Quantum Many-Body Topology of Quasicrystals

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In this paper, we characterize quasicrystalline interacting topological phases of matter, i.e., phases protected by some quasicrystalline structure. We show that the elasticity theory of quasicrystals, which accounts for both "phonon" and "phason" modes, admits nontrivial quantized topological terms with far richer structure than their crystalline counterparts. We show that these terms correspond to distinct phases of matter and also uncover intrinsically quasicrystalline phases, which have no crystalline analogs. For quasicrystals with internal U(1) symmetry, we discuss a number of interpretations and physical implications of the topological terms, including constraints on the mobility of dislocations in $d = 2$ quasicrystals and a quasicrystalline generalization of the Lieb-Schultz-Mattis-Oshikawa-Hastings theorem. We then extend these ideas much further and address the complete classification of quasicrystalline topological phases, including systems with point-group symmetry as well as noninvertible phases. We hence obtain the "quasicrystalline equivalence principle," which generalizes the classification of crystalline topological phases to the quasicrystalline setting.

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I. INTRODUCTION

Spurred by the theoretical prediction [1–5] and experimental discovery [6,7] of topological band insulators, there has been remarkable progress in the topological classification of gapped quantum matter [8,9]. By now, it is well established that topological insulators and other free-fermion topological states [10,11] belong to the much larger family of symmetry-protected topological (SPT) phases, which encompass strongly correlated systems with internal (on-site) symmetries [12–19]. SPT phases lack spontaneous symmetry breaking and are instead characterized by a bulk gap to all excitations, a unique ground state (with periodic boundary conditions), and nontrivial surface states which are robust against arbitrary local symmetry-preserving perturbations [20,21]. The ground state of a nontrivial SPT can be adiabatically connected to a trivial product wave function only if the bulk gap is closed or if the symmetries protecting the system are explicitly broken. Thus, two states which are smoothly connected absent any symmetries may belong to distinct SPT phases once a symmetry is enforced.

The topological classification has been further extended to include crystalline point-group and space-group symmetries, such as spatial reflection or rotation, which are of intrinsic interest, being crucial for understanding phenomena in solids. Noninteracting fermionic systems protected by such symmetries, also called topological crystalline materials (see Refs. [22,23], and references therein), are particularly well understood and include so-called higher-order topological insulators (HOTIs) [24]. As a consequence of crystalline symmetry, a d-dimensional higher-order topological phase is gapped everywhere except on a $(d-n)$-dimensional surface (with $n > 1$), such that a nontrivial 3D HOTI can host gapless hinge ($n = 2$) or corner ($n = 3$) modes. More recently, a general framework for interacting topological phases protected by crystalline symmetries—crystalline SPT (cSPT) phases—has emerged and is conjectured to give a complete classification of crystalline topological matter [25–28].

Given the progress in classifying crystalline topological phases, a natural question is whether topological phenomena can be protected by the structure of *quasicrystals*, which possess long-range orientational order but are

nonperiodic [29,30], and, hence, occupy a regime intermediate between periodic and amorphous structures. The long-range order of quasicrystals manifests in sharp Bragg peaks, and, although lacking translation symmetry, quasicrystals can still have a notion of rotation symmetries (often crystallographically forbidden). Moreover, quasicrystals can always be understood as an incommensurate projection of a higher-dimensional periodic lattice to the physical space. This latter fact has been exploited for understanding topological phases of quasicrystals, whose properties can sometimes be inferred from those of a higher-dimensional periodic system. For instance, certain 1D quasiperiodic systems have been shown [31] to inherit the topological indices and edge states of the ubiquitous 2D Harper-Hofstadter model [32,33]. Indeed, a plethora of intriguing phenomena have been unearthed in the context of noninteracting quasicrystalline topological phases [31,34–46], including HOTI phases without crystalline analogs [47–49].

Besides its intrinsic theoretical interest, the study of quasicrystalline topological matter has immediate experimental relevance, especially given the possibility of creating and manipulating synthetic lattices with quasiperiodic structures on various platforms, including ultracold atoms and optical cavities, among others [50–61]. A burgeoning platform for realizing electronic quasicrystals are layered materials [62–65], including twisted bilayer graphene, which provide an exciting avenue for exploring the interplay between quasicrystalline order, topology, and strong interactions. However, there has been little progress toward the classification and characterization of strongly correlated quasicrystalline topological phases. While some studies consider the stability of topologically nontrivial quasiperiodic spin chains to interactions [66–68], a general classification akin to that of cSPTs remains far from complete. In this paper, we fill this gap by proposing a general classification of quasicrystalline topological phases.

We proceed by first characterizing cSPT phases in terms of their response to elastic deformations, as captured by the presence of a quantized topological response term in the action. Espousing this *topological elasticity theory* perspective proves particularly efficacious when generalizing to quasicrystals, whose elasticity theory is distinguished by the presence of both "phonon" and "phason" modes [69–73]. In particular, we show that, *despite* the absence of any translation symmetry, the elasticity theory of quasicrystals admits quantized topological terms with a far richer structure than their crystalline counterparts.

For d-dimensional quasicrystalline SPTs with internal U(1) symmetry, we enumerate several allowed topological terms. These terms, which we show correspond to distinct quasicrystalline phases of matter, include terms describing the quantized response to a background U(1) gauge field; topological theta terms; and Wess-Zumino-type topological terms. The latter two classes, which do not require U(1) symmetry, are intrinsically quasicrystalline, i.e., have no crystalline analogs. We discuss several interpretations and consequences of the term describing the response to a background U(1) gauge field, including the mobility restrictions it imposes on dislocations in a $d = 2$ quasicrystal. We also show that our results reproduce known results in the classical limit and in the limit of noninteracting fermions. However, an important aspect of our approach is that it naturally incorporates the role of quantum fluctuations and strong correlations and, hence, also accounts for intrinsically interacting topological phases, i.e., those with no noninteracting analog.

We then extend these ideas much further and address the complete classification of quasicrystalline topological phases. As a first nontrivial extension, we consider quasicrystalline SPTs with only internal symmetries and no point-group symmetry. We show that, despite the absence of translation symmetry, such phases are partially classified by generalizations of the "weak invariants" [74,75] that appear in the classification of SPT phases with translation symmetry. We also find additional *intrinsically* quasicrystalline topological phases, which have no crystalline analog. Finally, we provide a general classification which accounts for point-group symmetry and also extends to symmetry-enriched topological phases; this is encapsulated in the "quasicrystalline equivalence principle," which extends the classification of crystalline topological states to the quasicrystalline setting and represents our main result.

The rest of this paper is organized as follows: In Sec. II, we give a pedagogical overview of our approach and summarize the main results. We discuss the topological elasticity theory for cSPTs with lattice translation and U(1) symmetry in Sec. III. Section IV reviews the primary concepts regarding quasicrystals that we need throughout this paper. This is where we define what we mean by a quasicrystalline topological phase and introduce the phase angle fields that parameterize elastic deformations in a quasicrystal; this differs from the usual description in terms of phonon and phason variables, so even those familiar with the elasticity theory of quasicrystals are encouraged not to skip this section. In Sec. V, we generalize the topological elasticity theory to quasicrystals and show that the topological terms correspond to distinct quasicrystalline phases of matter; their classification in terms of integral cohomology is also stated here. Sections VI–VII are devoted to physical implications and interpretations of the topological terms for quasicrystals with U(1) symmetry: In Sec. VI, we show how the topological term constrains the mobility of dislocations in 2D quasicrystals; in Sec. VII, we formulate a quasicrystalline version of the Lieb-Schultz-Mattis-Oshikawa-Hastings (LSMOH) theorem; and in Sec. VIII, we discuss some simple interpretations of the topological invariants. Finally, we present the general classification of quasicrystalline topological phases in Sec. IX, which includes systems with and without point-group symmetry as well as noninvertible phases. We conclude with a discussion of open questions and future directions in Sec. X.

II. SUMMARY OF APPROACH AND MAIN RESULTS

A. Definition of quasicrystalline topological phases

The usual definition of an SPT or symmetry-enriched topological (SET) phase is that of a family of gapped Hamiltonians that can be continuously deformed into each other while respecting the relevant symmetries and without closing the bulk gap [76]. If the symmetries include crystalline symmetries, then this constitutes a crystalline SPT or SET phase. The main point of this paper is to generalize these ideas to quasicrystals. However, in this case, the definitions become somewhat more subtle, because, unlike crystals, quasicrystals do not have any exact lattice translation symmetry, and, while point-group symmetry can be defined for a quasicrystal, it is not literally a symmetry of the Hamiltonian [77]. Instead, we consider families of gapped Hamiltonians that can be continuously deformed into each other while preserving some notion of quasicrystallinity. We give the precise definition later on (see Secs. IV A and IX).

Finally, let us note that a real quasicrystal, as in the crystalline case, always possesses gapless phonon modes due to the spontaneously broken continuous translation symmetry. Therefore, as in the case of crystalline topological phases, when we talk about gapped Hamiltonians, what we really mean is that there exists a gap for the remaining degrees of freedom after the phonons (and, in the case of quasicrystals, the phasons—see below) are gapped out by adding a pinning term to the Hamiltonian that explicitly breaks the continuous translation symmetry (for example, by applying a periodic or quasiperiodic potential to pin the atoms to particular locations in space). It is in this case that the classification of topological phases can be made totally precise. However, much of the physics we discuss is still relevant for the real crystal or quasicrystal where the phonons and phasons are gapless; we return to this point later on.

B. Understanding crystalline topological phases through elasticity theory

In general, symmetry-protected topological phases of matter can be characterized by their response to background gauge fields; for example, quantum Hall states are characterized by their Hall conductance. Attempts have been made [27,78,79] to generalize this to "crystalline gauge fields," which are gauge fields for crystalline symmetries; however, the results are rather formal and difficult to interpret, and certainly it is not clear how to generalize them to quasicrystalline systems, which do not have exact crystalline symmetries to "gauge."

A core idea of this paper is that we should instead characterize crystalline topological phases through their response to elastic deformations (i.e., to phonons), which is much more physically meaningful in any case. This immediately makes it clear how to generalize to quasicrystalline topological phases; one simply needs to take into account that the elasticity theory of quasicrystals is richer than that of crystals, since, in addition to phonon modes, they also host phason modes.

To illustrate the idea, let us recall that "integer" or "invertible" topological phases (i.e., those without topologically nontrivial excitations) in a d-dimensional crystal with d lattice translation symmetries and a U(1) charge conservation symmetry have an integer invariant ν, representing the charge per unit cell. For a system with invariant ν, we claim that the topological response of such a system to elastic deformations and to a background U(1) gauge field A_μ is characterized by a quantized topological term that appears in the action (previous works discussing quantized topological terms in elasticity theory include Refs. [80–84]). For example, in $d = 2$, this term takes the form

$$\int \frac{\nu}{8\pi^2}\epsilon^{\mu\nu\lambda}\epsilon_{IJ}A_\mu\partial_\nu\theta^I\partial_\lambda\theta^J d^2x dt, \tag{1}$$

where the phase angle fields $\theta^1(\mathbf{x}, t)$ and $\theta^2(\mathbf{x}, t)$ are related to the phonon modes and are defined below. Furthermore, we argue that, once the phason modes are included, these phase angles get generalized, for a two-dimensional system, to D phase angle fields $\theta^1, \ldots, \theta^D$ for some $D > 2$. Then, one can generalize the above topological term to [85]

$$\int \frac{1}{8\pi^2}C_{IJ}\epsilon^{\mu\nu\lambda}A_\mu\partial_\nu\theta^I\partial_\lambda\theta^J d^2x dt, \tag{2}$$

where C^{IJ} can be any antisymmetric D-dimensional integer matrix. Thus, such matrices give a partial classification of quasicrystalline topological phases with U(1) symmetry. In general space dimension d, the analogous partial classification is by D-dimensional rank-d integer tensors that are antisymmetric in each pair of indices. Another way to say this is that there is an independent integer-valued invariant for each size-d subset of the set $\{1, \ldots, D\}$.

