Analytical, Statistical Approximate Solution of Dissipative and Nondissipative Binary-Single Stellar Encounters

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We present a statistical approximate solution of the bound, nonhierarchical three-body problem, and extend it to a general analysis of encounters between hard binary systems and single stars. Any such encounter terminates when one of the three stars is ejected to infinity, leaving behind a remnant binary; the problem with binary-single-star scattering consists of finding the probability distribution of the orbital parameters of the remnant binary as a function of the total energy and the total angular momentum. Here, we model the encounter as a series of close, nonhierarchical, triple approaches, interspersed with hierarchical phases, in which the system consists of an inner binary and a star that orbits it; this series of approaches turns the evolution of the entire encounter to a random walk between consecutive hierarchical phases. We use the solution of the bound, nonhierarchical three-body problem to find the walker's transition probabilities, which we generalize to situations in which tidal interactions are important. Besides tides, any dissipative process may be incorporated into the random-walk model, as it is completely general. Our approximate solution can reproduce the results of the extensive body of past numerical simulations and can account for different environments and different dissipative effects. Therefore, this model can effectively replace the need for direct few-body integrations for the study of binary-single encounters in any environment. Furthermore, it allows for a simply inclusion of dissipative forces typically not accounted for in full N-body integration schemes.

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I. INTRODUCTION

The three-body problem—that of the interaction of three gravitating objects—is one of the oldest physical and astrophysical problems, and has been studied for centuries; solving it requires an understanding of the outcomes of encounters between binary systems and single stars. Such encounters play a key role in the evolution of binary systems, both in dense environments such as globular or nuclear clusters [1,2], and in the field [3,4]. These encounters can result in the formation of compact binaries and, consequently, engender a plethora of physical phenomena, ranging from mergers of stars and compact objects to the production of electromagnetic and gravitational-wave transients, such as type-Ia supernovæ, short gamma-ray bursts, etc. Currently, full numerical integrations of few-body systems are required in order to characterize the results of such interactions. However, the large number of such few-body simulations needed for one cluster comes with a significant computational cost; any change in the basic aspects of the problem (e.g., different masses, binary configuration, etc.) necessitates a new set of simulations; and likewise, different environments (e.g., external, outer potentials in clusters) require different sets of simulations (and, in many cases, are not even consistently accounted for). Moreover, realistic physical processes that may affect the outcomes, such as dissipative forces (e.g., tidal interactions or gravitational-wave dissipation) acting on the interacting stars, are rarely incorporated into simulations of the evolution of binary systems through binary-single encounters. Finally, such simulations, by their very nature, provide neither an explanation for their outcomes nor any direct insight into them, and they are used, to some extent, as black-box ingredients in simulations of large-scale systems. Despite much progress in the analytical understanding of the problem, the full solution for the outcomes of nonhierarchical three-body systems—and, in particular, the general understanding of binary-single encounters and their long-term end states under realistic conditions—remains unsolved. Furthermore, analytically accounting for dissipative aspects of this problem has not been done previously, to the best of our knowledge.

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Here, we provide a full analytical, statistical model that solves the nonhierarchical three-body problem in the limit of encounters with hard binaries. We show how our approach can account for the long-term evolution of a nonhierarchical triple system and characterize it as it goes through consecutive close binary-single encounters until the system is destroyed. Our solution provides the detailed cross section for the final outcomes, including both cases of ejection of one of the components or the collision or merger of two of the components, as well as, most importantly, the characterization of the final remnant binary properties. Our approach can also account for different spatial cutoffs due to external perturbations. Last but not least, it can generally account for dissipative forces—we exemplify its application for dissipative tidal forces during binary-single encounters. Thus, the use of our model, which is based on a random-walk approach, can potentially replace the need for direct integration of binary-single encounters and provide a general, robust tool for understanding the outcomes of binary-single encounters.

Binary-single encounters can generally be divided between the cases of encounters with hard or soft binaries, i.e., according to the ratio between the binary binding energy and the kinetic energy of the incoming third star. When the binary star is wide—when its binding energy is considerably smaller than the typical kinetic energy of a star in the host cluster, $k_B T$ (where T is proportional to the squared velocity dispersion of the cluster), the encounter is well described by a composition of two two-body interactions [1,5,6]. In the case where the binary is "hard," i.e., when its binding energy is much larger than $k_B T$, an analytical description of the encounter is more difficult. If the encounter stays hierarchical, it may be treated by means of perturbation theory [5], but if it does not, it becomes a so-called "resonant encounter," where there are phases when all three bodies are close to each other, and the evolution is inherently chaotic. One could still find some analytical insight by investigating the phase-space evolution of a set of various initial conditions during such a close encounter. For example, it was shown in the early 20th century that if the system starts as a binary and an unbound third star, then, eventually, it will end up with at least one unbound star (apart from a set of measure zero); see Arnold *et al.* (Secs. 2.4 and 2.6 of Ref. [7]) for a review of this theorem and related ones.

Here, we present an analytical, statistical model for the evolution of such encounters with hard binaries. We endeavour to derive a probability distribution function for the possible end states. The initial binary is hard, so the total energy of the system, E, must be negative, whence an end state with three stars unbound is forbidden—the set of all possible end states is therefore the set of all possible final binary configurations, multiplied by the set of configurations of the ejected star (which, of course, may be a different star from the initial unbound one—see Sec. III below). We derive a distribution function, $f_{\rm bin}(E_{\rm bin}, \mathbf{S}|E, \mathbf{J})$, for the resultant binary to have energy $E_{\rm bin}$ and angular momentum $\mathbf{S}$, given that the total energy is E and that the total angular momentum is $\mathbf{J}$.

Let m_1, m_2, and m_3 denote the masses of the three stars, which we take to be of similar magnitude, and let $M = m_1 + m_2 + m_3$ be the total mass. Further, let the subscript "bin" denote any quantity related to the binary, such as its total mass $m_{\rm bin}$, the reduced mass of its two components $\mu_{\rm bin}$, etc. Likewise, let a subscript s denote quantities pertaining to the ejected star, like $\mu_s = m_s m_{\rm bin}/M$. We denote the total energy of the triple by E or sometimes by E_0, and its total angular momentum by $\mathbf{J}$. More explicitly, then, the problem we investigate is as follows: We consider a single star with velocity v_0 at infinity, which is scattered on a hard binary, such that the total energy is E and the total angular momentum is $\mathbf{J}$. As the binary is hard, $E \approx -G m_a m_b/(2a_0)$, where a_0 is the initial semimajor axis of the binary, consisting, initially, of stars a and b. For a given impact parameter b of the single star, relative to the binary center of mass (as well as all the initial anomalies of the binary), it is possible, in principle, to predict the outcome of the encounter, i.e., the final values of $E_{\rm bin}$, $\mathbf{S}$, and E_s. However, because of the chaotic nature of the close three-body interaction, doing so is impracticable. One could study the problem either by performing numerical simulations (see, e.g., Refs. [8–21]), or, statistically, using various analytical approximations. Such analytical treatments presuppose that the results of such numerical scattering experiments may be treated as random variables, drawn from some distribution; in effect, it reduces the problem to finding that distribution. Numerical simulations have shown that the encounter proceeds as a sequence of many close triple approaches, after each of which a single star is ejected [11,12,19]: If the star is still bound to the other binary, it eventually returns, whereupon a new close triple approach ensues, and so on, until the single star is ejected to infinity with positive energy.

There are two dominant analytical approaches in the literature (to our knowledge): One, initiated by Monaghan in Ref. [22], relies on chaotic mixing during the close triple approach to argue that the distribution is ergodic in the relevant part of the system's phase space. This approach was refined later by various authors, e.g., Refs. [6,23–25], to more accurately account for angular-momentum conservation, until Stone and Leigh [26] succeeded in doing so fully, for the unbound case.

The other approach, which dates back to Heggie [5], was to draw upon the principle of detailed balance to deduce what the outcome distribution, $f_{\rm bin}(E_{\rm bin}, \mathbf{S}|E, \mathbf{J})$, must be in order for the number of bound triples formed by such encounters to be fixed, in a cluster in thermal equilibrium (see, e.g., Refs. [2,5,15]). In simulations, various authors also attempted to account not only for the Newtonian interaction of three point particles but also include other physical effects, such as collisions [17,27], and, more recently, also gravitational-wave emission and tidal dissipation [19,28]. The reader is referred to Chapters 7 and 8 of Ref. [6] for a review of some of the work on binary-single scattering.

The contribution of this work to the understanding of binary-single encounters is threefold: First, we present the first (to our knowledge) closed-form statistical approximate solution of the bound, nonhierarchical three-body problem that takes both energy and angular-momentum conservation into account, which complements the solution of the unbound case of Ref. [26]; using the results presented here, one can compute the outcome distribution of each intermediate close approach, not only of the final one. [29] Second, we show how to derive the solution using both detailed balance arguments and ergodic arguments, thereby unifying the two approaches. That they give the same result is hardly surprising since the principle of detailed balance is essentially a statement about phase-space volumes. Third, and most significantly, we elevate our solution of the bound case to a random-walk model of the entire encounter, in which the binary actions and the three-body conserved quantities perform a random walk, whose transition probabilities are related to $f_{\rm bin}(E_{\rm bin}, \mathbf{S}|E, \mathbf{J})$. This random-walk model is extremely general and can incorporate many physical processes beyond the Newtonian gravitational three-body interaction and in addition to it. Here, we provide, as an example, the important case of dissipation due to tidal forces, but our approach can be generalized to any other type of additional perturbations and physical processes, which will be done in future work.

The structure of this paper is as follows: We start with an explicit calculation of $f_{\rm bin}(E_{\rm bin}, \mathbf{S}|E, \mathbf{J})$ using ergodic arguments in Sec. II, while Sec. IV presents a computation of the same function using the principle of detailed balance. In Secs. III and V, we discuss two marginal distributions: the probability function of the ejected star's mass and the marginal energy distribution. Our paper culminates in Sec. VI, where we present the random-walk model in its full generality. In Sec. VII, we compare the results of Secs. III and V to past numerical simulations, and in Sec. VIII, we apply the random-walk model to tidal dissipation while also comparing it with relevant simulations. All of our results mesh well with the simulations.

II. CROSS SECTION VIA PHASE-SPACE INTEGRATION

All the system's phase space is divided into three parts: one where the system is hierarchical but the outer body is unbound; one where the system is still hierarchical but the outer body is bound; and a chaotic region in which all three bodies interact closely. Let us denote them by $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$, respectively. The latter is taken to be the collection of points where the maximum relative separation between any one body and the center of mass of the other two is no more than some value R, which is defined below. Then, if chaos is sufficiently strong to lead to phase-space mixing inside $\mathcal{C}$, the cross section of the outcome of a close approach is simply an integral over $\mathcal{C}$, as assumed in previous studies [6,22–24,26,30]:

$$\sigma = \left(\prod_{i=1}^{3}\int_{\mathcal{C}} \mathrm{d}^3\mathbf{r}_i \mathrm{d}^3\mathbf{p}_i\right)\delta(E-\mathcal{H})\delta\left(\mathbf{J}-\sum_{j=1}^{3}\mathbf{r}_j\times\mathbf{p}_j\right) \times\delta\left(\mathbf{P}_{\rm CM}-\sum_{j=1}^{3}\mathbf{p}_j\right), \quad (1)$$

where $\mathbf{r}_j$ and $\mathbf{p}_j$ are the position vector and the momentum of the jth particle, $\mathcal{H}$ is the Hamiltonian, and E, $\mathbf{P}_{\rm CM}$, and $\mathbf{J}$ are the total energy, momentum, and angular momentum, respectively. This equation is equivalent to the statement that the probability of the system exiting $\mathcal{C}$ through a point w is actually independent of w; we give a heuristic argument for this statement in the Appendix B.

One can calculate this integral by transforming to the coordinate system of a binary and a lone star orbiting its center of mass, which amounts to demanding that the system becomes hierarchical when it leaves $\mathcal{C}$; this requirement is what defines R below. Working in the center-of-mass frame,

$$\sigma = \int \mathrm{d}^3\mathbf{r}_s\mathrm{d}^3\mathbf{p}_s\mathrm{d}^3\mathbf{r}_{\rm bin}\mathrm{d}^3\mathbf{p}_{\rm bin}\delta(E-E_{\rm bin}-E_s)\delta(\mathbf{J}-\mathbf{L}-\mathbf{S}), \quad (2)$$

where $E_{\rm bin} = -Gm_1m_2/2a_{\rm bin}$ is the binary energy, and

$$E_s = \frac{|\mathbf{p}_s|^2}{2\mu_s} - \frac{G(m_1+m_2)m_3}{|\mathbf{r_s}-\mathbf{r}_{\rm cm,bin}|} = \pm\frac{G(m_1+m_2)m_3}{2a_s}, \quad (3)$$

where the sign is negative or positive when the system goes into $\mathcal{A}$ or $\mathcal{B}$, respectively. The spin of the binary is $\mathbf{S}$, and the angular momentum of the third body about the binary is $\mathbf{L}$. Below, the integral over the phase-space of the remaining binary is shortened to $\int_{\rm bin}$.

As there are three stars, with possibly different masses, there are three distinct outcomes, depending on which of the three bodies ends up being ejected. Since the original binary was hard, the final binary must also be hard. Formally, we weigh the binding energies between each of the three pairs, and the one whose binding energy is considerably lower (i.e., more negative) is the remaining binary. Consequently, the total cross section splits into a sum of three cross sections,

$$\sigma = \sigma_1 + \sigma_2 + \sigma_3, \quad (4)$$

where σ_i is the cross section for a breakup with star i ejected as the lone star in the hierarchical system. This split also naturally gives an infrared cutoff for $E_{\rm bin}$, forcing it to be less than $E/3$, for if it were more, one of the other pairs would actually be harder. Below, we calculate such a σ_i; because all three are symmetric, we ignore this complication for now and return to it in Sec. III.

