2$ (independent of β). (b) $1d$ SESRW with $\beta = 0.5$ and $k = 0.5$. (c) $1d$ repulsive SATW with $(1+e^{-\beta})^{-1} = 0.7$ and (d) $(1+e^{-\beta})^{-1} = 0.4$.

FPT distribution in confinement can be determined explicitly; as expected, this exact result can be recast under the scaling form (21) that we predict.

B. Noncompact (transient) case

We now turn to the noncompact case. As opposed to the compact case, in the regime $r \gg a$ that we consider below, the FPT distribution depends on a. Following Ref. [50], we call excursion a fraction of trajectory that starts from the sphere S of radius $R/2$ centered on the target, next hits the boundary, and eventually returns to S. The FPT distribution can then be written as a partition over the number n of excursions before the first passage to the target, where we introduce $\Phi_n(t)$ as the corresponding conditional FPT distribution:

$$F(t,a,r,R) = p_0\Phi_0(t) + (1-p_0)\sum_{n=1}^{\infty}\Phi_n(t)P(n). \qquad (24)$$

Here, $p_0 \sim (a/r)^{\psi}$ is the probability to hit the target before the boundary starting from r, and $P(n)$ the probability that the target is reached for the first time during the nth excursion. This can be written

$$P(n) = p_n\prod_{k=1}^{n-1}(1-p_k), \qquad (25)$$

where p_k is the probability that the target is found during the kth excursion, knowing that is has not been found before. Our analysis of increments and transience exponents ψ, ψ_c above [see Eq. (13)] shows that, in confinement, noncompact self-interacting random walks are diffusive for $t \gg 1$ in both regimes $T \ll R^{d_w}$ (with diffusion coefficient $D_<$) and $T \gg R^{d_w}$ (with diffusion coefficient $D_>$). We thus denote by $D_n \equiv D_> + \delta D_n$ the effective diffusion coefficient during the nth excursion, which verifies $|\delta D_n| \leq |D_< - D_>|$ and $\delta D_n \to 0$ for $n \gg 1$. In addition, one has $\psi_c = \psi$. We can thus write $p_k \sim (C_> + \delta C_k)(a/R)^{\psi}$, where $\delta C_k \to 0$ for $k \gg 1$. Note that, here, we have implicitly assumed (and checked numerically, see Ref. [65]) that the conditional probability p_k behaves as the unconditional probability that the target is found during the kth excursion. Last, a scaling argument (see Ref. [65]) shows that

$$\Phi_n(t) = \frac{1}{t}\phi(t/t_n), \qquad (26)$$

where t_n is the typical time elapsed before the nth excursion, which verifies

$$t_n = R^{d_w}\sum_{k=1}^{n-1}\frac{1}{D_k}. \qquad (27)$$

Finally, taking the $R \to \infty$ limit in Eq. (24) with r, a fixed, one finds that the rescaled variable $\eta = t/T_{\rm typ}$ with $T_{\rm typ} = R^d/a$ asymptotically admits the following distribution for $\eta \neq 0$ (see Ref. [65]):

$$\bar{F}(\eta,r,R) = \left(1 - C\left(\frac{a}{r}\right)^{\psi}\right)h(\eta). \qquad (28)$$

Thus, the corresponding survival probability reads

$$\bar{S}(\eta,r,R) = \left(1 - C\left(\frac{a}{r}\right)^{\psi}\right)\int_{\eta}^{\infty}h(\eta')d\eta'. \qquad (29)$$

Here, h is an undetermined process-dependent scaling function—not necessarily exponential—and C a process-dependent constant. Similarly to the compact case, this explicitly captures the dependence of the FPT distribution on the geometrical parameters r, R, and therefore of all its moments (when they exist). In particular, the mean FPT is given by

$$\langle T\rangle \sim \frac{R^d}{a^{\psi}}\left(1 - C\left(\frac{a}{r}\right)^{\psi}\right). \qquad (30)$$

In contrast to the compact case, the mean FPT thus scales linearly with the confining volume $V \sim R^d$. Remarkably, the asymptotic form of the FPT distribution (21) is comparable to that obtained in Ref. [50] in the absence