In this paper, we discuss various interpretations and implications of these invariants, such as the following.

(i) We show that these invariants determine which directions a dislocation in a quasicrystalline topological phase can move in. This generalizes the familiar observation that, in a crystal, dislocations can move only in the direction of their Burgers vector.

(ii) Since these invariants determine the average charge density as in the crystalline case, they allow one to formulate the quasicrystalline version of the LSMOH theorem [86–89]; recall that the crystalline version of this theorem states that, for an integer topological phase, the charge per unit cell must be an integer. The quasicrystalline version similarly specifies which average charge densities are permissible.

(iii) We give a generalization of the interpretation of ν as the charge per unit cell in the crystalline case; we argue that, in a construction of a quasicrystal by

tiling space through a set of tiles, the integer invariants C^{IJ} can be identified as the charge bound to each kind of tile.

We emphasize that, although when studying topological phases one traditionally considers only the electronic sector, the quasicrystalline invariants we discuss can also be applied to the atomic lattice of a quasicrystalline solid, where we identify the U(1) symmetry as representing conservation of a number of atoms (one can, of course, have a generalization in which there is more than one conserved species of atoms). The invariants then coincide with "stoichiometric coefficients for quasicrystalline compounds" that have long been studied in connection with quasicrystals [90,91]. However, while such quantities were previously studied assuming that the positions of the atoms are classical points, our new perspective makes it clear that the invariants are still well defined and quantized in the presence of quantum fluctuations of the atomic locations.

C. The general classification

In this paper, we also go further and extend these ideas to address the complete classification of quasicrystalline topological phases. One can distinguish between quasicrystals with nontrivial point-group symmetry and those without any point-group symmetry.

1. Invertible topological phases without point-group symmetry

In the case without point-group symmetry, we show that "integer," i.e., invertible, topological phases with internal symmetry G in a quasicrystalline system with D independent elastic modes (phonons and phasons) are classified by

$$\bigoplus_{k=0}^{D} \mathcal{Q}_{d-k}^{\times \binom{D}{k}}. \tag{3}$$

Here, $\mathcal{Q}_p$, for $p \geq 0$, is the classification of invertible topological phases with internal symmetry G in p spatial dimensions. In words, Eq. (3) is saying that, for each $0 \leq k \leq d$, there are topological invariants for d-dimensional topological phases that are related to the classification of $(d-k)$-dimensional topological phases; specifically, there are $\binom{D}{k}$ independent such invariants. In the crystalline case ($D = d$), Eq. (3) reduces to the familiar classification of so-called "weak invariants," which are based on stacking lower-dimensional topological phases. The invariants discussed in Sec. II B above correspond to the $k = d$ term in Eq. (3) when the internal symmetry is $G = \mathrm{U}(1)$ [recall that the classification of 0-dimensional topological phases with U(1) symmetry is $\mathcal{Q}_0 = \mathbb{Z}$, i.e., the U(1) charge; hence, those invariants are integer valued].

However, in the quasicrystalline case, there is an additional phenomenon that has no crystalline analog. For $D > d$, $d-k$ can become negative in the sum Eq. (3). Topological phases in negative spatial dimension might not seem to make much physical sense, but it turns out that to get the correct classification of quasicrystalline topological phases it is necessary to define

$$\mathcal{Q}_{-1} = \begin{cases} \mathbb{Z}_2 & G \text{ contains time reversal,} \\ 0 & \text{otherwise,} \end{cases} \tag{4}$$

as well as

$$\mathcal{Q}_{-2} = \begin{cases} 0 & G \text{ contains time reversal,} \\ \mathbb{Z} & \text{otherwise,} \end{cases} \tag{5}$$

while $\mathcal{Q}_p = 0$ for $p < -2$. We do not discuss the physical interpretations of these intrinsically quasicrystalline topological phases in the present work very much, but it seems important to study them further.

2. The general case (including noninvertible topological phases)

Finally, we describe the complete theory taking into account point-group symmetries and applied also to noninvertible topological phases. Our main result is the quasicrystalline equivalence principle, which states that the classification of quasicrystalline topological phases is in one-to-one correspondence with the classification of topological phases with only *internal* symmetry, with an effective symmetry group $\hat{G}$ related to the quasicrystalline structure. This is the quasicrystalline generalization of the "crystalline equivalence principle" for crystalline topological phases proposed in Ref. [27].

III. CRYSTALLINE SPTS AND "TOPOLOGICAL ELASTICITY THEORY"

In this section, we review the classification of crystalline SPT phases for the case where there are no lattice symmetries other than translation symmetry and the only internal symmetry is a U(1) charge conservation symmetry. In this case, there is an integer-valued invariant ν, which represents the average charge per unit cell. The LSMOH theorem ensures that, for a gapped SPT ground state, ν is an integer.

There is an important interpretation of this invariant ν: It is the coefficient of a *quantized topological term* that appears in the action describing the elasticity theory of the system (i.e., the dynamics of phonons), after integrating out all the degrees of freedom other than the long-wavelength elastic modes. The elastic modes can be described by a slowly varying displacement field $\mathbf{u}(\mathbf{x}, t)$ corresponding to the displacement of atoms from their equilibrium positions. The topological term is most naturally expressed in terms of the "phase angles" θ^I $(I = 1, \ldots, d)$, defined in terms of $\mathbf{u}$ as

$$\mathbf{u}(\mathbf{x},t)=\frac{1}{2\pi}\mathbf{a}_I\theta^I(\mathbf{x},t)-\mathbf{x}, \tag{6}$$

where we use the repeated index summation convention and $\mathbf{a}^I$ $(I=1,\ldots,d)$ are a set of primitive lattice vectors. Then, in the presence of a background gauge field A_μ for the U(1) symmetry, the Lagrangian can contain, in addition to the kinetic term, a topological term:

$$\mathcal{L}=\mathcal{L}_0+\frac{\nu}{2\pi}\epsilon^{\mu\nu}A_\mu\partial_\nu\theta^1 \quad (d=1), \tag{7}$$

$$\mathcal{L}=\mathcal{L}_0+\frac{\nu}{8\pi^2}\epsilon^{\mu\nu\lambda}\epsilon_{IJ}A_\mu\partial_\nu\theta^I\partial_\lambda\theta^J \quad (d=2), \tag{8}$$

$$\mathcal{L}=\mathcal{L}_0+\frac{\nu}{48\pi^3}\epsilon^{\mu\nu\lambda\sigma}\epsilon_{IJK}A_\mu\partial_\nu\theta^I\partial_\lambda\theta^J\partial_\sigma\theta^K \quad (d=3). \tag{9}$$

By demanding that the action be invariant (modulo 2π) under large gauge transformations of A on any space-time manifold, one can show that ν must be an integer.

From the topological term, one can derive various properties. Let us focus on the $d=2$ case for concreteness. Then, one finds that the charge density is

$$\rho=\frac{\delta S}{\delta A_0}=\frac{\nu}{8\pi^2}\epsilon^{ij}\epsilon_{IJ}\partial_i\theta^I\partial_j\theta^J. \tag{10}$$

In particular, in equilibrium, one has [setting $\mathbf{u}=0$ in Eq. (6)] that $\theta^I=K^I{}_ix^i$ [where $K^I{}_i$ is the inverse of the matrix $(1/2\pi)a^i{}_I$ whose columns are the vectors $(1/2\pi)\mathbf{a}^I$], which gives

$$\rho=\frac{\nu}{4\pi^2}\det K=\frac{\nu}{V_{\text{unit}}}, \tag{11}$$

where $V_{\text{unit}}=\det a$ is the volume of the unit cell. Note that, because the topological term is part of the effective theory that describes the system on length scales large compared with the unit cell size, the ρ that we calculate represents the *average* charge density over such a scale; of course, the *microscopic* charge density can vary rapidly over the scale of the unit cell, which is not captured by the effective theory. From Eq. (11), one finds that the charge per unit cell is ν, as expected. In fact, in general, one can show that Eq. (10) is equivalent to

$$\rho=\frac{\nu}{\tilde{V}_{\text{unit}}}, \tag{12}$$

where $\tilde{V}_{\text{unit}}$ is the volume of the deformed unit cell due to the local strain. One can also compute the current density

$$J^i=\frac{\delta S}{\delta A_i}=\frac{\nu}{4\pi^2}\epsilon_{IJ}\epsilon^{ij}\partial_j\theta^I\partial_0\theta^J, \tag{13}$$

which one can convince oneself is indeed the correct expression for the current density for elastic deformations of a crystal in which a charge ν is bound to each unit cell.

Here, we should mention some conceptual sleight of hand has taken place. Normally, one thinks of crystalline SPT phases characterized by the integer invariants ν as protected by the lattice translation symmetry. By contrast, phonons are the Goldstone modes arising from the spontaneous breaking of a continuous translation symmetry down to the aforementioned lattice translation symmetry. One might wonder what the topological term of the Goldstone modes has to do with the SPT phase with respect to the residual symmetry. However, in fact, one does expect there to be a general relation [92].

The topological elasticity point of view proves particularly fruitful when one passes from crystals, which have discrete translation symmetry, to quasicrystals, which have *no translation symmetry at all*. Nevertheless, what we show is that the elasticity theory of quasicrystals still admits quantized topological terms, in fact, with an even richer structure compared to the crystalline case, and that these topological terms correspond to distinct quasicrystalline phases of matter.

Finally, let us note that, if we want to take the view that we are classifying gapped topological phases of matter as described at the end of Sec. IV A, we can view the elastic deformations described by the field θ^I as being imposed externally, through spatial and temporal variations in the pinning potential that we impose to make the system gapped, in which case θ^I is a nondynamical probe field. If, on the other hand, we do not care about precise definitions of topological phases and simply want to study the physical properties of a real crystal, we are free to treat the θ^I as gapless dynamical fields and study the physical implications of the topological terms. In the latter case, the prerequisite property for the validity of our discussion is that the low-energy effective theory of the system should contain only the elastic modes; thus, metals, for instance, are excluded.

IV. REVIEW OF QUASICRYSTALS

A. Definition of quasicrystals and quasicrystalline topological phases

Let us describe the definition of a quasicrystal that we adopt in the present paper. We consider systems for which there is a countable set $\mathcal{L}$ of wave vectors such that the expectation value $\langle\hat{o}(\mathbf{x})\rangle$ of all local observables in the ground state can be expanded in a Fourier series:

$$\langle\hat{o}(\mathbf{x})\rangle=\sum_{\mathbf{Q}\in\mathcal{L}}a_{\mathbf{Q}}\exp(i\mathbf{Q}\cdot\mathbf{x}), \tag{14}$$

with coefficients $a_{\mathbf{Q}}$ that depend on the choice of operator $\hat{o}$. In a crystal, $\mathcal{L}$ is simply the reciprocal lattice of the crystal, and for a system in d spatial dimensions it can be generated by a finite set of d vectors (the primitive reciprocal lattice vectors), in the sense that all elements

of $\mathcal{L}$ can be written as an integer linear combination of primitive reciprocal lattice vectors. By contrast, we say that the system is a *quasicrystal* if $\mathcal{L}$ can be generated by a finite set of vectors, but the smallest such set has size greater than d.