A. $\mathcal{C}$ in angle-action variables

Performing this integral in angle-action variables is the best way to proceed [26], but first it is essential to determine the integration region explicitly: In the initial encounter, the third body must come close enough to the binary such that its pericenter distance is of the same order as R, which implies that

$$\begin{aligned} &a_s(1-e_s) \lesssim R \text{ for an elliptic orbit, or} \\ &a_s(e_s-1) \lesssim R \text{ for a hyperbolic orbit.} \end{aligned} \tag{5}$$

Equation (5) needs to be supplemented by another condition, which says that, while the system becomes hierarchical, the lone star's apoapsis is sufficiently larger than R. This condition is automatically verified for a hyperbolic orbit (where the lone star escapes and the preceding close encounter is the final one), but for an elliptic orbit (where the lone star eventually returns), one must have

$$a_s(1+e_s) \gtrsim \eta R, \tag{6}$$

for $\eta \approx 5$ (other values are acceptable, too).

Let us discuss how R is related to the binary parameters: R is defined as the critical distance between the would-be ejected star to the center of mass of the other two stars, where the problem becomes hierarchical. Conversely, for a hierarchical triple, one might write the Hamiltonian (in the center-of-mass frame) as

$$\mathcal{H} = E_{\rm bin} + E_s - \frac{G}{r_s}\sum_{n=2}^{\infty} M_n \frac{r_{\rm bin}^n}{r_s^n} P_n(\cos\Phi), \tag{7}$$

where

$$M_n = \frac{m_1 m_2 m_3 (m_a^{n-1} - (-m_b)^{n-1})}{m_{\rm bin}^n}, \tag{8}$$

where m_a and m_a are the inner binary's masses, $r_{\rm bin}$ is the distance between them, Φ is the angle between $\mathbf{r}_{\rm bin}$ and $\mathbf{r}_s$, and P_n is the nth Legendre polynomial. We refer the reader to Refs. [6,31] for details and for more references on the hierarchical three-body problem. Now, the triple ceases to be hierarchical if the multipole series becomes as large as the leading-order term in the Hamiltonian, namely, as big as the energy $E = E_{\rm bin} + E_s$. The leading term in this series is the quadrupole (for nonextreme mass ratios). Approximating $r_{\rm bin}^2 P_2(\cos\Phi) \approx a_{\rm bin}^2$, we find

$$R \le \beta\left(\frac{G\mu_{\rm bin}\mu_s M}{m_{\rm bin}|E|}\right)^{1/3} a_{\rm bin}^{2/3}, \tag{9}$$

where β is a constant of order unity. If the masses are unequal, this formula might lead to an overestimate since, then, it is possible that an exchange of a light star with a heavy star would yield $R \gg a_{\rm bin}$, in which case the problem is still quite visibly hierarchical. To account for that, we define R as

$$R = \beta \min\left\{\left(\frac{G\mu_{\rm bin}\mu_s M}{m_{\rm bin}|E|}\right)^{1/3} a_{\rm bin}^{2/3}, a_{\rm bin}\right\}, \tag{10}$$

where, again, $\beta \ge 1$ is of order unity; the cross section σ depends only weakly on β (cf. Ref. [26] for the unbound case). In future work, we will explore the consequences of the existence of this threshold for the stability of hierarchical triples. If the reader is concerned about the crude approximation of the Legendre polynomial, we offer a more refined one and test both in Appendix D, but we recommend that it be read only after the next section.

Both conditions (5) and (6) may be translated into conditions on the angular momentum L (they are, evidently, independent of its direction). Using energy conservation, one may express E_s in terms of $E_{\rm bin}$, whence conditions (5) and (6) may be written as $L \le A(E_{\rm bin})$. Let us start with the apoapsis condition: First, if $\eta R < a_s$, then this condition is fulfilled automatically. If not, it simplifies to

$$L^2 \le \mu_s^2 GM\eta R\left(2 - \frac{\eta R}{a_s}\right). \tag{11}$$

As the left-hand side is non-negative, this implies that $\eta R \le 2a_s$. Therefore, a_s can either be more than ηR or more than $\eta R/2$, so, in both cases, more than $\eta R/2$. Thus, one has a restriction on the energy difference $|E - E_{\rm bin}|$, viz.

$$|E - E_{\rm bin}| \le \frac{Gm_{\rm bin}\mu_s}{\eta R}. \tag{12}$$

The elliptic periapsis condition is satisfied trivially if $a_s < R$, but if $\eta > 2$, this cannot be the case. Otherwise, one has

$$L^2 \le \mu_s^2 GMR\left(2 - \frac{R}{a_s}\right). \tag{13}$$

Which of conditions (11) and (13) is more stringent depends on the masses, and on a_s. For the unbound, hyperbolic case, the periapsis condition is equivalent to

$$L^2 \le \mu_s^2 GMR\left(2 + \frac{R}{a_s}\right); \tag{14}$$

note that, as a consequence of the plus sign inside the brackets here, there is no restriction on a_s in this case, and it may be as small as one wants.

The function $A(E_{\mathrm{bin}})$ is therefore defined as

$$A(E_{\mathrm{bin}})^2 = \mu_s^2 GMR \begin{cases} \min\left\{2-\frac{R}{a_s(E_{\mathrm{bin}})}, \eta\left(2-\frac{\eta R}{a_s(E_{\mathrm{bin}})}\right)\right\} & \text{if } E < E_{\mathrm{bin}} \\ 2+\frac{R}{a_s(E_{\mathrm{bin}})} & \text{otherwise.} \end{cases} \tag{15}$$

The approximation made here—which includes a separation of phase space into the three regions $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$, and the different treatment of each—inadvertently induces some uncertainty. We attempt to gauge it in Appendix C, but we urge the reader to peruse Secs. III–V before turning to this Appendix.

B. Cross-section calculation

Now, one may perform the integration in Eq. (2). The goal is to find the final distribution of binary spin and energy, of the remaining binary, after the system stops being chaotic. Thus, one has to integrate over all of the lone star's phase space, as well as the angle variables of the binary. First, though, note that there are two possibilities for each encounter: either $E_s < 0$, or $E_s > 0$ after the encounter ($E_s = 0$ has zero measure). So, $\sigma = \sigma_{\mathrm{bd}} + \sigma_{\mathrm{ubd}}$, where each cross section pertains to each possible sign of E_s [32]. Note that σ_{ubd} has been calculated by Stone and Leigh [26]. They obtained

$$\sigma_{\mathrm{ubd}} \propto \int \frac{\mathrm{d}E_{\mathrm{bin}}\mathrm{d}J_a\mathrm{d}J_b}{|E-E_{\mathrm{bin}}|^{3/2}|E_{\mathrm{bin}}|^{3/2}|\mathbf{J}-\mathbf{S}|}\left[\sqrt{\frac{R^2}{a_s^2}+\frac{2R}{a_s}+1-e_s^2}-\operatorname{arccosh}\left(\frac{R+a_s}{a_s e_s}\right)\right]. \tag{16}$$

The other cross section, σ_{bd}, may be calculated using Delaunay variables for both the emergent binary and the binary formed by the star and the inner binary. We do so below, which is meaningful given that the whole encounter proceeds as a series of consecutive close approaches, each one having a cross section σ_{bd}. While this case is supported by numerical simulations, as mentioned in the Introduction, we also give a heuristic argument for it in Appendix B. The single close approaches are combined below in Sec. VI. The Delaunay variables are denoted by $(J_a, J_b, J_c, \theta_a, \theta_b, \theta_c)$ and are defined in, e.g., Ref. [1].

Using a superscript or a subscript s to denote variables pertaining to the outer binary, we have

$$L_z = J_a^s, \tag{17}$$

$$L_x = J_b^s \sin\theta_a^s \sin i_s, \tag{18}$$

$$L_y = -J_b^s \cos\theta_a^s \sin i_s, \tag{19}$$

which implies that the angular-momentum-conserving delta function is

$$\delta(\mathbf{J}-\mathbf{S}-\mathbf{L}) = \delta(J_x - S_x - J_b^s \sin\theta_a^s \sin i_s)\delta(J_y - S_y + J_b^s\cos\theta_a^s \sin i_s)\delta(J_z - S_z - J_a^s). \tag{20}$$

This equation, in turn, implies that the angular-momentum integral is independent of J_c^s, modulo the integration domain boundaries, which are

$$|J_a^s| \le J_b^s, \tag{21}$$

$$0 \le J_b^s \le \min\{A(E_{\mathrm{bin}}), J_c^s\} \equiv \alpha. \tag{22}$$

[Please note that, for the unbound case, α is defined simply as $A(E_{\mathrm{bin}})$.] The $\hat{\mathbf{z}}$-axis integral gives

$$\sigma_{\mathrm{bd}} = \int_{\mathrm{bin}} \int \mathrm{d}J_c^s \mathrm{d}J_b^s \mathrm{d}\theta_a^s \mathrm{d}\theta_b^s \mathrm{d}\theta_c^s \delta(\mathrm{energy}) \times \delta\left(J_x - S_x - J_b^s \sin\theta_a^s \sqrt{1 - \frac{(J_z - S_z)^2}{(J_b^s)^2}}\right) \delta\left(J_y - S_y + J_b^s \cos\theta_a^s \sqrt{1 - \frac{(J_z - S_z)^2}{(J_b^s)^2}}\right). \tag{23}$$

As in Ref. [26], we perform a change of variables

$$\begin{pmatrix} J_b^s \\ \theta_a^s \end{pmatrix} \mapsto \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} \sin\theta_a^s \sqrt{(J_b^s)^2 - (J_z - S_z)^2} \\ \cos\theta_a^s \sqrt{(J_b^s)^2 - (J_z - S_z)^2} \end{pmatrix}. \tag{24}$$

The Jacobian of this transformation is

$$\left| \frac{\partial(\theta_a^s, J_b^s)}{\partial(z_1, z_2)} \right| = \frac{1}{\sqrt{z_1^2 + z_2^2 + (J_z - S_z)^2}}. \tag{25}$$

Integrating over z_1, z_2 yields

$$\sigma_{\mathrm{bd}} = \int_{\mathrm{bin} \cap \{|\mathbf{J}-\mathbf{S}| \le \alpha\}} \int \mathrm{d}J_c^s \mathrm{d}\theta_c^s \mathrm{d}\theta_b^s \frac{\delta(E - E_{\mathrm{bin}} + \frac{G^2 M^2 \mu_s^3}{2(J_c^s)^2})}{|\mathbf{J} - \mathbf{S}|}. \tag{26}$$

The integral $\mathrm{d}\theta_b^s$ gives 2π, while the integral $\mathrm{d}\theta_c^s$—over the mean anomaly—gives a multiplicative factor of θ_{max}, which is the maximum mean anomaly the star may have and still stay in $\mathcal{C}$. Condition (5) implies that $\theta_c^s = 0$ is in $\mathcal{C}$, while condition (6) implies that $\theta_c^s = \pi$ is no longer in $\mathcal{C}$. Thus, θ_{max} restricts $|\mathbf{r}_s|$ to $|\mathbf{r}_s| \le R$, such that the integration is carried out in $\mathcal{C}$. The last lone-star integration over J_c^s may now be performed to remove the last delta function and give

$$\sigma_{\mathrm{bd}} = \frac{2\pi G M \mu_s^{3/2}}{\sqrt{8}} \int_{\mathrm{bin} \cap \{|\mathbf{J}-\mathbf{S}| \le \alpha\}} \frac{\theta_{\mathrm{max}}}{|\mathbf{J} - \mathbf{S}| |E_0 - E_{\mathrm{bin}}|^{3/2}}, \tag{27}$$

where

$$\theta_{\mathrm{max}} = \arccos\left(\frac{a_s - R}{e_s a_s}\right) - \sqrt{\frac{2R}{a_s} - \frac{R^2}{a_s^2} - 1 + e_s^2}. \tag{28}$$

One may perform the integration over the binary angles trivially to give an additional factor of $(2\pi)^3$, which yields a cross section

$$\sigma_{\mathrm{bd}} = \frac{(2\pi)^4 G M \mu_s^{3/2}}{\sqrt{8}} \int \frac{\theta_{\mathrm{max}} \mathrm{d}J_c \mathrm{d}J_b \mathrm{d}J_a}{|\mathbf{J} - \mathbf{S}| |E_0 - E_{\mathrm{bin}}|^{3/2}}. \tag{29}$$