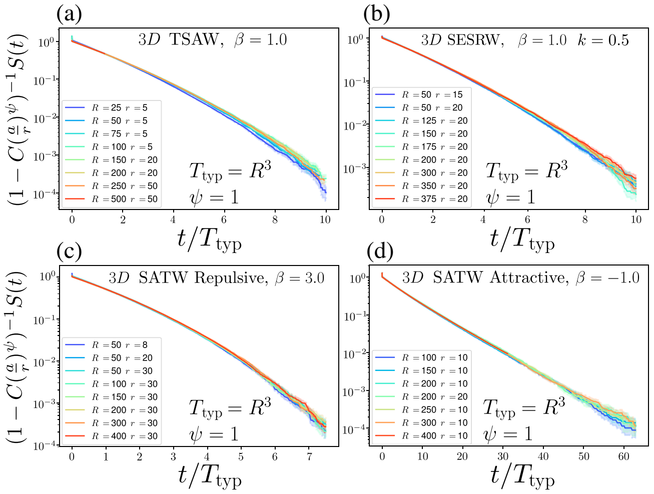

FIG. 6. Asymptotic FPT distribution of noncompact self-interacting random walks in confined domains for fresh initial conditions. Here, $S(t)$ is the survival probability of the random walker, whose scaling with geometrical parameters is deduced from Eq. (28). The collapse of numerical simulations after rescaling for different values of the geometrical parameters captures the dependence of the FPT distribution on r, R. Simulations are performed in $3d$ boxes of size R with reflecting boundary conditions, and the constant C is measured numerically (here, a is a cube of size 1). (a) $3d$ TSAW with $\beta = 1.0$. (b) $3d$ SESRW with $\beta = 1.0$ and $k = 0.5$. The $3d$ SATW repulsive with $\beta = 3.0$ (c) and with $\beta = -1.0$ (d).

of power-law aging and can be determined solely from the knowledge of d_w, ψ, which are defined in infinite space. This holds even if the dynamics of the process is impacted by the geometric confinement, as we have shown above. However, geometric confinement does not change the diffusive scaling of noncompact self-interacting walks; Eq. (28) shows that this is sufficient to preserve the dependence on r, R of the FPT distribution in confinement. Figure 6 shows an excellent quantitative agreement between numerical simulations and this analytical result. The data collapse of the properly rescaled FPT distribution shows that our approach fully captures its dependence on both r and R for all examples of noncompact self-interacting random walks that we have studied. Additional examples are provided in Ref. [65], and they further confirm the applicability of our approach.

VI. AGED INITIAL CONDITIONS

In this section, we analyze the impact of initial conditions on the FPT statistics of confined self-interacting random walks. As we have shown above, the dynamics of self-interacting random walks display aging properties, which can depend on geometric confinement. In other words, the dynamics is different if the random walk starts at $T = 0$ (fresh initial conditions studied above, for which the number of visits to any site i of the domain is set to $n_i = 0$) or at $T \gg R^{d_w}$ (aged initial conditions, for which $n_i \gg 1$). This dependence on initial conditions was clearly evidenced experimentally in the case of living cells, which perform attractive self-interacting random walks [32,33]; in both $1d$ and $2d$ settings, it was found that for aged initial conditions, the MSD increases with time much faster than for fresh initial conditions. In this section, we show that the FPT distribution can be readily obtained for aged initial conditions by adapting the approach developed above for fresh initial conditions, and we highlight the impact of initial conditions.