When we talk about a quasicrystalline topological phase, we mean a family of Hamiltonians such that the ground state is always gapped and satisfies the quasicrystallinity property stated above, *with the reciprocal lattice $\mathcal{L}$ held fixed throughout the whole family.* In the crystalline case, this would be equivalent to demanding that the ground state always respects the discrete translation symmetry of the lattice, i.e., to a topological phase protected by discrete translation symmetry. We return to the question of how to consider topological phases protected by the point-group "symmetry" of a quasicrystal later on. We also note that the definition of quasicrystalline topological phases is distinct from that of "local isomorphism" classes of quasicrystals—see the Appendix A for a discussion.

B. Phonons and phasons in quasicrystals

The main reason that the elasticity theory of quasicrystals is richer than that of crystals is that there are two independent types of elastic deformations in a quasicrystal, corresponding to phonon and phason modes. In this section, we review the physics of phonons and phasons, with a particular view to setting up the notation that we use in later sections.

Let us imagine a ground state in which the local observables satisfy Eq. (14). One can then argue that, if the Hamiltonian has continuous translation symmetry (that is, we are thinking of the quasicrystal as resulting from spontaneously breaking continuous translation symmetry), then there is a whole manifold of ground states with equal energy in the thermodynamic limit. Specifically, one can show that there is a ground state in which expectation values take the form

$$\langle \hat{o}(\mathbf{x})\rangle' = \sum_{\mathbf{Q}\in\mathcal{L}} a_{\mathbf{Q}} \exp(i\phi_{\mathbf{Q}} + i\mathbf{Q}\cdot\mathbf{x}), \tag{15}$$

with the same coefficients $a_{\mathbf{Q}}$ as the original state, for any choice of phases $\phi_{\mathbf{Q}}$ (*independent of* $\hat{o}$) satisfying

$$\phi_{\mathbf{Q}_1+\mathbf{Q}_2} = \phi_{\mathbf{Q}_1} + \phi_{\mathbf{Q}_2} \quad [\mathrm{mod}\ 2\pi] \tag{16}$$

for any $\mathbf{Q}_1, \mathbf{Q}_2 \in \mathcal{L}$. The proof is basically that a state that assigns expectation values Eq. (15) is either a translation of the original state or else can be *arbitrarily well locally approximated* by translations of that state (in the language of Appendix A, it is in the same "local isomorphism class" as the original state); either way, it must have the same energy.

The phases satisfying Eq. (16) can be thought of as the "order parameter" that labels the spontaneous symmetry-breaking ground states. The low-energy elastic deformations correspond to long-wavelength fluctuations of this order parameter. Thus, we replace $\phi_{\mathbf{Q}}$ in Eq. (15) with slowly varying functions $\phi_{\mathbf{Q}}(\mathbf{x}, t)$ [which are still required to satisfy Eq. (16) at each point $\mathbf{x}$]. A helpful way to formulate this is by introducing the phase fields $\theta_{\mathbf{Q}}(\mathbf{x}, t) = \phi_{\mathbf{Q}}(\mathbf{x}, t) + \mathbf{Q}\cdot\mathbf{x}$; then, we have

$$\langle \hat{o}(\mathbf{x})\rangle' = \sum_{\mathbf{Q}\in\mathcal{L}} a_{\mathbf{Q}} \exp[i\theta_{\mathbf{Q}}(\mathbf{x}, t)], \tag{17}$$

where $\theta_{\mathbf{Q}}(\mathbf{x}, t)$ satisfies

$$\theta_{\mathbf{Q}_1+\mathbf{Q}_2}(\mathbf{x}, t) = \theta_{\mathbf{Q}_1}(\mathbf{x}, t) + \theta_{\mathbf{Q}_2}(\mathbf{x}, t) \quad [\mathrm{mod}\ 2\pi] \tag{18}$$

and where, for a low-energy deformation, we must have

$$\nabla\theta_{\mathbf{Q}}(\mathbf{x}, t) \approx \mathbf{Q}, \tag{19}$$

while in the ground state Eq. (19) becomes an equality.

We can obtain a concrete parameterization of the solutions to Eq. (18) by introducing a set $\mathbf{K}^1, \ldots, \mathbf{K}^D$, of reciprocal vectors that generate $\mathcal{L}$. For a crystal, one would have $D = d$, whereas for a quasicrystal $D > d$. Specifically, we impose the following properties.

(1) Every $\mathbf{Q} \in \mathcal{L}$ can be written as an integer linear combination of the $\mathbf{K}^I$'s.
(2) The $\mathbf{K}$'s are linearly independent over the integers; if $n_I\mathbf{K}^I = 0$ for some integers n_I (here, we are using the repeated index summation convention), then all the n_I's are zero.

In some cases, it is traditional to drop the second requirement and have an overcomplete set of vectors. This is commonly done in order for the set of generating vectors to be closed under point-group symmetry, as with the pentagonal quasicrystal discussed in Ref. [70], for which an overcomplete set of five vectors is traditionally used even though one of them is an integer linear combination of the other four. However, in this paper, we do not adopt such an approach and our generating sets are never overcomplete. Note that, because there are infinitely many possible choices of generating sets for a given $\mathcal{L}$, related by $\mathrm{GL}(D, \mathbb{Z})$ transformations, there is always a $\mathrm{GL}(D, \mathbb{Z})$ gauge freedom associated with formulas that depend on the definition of the $\mathbf{K}^I$'s. In Appendix B, we discuss an example of the choice of $\mathbf{K}$ vectors for an octagonal quasicrystal in 2D.

The solutions to Eq. (18) can then be parameterized in terms of D phase angle fields $\theta^I(\mathbf{x})$ $(I = 1, \ldots, D)$ as

$$\theta_{\mathbf{Q}}(\mathbf{x}, t) = \theta^I(\mathbf{x}, t) n_I(\mathbf{Q}), \tag{20}$$

where $n_I(\mathbf{Q})$ is the unique integer vector such that $\mathbf{Q} = n_I\mathbf{K}^I$. Equation (19) can then be expressed as

$$\nabla\theta^I(\mathbf{x}, t) \approx \mathbf{K}^I. \tag{21}$$

Thus, the elastic deformations are parameterized by the D fields $\theta^I(\mathbf{x},t)$. In a crystal, we have that $D=d$, and these are just the usual phonon modes. On the other hand, for a quasicrystal, we have $D>d$, and there are more elastic modes than in a crystal. It is conventional [69–73] to decompose these modes into phonon and phason modes, corresponding to certain linear combinations of the θ^I's. However, in this paper, we find it more convenient to work with the θ^I's directly.

Finally, let us note an appealing interpretation of the θ fields. Define a function $\tilde{o}$ on $\mathbb{R}^D$ according to

$$\tilde{o}(\boldsymbol{\theta}) = \sum_{\mathbf{n}\in\mathbb{Z}^D} a_{n_I\mathbf{K}^I} \exp(i\mathbf{n}\cdot\boldsymbol{\theta}), \tag{22}$$

which is periodic with respect to 2π translations along any of the coordinate directions in $\mathbb{R}^D$ and where $a_{\mathbf{Q}}$ are the Fourier coefficients in Eq. (14). Then, we can write Eq. (15) as

$$\langle \hat{o}(\mathbf{x})\rangle' = \tilde{o}[\boldsymbol{\theta}(\mathbf{x},t)], \tag{23}$$

where $\boldsymbol{\theta}(\mathbf{x},t)$ is the vector in $\mathbb{R}^D$ whose components are $\theta^I(\mathbf{x},t)$. Thus, in the ground state, where $\boldsymbol{\theta}^I(\mathbf{x},t)=\theta^I_{(0)}+\mathbf{K}^I\cdot\mathbf{x}$, the expectation values of observables in a quasicrystal are given by linearly mapping d-dimensional physical space into a hyperplane slicing through a D-dimensional "crystal." Moreover, phonon and phason deformations correspond to keeping the D-dimensional crystal fixed (so the function $\tilde{o}$ remains fixed) but deforming the mapping from d-dimensional space to the D-dimensional "superspace."

V. TOPOLOGICAL TERM FOR THE ELASTICITY THEORY OF QUASICRYSTALS

The generalization to quasicrystals of the topological term discussed in Sec. III, expressed in terms of the fields θ^I introduced in Sec. IV B, can be written as

$$\mathcal{L} = \mathcal{L}_0 + \frac{1}{2\pi} C_I \epsilon^{\mu\nu} A_\mu \partial_\nu \theta^I \quad (d=1), \tag{24}$$

$$\mathcal{L} = \mathcal{L}_0 + \frac{1}{8\pi^2} C_{IJ} \epsilon^{\mu\nu\lambda} A_\mu \partial_\nu \theta^I \partial_\lambda \theta^J \quad (d=2), \tag{25}$$

$$\mathcal{L} = \mathcal{L}_0 + \frac{1}{48\pi^3} C_{IJK} \epsilon^{\mu\nu\lambda\sigma} A_\mu \partial_\nu \theta^I \partial_\lambda \theta^J \partial_\sigma \theta^K \quad (d=3). \tag{26}$$

Here, C_I is a D-dimensional vector, C_{IJ} is a $(D\times D)$ antisymmetric matrix, and C_{IJK} is a rank-3 tensor of dimension D which is antisymmetric in all pairs of indices. By demanding that the action be invariant (modulo 2π) under large gauge transformations of A on any space-time manifold, one can show that each entry of C is quantized to be an integer. Therefore, we obtain a partial classification of quasicrystalline SPTs with U(1) symmetry by integral antisymmetric rank-d tensors of dimension D. In the crystalline case, where $D=d$, all such tensors are simply integer multiples of the Levi-Civita tensor, and we recover Eq. (1) in $d=2$, for example.

Note that, for a given quasicrystalline SPT phase with underlying reciprocal lattice $\mathcal{L}$, the entries of C depend on the arbitrary choice of the generating reciprocal vectors $\mathbf{K}^I$'s for $\mathcal{L}$. We see later that a more abstract way to state the classification, that has the advantage of being gauge invariant, is as follows: The SPT phases are classified by integral cohomology $H^d(\mathcal{L}^*,\mathbb{Z})$, where $\mathcal{L}$ is the space of homomorphisms from $\mathcal{L}^*$ into U(1), that is, the space of solutions to Eq. (16) (note that we are taking singular cohomology of the topological space $\mathcal{L}^*$, discarding its group structure, *not* group cohomology). We can recover the concrete classification in terms of integral antisymmetric tensors if we observe that $\mathcal{L}^*$ is topologically a D torus.

For quasicrystals, there are, in fact, additional possible topological terms that one can write, which do not depend on the U(1) symmetry. First, if $D\geq d+1$, then we can write the term

$$\mathcal{L}_{\text{Theta}} = \Theta_{IJ}\frac{1}{8\pi^2}\epsilon^{\mu\nu}\partial_\mu\theta^I\partial_\nu\theta^J \quad (d=1), \tag{27}$$

$$\mathcal{L}_{\text{Theta}} = \Theta_{IJK}\frac{1}{48\pi^3}\epsilon^{\mu\nu\lambda}\partial_\mu\theta^I\partial_\nu\theta^J\partial_\lambda\theta^J \quad (d=2), \tag{28}$$

and so on, where Θ is an antisymmetric tensor in dimension D. One can show that, if the entries of Θ are integer multiples of 2π, then the action is a multiple of 2π on any closed manifold; therefore, the entries of Θ are defined only modulo 2π. Hence, since under time-reversal symmetry we have $\Theta\to-\Theta$, we see that, if we impose time-reversal symmetry, then the components of Θ are quantized to be 0 or π (modulo 2π). Thus, for each independent component of Θ, we obtain a $\mathbb{Z}_2$ quantized topological invariant (these arguments should, of course, be very reminiscent of those for a topological insulator in three spatial dimensions [93]). In the more abstract language, these topological terms (and, hence, the corresponding quasicrystalline SPTs) are classified by $H^{d+1}(\mathcal{L}^*,\mathbb{Z}_2)$.