Taking the $\hat{\mathbf{z}}$ axis in this integral to be along $\mathbf{J}$ implies that the integration domain is

$$\begin{aligned} &(J_a, J_b) \in \{(J_a, J_b) : 0 \le J^2 + J_b^2 - 2 J J_a \le \alpha^2\} \cap \{|J_a| \le J_b\}, \\ &J_c \in \{E \le E_{\mathrm{bin}}(J_c) \le E_{\mathrm{min}}\}, \end{aligned} \tag{30}$$

where E_{min} is some minimum cutoff on the binary energy. This form is the same as that of σ_{ubd} of Ref. [26], which implies that

$$\sigma = \frac{(2\pi)^4 G M \mu_s^{3/2}}{\sqrt{8}} \int \frac{\theta_{\mathrm{max}} \mathrm{d}J_c \mathrm{d}J_b \mathrm{d}J_a}{|\mathbf{J} - \mathbf{S}| |E_0 - E_{\mathrm{bin}}|^{3/2}}, \tag{31}$$

where now the integration domain is

$$(J_a, J_b) \in \Omega \equiv \{(J_a, J_b) : 0 \le J^2 + J_b^2 - 2JJ_a \le \alpha^2\} \cap \{|J_a| \le J_b\}$$
$$J_c \in \left\{E_{\rm bin}(J_c) \le \frac{E}{3}\right\}, \tag{32}$$

and $\theta_{\max}$ is defined by Eq. (28) for the bound case and by Eq. (16) for the unbound case. Equation (31), together with Eqs. (32), specify the probability that the resultant binary has Delaunay actions J_a, J_b, J_c, given conserved quantities $E, \mathbf{J}$—their probability density function is simply the integrand in Eq. (31). It might be more useful to express Eq. (31) in terms of $E_{\rm bin}$, rather than J_c; the Jacobian for this transformation is simply $\propto E_{\rm bin}^{-3/2}$, which gives a distribution function for the outcome of one binary-single close approach,

$$f_{\rm bin}(E_{\rm bin}, \mathbf{S}|E, \mathbf{J}) \propto \frac{E_{\rm bin}^{-3/2}}{|\mathbf{J}-\mathbf{S}||E_0 - E_{\rm bin}|^{3/2}} \theta_{\max}(E_{\rm bin}, E_0, \mathbf{J}-\mathbf{S}). \tag{33}$$

III. DIFFERENT MASSES

In fact, Eq. (33) is proportional to the probability that a star $s \in \{1,2,3\}$ escapes, leaving a binary with energy $E_{\rm bin}$ and spin $\mathbf{S}$. What remains is the coefficient, which depends on the masses, as in Eq. (31). Therefore, the probability density that star s is ejected after a close interaction and that the remaining binary has energy $E_{\rm bin}$ and spin $\mathbf{S}$ is

$$f_{\rm bin}(E_{\rm bin}, \mathbf{S}, s|E, \mathbf{J}) = N(s)(\mu_s \mu_{\rm bin})^{3/2} \frac{m_{\rm bin} E_{\rm bin}^{-3/2}}{|\mathbf{J}-\mathbf{S}||E_0 - E_{\rm bin}|^{3/2}} \theta_{\max}(E_{\rm bin}, E_0, \mathbf{J}-\mathbf{S}), \tag{34}$$

where $N(s)$ is a normalization constant, which depends on s through the integration over Eq. (32). To obtain this constant, one should change the integrands in σ to ones that are independent of the masses, i.e., from (J_a, J_b, J_c) in Eq. (31) to semimajor axes, eccentricities, and inclinations. The lone star's angular momentum and energy simply contribute a factor of $\mu_s^{-5/2}$, and the measure contributes an additional factor of $\mu_{\rm bin}^3 m_{\rm bin}^{3/2}$. Thus, up to dimensionless quantities,

$$N(s)\mu_s^{3/2} \propto \frac{\mu_{\rm bin}^3 m_{\rm bin}^{3/2}}{\mu_s}, \tag{35}$$

where, now, the proportionality constant is independent of the identity of the ejected star (but may still depend on the total mass or on conserved quantities). One immediate prediction of Eq. (35) is that, if one of the masses is much smaller than the other two, then the probability that each close interaction ends with the lighter one being shot out, rather than one of the others, dominates. By dimensional analysis, the probability that mass m escapes is therefore

$$P(m) \approx \frac{m_a^4 m_b^4}{(m_a + m_b)^{5/2}\left[m_a^4\left(\frac{m_b^4}{(m_a+m_b)^{5/2}} + \frac{m^4}{(m_a+m)^{5/2}}\right) + \frac{m^4 m_b^4}{(m_b+m)^{5/2}}\right]}. \tag{36}$$

We emphasize that Eq. (36) is an approximation, and for accurate results, one should integrate Eq. (34) over remnant binary energies or angular momenta or the marginal energy distribution, Eq. (48) below, over the allowed energies [33].

One can also compute the exchange cross section, given by

$$\sigma(\text{exchange}) = \sigma_{\rm tot} \times P(\text{exchange}) = \frac{2\pi G M a_0}{v_0^2} P(\text{exchange}), \tag{37}$$

as the total cross section is, of course,

$$\sigma_{\rm tot} = \frac{2\pi G M a_0}{v_0^2}, \tag{38}$$

where a_0 is the initial binary semimajor axis, and v_0 is the perturber's initial velocity (see, e.g., Heggie and Hut [2]).

IV. DETAILED BALANCE

Let us try to obtain Eq. (31) by an easier means. One might, for instance, assume that the triple is part of a globular cluster that is in thermal equilibrium and that contains binaries, single stars, and triples. Of course, the desired cross section σ does not depend on whether there exists such a cluster or on whether this hypothetical cluster is indeed in thermal equilibrium. Making those auxiliary assumptions would simplify the calculation because then we could use the principle of detailed balance (cf. Refs. [2,5,15]).

If the cluster is in thermal equilibrium, then the numbers of binaries with energy $E_{\rm bin}$ and stars with energy E_s must be constant. This number density is proportional (in the canonical ensemble, neglecting collisions and stellar evolution) to $\exp(-\beta^*\mathcal{H})$, where $\beta^* = 1/(k_B T)$. Thus, one may deduce that the rates at which encounters between stars and binaries transfer these systems from one energy to another, and those of the reverse process, have to be equal. This is the principle of detailed balance.

Let $\Gamma(E_{\rm bin}, \mathbf{S}, \mathbf{p} \to E_0, \mathbf{J})\mathrm{d}E_0\mathrm{d}\mathbf{J}\mathrm{d}E_{\rm bin}\mathrm{d}J_a\mathrm{d}J_b$ be the differential rate at which binaries and single stars combine to form bound triples with energy E_0 and total angular momentum $\mathbf{J}$, and likewise, let $\Gamma(E_0, \mathbf{J} \to E_{\rm bin}, \mathbf{S})\mathrm{d}E_0\mathrm{d}\mathbf{J}\mathrm{d}E_{\rm bin}\mathrm{d}J_a\mathrm{d}J_b$ be the rate of disintegration of such triples. Let $n_{\rm triple}(E_0, \mathbf{J})$ denote the number density of such triples, and let $n_{\rm bin}(E_{\rm bin}, \mathbf{S})$ and $n(\mathbf{p}_3)$ denote the phase-space densities of binaries and single stars, respectively. Detailed balance amounts to the requirement that the number of bound triples remain constants, i.e., that the rate of the forward and backward reactions, weighed by the relevant densities, cancel each other out. Symbolically, in barycentric coordinates,

$$n_{\rm triple}(E_0, \mathbf{J})\Gamma(E_0, \mathbf{J} \to E_{\rm bin}, \mathbf{S}) = \int \mathrm{d}^3\mathbf{R}_{\rm cm}\mathrm{d}^3\mathbf{P}\mathrm{d}^3\mathbf{p}\delta(\mathbf{R}_{\rm cm})\delta(\mathbf{p}_{\rm bin} + \mathbf{p}_3) \times n(\mathbf{p}_3)n_{\rm bin}(E_{\rm bin}, \mathbf{S})\Gamma(E_{\rm bin}, \mathbf{S}, \mathbf{p} \to E_0, \mathbf{J}), \tag{39}$$

where $\mathbf{p}$ is the momentum of the relative motion between the third star and the binary center of mass. (See, e.g., Ref. [15] for a derivation of a similar expression for the reaction rate.)

The advantage of Eq. (39) is that the rate of formation of bound triples may be simpler to compute; one could approximate it as

$$\begin{aligned} n_{\rm bin}(E_{\rm bin}, \mathbf{S})\Gamma(E_{\rm bin}, \mathbf{S} \to E_0, \mathbf{J}) &= \int \mathrm{d}^3\mathbf{p} n(\mathbf{p}_3)n_{\rm bin}(E_{\rm bin}, \mathbf{S})\Gamma(E_{\rm bin}, \mathbf{S}, \mathbf{p} \to E_0, \mathbf{J}) \\ &= \int \mathrm{d}^3\mathbf{p} \iint_{D(v)} \mathrm{d}^2\mathbf{b}v\delta(E_0 - E_{\rm bin} - E_s)\delta(\mathbf{J} - \mathbf{S} - \mu_s bv\hat{\phi})n(\mathbf{p}_3)n_{\rm bin}(E_{\rm bin}, \mathbf{S}), \end{aligned} \tag{40}$$

where $D(v)$ is a disc in the $\hat{\mathbf{x}}$-$\hat{\mathbf{y}}$ plane at infinity, with radius $a_{\rm bin}\sqrt{1 + (GM/\mu_s v^2 a_{\rm bin})}$ (see, e.g., Ref. [15]). [34] The direction of the $\hat{\mathbf{z}}$ axis is chosen so that it points along $\mathbf{v}$ (which is assumed not to be away from the binary, for then there would be no encounter). The integral over $\mathbf{b}$, including the angular-momentum delta function, gives

$$\delta(\theta_{\mathbf{J}-\mathbf{S}} - \pi/2) \times \begin{cases} \frac{1}{\mu_s^2 v^2 |\mathbf{J}-\mathbf{S}|} & \text{if } (\mathbf{J} - \mathbf{S})^2 < A^2 \\ 0 & \text{otherwise.} \end{cases} \tag{41}$$

The remaining delta function should not really be there—it is just a mathematical artifact, which came from the way we defined the axes, so it is safe to omit it. Alternatively, one could justify its omission by integrating over all possible orientations of the axes: This delta function then picks out the orientation where the $\hat{\mathbf{z}}$ axis is perpendicular to $\mathbf{J} - \mathbf{S}$; such an integration is, in turn, justified by the fact that a choice of axis is meaningful only for the right-hand side of Eq. (39) and not for the left-hand side. The factor of $|\mathbf{J} - \mathbf{S}|$ in the denominator comes from expressing the angular-momentum delta function in spherical coordinates, which turns the three-dimensional Dirac delta function $\delta(\mathbf{J} - \mathbf{S} - \mathbf{L})$ (recall that $\mu_s bv\hat{\phi} = \mathbf{L}$) into a product of three 1D delta functions—one for the magnitudes of $\mathbf{J} - \mathbf{S}$ and $\mathbf{L}$, one for one angle, and one for the other angle—divided by the appropriate Jacobian $|\mathbf{L}|^2 \sin\theta_{\mathbf{J}-\mathbf{S}}$. Since the other delta function sets $\theta = \pi/2$, we are left with $|\mathbf{L}|^2$ in the denominator, and in the numerator, two angular delta functions are multiplied by $\delta(|\mathbf{J} - \mathbf{S}| - |\mathbf{L}|)$. We now perform the integral $\mathrm{d}^2\mathbf{b}$ in polar coordinates b and φ, by writing $|\mathbf{L}| = \mu_s bv$ and changing variables from $|\mathbf{L}|$ to b. As $\mathrm{d}^2\mathbf{b} = b\mathrm{d}b\mathrm{d}\varphi$, one b cancels one power of $|\mathbf{L}|$ from the denominator, and the integral $\mathrm{d}b$ removes the absolute-value delta function, replacing the other $|\mathbf{L}|$ with $|\mathbf{J} - \mathbf{S}|$. The integration $\mathrm{d}\varphi$ removes one of the angular delta functions, and the second one is removed as explained above.

The density $n(\mathbf{p}_3)$ is a Maxwell-Boltzmann distribution, which is proportional to $\rho m_3^{-3/2}\exp(-\beta^* E_s)$. After performing the integral over $\mathbf{p} = \mu_s\mathbf{v}$, keeping track of the energy-conserving delta function, one has $\Gamma(E_{\rm bin}, \mathbf{S} \to E_0, \mathbf{J}) \propto (\mu_s^3 m_3^{-3/2}/\mu_s^3|\mathbf{J} - \mathbf{S}|)e^{\beta^*(E_{\rm bin} - E_0)}$. Heggie [5] gives a formula for the density of binaries with energy $E_{\rm bin}$, eccentricity e, and inclination i (see Eq. 2.12 in Ref. [5][]), from which the phase-space density $n_{\rm bin}(E_{\rm bin}, \mathbf{S})$ is determined to be $\propto \rho^2(m_1 m_2)^{-3/2}\mu_{\rm bin}^{3/2} m_{\rm bin} e^{-\beta^* E_{\rm bin}} E_{\rm bin}^{-3/2}$. Hence, by detailed balance,

$$\Gamma(E_0, \mathbf{J} \to E_{\rm bin}, \mathbf{S}) \sim \frac{\rho^3 e^{-\beta^* E_0}(m_1 m_2 m_3)^{-3/2} m_{\rm bin}\mu_{\rm bin}^{3/2}}{n_{\rm triple}(E_0, \mathbf{J}) E_{\rm bin}^{3/2}|\mathbf{J}-\mathbf{S}|}, \tag{42}$$

provided that the condition $(\mathbf{J}-\mathbf{S})^2 < \alpha^2$ is obtained.

How is this rate related to $f_{\rm bin}(E_{\rm bin}, \mathbf{S}|E_0, \mathbf{J})$? By definition, it is the number of disintegrations per unit time; i.e., it is the number of triples in which star s is between θ_c^s and $\theta_c^s + \Omega_c^s \mathrm{d}t$, divided by $\mathrm{d}t$, where $\theta_c^s \in [0, \theta_{\max}]$. In other words,

$$\Gamma(E_0, \mathbf{J} \to E_{\rm bin}, \mathbf{S}) \propto \Omega_c^s \frac{\mathrm{d}f_{\rm bin}(E_{\rm bin}, \mathbf{S}|E_0, \mathbf{J})}{\mathrm{d}\theta_c^s}, \tag{43}$$

on the one hand, and on the other hand, we have Eq. (42). Together, these imply that, upon division, by the lone star's orbital frequency $\Omega_c^s \propto |E_s|^{3/2}/(M\mu_s^{3/2})$ and integration over θ_c^s,

$$f_{\rm bin}(E_{\rm bin}, \mathbf{S}|E_0, \mathbf{J}) \propto \begin{cases} \frac{m_{\rm bin}\theta_{\max}|\mathbf{J}-\mathbf{S}|^{-1}}{|E_s|^{3/2}|E_{\rm bin}|^{3/2}} & \text{if } |\mathbf{J}-\mathbf{S}| \le \alpha \\ 0 & \text{otherwise} \end{cases} \tag{44}$$

up to a function symmetric in all the particle masses, as in Eq. (33). (The expressions $\mu_s\mu_{\rm bin}$, $m_1m_2m_3$, and M are all such symmetric functions.)