A. Compact (recurrent) case

For aged initial conditions, because $T \gg R^{d_w}$, the only relevant regime is $t \ll T$. In this regime, we have found the following behavior of the increments:

$$\Delta^2(T, t) \sim 2D_<(t) t^{2/d_w} \propto t^{2/d_{w,c}}, \tag{31}$$

where the effective walk dimension $d_{w,c}$ can be different from d_w (see SESRW in Fig. 3). The relevant persistent exponent is clearly θ_c in this regime. All steps leading to the derivation of the FPT distribution (see previous section) can then be reproduced. It is found that the rescaled variable $\eta = t/T_{\text{typ}}$, with $T_{\text{typ}} = R^{d_{w,c}}$, is asymptotically distributed according to

$$\bar{F}_c(\eta, r, R) = \left(\frac{r}{R}\right)^{d_{w,c}\theta_c} h_c(\eta), \tag{32}$$

so the corresponding survival probability reads

$$\bar{S}_c(\eta, r, R) = \left(\frac{r}{R}\right)^{d_{w,c}\theta_c} \int_\eta^\infty h_c(\eta') d\eta'. \tag{33}$$

Here, h_c is an undetermined function that depends on the process. Initial conditions can thus deeply impact the FPT distribution and even its scaling form: They can modify the walk dimension $d_{w,c}$, the persistence exponent θ_c, and the scaling function h_c. This result is confirmed by numerical simulations (see Fig. 7). For the SATW, one has $\theta_c \neq \theta$ and $d_{w,c} = d_w = 2$, while for the SESRW, one has $\theta_c \neq \theta$ and $d_{w,c} \neq d_w$; the scaling of the FPT distribution is thus modified by initial conditions for these processes. In contrast, for the TSAW, one has $\theta_c = \theta$ and $d_{w,c} = d_w$, and the scaling of the FPT distribution is not modified by initial conditions.

B. Noncompact (transient) case

In the regime $T \gg R^{d_w}$ and $1 \ll t \ll T$, we have found the following diffusive scaling of increments for noncompact self-interacting random walks:

$$\Delta^2(T, t) \underset{t \gg 1}{\sim} 2D_> t. \tag{34}$$

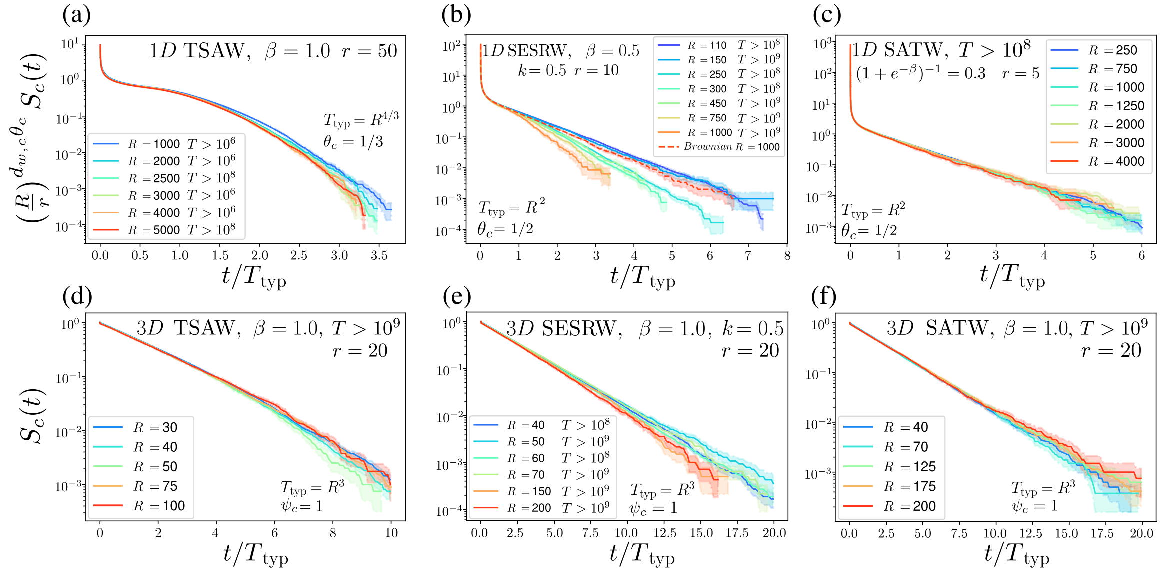

FIG. 7. Asymptotic FPT distribution of self-interacting random walks in confined domains for aged initial conditions. The search process starts at $t = 0$, but the random walker is assumed to have explored the domain from $t = -T$ to $t = 0$. Here, $S(t)$ is the survival probability of the random walker at time t. The scaling of $S(t)$ with geometrical parameters is deduced from Eqs. (32) and (35) for compact and noncompact processes, respectively. The collapse of numerical simulations after rescaling captures the dependence of the FPT distribution on geometrical parameters. Simulations are performed in $1d$ and $3d$ boxes of size R with reflecting boundary conditions for fixed r and a. Compact cases: (a) $1d$ TSAW with $\beta = 1.0$; (b) $1d$ SESRW with $\beta = 1.0$ and $k = 0.5$ [the FPT distribution of the simple random walk is added for comparison (dashed curve)]; (c) $1d$ SATW repulsive with β defined in inset. Noncompact cases: (d) $3d$ TSAW with $\beta = 1.0$; (e) $3d$ SESRW with $\beta = 1.0$ and $k = 0.5$; (f) $3d$ SATW with $\beta = 1.0$.

In addition, we have shown that $\psi_c = \psi$. All steps leading to the derivation of the FPT distribution (see previous section) can then be straightforwardly reproduced. It is found that

$$\bar{F}_c(\eta, r, R) = \left(1 - C\left(\frac{a}{r}\right)^{\psi}\right) h_c(\eta), \qquad (35)$$

so the corresponding survival probability reads

$$\bar{S}_c(\eta, r, R) = \left(1 - C\left(\frac{a}{r}\right)^{\psi}\right) \int_{\eta}^{\infty} h_c(\eta') d\eta'. \qquad (36)$$

Here, $\eta = t/T_{\rm typ}$, with $T_{\rm typ} = R^d/a$, and h_c is an undetermined function that depends on the process. In the case of noncompact self-interacting random walks, initial conditions thus do not modify the scaling of the FPT distribution; however, they can change the scaling function h_c. This result is confirmed by numerical simulations (see Fig. 7). Note that for the SESRW, the time needed to reach the predicted stationary (Brownian-like) behavior in confinement becomes extremely large when R is increased; in practice, this asymptotic regime could not be reached in our numerical simulations for larger values of R within reasonable aging times, which explains the departure from the predicted asymptotic form in this case [panels (b) and (e)].

VII. DISCUSSION AND CONCLUSION

A. Summary of the results

Our joint analytical and numerical analysis shows that long-range memory effects can have deep consequences on the dynamics of generic self-interacting random walks; they can induce aging [quantified by $D(T, t)$ and d_w] and nontrivial persistence and transience exponents θ and ψ, which we characterized quantitatively. In striking contrast with other non-Markovian processes, we have shown that geometric confinement can strongly modify the dynamic properties of self-interacting random walks, beyond imposing locally reflecting boundary conditions: The dynamics of increments can be modified (in the noncompact case) as can persistent exponents (in the compact case).

Based on this systematic quantitative analysis, we have shown that universal scaling forms of the FPT distributions of general self-interacting random walks in confinement can be derived in the large volume limit by generalizing the approach introduced in Ref. [50]. For "fresh" initial conditions, we have found that the FPT distribution in confinement can be asymptotically predicted from the knowledge of the process in infinite space only (via the infinite space exponents d_w, θ, ψ): Geometric confinement ultimately modifies the dynamics of the process and even changes the corresponding exponents $d_{w,c}, \theta_c, \psi_c$ defined in confinement, but this occurs only at timescales larger than the typical FPT and thus only mildly impacts the FPT statistics. In contrast, for aged initial conditions, the exponents that determine the FPT distribution are those defined in confinement, $d_{w,c}, \theta_c, \psi_c$, and can thus be different from the classical infinite space exponents d_w, θ, ψ. In that case, the process in confinement must therefore be characterized to determine the FPT distribution.

B. Search efficiency of self-interacting random walkers

These results allow us to assess the efficiency of space exploration of self-interacting random walks and, in particular, to discuss the impact of memory effects on target search kinetics.