Finally, if time-reversal symmetry is broken and $D\geq d+2$, then there is an additional class of topological terms of the Wess-Zumino type [94]. These ones are not canonically expressible as a local Lagrangian in $d+1$ space-time dimensions. Instead, the action is written in terms of an extension of the θ fields into one higher dimension:

$$S_{\text{WZ}}[\theta] = \int_{M_{d+2}} \mathcal{L}^{\Theta}_{d+2}[\tilde{\theta}], \tag{29}$$

where M_{d+2} is a $(d+2)$-dimensional manifold whose $(d+1)$-dimensional boundary corresponds to the physical space-time and $\tilde{\theta}$ is an extension of the θ fields onto M_{d+2}. $\mathcal{L}_{d+2}$ is an action of the form analogous to Eqs. (27) and (28) (but in space-time dimension $d+2$), parameterized by a rank-$d+2$, dimension D antisymmetric integer tensor. In order for the action to be independent of the choice of extension $\tilde{\theta}$, we require that all the components of Θ are an integer multiple of 2π. In the abstract language, these terms are classified by $H^{d+2}(\mathcal{L}^*, \mathbb{Z})$.

In this paper, we focus on the topological terms of the form Eqs. (24)–(26), leaving a detailed examination of the consequences of the other topological terms described above for future work.

VI. MOBILITY OF DISLOCATIONS IN A QUASICRYSTAL

A well-known fact about defects in crystals is that dislocations can move only in the direction of their Burgers vector. Specifically, this applies to systems with a conserved charge, such that the charge ν per unit cell (see Sec. II B) is nonzero. One can then show that moving the dislocation in a direction not parallel to the Burgers vector violates charge conservation. For crystals, an approach based on topological terms in the elasticity theory is given in Ref. [80]. More recently, these mobility constraints have been interpreted in terms of a duality between the elasticity theory of a crystal and higher-rank tensor gauge theory [95–97].

In this section, we show that a generalization of this mobility constraint applies to dislocations in two-dimensional quasicrystals. A dislocation in a quasicrystal is characterized by a nontrivial winding number of the θ field defined in Sec. IV B above as the dislocation is encircled:

$$\frac{1}{2\pi}\oint_C dx^i \partial_i \theta^I = b^I \in \mathbb{Z}^D, \tag{30}$$

where C is a loop surrounding the dislocation and b^I is the Burgers vector. The topological mobility constraint of a dislocation can be derived from the charge conservation, as we now show.

From Eq. (25), we see that there is a contribution to the current given by

$$J^\mu = \frac{1}{8\pi^2} C_{IJ}\epsilon^{\mu\nu\lambda}\partial_\nu\theta^I\partial_\lambda\theta^J. \tag{31}$$

From this, we find that the condition for local charge conservation takes the form

$$\partial_\mu J^\mu = C_{IJ}\epsilon^{\mu\nu\lambda}(\partial_\mu\partial_\nu\theta^I - \partial_\nu\partial_\mu\theta^I)\partial_\lambda\theta^J = 0. \tag{32}$$

Assuming that we are close to the equilibrium configuration, we have

$$\partial_0\theta^I \approx 0, \qquad \partial_i\theta^I \approx K^I{}_i. \tag{33}$$

We then expand Eq, (32) to the leading order in $\partial_i\theta^I - K^I{}_i$, giving

$$\partial_\mu J^\mu = C_{IJ}\epsilon^{ij}(\partial_0\partial_i\theta^I - \partial_i\partial_0\theta^I)K^J{}_j = 0. \tag{34}$$

Now consider a dislocation with Burgers vector b^I at position $x^i(t)$ moving at velocity v^j. Then, one finds that

$$(\partial_0\partial_i - \partial_i\partial_0)\theta^I = \epsilon_{ij}v^j\delta^2[\mathbf{x} - \mathbf{v}(t)], \tag{35}$$

and, hence, the local charge conservation requires that

$$C_{IJ}b^I K^J{}_j v^j = 0. \tag{36}$$

Equation (36) gives the topological mobility constraint of dislocations in a two-dimensional quasicrystal. We see that a dislocation can move only along the direction set by a combination of the reciprocal lattice vectors, the SPT invariant C_{IJ}, and the Burgers vector b^I.

One can, of course, raise an objection regarding the rigor of the above argument, since it is based on linearizing about the equilibrium configuration, while, in fact, θ^I always becomes singular at the dislocation core. Moreover, one does not expect the continuum field theory description that we are using to be valid near the dislocation core. In Appendix E, we give a more careful argument for the constraint Eq. (36) by invoking the continuum field theory description only far away from the dislocation core, where one expects such a description to be valid.

Note that there is always at least one direction v^j that satisfies Eq. (36). Thus, as in crystals, dislocations in quasicrystals are never completely immobilized but can move in some direction. On the other hand, at low temperatures, quasicrystals are found experimentally to be quite brittle, reflecting the fact that, for dynamical reasons, it is more difficult (but not impossible) for dislocations to move in quasicrystals than in crystals [98], even though there is no topological constraint that fully immobilizes them.

VII. AVERAGE CHARGE DENSITY FOR QUASICRYSTALS AND THE LSMOH THEOREM

Another consequence of the topological terms in Eqs. (24)–(26) is for the overall charge density of the system. Since the charge density can be evaluated as $\rho = \delta S/\delta A_0$ and using the fact that, in the ground state $\partial_i\theta^I = K^I{}_i$, we find the average density

$$\rho = \frac{1}{2\pi}C_I K^I \quad (d=1), \tag{37}$$

$$\rho = \frac{1}{8\pi^2}C_{IJ}\epsilon^{ij}K^I{}_i K^J{}_j \quad (d=2), \tag{38}$$

$$\rho = \frac{1}{48\pi^2} C_{IJK} \epsilon^{ijk} K^I{}_i K^J{}_j K^K{}_k \quad (d = 3). \tag{39}$$

Here, we assume that the nontopological part of the action does not contribute to the average charge density. This is a reasonable assumption, because the phonon and phason fields do not transform nontrivially under U(1), so they do not minimally couple to the gauge field (and one can verify that a nonminimal coupling does not give any contribution to the ground state charge density).

In the crystalline case, where C is always an integer multiple ν of the Levi-Civita symbol, then this simply amounts to saying that the charge per unit cell is ν. More generally, a helpful way to interpret these expressions is in terms of the tile picture in Sec. VIII A; these densities are precisely the average charge densities that one expects from having integer charges bound to the tiles (with the charge bound to each class of tiles given by the appropriate entry of C), once one takes into account how often each tile class appears in a tiling of space.

The set $\mathcal{S}(\mathcal{L})$ of all allowed densities, generated by varying C in Eqs. (37)–(39) over all integer antisymmetric tensors, determines the quasicrystalline version of the LSMOH theorem: Only densities in this set are allowed in an insulating quasicrystal with reciprocal lattice $\mathcal{L}$ without noninvertible topological order (i.e., without ground state degeneracy on the torus). Unlike in the crystalline case, the set $\mathcal{S}(\mathcal{L})$ is typically *dense* in the space of all real numbers; that is, for any $\rho \in \mathbb{R}$, we can find allowed densities arbitrarily close to ρ. However, the allowed densities are not *continuous*—there is no interval $[\rho_0, \rho_1]$ with $\rho_0 < \rho_1$ such that $[\rho_0, \rho_1]$ is a subset of $\mathcal{S}(\mathcal{L})$—as can be seen from the fact that $\mathcal{S}(\mathcal{L})$ is a countable set. Note that, in order for $\mathcal{S}(\mathcal{L})$ to be dense, one must allow the integer entries of C to take arbitrary large values. In fact, we expect that systems with large entries of C are probably very hard to realize in practice.

Finally, let us remark that in the case of *free-fermion* insulators, Eqs. (37)–(39), involving the quantized integer "gap labels" C, have long been known [99,100]. Our work can, thus, be seen as a demonstration that these free-fermion topological invariants are robust to arbitrary local interactions that respect the quasicrystalline structure.

VIII. SIMPLE INTERPRETATIONS OF THE SPT INVARIANTS

A. Tiling interpretation

We now discuss a simple interpretation of the SPT invariants C in Eqs. (24)–(26) in terms of the tiles forming the quasicrystal. Recall first the well-known fact that a quasicrystalline structure can be obtained by tiling space in such a way that each tile T_i can, by translation, be related to one of a finite set $\{T_i\}_{i=1}^n$ of primitive tiles. In fact, there is a canonical way to generate such a tiling for any reciprocal lattice $\mathcal{L}$ [101], such that, for an $\mathcal{L}$ with D primitive vectors, there are $n = \binom{D}{d}$ primitive tiles. In the case of a periodic crystal, we have $D = d$ and, hence, $n = 1$, so that only a single primitive tile is required (the unit cell).

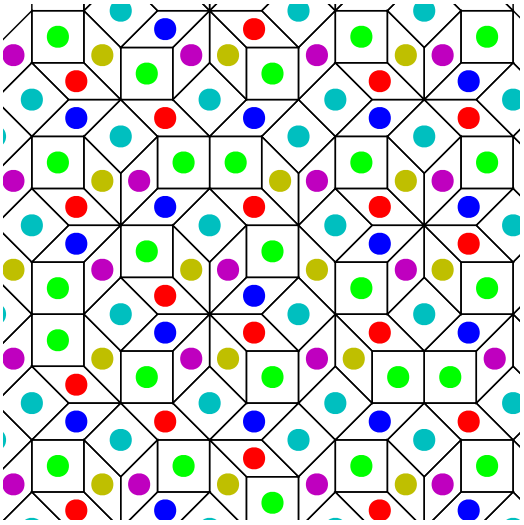

FIG. 1. A construction of the $k = 2$ quasicrystalline topological phases for a $d = 2$, $D = 4$ quasicrystal. Each color (of which there are six) is assigned a topological invariant that can be chosen independently (if no rotational symmetry is imposed) and represents the charge bound to the corresponding tile. If eightfold rotational symmetry is imposed, then there are two independent invariants, corresponding to the square and rhombus tiles.

Now, since C is a rank-d antisymmetric integer tensor of dimension D, it has $\binom{D}{d}$ independent entries. The basic idea is that, in a suitable limit, we can interpret these integer entries of C as the charge bound to the corresponding tiles of the quasicrystal. For example, Fig. 1 illustrates this picture for the case of a $d = 2$, $D = 4$ quasicrystal, for which there are six independent invariants (for a discussion of how the tiling shown in Fig. 1, known as Ammann-Beenker tiling, is constructed, see Appendix D). In the case $D = d$, where there is only one kind of tile, this evidently reduces to just the total charge per unit cell. Of course, this picture is valid only in the limit where the charges are tightly bound to the tiles; however, the discussion in the previous section demonstrates that the SPT invariants are still robust in the presence of charge fluctuations, as long as the system remains gapped.