V. MARGINAL ENERGY DISTRIBUTION

Let us compute the marginal energy distribution

$$f_{\rm bin}(E_{\rm bin}|E, \mathbf{J}) = \int_\Omega \mathrm{d}J_a \mathrm{d}J_b f_{\rm bin}(E_{\rm bin}, \mathbf{S}|E, \mathbf{J}). \tag{45}$$

For this purpose, let

$$\mathcal{I}(\alpha) = \int_\Omega \frac{\mathrm{d}J_b \mathrm{d}J_a}{|\mathbf{J}-\mathbf{S}|} \frac{\theta_{\max}}{\theta_{\max}(e_s = 1)}, \tag{46}$$

where $\mathcal{I}$ is the integral one must evaluate. The calculation is performed in Appendix A, and the outcome is that $\mathcal{I}$ is well approximated by a power law, proportional to $E_{\rm bin}^{-1/2}$ for low J, but to $E_{\rm bin}^{-1}$ for large values of J.

A consequence of Sec. II A [inequality (12)] is that for $E_{\rm bin} > E_0$—i.e., in the bound case—the final binary energy is constrained to lie close to the total energy. In this neighborhood, $\theta_{\max} \sim |E - E_{\rm bin}|^{3/2}$, which cancels the existing $|E - E_{\rm bin}|^{-3/2}$. Outside this region, $\theta_{\max}$ is approximately constant; thus, one may remove its eccentricity dependence by approximating $e_s \approx 1$, writing

$$\theta_{\max} \approx \theta_{ap}(E_{\rm bin}) \equiv \begin{cases} \arccos\left(1 - \frac{R}{a_s}\right) - \sqrt{2\frac{R}{a_s} - \frac{R^2}{a_s^2}} & \text{bound case} \\ \sqrt{2\frac{R}{a_s} + \frac{R^2}{a_s^2}} - \operatorname{arccosh}\left(1 + \frac{R}{a_s}\right) & \text{unbound.} \end{cases} \tag{47}$$

This implies that the marginal energy distribution is

$$f_{\rm bin}(E_{\rm bin}|E, \mathbf{J}) \propto m_{\rm bin} \begin{cases} \frac{\mathcal{I}[\alpha(E_{\rm bin})]\theta_{ap}(E_{\rm bin})}{|E_{\rm bin}|^{3/2}|E-E_{\rm bin}|^{3/2}} & \text{if } E_{\rm bin} > E, |E - E_{\rm bin}| \le \frac{Gm_{\rm bin}\mu_s}{\eta R} \\ \frac{\mathcal{I}[A(E_{\rm bin})]\theta_{ap}(E_{\rm bin})}{|E_{\rm bin}|^{3/2}|E-E_{\rm bin}|^{3/2}} & E_{\rm bin} \le E. \end{cases} \tag{48}$$

VI. RANDOM-WALK DESCRIPTION

We have now reached the point where we may introduce a random-walk description of the evolution between consecutive close triple approaches. Suppose that during each close approach, the constants of motion $(E, \mathbf{J})$ might change by some amount, according to some probability distribution, which depends on the state of the system at the beginning of the close approach (i.e., at the end of the previous one) because of some additional astrophysical process. We denote this distribution by $f_c(E^k, \mathbf{J}^k|E_{\rm bin}^{k-1}, E^{k-1}, \mathbf{S}^{k-1}, \mathbf{J}^{k-1})$, where E^j, etc., denote quantities at the end of the jth close approach. A definition of such an additional astrophysical process one would like to incorporate in the study of binary-single encounters therefore amounts to providing f_c. Otherwise, $f_c = \delta(E^k - E^{k-1}) \times \delta(\mathbf{J}^k - \mathbf{J}^{k-1})$, by default.

The total energy E, the total angular momentum $\mathbf{J}$, the spin $\mathbf{S}$, and the binary energy $E_{\rm bin}$ thus perform a random walk, where the probabilities for the jth value are dictated by the values of these quantities at the $j-1$th step—this random walk has a one-step memory. For example, this process may describe a tidal interaction between two stars during the encounter (see Sec. VIII below).

The beauty of this description is that now one can use it to find the ultimate binary parameter distribution $P(E_{\rm bin}, \mathbf{S})$, when the single star leaves, never to return. Let $x = (E_{\rm bin}, E, \mathbf{S}, \mathbf{J})$, and let $h(x|x')$ denote the probability of the walker (i.e., the binary + single) moving from x' to x in one step—that is, the probability that it started the close approach at x' and left it at x. Explicitly,

$$h(x|x') = f_{\rm bin}(E_{\rm bin}, \mathbf{S}|E, \mathbf{J}) f_c(E, \mathbf{J}|E', E'_{\rm bin}, \mathbf{S}', \mathbf{J}'). \qquad (49)$$

The ultimate reason we spent so much effort above computing $f_{\rm bin}$ for the bound three-body problem is precisely so that we would know what $h(x|x')$ looks like. The mixing hypothesis ensures that the functional form of the way h depends on the $E_{\rm bin}, \mathbf{S}$ components of x is only through $f_{\rm bin}$ as calculated in Sec. II. In particular, it also allows us to account for spatial cutoffs; e.g., in a dense cluster environment, an ejected but still bound third star, which would have otherwise eventually fallen back into $\mathcal{C}$, would now be met with an external perturbation by other stars, if its separation became comparable to the distance between stars in the cluster. In other words, the environment could potentially dictate an effective binding energy limit, which would be different from the clean case of an isolated interacting triple. Our model can easily incorporate this aspect.

Let us also introduce the following linear, integral operators, acting on a function $\varphi(x)$:

$$(W_{\rm lim}\varphi)(x) = \int_{\{E'_{\rm bin}\geq E'\}} {\rm d}x' h(x|x')\varphi(x'), \qquad (50)$$

$$(W_{\rm unlim}\varphi)(x) = \int_{\{E'_{\rm bin}< E'\}} {\rm d}x' h(x|x')\varphi(x'). \qquad (51)$$

The first describes an encounter that ends with the third star bound, and the second is the final encounter. Suppose we start with the initial probability

$$p_i(x) = N f_{\rm bin}(E_{\rm bin}, \mathbf{S}|E_0, \mathbf{J}_0)\delta(E - E_0)\delta(\mathbf{J} - \mathbf{J}_0), \qquad (52)$$

where E_0 is the initial total energy and $\mathbf{J}_0$ is the initial total angular momentum. Now,

$$W_{\rm lim}(p_i) = N f_{\rm bin}(E_{\rm bin}|E, \mathbf{J}) \int {\rm d}E' {\rm d}^3\mathbf{J}' {\rm d}E'_{\rm bin} {\rm d}^2\mathbf{S}' f_{\rm bin}(E'_{\rm bin}, \mathbf{S}'|E', \mathbf{J}')\delta(E' - E_0)\delta(\mathbf{J}' - \mathbf{J}_0) f_c(E, \mathbf{J}|x') \qquad (53)$$

$$= N f_{\rm bin}(E_{\rm bin}|E, \mathbf{J}) \int {\rm d}E'_{\rm bin} {\rm d}^2\mathbf{S}' f_{\rm bin}(E'_{\rm bin}, \mathbf{S}'|E_0, \mathbf{J}_0) f_c(E, \mathbf{J}|E_0, E'_{\rm bin}, \mathbf{S}', \mathbf{J}_0). \qquad (54)$$

This result means that the only dependence on $E_{\rm bin}, \mathbf{S}$ is outside the integral, inside an $f_{\rm bin}$—just as in Eq. (52). Insofar as the binary actions are concerned, the action of $W_{\rm lim}$ does not alter the functional form of the probability distribution.

Now, suppose that f_c may be expanded as a sum of changes in energy and angular momentum, relative to E_0, $\mathbf{J}_0$, whose probabilities depend on the previous round:

$$f_c(E, \mathbf{J}|E', E'_{\rm bin}, \mathbf{S}', \mathbf{J}') = \int {\rm d}\lambda {\rm d}^3\chi \delta[E - (E_0 - \lambda)]\delta[\mathbf{J} - (\mathbf{J}_0 - \chi)] p_E(\lambda, \chi|E'_{\rm bin}, \mathbf{S}', E_0, \mathbf{J}_0) \qquad (55)$$

(this is the law of total probability in disguise); if this is the case, then

$$W_{\rm lim}(p_i) = N f_{\rm bin}(E_{\rm bin}, \mathbf{S}|E, \mathbf{J}) \int {\rm d}\lambda {\rm d}^3\chi \tilde{p}_E(\lambda, \chi; E_0, \mathbf{J}_0)\delta[E - (E_0 - \lambda)]\delta[\mathbf{J} - (\mathbf{J}_0 - \chi)], \qquad (56)$$

where we have defined

$$\tilde{p}_E(\lambda, \chi; E_0, \mathbf{J}_0) = \int {\rm d}E'_{\rm bin} {\rm d}^2\mathbf{S}' p_E(\lambda, \chi|E'_{\rm bin}, \mathbf{S}', E_0, \mathbf{J}_0). \qquad (57)$$

Equation (56) implies that one encounter—if the initial probability distribution is some constant time $f_{\rm bin}$ and a total-energy–total-angular-momentum delta function—turns this form into a sum of terms of similar structure. This fact helps us find the final distribution of binary energies and spins, when tidal interactions are taken into account, as we do below.

Using the particularly special form of the action of $W_{\rm lim}$ on p_i (the action of $W_{\rm unlim}$ is quite similar), we may determine the final distribution in terms of the initial total energy and angular momentum. This is done in a perturbative manner, assuming that the probability of a nonzero change in these quantities is small. [35] Let $P_n(x|x_0)$ be the probability that the full close interaction ends after exactly

n steps, at x, given that it started out initially at x_0. By the properties of random walks (see, e.g., Hughes [36]),

$$P_n(x) = \int \mathrm{d}x_1 \ldots \mathrm{d}x_{n-1} h(x|x_{n-1}) \cdot \ldots \cdot h(x_1|x_0) p_i(x_0), \tag{58}$$

integrated over

$$\{E^k_{\mathrm{bin}} \geq E^k\}^{n-1}_{k=1}. \tag{59}$$

Therefore, P_n is given by W_{unlim} acting once after $n-1$ actions of W_{lim}, on p_i. The final probability is

$$P(x) = \sum_{n=0}^{\infty} P_n(x). \tag{60}$$

Equation (60) is the ultimate distribution of binary parameters, after a complete binary-single encounter, incorporating the physical process described by f_c, in addition to the classical three-body dynamics. We apply this formalism to include the effects of tides and collisions in Sec. VIII. Equation (60) implies that if the conserved quantities do not change, then $P(x)$ is just f_{bin}, and the entire process is rendered memoryless. We now compare the theoretical predictions made in this paper with results of numerical simulations.

VII. COMPARISON WITH SIMULATIONS—MARGINAL DISTRIBUTIONS

Let us start by comparing some marginal distributions of f_{bin} to numerical simulations, before testing the full random-walk model in Sec. VIII. All numerical integrations in this paper were done using MATLAB's integral functions.

We start by testing the ejected mass probability, which is given by Eq. (36), to two simulations, by Saslaw *et al.* [8] and by Hills [14], in Fig. 1. Please bear in mind that when the perturbing star's mass is much larger than the initial binary members' masses, a_0 in Eq. (37) needs to be modified by another multiplicative factor of $[M/(m_a+m_b)]^{1/3}$, due to an increased effective total cross section, which is what we show in the right panel of Fig. 1.

We also test the predictions of our model by comparing them to the simulation results of Ref. [37], which calculated exchange cross sections for a wide range of masses. In Fig. 2, we plot the predictions obtained by integrating Eq. (48) over the allowed range, as well as those of the approximate equation (36), and those of Appendix A of Ref. [25], for the following situation: The initial binary consists of masses $m_1 = 1\ M_\odot$ and m_2, and the incoming star has mass $m_3 = m_1$; we plot the branching ration, defined by $\mathrm{BR} = [\sigma_{\mathrm{exch}(1)}/\sigma_{\mathrm{exch}}(2)]$, i.e., by the ratio of the exchange cross section for ejecting star 1 and that of ejecting star 2. [38] Reference [37] provides an analytical fit, which is also plotted, as well as data from Ref. [17], for comparison. This fit is based on the entirety of the numerical simulations in Ref. [37], which cover an extensive range of mass ratios. The data from Ref. [37] in this figure are for resonant cross sections, while the fit is, to our understanding, for the total one, which is dominated by the resonant cross section everywhere, especially for large mass ratios.