In infinite space, attractive self-interactions ($\beta < 0$) can have drastic consequences on space exploration: For bounded effective interaction potentials $V(n)$ (SATW), the random walk is subdiffusive for $d = 2$, 3 and $|\beta| > |\beta_c|$ and characterized by $\theta = \infty$, so all moments of the FPT to a target are finite. In this case, memory effects thus give a decisive advantage to attractive SATW ($|\beta| > |\beta_c|$) as compared to normal random walks ($\beta = 0$) or repulsive self-interacting walks ($\beta > 0$). Note that, as stated in the Introduction, it has recently been shown that examples of migrating cells perform such attractive self-interacting random walks [32,33]; indeed, subdiffusion was reported in this case, in agreement with our results. In this context, our findings thus suggest that cells can be efficient local explorers thanks to self-interactions.

In confined domains, the discussion is very different. If no prior information on the target position is available, the relevant observable to quantify the search kinetics is the position averaged mean FPT $\langle \bar{T} \rangle$. For compact processes, our results yield $\langle \bar{T} \rangle \propto R^{d_w}$. Search kinetics is thus enhanced by lowering d_w, which amounts to maximizing the scaling of the MSD with time. In that case, memory effects give a decisive advantage to repulsive $1d$ TSAW and $1d$ SESRW, which both show a superdiffusive exponent $d_w < 2$ for all values of $\beta > 0$. For noncompact processes, we obtain $\langle \bar{T} \rangle \propto R^d$, which is consistent with the large time diffusive limit of noncompact (repulsive or attractive) self-interacting random walks. The scaling of $\langle \bar{T} \rangle$ with R is thus independent of memory effects, which, however, modify the effective diffusion coefficient and are thus favorable in the repulsive case.

If the starting distance r from the target is known, the full FPT distribution is needed to analyze the search kinetics. For compact processes, our results (21) show that the set of trajectories that hit the target can be decomposed into a set of fast trajectories, with a timescale proportional to r^{d_w} and weight $1 - \alpha(r/R)^{d_w\theta}$ (where α is a constant), and a set of slow trajectories that typically hit the domain boundaries before the target, with a timescale proportional to R^{d_w} and weight proportional to $(r/R)^{d_w\theta}$. The exponents d_w, θ thus appear as key parameters that control the respective weight of fast and slow trajectories, as well as the typical timescale of slow trajectories. For random processes with stationary increments, it has been proposed that the exponents are not independent and that they satisfy $\theta = 1 - d/d_w$ [50,52,67]; in that case, increasing the weight of direct, fast trajectories by increasing d_w comes at the cost of increasing the timescale of indirect trajectories. This is also the case of the $1d$ repulsive TSAW, for which we found numerically that $\theta = 1 - d/d_w$. In the case of the $1d$ SATW, however, we found that d_w and θ are independent (for the SESRW, we note that $\theta \neq 1 - d/d_w$), with a dependence of θ only on the coupling parameter β. This shows that repulsive self-interactions can be favorable for large starting distances because they diminish the timescale of indirect trajectories by lowering d_w ($1d$ TSAW and $1d$ SESRW); however, in all cases, they reduce the weight of direct trajectories and are thus detrimental at short distances. In turn, attractive interactions (SATW) are detrimental for $d = 2$, 3 because they increase the timescale of indirect trajectories by increasing d_w (subdiffusive SATW), while they preserve the diffusive scaling for $d = 1$; however, they can significantly increase the weight of direct trajectories by increasing θ ($d = 1$, 2, 3) and are thus favorable at short distances. Finally, in the noncompact case, our results (28) show that the FPT statistics is characterized by the single timescale R^d as long as $r \gg a$. As in the case of the position averaged mean FPT, memory effects modify only the effective diffusion coefficient; they are thus favorable in the repulsive case but do not impact scaling properties of the FPT distribution.

Finally, this analysis shows that memory effects induced by self-interactions can have a deep impact on space exploration, as quantified by various observables. Qualitatively, attractive self-interactions, as reported recently for living cells [32,33], have dramatic effects and can lead to subdiffusion with compact exploration, which is favorable for local exploration, and even to self-trapping. Repulsive self-interactions have important effects for compact random walks, for which they modify the walk dimension d_w and thus the scaling of the position averaged mean FPT with the size of the confining domain; thus, this is favorable for global exploration of confined domains with no prior information on the target position.

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