This picture corresponds to the $k = d$ invariants in the general classification discussed in Sec. II C 1. The picture can also be generalized to the other SPT invariants in that classification, at least for $k \leq d$. For example, consider the case of $k = d - 1$. Then, there are $\binom{D}{d-1}$ independent SPT invariants, each of which takes values in the classification group of one-dimensional SPTs with respect to the internal symmetry. $\binom{D}{d-1}$ is the number of different kinds of codimension-1 faces of the tiles. Thus, the SPT states correspond to piercing the codimension-1 faces with the

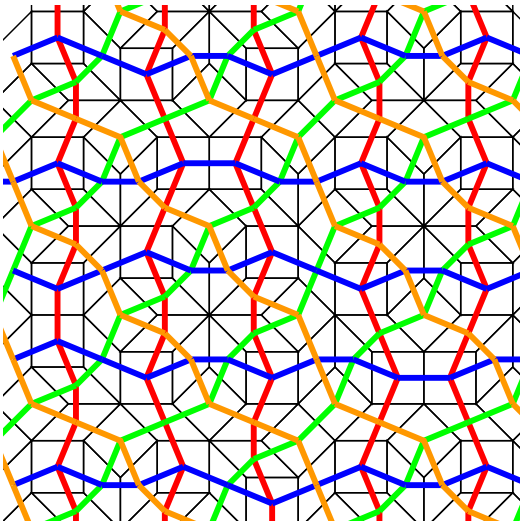

FIG. 2. A construction of the $k = 1$ quasicrystalline topological phases for a $d = 2$, $D = 4$ quasicrystal. Each color (blue, orange, green, and red) is assigned a 1D SPT invariant that can be chosen independently (if no rotational symmetry is imposed) and represents the 1D SPT phases placed along the corresponding paths. If eightfold rotational symmetry is imposed, there is only one independent SPT invariant.

corresponding one-dimensional SPTs. This is illustrated in Fig. 2.

In order to demonstrate the validity of this tile picture in the $k = d$ case for a system with U(1) symmetry, it is instructive to compute the average charge density ρ of the system. Let us focus on the $d = 2$ case for concreteness. The tiles' classes are then labeled by pairs of indices (IJ). Then, since each tile of type (IJ) carries charge $C_{IJ}\Sigma_{IJ}$ (where $\Sigma_{IJ} = \pm 1$ defines some sign convention, such that $\Sigma_{IJ} = -\Sigma_{JI}$), we find that

$$\rho = \frac{1}{2} C_{IJ} \rho^{IJ}, \tag{40}$$

where $\rho^{IJ} = \Sigma_{IJ}|\rho^{IJ}|$ and $|\rho^{IJ}|$ is the average number of (IJ)-type tiles in the tiling per unit volume. Moreover, we show in Appendix C that

$$\rho^{IJ} = \frac{1}{(2\pi)^2} K^I{}_i K^J{}_j \epsilon^{ij}. \tag{41}$$

Hence, we find

$$\rho = \frac{1}{8\pi^2} C_{IJ} K^I{}_i K^J{}_j \epsilon^{ij}, \tag{42}$$

which agrees with Eq. (38).

1. Example: Fibonacci tiling

To further illuminate the tile picture (for the $k = d$ case), let us consider for simplicity the case of a Fibonacci quasicrystal in $d = 1$ dimension. In this case, we have $D = 2$ and so $\binom{D}{d} = 2$ distinct tiles. A typical Fibonacci quasicrystal is obtained by putting particles at positions

$$x_n = x_0 + n(3/\tau - 1) + (1/\tau - 1)\mathrm{frac}(n/\tau), \tag{43}$$

where $\mathrm{frac}(x)$ is the fractional part of x and $\tau = (1 + \sqrt{5})/2$ is the golden ratio. The distances of neighboring points $x_n - x_{n-1}$ are either 1 or $1/\tau$. One can, thus, equivalently view the Fibonacci quasicrystal as consisting of a sequence of intervals with length 1 and $1/\tau$. These two types of intervals are called a long (L) and a short (S) tile, respectively—a typical Fibonacci quasicrystal is shown in Fig. 3, where the L and S tiles are colored in blue and green, respectively. The primitive reciprocal vectors K^1 and K^2 (actually just scalars, since $d = 1$) for the Fibonacci quasicrystal are given by [102]

$$\frac{1}{2\pi}\begin{pmatrix} K^1 \\ K^2 \end{pmatrix} = \frac{1}{2 - 1/\tau}\begin{pmatrix} 1 \\ 1/\tau \end{pmatrix}. \tag{44}$$

From Eq. (37), we obtain the charge density of the Fibonacci quasicrystal:

$$\rho = \frac{1}{2\pi} C_I K^I \tag{45}$$

$$= \frac{C_1}{2 - 1/\tau} + \frac{C_2/\tau}{2 - 1/\tau}. \tag{46}$$

We now show that we can interpret the SPT invariants C_1 and C_2 as the integer charges bound to the L and S tiles,

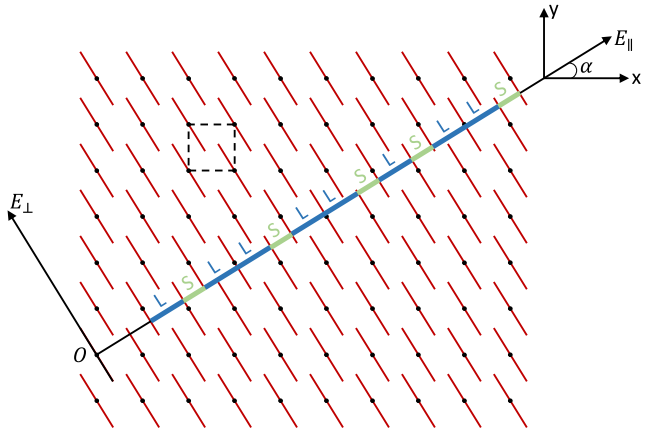

FIG. 3. Generating the $d = 1$ Fibonacci quasicrystal from a $D = 2$ square lattice decorated with $d = 1$ atomic surfaces σ (red line segments). Black dots denote lattice sites, and dashed black lines indicate the superspace unit cell. Intersections between the atomic surfaces and physical space $\mathbf{E}_\parallel$ generate the quasicrystal; the intersections divide $\mathbf{E}_\parallel$ into a sequence of long (L, in blue) and short (S, in green) intervals.

respectively. Consider a large region with length V in the physical space, and let C_1 and C_2 be the integer charges bound to the L and S tiles, respectively. The total charge Q inside the region V is given by

$$Q = n_L C_1 + n_S C_2, \tag{47}$$

where n_L and n_S denote the number of L and S tiles, respectively, inside the region V. Let l_L and l_S be the lengths of L and S tiles. The total length of this region is then given by

$$V = l_L n_L + l_S n_S. \tag{48}$$

Combining the above, we find the average charge density:

$$\frac{Q}{V} = \frac{n_L C_1 + n_S C_2}{l_L n_L + l_S n_S} \tag{49}$$

$$= \frac{C_1 + C_2/\tau}{2 - 1/\tau}\frac{1}{l_L}. \tag{50}$$

To obtain the second equality, we use the fact that the L tile appears more frequently than the S tile and that the ratio of their frequencies converges to the golden ratio τ. Hence, we have $n_L = \tau n_S$. We also use that the lengths of the tiles satisfy $l_L = \tau l_S$ and $1 + (1/\tau)^2 = 2 - 1/\tau$. In calculating the average charge density, we see that it is important to take the frequency of each class of tiles into account, which is a unique feature of quasicrystals. Comparing Eq. (50) with Eq. (46), we further see that the density ρ given in Eq. (46) is the average charge density over a scale much larger than the length of the L tile. We can, therefore, interpret the SPT invariants C_1 and C_2 as the integer charges bounds to the L and S tiles.

B. Atomic surfaces

In this section, we provide another interpretation of the SPT invariants introduced in Eqs. (24)–(26). Specifically, we describe how to assign a quantized topological invariant to any quasicrystal such that the location of particles whose number is conserved forms a classical point set (they no longer need to be rigidly bound to the center of tiles as they were in the previous interpretation). In this context, the invariant is a well-known one in the quasicrystal literature [90,91]. In this section, we refer to the particles as "atoms" in line with previous literature. As always, the considerations of the present paper demonstrate that the SPT invariants do remain well defined beyond the limit of classical point particles, but in the current section we focus on this limit.

First, we need to introduce some important concepts of quasicrystals from the superspace perspective. Recall that a quasiperiodic arrangement of point atoms can be described as a d-dimensional section of a D-dimensional periodic measure (where there exists a unique minimal value for D [103]). More specifically, a function of d real variables on an affine d-dimensional space $\mathbf{E}_\parallel$ is said to be quasiperiodic if it is the restriction to $\mathbf{E}_\parallel$ of some periodic function of D real variables defined in a higher-dimensional space $\mathbb{R}^D$. Of course, when the physical space $\mathbf{E}_\parallel$ (also referred to as the "embedded" space or the "cut") is oriented rationally with respect to the lattice of periods of the periodic function, the restriction of this function to $\mathbf{E}_\parallel$ is itself a periodic function. However, a quasiperiodic function is obtained upon restriction of the periodic function to a cut that is irrationally oriented.

As discussed in Ref. [91], to describe a quasiperiodic arrangement of point atoms, consider a D-dimensional periodic measure μ such that its restriction to $\mathbf{E}_\parallel$ results in a quasiperiodic configuration of Dirac delta functions. Let Λ be the lattice of periods of this measure; then, the support of μ defines a Λ-periodic geometric locus Σ. We refer to Σ as the set of atomic surfaces. The quasiperiodic atomic configuration is then generated by the intersection points of Σ with $\mathbf{E}_\parallel$.

As an example, let us consider a two-dimensional ($D = 2$) square lattice (with lattice constant unity) and $\mathbf{E}_\parallel$ given by a $d = 1$ subspace oriented at an angle α with respect to the x axis (see Fig. 3). Each vertex of the square lattice is decorated with an atomic surface of length $\ell = \sin(\alpha) + \cos(\alpha)$. The intersection points of Σ with $\mathbf{E}_\parallel$ generate a quasicrystalline structure iff $\tan\alpha$ is irrational; in particular, one obtains the paradigmatic Fibonacci quasicrystal when $\tan(\alpha) = 1/\tau$, where $\tau = (1 + \sqrt{5})/2$ is the golden ratio. As shown in Fig. 3, Σ divides $\mathbf{E}_\parallel$ into a sequence of long (L) and short (S) intervals—or "tiles"—of length 1 and $1/\tau$, respectively.

We can imagine that there are two different species of atoms, corresponding to particles that appear to the right of an L tile and atoms that appear to the right of an S tile; we refer to these as L and S atoms, respectively. Therefore, we can classify segments of atomic surface into two different "types," depending on which kind of particle they produce upon intersection with $\mathbf{E}_\parallel$, as shown in Fig. 4. Moreover, while the atomic surfaces are originally discontinuous, we introduce segments that are parallel to the physical space to make the atomic surfaces of each type closed curves without affecting the quasicrystalline structure on $\mathbf{E}_\parallel$. The property that the atomic surfaces should be closed is referred to as the "closeness condition" [91]. Physically, this condition reflects the conservation of the two species of atoms, since it ensures that a phason deformation (which corresponds to moving $\mathbf{E}_\parallel$ in the superspace) at most causes particles to jump to nearby positions but not to be created or destroyed. Henceforth, we take the term "atomic surface" to mean a continuous atomic surface.