As one is concerned with exchange cross sections, they decay to zero at sufficiently large impact parameters. The impact parameter serves only to determine the total angular momentum, and as the exchange cross section tends to zero as $J \to \infty$, one can integrate equivalently over $\mathbf{J}$. There is a natural cutoff J_*, which is the maximum angular momentum for which $\mathcal{I}$ does not vanish (for any E_{bin}), minimized over all ejected masses. Above J_*, there is no configuration in which the triple could have been in a nonhierarchical phase before separating (cf. Sec. VIII C below). The reader should bear in mind that if m_2 is considerably larger than m_1, then $\mathrm{BR} \gg 1$ since there is a much higher probability

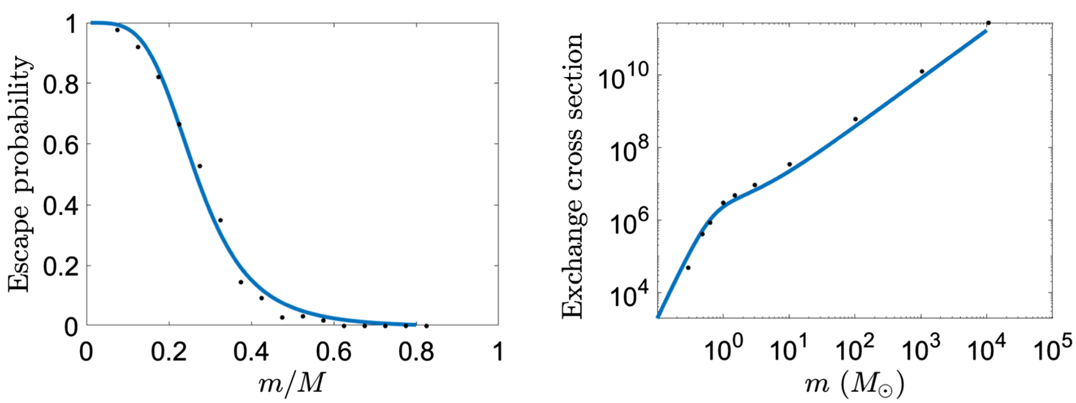

FIG. 1. Left: probability that a star of mass m escapes, where $m_a = m_b = 1\ M_\odot$, compared with data from Ref. [8]. Right: exchange cross section, in units of the geometric cross section, πa_0^2, for binary masses equal to $1\ M_\odot$, and third star mass m, which arrives with initial velocity $v = 0.001 v_{\mathrm{orb}}$, compared with data from Ref. [14].

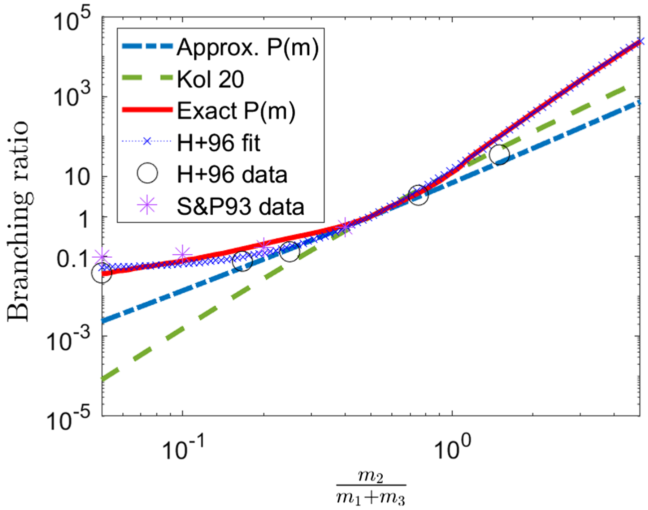

FIG. 2. Predictions of Eq. (36) (sea blue, dash-dotted line), Eq. (48) (red), integrated over the relevant range, and Eq. (A.16) of Kol [25] (green dashed line), compared with the semianalytical fit of Heggie *et al.* [37] (blue, dotted with crosses) and numerical simulation data from Heggie *et al.* [37] (black circles) and Sigurdsson and Phinney [17] (purple asterisks). The initial binary masses are $m_1 = 1\ M_\odot$ and m_2, and the incoming star's mass, $m_3 = m_1$. Its velocity is $v_0 = 0.1 v_c$ (where v_c is defined in the caption of Fig. 3) to ensure that the binary is hard. The y axis shows the branching ratio of the cross section for ejecting m_1 relative to the cross section for ejecting m_2. There is excellent agreement with Eq. (48).

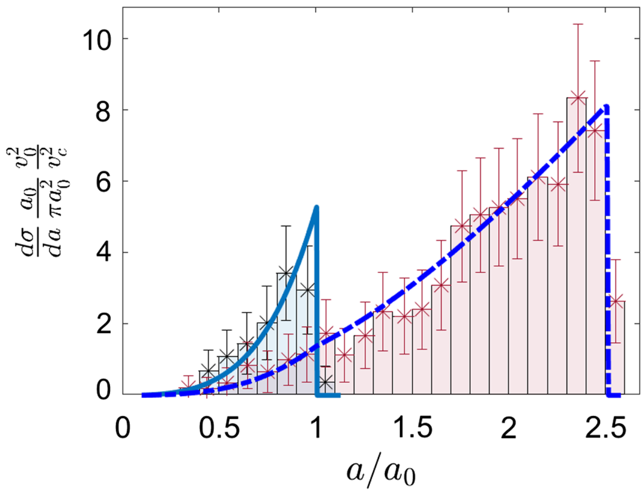

FIG. 3. Comparison between Eq. (48) and the numerical simulations of Sigurdsson and Phinney [see Fig. 2(b) in Ref. [17]]. The initial semimajor axis is $a_0 = 0.1$ AU, and the initial eccentricity is $e_0 = 0$; the initial binary masses are $m_1 = 1.4\,M_\odot$, $m_2 = 0.56\ M_\odot$, and the incoming third star's mass is $m_3 = 1.4\ M_\odot$. Its initial velocity v_0 is uniformly distributed between $0.05 v_c$ and $0.15 v_c$, where $v_c = \sqrt{GM\mu_{\rm bin}/(m_3 a_0)}$. The impact parameter is uniformly distributed in a disc $D(v_0)$ at infinity, whose radius is $b_{\max}(v_0) = a_0[4v_c/v_0 + 0.6(1+e_0)]$. All these parameters were chosen to match those of Ref. [17]. We use $\beta = 1.5$, but the results are insensitive to β. The straight line and the data marked by blue bars pertain to the exchange $(1,2)+(3) \to (1)+(2,3)$, while the dashed-dotted line and the data marked by purple bars are for the process $(1,2)+(3) \to (1,3)+(2)$, where the lighter star is ejected. Error bars correspond to 3σ statistical (Poisson) errors, arising from a total integration number of 4000 (see Fig. 9 in Ref. [17]). As expected by Sigurdsson and Phinney [17], the cutoff at large values of a is faster than exponential, occurring almost instantaneously. Data for flybys is not shown here, as in Sigurdsson and Phinney [17], it is dominated by adiabatic flybys, and the effect of resonant scattering is hard to disentangle from it (Observe that as the masses are different, the factor of $m_{\rm bin}$ in Eq. (31) must be incorporated as well. Thus, the normalization of Eq. (48) is the sum, over all three possible final states, of the integrals of $f_{\rm bin}(E_{\rm bin}|E, \mathbf{J})$ over $E_{\rm bin}$. Also note that the graphs shown here are normalized such that the cross sections (integrated over a) match the values of Sigurdsson and Phinney (see Table 3B in Ref. [17]). This is necessary, as the total cross section we calculate only takes close encounters into account, while the total cross section of Sigurdsson and Phinney [17] also includes weak interactions).

to eject the light particle; likewise, for $m_2 \ll m_1, m_3$, BR should decay to zero. All theoretical predictions plotted in Fig. 2 satisfy these limits, but only the exact prediction of Eq. (48) meshes well with the data and with the semianalytical fit.

Next, we show the semimajor axis distribution. Reference [26] already obtained a good agreement between numerical simulations and the unbound cross section, which has a form similar to Eq. (31). To check whether the bound one is also correct, we compare $f_{\rm bin}$ to the numerical results of Sigurdsson and Phinney [17], who imposed an energy cutoff on the ejected star. As mentioned above, in that work, even some encounters with the lone star ejected with negative energy were deemed to be concluded, if its semimajor axis was large enough. This result was meant to mimic the environmental effect of the globular cluster, where the triple resides. Clearly, once a single star is sufficiently far from the binary, it feels the cluster's potential more strongly and ceases to be bound to the binary, even if its energy is negative. This environmental cutoff implies that one has to use both $\sigma_{\rm bd}$ and $\sigma_{\rm ubd}$ to match the numerical results of Ref. [17].

We do so by modifying the marginal energy distribution in Eq. (48) to account for the external cutoff criterion. Explicitly, Sigurdsson and Phinney [17] took a cutoff of $a_s(1+e_s) = 960 a_0$, where a_0 is the initial semimajor axis of the binary. When the apoapsis was larger than this value, they considered the third body to be unbound from the binary. This may be incorporated into $f_{\rm bin}$ simply by modifying the apoapsis criterion in Eq. (5) accordingly. The result is compared with their simulation results in Fig. 3.

As one can tell from Figs. 1–3, the function $f_{\rm bin}$ calculated above agrees well with simulation results. We now proceed to test the random-walk model in more detail in the next section.

VIII. COMPARISON WITH SIMULATIONS—TIDES

To test the random-walk model described in Sec. VI, we assume that dissipation is caused by tides, and we compare with the extensive three-body simulations conducted by Samsing *et al.* [28], which include both tidal forces and relativistic corrections. They investigated many cases, but we choose the equal mass case $m = 1.2\,M_\odot$; the initial binary is comprised of a white dwarf and a compact object (i.e., point particle), and the incoming lone star is also a point particle. The white dwarf's radius is $r_* = 0.006 R_\odot$. The initial orbital separation is a_0, and the initial speed of the third body is $v_0 = 10\ \mathrm{km\,s^{-1}}$. Its origin is sampled uniformly from a disc $D(v_0)$ whose radius is [19]

$$b_{\max} = \frac{a_0}{2}\sqrt{1+\frac{4GM}{a_0 v_0^2}}. \tag{61}$$

Before proceeding, let us note that Eq. (56), together with the form of f_{bin}, suffices to explain the main features of numerical simulations; inserting Eq. (56) into Eq. (60) implies that

$$P(x) = f_{\mathrm{bin}}(E_{\mathrm{bin}}, \mathbf{S}|E, \mathbf{J}) \int \mathrm{d}\tilde{\lambda}\, \tilde{p}(\tilde{\lambda}|E_0)\delta[E-(E_0-\tilde{\lambda})], \tag{62}$$

where the upper $\sim$ signs indicate that the energy shifts and their associated probabilities may be different from those obtained in a single action of W_{lim}, but the structure is still the same. The final energy is "traced out," while the initial total energy is fixed.

Below, we start by describing the tidal model we adopt in this paper; next, we compute $P(x)$ perturbatively; and then, we compute the cross section for a collision and a tidal inspiral. The latter event is an inspiral of two stars into one another due to orbital energy loss to tides. We back our analysis up by comparing its results to the numerical simulations of Ref. [28].

A. Tidal model

We adopt the tidal model of Ref. [39]. We further deem any energy that goes into the tidal oscillations of the white dwarf as lost from the system (that is, the timescale on which it might return to orbital energy is much larger than the relevant dynamical timescales), and we approximate the total angular momentum as fixed. [40] We take a tidal dissipation event in a single close approach to occur with probability

$$p_{\mathrm{tide}}(a_{\mathrm{bin}}, y) = \frac{12 a_0 y r_*}{a_{\mathrm{bin}}^2}, \tag{63}$$

where $1 < y$ is a free parameter, which describes, roughly, the maximum separation between two bodies that would engender sizable tidal effects. Equation (64) is strictly correct in the limit where $yr_* \ll a_{\mathrm{bin}}$ [27], which is the limit we consider here; at larger yr_*, it has to be modified [17,27]. It originates from the following reasoning: Let $u < y$. For $ur_* \ll a_{\mathrm{bin}}$, the cross section for star 1 (which we choose to be the white dwarf) to come within a distance ur_* of one of the other two stars is approximately described by a two-body interaction with either one of them. Gravitational focusing thus implies that the cross section for this event is $4\pi G m u r_*/v^2$, where v^2 is the "initial" velocity of star 1. As the tidal interaction occurs during the chaotic three-body-close-interaction phase, v should be, roughly, given by the virial speed, multiplied by $\sqrt{2}$ because it is a relative velocity, i.e., $v^2 \approx \frac{4}{3}(|E|/m)$, with $E \approx -(Gm^2/2a_0)$, since the original binary was hard. The cross section should be divided by the total area available for star 1, which is approximately πR^2, and multiplied by 2 to account for the two possible partners 2 and 3. Thus,

$$p_{\mathrm{tide}}(a_{\mathrm{bin}}, u) = 2 \times \frac{4\pi G m u r_*}{\pi R^2}\frac{3a_0}{2Gm} = \frac{12 a_0 u r_*}{R^2}, \tag{64}$$

as in Eq. (63). Therefore, $p_{\mathrm{tide}}(a_{\mathrm{bin}}, u)$ is the probability of star 1 coming within ur_* from star 2 or star 3. The probability density function is

$$\frac{\mathrm{d}p_{\mathrm{tide}}}{\mathrm{d}u} = \begin{cases} \frac{12a_0 r_*}{R^2} & \text{if } 0 < u < \frac{R^2}{12 a_0 r_*} \\ 0 & \text{otherwise.} \end{cases} \tag{65}$$

If such an event does occur [i.e., if star 1 arrives at $(u + \mathrm{d}u)r_*$ from either of its companions, but not below ur_*], the energy loss is given by [see Ref. [39], with $T_2(x) \sim x^{8/3}$]

$$\Delta E(u) = -2.995\frac{Gm^2}{r_*}u^{-10}. \tag{66}$$

To gauge the error on our computations, we also use the simpler model of Ref. [42], in which

$$\Delta E(u) = -\frac{Gm^2}{r_*}u^{-6}. \tag{67}$$

Needless to say, the technique described in Sec. VI applies to more sophisticated tidal models, too.