Figure 4 shows an S atom (yellow dot) with the corresponding S-type atomic surface represented by the

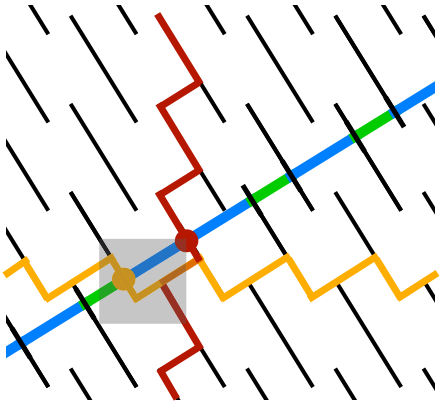

FIG. 4. Atomic surfaces (black lines) of a Fibonacci quasicrystal. The vertices of the Fibonacci quasicrystal are determined by the intersection points of the physical space and the atomic surfaces. The blue and green regions are the long (L) and the short (S) tiles that tile the $d = 1$ physical space. The yellow line is an S-type atomic surface associated to the yellow particle. The red line is an L-type atomic surface associated to the red particle.

yellow line. Figure 4 also shows an L atom (red dot), with the corresponding S-type atomic surface shown in red. As one can see from Fig. 4, the S-type atomic surface winds nontrivially on the 2-torus along the θ^1 direction; that is, it has a "winding number" (1,0), while the S-type atomic surface winds nontrivially on the 2-torus along the θ^2 direction; that is, it has a winding number (0,1). These winding numbers precisely determine the SPT invariants with respect to the two U(1) symmetries corresponding to the conservation of S and L atoms, respectively.

Similarly, one can give a general statement of the topological invariant for any configuration of atomic surfaces corresponding to a conserved species of particles. We represent an atomic surface as a $\mathbb{Z}$-valued $(D-d)$ cycle on $\mathbb{T}^D$. Topologically nontrivial configurations of the atomic surfaces are then classified by the homology group $H_{D-d}(\mathbb{T}^D, \mathbb{Z})$. By applying the Poincaré duality, we see that these are then equivalently classified by the cohomology group $H^d(\mathbb{T}^D, \mathbb{Z})$. As discussed in Sec. V, the SPT invariants are classified by the same cohomology group $H^d(\mathcal{L}^*, \mathbb{Z})$, with $\mathcal{L}^*$ topologically equivalent to $\mathbb{T}^D$; therefore, the SPT invariants are determined by the (co)homology classes of the atomic surfaces on the D torus.

This concludes our second interpretation of the SPT invariants C in terms of the atomic surfaces. We finally note that this picture can be generalized in an obvious way to the other SPT invariants discussed in Sec. II C 1 with $k < d$, where we replace the assumption of classical point particles in d spatial dimensions with the assumption that there is some configuration of $(d-k)$-dimensional surfaces in d spatial dimensions that host $(d-k)$-dimensional SPTs.

IX. THE GENERAL CLASSIFICATION OF QUASICRYSTALLINE TOPOLOGICAL PHASES

A. Without point-group symmetry

As we discuss in Sec. IV B above, the "order parameter manifold" for a quasicrystal is $\mathcal{L}^*$, the group of homomorphisms from the reciprocal lattice $\mathcal{L}$ into U(1). As previously discussed, for example, in Ref. [92], in general, "topological terms for Goldstone modes" arise when the symmetry-breaking ground states, viewed as a family of gapped ground states parameterized by the order parameter manifold, are a topologically nontrivial family. Thus, in order to classify quasicrystalline topological phases, which we view as being characterized by the topological term of the phonons and phasons, we want to classify topological families of ground states parameterized by $\mathcal{L}^*$.

Let us briefly note one subtlety, which is that, for quasicrystals, the ground states are not necessarily continuous functions of the order parameter. For example, in Fig. 3, varying the order parameter corresponds to moving the location of the cut in superspace, and one readily sees that this sometimes results in discontinuous jumps of the locations of particles. This is a well-known fact about phasons in quasicrystals. Nevertheless, these jumps always involve only local rearrangements; this implies, for example, that they can be effected by a local unitary. We conjecture, therefore, that the classification of topological families remains unchanged. In any case, these discontinuities are likely an artifact of treating the particle locations as totally classical variables and would likely get smoothed out once one allows these variables to quantum fluctuate.

The problem of classifying topological classes of families of gapped ground states parameterized by a space $\mathcal{L}^*$, in the presence of internal symmetry G_{int}, has previously been considered in Refs. [27,104–110]. Although not rigorously proven, it is believed that, for *invertible* states, such families are classified by $h^d(\mathcal{L}_* \times BG_{\text{int}})$, where BG_{int} is the classifying space of G_{int} and $h^\bullet$ is some generalized cohomology theory, probably cobordism for bosonic systems and spin cobordism for fermionic systems.[2] From axioms of generalized cohomology and using the fact that $\mathcal{L}^* \cong \mathbb{T}^D$, one can show [111] that

$$h^d(\mathcal{L}^* \times BG_{\text{int}}) = \bigoplus_{k=0}^{D} H^k[\mathcal{L}^*, h^{d-k}(BG_{\text{int}})]. \quad (51)$$

To interpret this equation, it is helpful to recall that, for $d-k \geq 0$, $h^{d-k}(BG_{\text{int}})$ is the classification of SPTs (and invertible topological phases) with internal symmetry G_{int} in $d-k$ spatial dimensions. Thus, this result reduces to the

[2]For systems with antiunitary symmetries or nontrivial extensions of G_{int} by fermion parity, the precise statement is slightly modified; we disregard this subtlety, since it does not affect the conclusions.

classification Eq. (3) stated in Sec. II C 1. As mentioned there, we can think of the terms of Eq. (51) (for $0 < k \leq d$) as a generalization of the "weak invariants" that classify SPT phases with translation symmetry, associated with stacking lower-dimensional SPT phases.

Intriguingly, however, the sum in Eq. (51) can also contain terms with $k > d$, which are not a straightforward generalization of weak invariants. In particular, one can argue that, if G_{int} contains an antiunitary symmetry, then $h^{-1}(BG_{\text{int}}) = \mathbb{Z}_2, h^{-2}(BG_{\text{int}}) = 0$; whereas, if G does *not* contain an antiunitary symmetry, then $h^{-1}(BG_{\text{int}}) = 0$, $h^{-2}(BG_{\text{int}}) = \mathbb{Z}$. To show this, one can consider the $d = 0$ case of the classification Eq. (51); here, we no longer interpret $\mathcal{L}^*$ as reflecting actual phonon and phason modes, just some manifold that parameterizes the ground state of the system. If we set $\mathcal{L}^* = S^1$, then, for time-reversal invariant systems, there must be a $\mathbb{Z}_2$ topological invariant corresponding to the Berry's phase of the ground state around the circle being 0 or π; while if we set $\mathcal{L}^*$ to be a 2-torus, then for non-time-reversal-invariant systems there be must be a $\mathbb{Z}$ topological invariant corresponding to the Chern number of the Berry connection over the torus. One can further argue that there are no new invariants arising in the $d = 0$ case if $\mathcal{L}^*$ is a higher-dimensional torus, so we can conclude that $h^r(BG_{\text{int}}) = 0$ for $r < -2$. The terms in Eq. (51) corresponding to $k > d$ reflect the theta and Wess-Zumino terms that can appear in the elasticity theory, as we discuss in Sec. V; see, in particular, Ref. [108] for a microscopic discussion of the topological invariant for ground states parameterized by a space X that gives rise to a Wess-Zumino term on X.

Finally, let us note that it is not necessary to restrict ourselves only to invertible states. We can also discuss quasicrystalline enrichment for noninvertible topological phases; we discuss this below as part of the general theory, taking into account point-group symmetries as well.

B. With point-group symmetry

We also can extend the above considerations to incorporate point-group symmetry. We first need to recall the somewhat subtle notion of point-group symmetry in a quasicrystal [77]. Let $R \in \mathrm{O}(d)$ be some (possibly improper) rotation, and recall from the previous sections that, for a quasicrystal, there is a family of ground states labeled by the "order parameter manifold" $\mathcal{L}^*$. We say that a quasicrystal has point-group symmetry R if there is an action of R on the Hilbert space that permutes these ground states among themselves. This ensures, for example, that the intensity of the peaks of the Fourier transform of the expectation value of any local observables is invariant under R.

Specifically, let $|\Psi[\phi]\rangle$ be the family of quasicrystalline ground states labeled by $\phi \in \mathcal{L}^*$. Let G be some group, let $\rho: G \to \mathrm{O}(d)$ and $\gamma: G \to \mathcal{L}^*$ be homomorphisms, and let $U(g)$, $g \in G$, be an (anti)unitary representation of G on the Hilbert space of the system that maps operators supported near the point $\mathbf{x}$ to operators supported near the point $\rho(g)\mathbf{x}$. Then, we say that the system has the symmetry G if

$$U(g)|\Psi[\phi]\rangle = |\Psi[\phi + \gamma(g)]\rangle. \tag{52}$$

Let us remark that, upon identifying $\mathcal{L}^*$ with $\mathbb{R}^D/\mathbb{Z}^D$, it is always possible to find a group $\hat{G}$ and an affine-linear action of $\hat{G}$ on $\mathbb{R}^D$, i.e.,

$$\mathbf{x} \mapsto R(g)\mathbf{x} + \hat{\gamma}(g), \qquad g \in \hat{G}, \tag{53}$$

where $R(g) \in \mathrm{GL}(D, \mathbb{Z})$ and $\hat{\gamma}(g) \in \mathbb{R}^D$, with the following properties: (a) $\hat{G}$ contains all the unit translations $\mathbb{Z}^D$ as a normal subgroup, and $\hat{G}/\mathbb{Z}^D \cong G$; (b) $\sigma[\hat{\gamma}(g)] = \gamma[\pi(g)]$ for all $g \in \hat{G}$, where $\sigma: \mathbb{R}^D \to \mathbb{R}^D/\mathbb{Z}^D$ and $\pi: \hat{G} \to G$ are the projection maps; and (c) the action of $R(g) \in \mathrm{GL}(D, \mathbb{Z})$ on the torus $\mathcal{L}^* = \mathbb{R}^D/\mathbb{Z}^D$ agrees with the action induced on $\mathcal{L}^*$ from the action of $\rho(g)$ on $\mathcal{L}$. In terms of the superspace picture, we can think of $\hat{G}$ as the space group of the D-dimensional crystal through which we take a cut to obtain a d-dimensional quasicrystal. The subgroup $\mathbb{Z}^D \leq \hat{G}$ is the translation symmetry of the D-dimensional crystal, and $G = \hat{G}/\mathbb{Z}^D$ is its point group.

It turns out that $\hat{G}$ can be thought of as the "effective symmetry group" of the quasicrystal from the point view of the classification, even though there is no sense in which $\hat{G}$ can literally be interpreted as a symmetry of the quasicrystal. To show the classification, we note that the problem of how to classify topological families satisfying Eq. (52) is precisely the problem considered in Ref. [27] (some subtleties glossed over in Ref. [27] are clarified in Ref. [28]), even though the physical interpretation given in Refs. [27,28] is different. Moreover, Refs. [27,28] focus on the case where the family is parameterized by a d-dimensional torus $\mathbb{T}^d$ or, equivalently, by Euclidean space $\mathbb{R}^d$. Here, the family is parameterized by the space $\mathcal{L}^* \cong \mathbb{T}^D$ of dimension $D > d$. However, it is not, in fact, essential for any of the arguments of Refs. [27,28] that the parameterizing space have dimension d. Therefore, by similar arguments to Refs. [27,28], we obtain the following principle.

1. *Quasicrystalline equivalence principle*

The classification of quasicrystalline topological phases in d spatial dimensions with "effective symmetry" $\hat{G}$ is in one-to-one correspondence with the classification of topological phases with *internal* symmetry $\hat{G}$ in d spatial dimensions.

This is the quasicrystalline generalization of the crystalline equivalence principle of Ref. [27] and holds for both symmetry-protected and symmetry-enriched topological phases. As in the crystalline case, there are some "twists";

for example, a unitary spatial orientation-reversing symmetry maps to an antiunitary internal symmetry.

We emphasize that we are *not* saying that quasicrystallinity is equivalent to additional spatial dimensions. Both sides of the correspondence relate to topological phases in d spatial dimensions; it is only the effective symmetry group $\hat{G}$ that can be interpreted, if one likes, as the space group of a fictitious crystal in D spatial dimensions.

Moreover, from the arguments in Ref. [28], we can also derive a "defect network" picture for quasicrystalline topological phases. The defect network lives in the D-dimensional superspace and is required to be invariant under the symmetry $\hat{G}$, but it is *not* equivalent to a defect network for a D-dimensional crystalline topological phase. The difference is that, in the latter, an r-dimensional defect carries the data of an r-dimensional SPT (in the invertible case, say), whereas, in the former, an r-dimensional defect carries the data of an $[r-(D-d)]$-dimensional SPT.

X. CONCLUSIONS

In this work, we describe a general approach for understanding many-body topological phases of matter protected by quasicrystallinity, leading ultimately to a general classification result. More concretely, we focus on the physical implications of a particular class of quasicrystalline topological phases, namely, those protected by U(1) charge conservation. For such phases, we provide various interpretations and implications through the lens of topological elasticity theory. In the future, it will be desirable to explore the physical characteristics of other quasicrystalline topological phases that result from our general classification, particularly those with the "Wess-Zumino"-type response discussed in Sec. V; such phases are fundamentally new to quasicrystals and do not occur in crystals. Another important direction will be to search for these phases of matter in experimental platforms. As a step in this direction, it would also be useful to find concrete microscopic models realizing such phases.

Here, we focus on the bulk manifestations of quasicrystalline topological phases. It would also be interesting to consider whether the surfaces of these phases host protected gapless edge modes. For phases protected by quasicrystalline point-group symmetries, one presumably expects some kind of corner or hinge modes, as in the case of crystalline HOTIs. For noninteracting systems, this has already been verified in Refs. [47–49]. Meanwhile, phases such as the one depicted in Fig. 2 [more generally, any of the phases classified by terms of Eq. (3) with $0 < k < d$] are generalizations of crystalline weak SPT phases that host gapless edge modes provided that the surface preserves some subgroup of lattice translation symmetry. In order to extend this to quasicrystals, one would first need to develop a theory of quasicrystalline surfaces, including formulating a surface property that generalizes "preserving a subgroup of lattice translation symmetry" to the quasicrystalline case.

It would be interesting to generalize the quasicrystalline LSMOH result discussed in Sec. VII above. For crystalline systems, in cases where LSMOH forbids a trivial gapped ground state, a gapless ground state must still obey constraints such as Luttinger's theorem; such constraints were recently placed on a general footing in Ref. [112]. It might be possible to combine our results with those of Ref. [112] to obtain analogs of such constraints on gapless systems for the quasicrystalline case.

With regards to the general classification scheme discussed here, further developing the "defect network" picture (discussed in Sec. IX B) as well as the "building block" picture (developed for cSPT phases in Refs. [25,26]) could shed further light on the nature of quasicrystalline topological phases. Finally, it will be instructive to study noninvertible quasicrystalline phases in more detail as well, since the interplay of quasicrystallinity with topological order has yet to receive much attention.

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APPENDIX A: TOPOLOGICALLY EQUIVALENT LOCAL ISOMORPHISM CLASSES

In this Appendix, we comment on how the classification discussed in the main text relates to existing notions of equivalence classes for quasicrystals. First, it is important to understand which quasicrystals are physically indistinguishable, i.e., have the same diffraction pattern and correlation functions. For periodic crystals, there is a unique arrangement of the fundamental repeating units (atoms or tiles/unit cells) that forms the ideal crystal, up to translations and rotations. In contrast, for quasicrystals there is an infinite number of ways of arranging the repeating units to form the ideal structure—the choice of orientational symmetry and the fundamental repeating units (be they atoms or tiles) does not suffice to uniquely specify a quasicrystal [98]. Indeed, there exists an uncountable infinity of distinguishable arrangements of the same repeating units whose diffraction patterns (Fourier spectra) are given by the same set of reciprocal wave vectors with different Bragg peak intensities.

It is, hence, desirable to organize quasicrystals into equivalence classes. One such existing notion is that of *local isomorphism* (LI) classes [113,114], where two

quasicrystals are said to be in the same LI class iff any bounded configuration of repeating units present in one appears, up to finite translations (plus global rotations and inversions), in the other with the same frequency. Quasicrystals within the same LI class cannot be distinguished by measurements made on any finite length scale. In fact, one can show that two quasicrystals are locally isomorphic if and only if they have identical diffraction patterns: The locations of the Bragg peaks *and* the peak intensities of locally isomorphic quasicrystals match [113,114]. On the other hand, two quasicrystals whose diffraction patterns have Bragg peaks in the same locations but with differing peak intensities belong to distinct LI classes. Consequently, quasicrystals have the same free energy if they belong to the same LI class. Note that for periodic crystals, for which there exists a unique arrangement (up to translations) of the fundamental repeating units, each LI class contains a single element.

In the atomic surface description of quasicrystals discussed in Sec. VIII B, notice that, once we fix an orientation for $\mathbf{E}_\parallel$, we can automatically generate an infinite number of different quasiperiodic arrangements. However, for a generic atomic surface, arrangements obtained from two cuts belong to the same LI class if and only if they can be mapped onto each other by a translation in superspace, as these shifts do not alter the diffraction pattern [91]. For example, we can consider a translation of the two-dimensional periodic structure in Fig. 3 with respect to the origin O of $\mathbf{E}_\parallel$ by a vector $\mathbf{v}$. It can be shown that the resulting atomic configurations on $\mathbf{E}_\parallel$ before and after the translation overlap out to arbitrary finite distances by a finite translation along $\mathbf{E}_\parallel$; i.e., they belong to the same LI class and are physically indistinguishable.

Hence, LI classes subdivide quasicrystals such that elements of a given LI class are physically equivalent; i.e., their diffraction patterns have Bragg peaks at identical locations and have the same intensities. As mentioned earlier, this classification is not particularly meaningful for periodic crystals where each LI class contains a single element. Nevertheless, one can still define *topological* equivalence classes for periodic crystals. As discussed in Sec. III, for the case where the only symmetries are translation and U(1) charge conservation, the integer-valued invariant ν [see Eqs. (7)–(9)] labels the topologically inequivalent classes, where elements within a class can be smoothly deformed into each other without changing ν.

This notion of topological equivalence classes generalizes to quasicrystals. As discussed earlier, there exist infinitely many arrangements of the same repeating units whose Fourier spectra have Bragg peaks in the same positions but with different peak intensities. We subdivide this space of quasicrystals into topological classes such that each class consists of LI classes that can be smoothly deformed into each other. In particular, two physically inequivalent LI classes are topologically equivalent iff one is smoothly connected to the other via deformations that leave invariant the positions of the Bragg peaks but are allowed to modify the peak intensities. Recall that deformations that also do not change the peak intensities do not change the LI class. Hence, our classification can be understood as further defining an equivalence relation over the space of LI classes of quasicrystals (see Fig. 5).

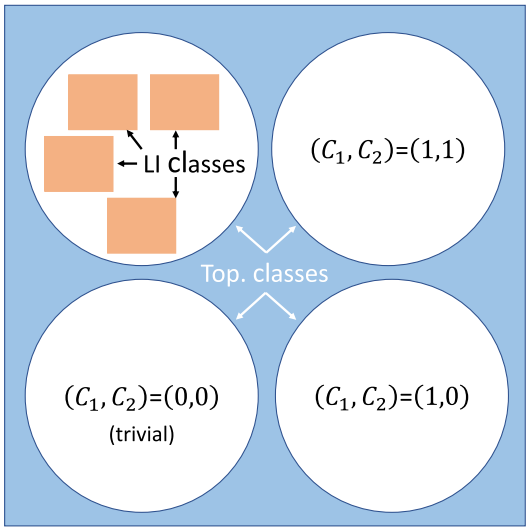

FIG. 5. Topological equivalence classes for $d = 1$ quasicrystals are labeled by a pair of integers (C_1, C_2) (see Secs. V and VIII). Each class consists of physically inequivalent LI classes which are equivalent topologically.

APPENDIX B: 2D OCTAGONAL QUASICRYSTALS

In this Appendix, we show how the reciprocal lattice vectors $\mathbf{K}^I$ are obtained for a 2D octagonal quasicrystal. The idea is to obtain a decomposition $\mathbb{R}^4 = \mathbf{E}_\parallel \oplus \mathbf{E}_\perp$, where in the four-dimensional "superspace" the physical system lives on the two-dimensional plane $\mathbf{E}_\parallel$.

Let a^I $(I = 1, \ldots, 4)$ denote the basis vectors of the unit hypercubic lattice $\mathbb{Z}^4$. The C_8 rotation matrix in this basis is given by

$$R_8 = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{pmatrix}. \tag{B1}$$

Diagonalizing R_8, we find that its eigenvalues are given by $\xi^n = \exp(2\pi i n/8)$, where $n = 1, \ldots, 4$. Note that there are two conjugate pairs: (ξ, ξ^{-1}) and (ξ^2, ξ^{-2}). This implies that the superspace $\mathbb{R}^4$ decomposes into two orthogonal two-dimensional planes, with both planes invariant under C_8 rotation. The eigenvectors corresponding to ξ^n $(n = 1, \ldots, 4)$ are

$$v_n = (\xi^n, \xi^{2n}, \xi^{3n}, \xi^{4n}). \tag{B2}$$

Although these eigenvectors have complex entries, it is possible to find a basis such that they contain only real entries. The basis vectors for one of the two-dimensional planes, which we choose to be our physical plane $\mathbf{E}_\parallel$, are given by

$$v_1 = \left(-\frac{1}{\sqrt{2}}, 0, -\frac{1}{\sqrt{2}}, 1\right), \tag{B3}$$

$$v_2 = \left(\frac{1}{\sqrt{2}}, 1, -\frac{1}{\sqrt{2}}, 0\right). \tag{B4}$$

The basis vectors for the perpendicular plane $\mathbf{E}_\perp$ are

$$v_3 = \left(-\frac{1}{\sqrt{2}}, 0, -\frac{1}{\sqrt{2}}, -1\right), \tag{B5}$$

$$v_4 = \left(-\frac{1}{\sqrt{2}}, 1, \frac{1}{\sqrt{2}}, 0\right). \tag{B6}$$

Let $\boldsymbol{M}$ be the transformation matrix between the two sets of basis vectors: $a^I = M^I{}_J v_J$, where a^I span a four-dimensional hypercubic reciprocal lattice and v_J are orthonormal unit vectors of the four-dimensional superspace. We find that the transformation matrix is

$$\boldsymbol{M} = \begin{pmatrix} -1/\sqrt{2} & 1/\sqrt{2} & -1/\sqrt{2} & -1/\sqrt{2} \\ 0 & 1 & 0 & 1 \\ -1/\sqrt{2} & -1/\sqrt{2} & -1/\sqrt{2} & 1/\sqrt{2} \\ 1 & 0 & -1 & 0 \end{pmatrix}. \tag{B7}$$

The corresponding direct lattice is spanned by four vectors $a^{*I} = a^I/2$. By definition, the reciprocal lattice vectors are given by $K^I{}_i = 2\pi M^I{}_i$ for $i = 1, 2$:

$$\boldsymbol{K} = \begin{pmatrix} -1/\sqrt{2} & 1/\sqrt{2} \\ 0 & 1 \\ -1/\sqrt{2} & -1/\sqrt{2} \\ 1 & 0 \end{pmatrix}. \tag{B8}$$

The projections of the basis vectors $\boldsymbol{a}^*$ of the direct lattice $\mathbb{Z}^4$ into the physical plane $\mathbf{E}_\parallel$ are given by

$$\begin{aligned} \pi(a_1^*) &= -\frac{1}{2\sqrt{2}} v_1 + \frac{1}{2\sqrt{2}} v_2, \\ \pi(a_2^*) &= \frac{1}{2} v_2, \\ \pi(a_3^*) &= -\frac{1}{2\sqrt{2}} v_1 - \frac{1}{2\sqrt{2}} v_2, \\ \pi(a_4^*) &= \frac{1}{2} v_1. \end{aligned} \tag{B9}$$

This information is useful in constructing the tiles of the octagonal quasicrystals in a later Appendix.