B. Perturbative calculation of $P(x)$

Let us define the following functions of the total triple energy E (and, albeit suppressed, total angular momentum $\mathbf{J}$), for an initial total energy E_i (please bear in mind that $\Delta E < 0$):

$$p(E,u)f_{\mathrm{bin}}(E_{\mathrm{bin}},\mathbf{S}|E,\mathbf{J})\delta\{E-[E_i+\Delta E(u)]\} = \int_{\{E'_{\mathrm{bin}}\geq E'\}} \mathrm{d}x' h(x|x') \frac{\mathrm{d}p_{\mathrm{tide}}(E'_{\mathrm{bin}})}{\mathrm{d}u}, \tag{68}$$

$$q(E)f_{\mathrm{bin}}(E_{\mathrm{bin}},\mathbf{S}|E,\mathbf{J})\delta(E-E_i) = \int_{\{E'_{\mathrm{bin}}\geq E'\}} \mathrm{d}x' h(x|x')[1-p_{\mathrm{tide}}(E'_b,y)]. \tag{69}$$

Suppose that $r_* \ll a_{\mathrm{bin}}$. Then, $p_{\mathrm{tide}} \ll 1$, and most of the steps conserve the total binary energy. Then, one can view this process as two random walks—a "small-scale" random walk in E_{bin}, and a larger-scale one in E, on top of it—and derive the equation perturbatively. What this means is that, in expanding P_n, one may truncate the series at some small power of $\varepsilon = (r_*|E_i|/GM\mu_s)$, where E_i is the initial total energy. Such a truncation corresponds to resumming the series of $P(x)$, as a series in powers of ε. Suppose one stops at first order; then,

$$\begin{aligned} P_n(E_{\mathrm{bin}},E|E_{b,i},E_i) = \int_0^y \mathrm{d}u \Bigg\{ W_{\mathrm{unlim}}\Bigg[f(E_{\mathrm{bin}}|E)\Bigg(\delta(E-E_i)\frac{q_{\mathrm{tide}}(a_0)}{y} q(E_i)^{n-1} \\ + \delta[E-E_i+\Delta E(u)] \sum_{k=1}^{n-1} q_{\mathrm{tide}}(a_0) q(E_i)^{k-2} p(E_i,u) q[E_i-\Delta E(u)]^{n-k} \\ + \delta[E-E_i+\Delta E(u)] \frac{\mathrm{d}p_{\mathrm{tide}}(a_0,u)}{\mathrm{d}u} q[E_i-\Delta E(u)]^{n-1} \Bigg)\Bigg]\Bigg\}. \end{aligned} \tag{70}$$

The first line in this equation corresponds to the occurrence of no tidal interactions, the last to a tidal interaction in the very first close approach, and the second to a tidal interaction in another close approach.

The sum over k is a geometric sum and may be computed analytically. Then, using Eq. (60) (summing from $n=1$, as we assume that the first, initial close approach always happens), one may sum over n analytically, too (this sum may be exchanged with the action of W_{unlim} by its linearity). Then, one may act with W_{unlim} to find that the probability distribution for the final binary energy reads

$$\begin{aligned} P(E_b,\mathbf{S},E|E_i,\mathbf{J}) \propto f_{\mathrm{bin}}(E_b,\mathbf{S}|E,\mathbf{J}) \\ \times \left\{ \frac{q_{\mathrm{tide}}(a_0)\delta(E-E_i)}{1-q(E_i)} + \int_0^y \mathrm{d}u \frac{\delta[E-E_i+\Delta E(u)]}{1-q[E_i-\Delta E(u)]} \left[\frac{q_{\mathrm{tide}}(a_0)p(E_i,u)}{1-q(E_i)} + \frac{\mathrm{d}p_{\mathrm{tide}}(a_0,u)}{\mathrm{d}u} \right] \right\} \end{aligned} \tag{71}$$

up to an overall normalization and $O(\varepsilon^2)$ corrections.

C. Inspiral cross section

Samsing *et al.* [28] define the result of an encounter to be deemed an inspiral if $a_{\mathrm{bin}} \leq 6r_*$ and if a collision has not occurred. Therefore, the final possible outcomes are as follows: a collision, an inspiral, an exchange, or a flyby. The exchange or flyby cross section may be computed by including a Heaviside function in p, q, p_u, q_u, which ensures that there is no collision and no inspiral. The inspiral or collision cross section is simply the total cross section minus the exchange or flyby cross section.

There is another possible outcome: If $\Delta E(u)$ is large enough relative to E_i, then for large enough J, upon losing energy to tides, there is too much angular momentum for the triple system to interact closely again, which results in none of the conditions in Sec. II A being satisfied. Denote the minimum J for which this occurs by J_*. In this case, the triple becomes hierarchical automatically, and the encounter ends. The inevitable fate of this triple is a tidal inspiral: During each pericenter approach of the inner binary, more energy is lost to tides, until the two stars collide. This process is just another route to a tidal inspiral, which does not require the triple to have ejected the third star.

As in both models $\Delta E(u)$ decays quite fast with u, the dependence on y is very weak, and the cross sections converge for sufficiently large y (see Fig. 4).

One may compute the tidal inspiral cross section as follows: Compute the cross section for cases in which there is no collision and no tidal inspiral, and then subtract the result from the total cross section, which we take, in this section, to be

$$\sigma_{\mathrm{tot}} = \frac{\pi G M a_0}{v_0^2}, \tag{72}$$

and not twice this value, consistent with Ref. [28]. The former is given by first computing the differential cross section for a final periapsis r_p, by integrating Eq. (71) over the disc $D(v_0)$—this is effectively an integration over

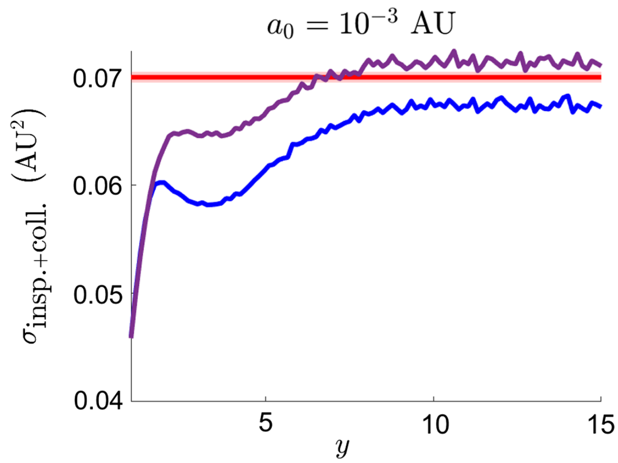

FIG. 4. Inspiral or collision cross section, as a function of y, for $\beta = 1.3$, $a_0 = 10^{-3}$ AU, and $v_0 = 10\ \mathrm{km\,s^{-1}}$. The purple line corresponds to the model of Eq. (67), while the blue line is for Eq. (66); the red line is the numerical result of Samsing *et al.* [28]. As expected, the cross section converges to a constant for large enough y, despite some noise due to numerical integration.

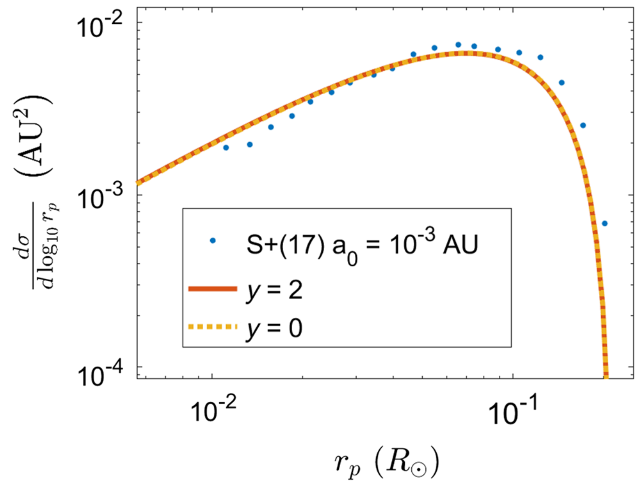

FIG. 5. Differential cross section for obtaining a final pericenter distance r_p in an exchange or flyby, which is the integral over $(E_{\rm bin}, \mathbf{S})$ of Eq. (71), multiplied by the relevant Dirac delta function. Note that y is defined below Eq. (63). This is compared with data from Samsing *et al.* [28] for $a_0 = 10^{-3}$ AU and $v_0 = 10\ \mathrm{km\,s^{-1}}$, and both are normalized to give the same total exchange or fly-by cross section. To reduce computational difficulty here, $\Delta E(u)$ is approximated to be equal to $\Delta E((1+y)/2)$ for $u < y$, and zero otherwise. The $y = 2$ case is indiscernible from the tideless case, $y = 0$, in agreement with Samsing *et al.*'s result that tides do not change the exchange or flyby cross section significantly.

the allowed range of total angular momenta—as well as over the binary actions with a Dirac delta function $\delta[r_p - a_{\rm bin}(1 - e_{\rm bin})]$, all the while enforcing the no-inspiral-and-no-collisions condition both in Eq. (71) and in the definitions of $p(E), q(E), p_u(E), q_u(E)$, by restricting the integration to the domain $\{a'_{\rm bin} \geq 6r_*\}$. This differential cross section is shown in Fig. 5. In this figure, it does not matter if one sets $R = \beta a_{\rm bin}$ for simplicity—this deviation from Eq. (10) does not change the results significantly since all three masses are equal, whence Eq. (10) reduces to $R = \beta \min\{a_0^{1/3} a_{\rm bin}^{2/3}, a_{\rm bin}\}$, and the distribution of $a_{\rm bin}$ is peaked sharply about a_0. The total exchange or flyby cross section is then found by integrating this differential cross section over $r_p \geq r_*$ (to preclude collisions); by the special nature of Eq. (62), this restriction automatically precludes collisions during all close approaches, not just the final one.

Explicitly, let

$$N_{\rm ubd}(E, \mathbf{J}) = \int \mathrm{d}E_{\rm bin} \mathrm{d}^2\mathbf{S} f_{\rm bin}(E_{\rm bin}, \mathbf{S}|E, \mathbf{J}), \tag{73}$$

$$N_{\rm ubd}^{ni}(E, \mathbf{J}) = \int \mathrm{d}E_{\rm bin} \mathrm{d}^2\mathbf{S} f_{\rm bin}(E_{\rm bin}, \mathbf{S}|E, \mathbf{J}) \Theta\left(\frac{G m_{\rm bin} \mu_{\rm bin}}{2 \times 6 r_*} - E_{\rm bin}\right); \tag{74}$$

these quantities, when evaluated at $E = E_i - \Delta E$, are only defined for $J \leq J_*$, so for $J > J_*$, we define them to be zero. Furthermore, define

$$B = \int_{D(v_0)} \mathrm{d}^2\mathbf{b} \left[\frac{\mathrm{d}p_{\rm tide}(a_0)}{\mathrm{d}u} + \frac{q_{\rm tide}(a_0) p(E_i, u)}{1 - q(E_i)}\right] \times \Theta(J(\mathbf{b}) - J_*), \tag{75}$$

where $J(\mathbf{b})$ is the magnitude of the total angular momentum as a function of the position $\mathbf{b}$ of the incoming star on the disc $D(v_0)$. Let us denote

$$U = \int_{D(v_0)} \mathrm{d}^2\mathbf{b} \frac{N_{\rm ubd}^{ni}(E_i, \mathbf{J}) q_{\rm tide}}{1 - q_{ni}(E_i)} + \frac{N_{\rm ubd}^{ni}(E_i - \Delta E, \mathbf{J})}{1 - q_{ni}(E_i - \Delta E)} \left(p_{{\rm tide},nc}(a_0) + \frac{q_{\rm tide}(a_0) p_{ni}(E_i)}{1 - q_{ni}(E_i)}\right), \tag{76}$$

$$V = \int_{D(v_0)} \mathrm{d}^2\mathbf{b} \frac{N_{\mathrm{ubd}}(E_i, \mathbf{J}) q_{\mathrm{tide}}}{1 - q(E_i)} + \frac{N_{\mathrm{ubd}}(E_i - \Delta E, \mathbf{J})}{1 - q(E_i - \Delta E)} \left(p_{\mathrm{tide}}(a_0) + \frac{q_{\mathrm{tide}}(a_0) p(E_i)}{1 - q(E_i)} \right), \tag{77}$$

where $p_{\mathrm{tide},nc}(a_{\mathrm{bin}}) = 12 a_0 (y-1) r_* / a_{\mathrm{bin}}^2$, to preclude collisions; $p_{ni}(E)$ is the analogue of $p(E)$ but with the same Heaviside theta function inserted as in the definition of N_{ubd}^{ni} and with $p_{\mathrm{tide},nc}(a_{\mathrm{bin}})$ used instead of p_{tide}; and $q_{ni}(E)$ is the same as $q(E)$ but also with the aforementioned Heaviside function. It follows from Eq. (71) that the inspiral or collision cross section is given by

$$\sigma_{\mathrm{insp+coll}} = \sigma_{\mathrm{tot}} \left(1 - \frac{U}{V + B} \right). \tag{78}$$

We find, from Eq. (78), that for $\beta \approx 1.3$, $\sigma_{\mathrm{insp+coll}}$ agrees with the findings of Ref. [28] for both $a_0 = 10^{-3}$ and 10^{-2} AU. The former improves upon the analytic estimate of 0.155 AU^2 considerably. This cross section depends on β, as shown in Fig. 6. The fact that $\beta = 1.3$ fits both cases, with initial semimajor axes differing by an order of magnitude, implies that indeed Eq. (78) is an adequate model to describe a binary-single encounter with tides.

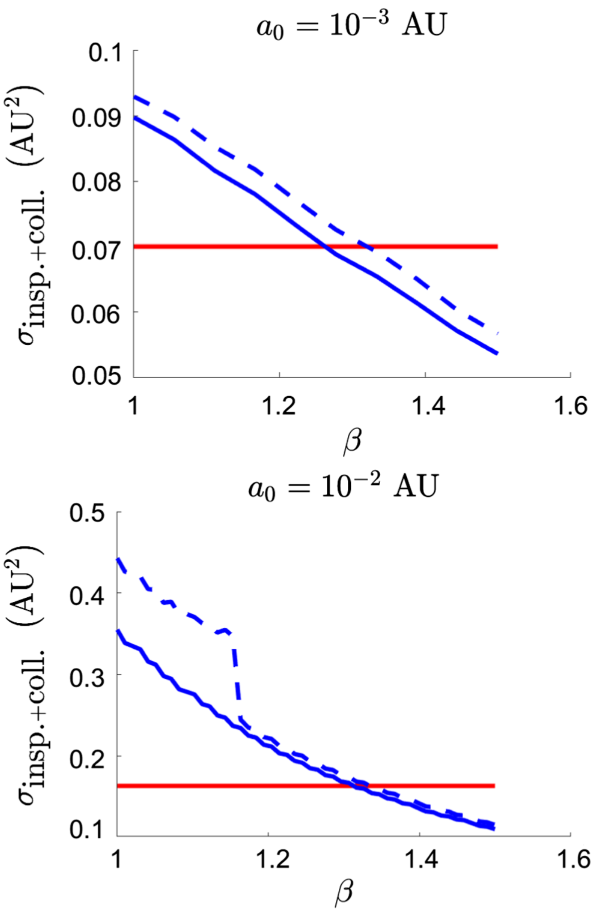

FIG. 6. Inspiral or collision cross section for the case described in the text, for different values of β [defined in Eq. (10)]. The red line is the result of the simulations of Samsing *et al.* [28]; the blue line shows our result, using the tidal model of Eq. (66), while the dashed blue line shows the cross section if the model in Eq. (67) is employed. There is some noise due to numerical integration. Top panel: $a_0 = 10^{-3}$ AU. Bottom panel: $a_0 = 10^{-2}$ AU. In both cases, $v_0 = 10\ \mathrm{km\,s^{-1}}$, and y is chosen to be very large.

IX. DISCUSSION AND SUMMARY

In this paper, we introduced a random-walk model for binary-single encounters in globular clusters. An encounter is viewed as a sequence of chaotic, close triple approaches, interspersed with hierarchical phases. The orbital parameters of the binary and the single star, as well as the triple system's constants of motion, perform a random walk: Each step of the walk corresponds to one close approach and its subsequent hierarchical phase. We calculated the transition probabilities between steps of the walk, both in the Newtonian, point-mass approximation [in which the walk is memoryless, and the final outcome distribution is simply given by the formula of Ref. [26], while the transition probabilities between intermediate steps are given by Eq. (33)], and in the case where there is some dissipative process involved, when Eq. (62) holds.

We have shown that this model reproduces numerical results well, as we exemplified for aspects such as the semimajor axis distribution, the escaper's mass distribution, and the final periapsis distribution. Including tides and collisions, our predictions match the inspiral or collision cross section measured by numerical simulations, which validates the prescription of R and the solution to the bound problem; besides, including a tide allowed us to perform a nontrivial test on the extra step involved in elevating the statistical solution of the scattering problem to the random-walk model, which it passed.

In some cases, the probability distribution can be computed completely analytically, while in others, it only involves a relatively simple calculation of a few integrals. Note that the formula for the escaper's mass distribution is valid in every intermediate step of the encounter. In conjunction with the random-walk model, it implies, *inter alia*, that if one of the stars is considerably lighter than the other two, then it will be the one ejected after each close approach—not just the final one. The only free parameters in our analysis that do not influence the results of the dissipation-free problem significantly are β and η; the former was found to be equal to 1.3—a single value that agrees with both cases considered in Sec. VIII—and the latter does not change the results much in either case. The fact that one value of β fits both parameters further supports the random-walk approach.

The random-walk model described here may be used to address a plethora of astrophysical phenomena using the analytical, statistical model we described in this paper: Apart from incorporating tidal interactions as was done here, one could also include gravitational-wave dissipation and the effects of stellar evolution. One could also investigate external effects, for example, the tidal influence of an external gravitational potential. The main change would be a modification of the largest value $a_s(1+e_s)$ that corresponds to a bound binary. Such cutoffs are important in globular clusters, which are the main places one expects to find significant rates of binary-single encounters. This investigation is made possible by our statistical approximate solution of the bound, nonhierarchical, three-body problem in Eq. (33). This approach also allows one to calculate the distribution of the number of consecutive close approaches before the disruption of the triple and hence the distribution of timescales of temporary captures, which could be relevant for various astrophysical capture processes by gravitating stars and planets. Other applications include—but are not limited to—binary-single encounters in nuclear star clusters around supermassive black holes at the centers of galaxies (or, equivalently, binary-single encounters of planets, dwarf planets, moons, or asteroids in the solar system), where the Hill radius effectively provides a limiting separation (as mentioned above) during consecutive encounters, as well as the velocity distribution of fast, runaway stars due to ejections through binary-single encounters, which will be treated in a separate paper.

Let us present an algorithm for how this would be done for *any* given astrophysical process involved in three-body physics. Suppose that in addition to Newtonian, point-particle motion, one wishes to incorporate another physical phenomenon, say, gravitational-wave emission (which we will address in future work). One would have to be able to calculate how this phenomenon changes the total energy and angular momentum $E, \mathbf{J}$ during a single close approach and subsequent hierarchical phase. With these data, one could compute $f_c(E,\mathbf{J}|E',E'_{\mathrm{bin}},\mathbf{S}',\mathbf{J}')$ from Sec. VI. Using the random-walk model, we immediately get the transition probabilities $h(x|x')$, and then all that remains is to sum up the series (60) to obtain the final probability—the probability density function of final binary parameters, given the initial total energy and angular momentum. If there is a small parameter, e.g., if the probability for a nonzero change in the constants of motion is small, one can resum Eq. (60) and expand in the small parameter. We summarize as follows:

(1) Compute the single-step (one close approach and one hierarchical phase) probabilities for a given change in total energy and angular momentum, $f_c(E,\mathbf{J}|E',E'_{\mathrm{bin}},\mathbf{S}',\mathbf{J}')$.
(2) Compute the transition probabilities $h(x|x')$ as in Sec. VI.
(3) Compute Eq. (60) to obtain the final distribution.
(4) If $h(x|x')$ is small $[O(\varepsilon)]$ for $(E',\mathbf{J}')\neq(E,\mathbf{J})$, resum Eq. (60) and expand in ε to the desired accuracy.

Here, we have brought the endeavor of statistical modeling of binary-single encounters closer to completion. By viewing them as concatenations of close triple approaches, we were able to model, statistically, the *bound* nonhierarchical three-body problem and to use the solution to model the entire encounter as a random walk. This model has the potential to facilitate simulations of globular clusters significantly. Instead of having to resolve binary-single encounters individually using a high-resolution few-body code, one could simply implement the analytical random-walk model as a probabilistic solution of these encounters, with the additional possibility of incorporating any astrophysical process one chooses. By the law of large numbers, if the number of encounters per cluster is sufficiently large, the few-body resolutions of binary-single encounters will be rendered unnecessary. We hope that this gain in speed will enable astrophysicists to study many more phenomena in clusters and planetary systems, and in the field, in much better detail.

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APPENDIX A: MARGINAL ENERGY DISTRIBUTION—EVALUATION OF THE ANGULAR-MOMENTUM INTEGRAL

Let us compute $\mathcal{I}(t)$, as defined in Eq. (46), first approximating $\theta_{\max}=\theta_{ap}$. There are two possibilities for Ω: either $J>t$ or $J\le t$. We denote $\mathcal{I}$ as $\mathcal{I}_+$ and $\mathcal{I}_-$, respectively, in these cases. Writing $J_b=S$, $J_a=Sx$, with $|x|\le 1$, one finds that, in both cases, Ω is a simple domain with respect to x, while S is integrated from 0 to $S_{\max}=\min\{J+t,J_c\}$. Note that Ω is shown in Fig. 7.

For the case $t\le J$, $\mathcal{I}(t)=\mathcal{I}_+(t)$, which is defined as

$$\mathcal{I}_+(t)=\frac{1}{J}\int_{J-t}^{S_{\max}}(t-|J-S|)\mathrm{d}S. \tag{A1}$$

Evaluating the last integral yields

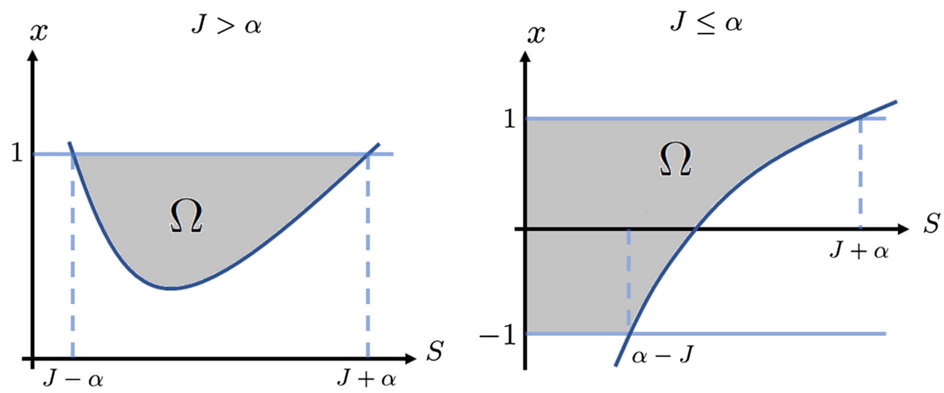

FIG. 7. Two options for Ω, with $t = \alpha$.

$$\mathcal{I}_{+}(t) = \frac{1}{J}\begin{cases} \frac{1}{2}(2t(S_{\max} - J) - (J - S_{\max})^2 + t^2) & \text{if } J < S_{\max} \wedge J - t \leq S_{\max} \\ \frac{1}{2}(-J + S_{\max} + t)^2 & \text{if } J \geq S_{\max} \wedge J - t \leq S_{\max} \\ 0 & \text{otherwise.} \end{cases} \tag{A2}$$

For the case $t > J$, $\mathcal{I}_{-} = \mathcal{I}_{-}^{(1)} + \Theta(S_{\max} + J - t)\mathcal{I}_{-}^{(2)}$, where Θ is the Heaviside theta function,

$$\mathcal{I}_{-}^{(1)}(t) = \frac{1}{J}\begin{cases} 2J \min\{t - J, S_{\max}\} - J^2 & \text{if } \min\{t - J, S_{\max}\} \geq J \\ \min\{t - J, S_{\max}\}^2 & \text{otherwise,} \end{cases} \tag{A3}$$

and

$$\mathcal{I}_{-}^{(2)}(t) = \frac{1}{J}\int_{t-J}^{S_{\max}} (t - |J - S|)\mathrm{d}S. \tag{A4}$$

To compute $\mathcal{I}_{-}^{(2)}$, one needs to consider the signs carefully. The result is

$$\mathcal{I}_{-}^{(2)}(t) = \frac{1}{J}\begin{cases} -\frac{(t-J)^2}{2} + (t + J)S_{\max} - \frac{S_{\max}^2}{2} - J^2 & \text{if } t - J \leq J < S_{\max} \\ (t + J)S_{\max} - \frac{S_{\max}^2}{2} - t^2 + J^2 + \frac{(t-J)^2}{2} & \text{if } t > 2J \\ (t - J)S_{\max} + \frac{S_{\max}^2}{2} - \frac{3(t-J)^2}{2} & \text{if } J \geq S_{\max}. \end{cases} \tag{A5}$$

Inserting R from Eq. (10), where $1 \lesssim \beta \lesssim 2$, the exact value of $\mathcal{I}[\alpha(E_{\mathrm{bin}})]$ roughly follows a power law until $J = \alpha$, where it begins to fall sharply (like $\mathcal{I}_{+}$). The dominant contribution to $\mathcal{I}_{-}$ is $\mathcal{I}_{-}^{(1)}$, whence, when $|E_{\mathrm{bin}}| \gg |E|$ but $\alpha > J$, one may approximate $\mathcal{I}$ by $\mathcal{I}_{-}^{(1)}$, which goes like $|E_{\mathrm{bin}}|^{-1}$ for high angular momentum but like $|E_{\mathrm{bin}}|^{-1/2}$ for low angular momentum.