APPENDIX C: DENSITY OF TILES IN A QUASICRYSTALLINE TILING

In this Appendix, we derive Eq. (41) for the density of a given tile in a quasicrystalline tiling generated according to the scheme in Ref. [101]. As shown in Ref. [101], such a tiling can be obtained via a cut-and-project scheme. That is, one considers some d-dimensional hyperplane in a D-dimensional superspace, which can be identified with the physical space $\mathbb{R}^d$ via a linear mapping $K : \mathbb{R}^d \to \mathbb{R}^D$. Then, one defines an acceptance window surrounding this hyperplane. All the d-dimensional facets of the D-dimensional hypercubic lattice (with lattice constant 2π) that lie wholly within the acceptance window are projected onto the image of K, and the preimage of these projections in $\mathbb{R}^d$ forms the tiles in a quasiperiodic tiling of $\mathbb{R}^d$. There are $\binom{D}{d}$ different facet orientations in the D-dimensional hypercubic lattice, labeled by unordered d-tuples of indices that can range over $1, \ldots, D$. The projected version of these facets then form the different kinds of tiles in the quasicrystalline tiling.

Here, what we want to prove is that, in $d = 2$, the density ρ^{IJ} of such tiles satisfies (in some sign convention)

$$\rho^{IJ} = \frac{1}{(2\pi)^2} K^I{}_i K^J{}_j \epsilon^{ij}. \tag{C1}$$

We invoke the fact that, as shown in Ref. [101], the *unprojected* versions of the facets contained within the acceptance window define a closed d-dimensional surface in superspace, which we call the *unprojected surface*.

As a warmup, let us consider the $d = 1$ case, where the analogous formula to Eq. (C1) is

$$\rho^I = \frac{1}{2\pi} K^I. \tag{C2}$$

Now, consider some interval of length V within the one-dimensional physical space. Then, after mapping the physical space into superspace, the end points of this interval are separated by a displacement vector in superspace of

$$\vec{K} V, \tag{C3}$$

where $\vec{K}$ is the vector in superspace whose components are K^I. Meanwhile, the portion of the unprojected surface that is near the mapped interval consists of a curve with end points, where the end points are separated by a displacement

$$2\pi N^I \vec{e}_I, \tag{C4}$$

where $\vec{e}_I$ is the basis vector in the Ith coordinate direction and N^I is the number of tiles of type I in the quasicrystalline tiling

that are contained within the interval of length V. Because the unprojected surface lies within the acceptance window, whose width is independent of V, we conclude that Eqs. (C3) and (C4) must be equal up to a correction that is $O(1)$ in V. Hence, by taking the Ith component, we find

$$\frac{N^I}{L} = \frac{1}{2\pi} K^I + O\left(\frac{1}{V}\right), \tag{C5}$$

and taking the limit $V \to \infty$ gives Eq. (C2).

The $d = 2$ version of the argument proceeds similarly, except that the quantity that we need to equate between the physical plane and the unprojected surface is the "net directed area," as measured by a bivector (i.e., a rank-2 element of the exterior algebra of $\mathbb{R}^D$). Thus, we require that

$$(2\pi)^2 N^{IJ} \vec{e}_I \wedge \vec{e}_J = \epsilon^{ij} V \vec{K}_i \wedge \vec{K}_j + O(1), \tag{C6}$$

where $\vec{K}_i$ is the vector in $\mathbb{R}^D$ given by taking the ith column of the matrix K. Taking components of this equation gives Eq. (C1).

APPENDIX D: AMMANN-BEENKER TILING

A 2D octagonal quasicrystal can be constructed by the Ammann-Beenker tilings. In an Ammann-Beenker tiling, there are two kinds of tiles: a rhombus (with the small angle equal to $\pi/4$) and a square. Here, we briefly explain how to obtain these tiles by the projection method. We are going to follow Ref. [101] closely.

The tiles are simply given by the projection of the 2-facets of a $D = 4$ hypercubic unit cell onto the physical plane. Let $M_2 = \{I = \{i_1, i_2\} \subset \{1, 2, 3, 4\}\}$ be the set of subsets of $\{1, 2, 3, 4\}$. The 2-facets of the unit hypercube, indexed by M_2, are defined as follows:

$$\gamma_I = \left\{ \sum_{i \in I} \lambda_i a_i^* | \lambda_i \in [0, 1] \right\} \tag{D1}$$

for all $I \in M_2$. Now, we use Eq. (B9) to obtain the projections of the facets γ onto the physical plane:

$$D_I = \pi(\gamma_I) = \left\{ \sum_{i \in I} \lambda_i v_i^P | \lambda_i \in [0, 1] \right\}, \tag{D2}$$

where we define $v_i^P \coloneqq \pi(a_I^*)$. Since a tile in 2D requires two coordinates, there are $\binom{4}{2} = 6$ prototiles. For example, a rhombus indexed by $I_{12} \coloneqq \{1, 2\}$ is

$$D_{I_{12}} = \lambda_1 v_1^P + \lambda_2 v_2^P. \tag{D3}$$

A square indexed by $I_{24} \coloneqq \{2, 4\}$ is

$$D_{I_{24}} = \lambda_2 v_2^P + \lambda_4 v_4^P. \tag{D4}$$

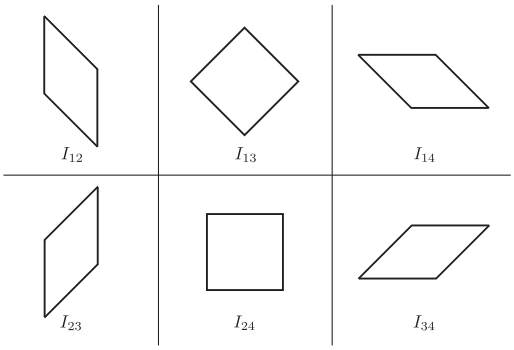

FIG. 6. Prototiles of the Ammann-Beenker tiling and their indices.

Figure 6 shows the prototiles of the Ammann-Beenker tiling along with their index. Note that, if we include the C_8 rotation in the definition of the congruence classes, there are only two prototiles: the rhombus and the square. The Ammann-Beenker tiling shown in Fig. 1 can be constructed by using the prototiles shown in Fig. 6.

APPENDIX E: DERIVING THE MOBILITY CONSTRAINT OF DISLOCATIONS IN A 2D QUASICRYSTAL

In this Appendix, we give a more precise argument for the constraint Eq. (36) on the motion of dislocations in a 2D quasicrystal.

The θ field for a dislocation moving uniformly in space with its core at position $\mathbf{a}(t)$ [we assume that $\mathbf{a}(0) = \mathbf{0}$] can be written as

$$\theta^I(\mathbf{x}, t) = \Theta^I(\mathbf{x}, t) + \phi^I(\mathbf{x}, t), \tag{E1}$$

where we define $\Theta^I(\mathbf{x}) = K_i^I x^i$ and ϕ satisfies

$$\phi^I(\mathbf{x}, t) = \phi^I[\mathbf{x} - \mathbf{a}(t), 0]. \tag{E2}$$

Let Σ be a loop in 2D space encircling the dislocation at time $t = 0$, and let $\Sigma(t) = \Sigma + \mathbf{a}(t)$. Now define $\Sigma_+[0, T]$ to be the 2D world sheet in 3D space-time swept out by $\Sigma(t)$ between time $t = 0$ and time $t = T$. We can evaluate the amount of charge that must be created at the dislocation core during the motion from time $t = 0$ to time $t = T$ by evaluating the integral

$$\Delta Q = \int_{\Sigma_+[0,\tau]} (*J), \tag{E3}$$

where J is the 1-form in 3D space-time constructed by using the metric to lower the index of the current 3-vector J^μ and $*$ is the Hodge star operator. Here and in what follows, we use the abstract notation for differential forms to simplify the derivation. From Eq. (31), we have that

$$*J = \frac{1}{8\pi^2} C_{IJ} d\theta^I \wedge d\theta^J \tag{E4}$$

$$= \frac{1}{8\pi^2}(C_{IJ} K^I \wedge K^J + 2C_{IJ} K^I \wedge d\phi^I + d\phi^I \wedge d\phi^J). \tag{E5}$$

The integral of the first term gives zero, while the integral of the third term can also be shown to be zero by first showing that it is invariant under small deformations of ϕ^I [provided that Eq. (E2) remains satisfied] and then deforming ϕ^I to a reference configuration for a particular winding number sector and evaluating the integral.

This leaves only the integral of the second term, and, hence, we find

$$4\pi^2 \Delta Q = \int_{\Sigma_+[0,T]} C_{IJ} K^I \wedge d\phi^J \tag{E6}$$

$$= \int_{\Sigma_+[0,T]} C_{IJ} d\Theta^I \wedge d\phi^J \tag{E7}$$

$$= \int_{\Sigma_+[0,T]} C_{IJ} d(\Theta^I d\phi^J) \tag{E8}$$

$$= \int_{\partial\Sigma_+[0,\tau]} C_{IJ} \Theta^I d\phi^J \tag{E9}$$

$$= C_{IJ}\left[\int_{\Sigma(T)} K^I{}_i x^i [\nabla\phi^J(\mathbf{x}, T)] \cdot d\mathbf{x} - \int_{\Sigma} K^I{}_i x^i [\nabla\phi^J(\mathbf{x}, 0)] \cdot d\mathbf{x}\right] \tag{E10}$$

$$= C_{IJ} K^I{}_i a^i(T) \int_{\Sigma} [\nabla\phi^J(\mathbf{x}, 0)] \cdot d\mathbf{x} \tag{E11}$$

$$= C_{IJ} K^I{}_i a^i(T) b^J, \tag{E12}$$

where in Eq. (E9) we use Stokes' theorem and to go from Eq. (E10) to Eq. (E11) we invoke Eq. (E2) and make the change of variables $\mathbf{x} \to \mathbf{x} + \mathbf{a}(T)$ in the first term. Here, b^J is the "Burgers vector" of the dislocation defined by Eq. (30). If we demand that $\Delta Q = 0$ and take the limit of infinitesimal T, we recover the constraint Eq. (36).

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