In the unbound case, when $|E_{\mathrm{bin}}| \gg |E|$, one has $a_s \approx a_{\mathrm{bin}} m_s/\mu_{\mathrm{bin}}$, and

$$\frac{A^2}{J_c^2} \approx \left(2 \pm \frac{\beta\mu_{\mathrm{bin}}}{m_s}\right)\frac{\beta m_s \mu_s}{\mu_{\mathrm{bin}}^2}. \tag{A6}$$

An even better approximation would be to take into account the eccentricity dependence of $\theta_{\max}$, when computing $\mathcal{I}$. The angle $\theta_{\max}$ is equal to θ_{ap} for $e_s = 1$, and it vanishes when L^2 saturates the periapsis bound of Sec. II A. Let us define $\xi = R/a_s$ and

$$b^2 = \frac{L^2}{GM\mu_s^2 R}\begin{cases} \frac{1}{2 - R/a_s} & \text{bound case} \\ \frac{1}{2 + R/a_s} & \text{unbound case,} \end{cases} \tag{A7}$$

so that

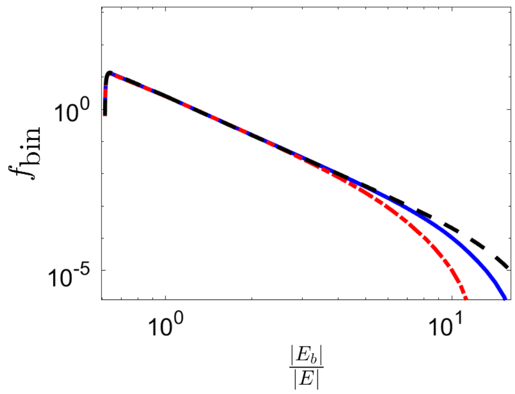

FIG. 8. Plots of the marginal energy distribution (for $E_b = E_{\rm bin}$) in Eq. (48) for $m_1 = m_2 = m_3$ for different values of β: $\beta = 1.5$ (blue line), $\beta = 1$ (red, dash-dotted line), and $\beta = 2$ (black, dashed line). One can see that except for at energies where $f_{\rm bin}$ is minuscule anyway, the three plots coincide.

$$e_s^2 = \begin{cases} 1 - b^2\xi(2-\xi) & \text{bound case} \\ 1 + b^2\xi(2+\xi) & \text{unbound case.} \end{cases} \tag{A8}$$

Then, $\theta_{\max}(b=0) = \theta_{ap}$, while $\theta_{\max}(b=1) = 0$. One could write $\theta_{\max} = \theta_{ap}(\theta_{\max}/\theta_{ap})$ and then approximate the fraction. While at $b=0$, $\xi = 0$, this fraction is unity, it differs from 1 for nonzero b even at $\xi = 0$. So, let us expand $\theta_{\max}/\theta_{ap}$ in powers of ξ; keeping only the leading term, we find

$$\frac{\theta_{\max}}{\theta_{ap}} = \sqrt{1-b^2}(1+2b^2) + \sqrt{1-b^2}O(\xi), \tag{A9}$$

uniformly in b. We keep only the leading term, and, since we have shown above that $\mathcal{I} \approx \mathcal{I}_-^{(1)}$ for most values of $E_{\rm bin}$ (except possibly those where $f_{\rm bin}$ is very small anyway), we only compute a correction to $\mathcal{I}_-^{(1)}$ here, but corrections to $\mathcal{I}_+$ and $\mathcal{I}_-^{(2)}$ may be obtained in a similar manner. The integral we need to compute is

$$\mathcal{I}(t) = \iint_\Omega \frac{S\mathrm{d}S\mathrm{d}x}{L}\frac{\theta_{\max}}{\theta_{ap}} \approx \iint_\Omega \frac{S\mathrm{d}S\mathrm{d}x}{L}\sqrt{1-b^2}(1+2b^2). \tag{A10}$$

Changing variables from x to L gives

$$\mathcal{I}(t) = \frac{1}{J}\iint \mathrm{d}S\mathrm{d}L\sqrt{1-b^2}(1+2b^2). \tag{A11}$$

Fortunately, this integral may be computed analytically in terms of inverse trigonometric functions. Let

$$\phi(u) = \frac{1}{20}\sqrt{1-u^2}[12 + u^2(1+2u^2)] + \frac{3}{4}u\arcsin u, \tag{A12}$$

and let us focus on $\mathcal{I}_-^{(1)}$. Performing the integrations, and keeping in mind the limits (here, S ranges from 0 to $\min\{t-J, S_{\max}\}$ and L goes from $J+S$ to $|J-S|$), yields, for the bound case,

$$\frac{J}{A_p^2}\mathcal{I}_{-,\rm bd}^{(1)} = \phi\left(\frac{J+\min\{t-J,S_{\max}\}}{A_p}\right) - 2\phi(J/A_p) + \begin{cases} \phi\left(\frac{J-\min\{t-J,S_{\max}\}}{A_p}\right) & \text{if } \min\{t-J,S_{\max}\} \le J \\ 2\phi(0) - \phi\left(\frac{\min\{t-J,S_{\max}\}-J}{A_p}\right) & \text{otherwise,} \end{cases} \tag{A13}$$

where $A_p = \mu_s\sqrt{GMR}\sqrt{2-R/a_s}$. For the unbound case,

$$\frac{J}{t^2}\mathcal{I}_{-,\rm ubd}^{(1)} = \phi\left(\frac{J+\min\{t-J,S_{\max}\}}{t}\right) - 2\phi(J/t) + \begin{cases} \phi\left(\frac{J-\min\{t-J,S_{\max}\}}{t}\right) & \text{if } \min\{t-J,S_{\max}\} \le J \\ 2\phi(0) - \phi\left(\frac{\min\{t-J,S_{\max}\}-J}{t}\right) & \text{otherwise.} \end{cases} \tag{A14}$$

This improved, more cumbersome approximation differs significantly from the simpler one, made at the beginning of this section, when the masses are significantly different from each other. Therefore, we use it only for Figs. 2 and 8 since the masses are considerably different or there is a need for a high degree of accuracy; only there does it make a difference.

APPENDIX B: CHAOS AS PHASE-SPACE DIFFUSION

In this Appendix, we endeavor to give a heuristic justification of Eq. (2), i.e., of the mixing assumption inside $\mathcal{C}$. We do so by defining $\mathcal{C}$ as the region in phase space in which the system is completely nonhierarchical.

By energy considerations, the spatial extent of this region must be related to the binary's initial semimajor axis (recall that the binary is hard, so the amount of energy contributed by the incoming star is negligible), so the spatial size of $\mathcal{C}$ must be approximately R, with β of order unity. If the system is nonhierarchical inside $\mathcal{C}$, then, by dimensional analysis, the relevant timescale must also be a function of the energy alone; i.e., it must be the virial timescale $\tau_{\rm vir} = G\sqrt{m_1 m_2 + m_1 m_3 + m_2 m_3} M/(2|E|^{3/2})$. Indeed, Heinämäki *et al.* [43] found that the Lyapunov time λ^{-1} is roughly

$$\frac{1}{27\sqrt{6}}\frac{GM^{5/2}}{|E|^{3/2}} = \frac{\sqrt{6}}{27}\tau_{\rm vir} \tag{B1}$$

for the equal mass case. [44] While this indeed consolidates the assumption that R is given by Eq. (10), with β of order unity, the weak dependence of σ on β in Eq. (31)—only through $\theta_{\rm max}$—implies that one cannot specify the precise value of β from the simulations of Ref. [43], but this could be theoretically performed with a similar simulation with very high resolution. In this Appendix, we normalize the units of time by this timescale and the units of mass by the total mass (recall that all three bodies are taken to have masses of the same order of magnitude), such that both momenta and distances have units of length. We also assume, for simplicity, that all three masses are equal.

Outside $\mathcal{C}$, f simply satisfies a Liouville equation

$$\frac{\partial f}{\partial t} = \{\mathcal{H}, f\}, \tag{B2}$$

where, now, one may write $\mathcal{H}$ in the coordinates of the inner binary, and those of the two-body system formed by the outer body and the center of mass of the inner binary. Both in $\mathcal{A}$ and in $\mathcal{B}$, the Hamiltonian $\mathcal{H}$ admits angle-action variables. Given an initial condition $f_0(\theta, J)$, the solution is

$$f(\theta, J, t) = f_0(\theta - \Omega t, J). \tag{B3}$$

Let us assume that, in $\mathcal{C}$, the motion is practically stochastic. Motion is deterministic throughout the evolution, but the chaotic dependence on initial conditions implies that, practically, it is random on short times (cf. Lichtenberg and Lieberman [45]). What we mean is that if the system is inside a small region $\mathcal{R}$ in $\mathcal{C}$ of size ε at one instant, it may jump, at the following instant, to any place in $\exp(\lambda\delta t)\mathcal{R}$, where δt is the time difference between the two instants, with λ being the Lyapunov exponent. This evolution is assumed to be valid for times that are of the same order as the Lyapunov time λ^{-1} and as long as the system is in $\mathcal{C}$. Suppose that one starts with a phase-space distribution that is uniform on some ε neighborhood of some $w_0 \in \mathcal{C}$, namely, uniform in $\mathcal{R} = B_\varepsilon(w_0) \subseteq \mathcal{C}$; then, after a time t, this phase-space density evolves to a uniform distribution in a sphere (in the metric in which the Lyapunov exponent is calculated) of volume about $V[B_\varepsilon(w_0)]e^{\lambda t}$.

Now, suppose that we wish to start with an initial condition in $\mathcal{C}$ very close to a delta function. Then, as time goes by, f spreads over $\mathcal{C}$, until some parts of it reach $\mathcal{C}$'s boundaries and enter $\mathcal{A}$ or $\mathcal{B}$. The timescale of evolution in $\mathcal{C}$ is roughly the virial timescale of the three-body system, while the timescales for interesting evolution in $\mathcal{A}$ and $\mathcal{B}$ are set by the frequencies Ω of the outer binary. *Per definitionem*, these are much longer than the virial timescale; otherwise the system would not be hierarchical. Thus, those parts of f that have left $\mathcal{C}$ are, from the point of view of the $\mathcal{C}$, stuck at the boundary, so to speak. The distribution continues to spread, until virtually all of f is on $\partial\mathcal{C}$. The proportion that arrives at $\mathcal{A}$ never returns to $\mathcal{C}$, but the rest, being in $\mathcal{B}$, eventually does return to $\mathcal{C}$ and starts all over. This process goes on *ad infinitum* or until all of the probability mass is in $\mathcal{A}$.

Separation of scales thus ensures that the evolution of the system proceeds as a sequence of close three-body, chaotic encounters, and between them, hierarchical phases, until one of the three bodies is ejected. This picture meshes well with simulations [11,12,19], but here we have given it some theoretical credence.

If w_0 is a typical initial condition inside $\mathcal{C}$, then its phase-space distance from the $\partial\mathcal{C}$ is roughly R. According to the evolution described above, the time it would take the system to arrive at $\partial\mathcal{C}$ is then

$$t \sim \frac{1}{\lambda}\ln\left(\frac{R^8}{V[B_\varepsilon(w_0)]}\right) \sim \frac{\ln R - \ln\varepsilon}{\lambda}. \tag{B4}$$

The power of 8 is due to the conserved quantities: The three-body phase space is 18 dimensional, but conservation of linear momentum in the center-of-mass frame, angular momentum, and energy reduces this dimension to 8. Requiring a resolution $\varepsilon/R \ll 1$ implies that the time it takes the system to leave $\mathcal{C}$ is larger than the Lyapunov time, consolidating the assumption of efficient phase-space mixing inside it. The simulations of Ref. [21] demonstrated that the half-life time of chaotic three-body systems is

$$2.6 \times \frac{3}{\sqrt{2}} \approx 30.4\lambda^{-1} \tag{B5}$$

(for the equal mass case, in units of $\tau_{\rm vir}$, see their Table 4). This value is indeed sufficient for chaotic mixing.

The next step is to see how much of the probability mass arrives at each point in $\partial\mathcal{C}$ in a single close encounter. (In a close approach that is not the first one, by linearity, the solution is a superposition of solutions that start out as delta functions around some point in $\mathcal{C}$.) Because of the scale separation, the evolution equation for the distribution function in $\mathcal{C}$ should have (approximately) Dirichlet boundary conditions on $\partial\mathcal{C}$, whence the probability of reaching a point $w \in \partial\mathcal{C}$—which leads immediately to the outcome probability in $\mathcal{A}$ and $\mathcal{B}$ via Liouville's equation—is given by the so-called "eventual hitting probability" of w [46], which is simply the time integral of the scalar product of ∇f with the unit normal to $\partial\mathcal{C}$, $\hat{\mathbf{n}}$ (the gradient ∇ is a phase-space gradient). As f effectively evolves like a top hat that expands, $\nabla \int f \mathrm{d}t$ is very large at $\partial\mathcal{C}$, but its magnitude is independent of w_0. The scalar product $\hat{\mathbf{n}} \cdot \nabla f$ gives an additional cosine, so that

$$f_{\text{eventual}}(w) \propto \cos\theta(w, w_0), \tag{B6}$$

where $\theta(w, w_0)$ is the angle between $\hat{\mathbf{n}}$ and the vector pointing from w_0 to w. However, the boundary between $\mathcal{C}$ and the other two sets is not well defined, so instead, it would be useful to think of $\partial\mathcal{C}$ as a "fuzzy" boundary—with some width. This implies that the cosine should be removed as unphysical and replaced by its average $\int_{-\pi/2}^{\pi/2} \cos\theta \mathrm{d}\theta$. Thus, the eventual hitting probability is well approximated by a function independent of both w_0 and w.

What we are interested in is the probability distribution of an outcome in $\mathcal{A}$ or $\mathcal{B}$ and, specifically, in the action distribution there. Consider the set $\mathcal{S}(J_a, J_b, J_c)$, which is the set of all points $w \in \mathcal{A} \cup \mathcal{B}$ such that the (Delaunay) actions of the inner binary are between (J_a, J_b, J_c) and $(J_a + \mathrm{d}J_a, J_b + \mathrm{d}J_b, J_c + \mathrm{d}J_c)$. We are interested in the measure of $\mathcal{S}(J_a, J_b, J_c)$. Given that these actions correspond to something that has been in a close triple system, one can use Liouville's theorem to translate this question into a question of finding the probability distribution function of $\partial\mathcal{C}$. This is independent of the initial condition, which implies that one can simply compute it by assuming a uniform distribution of initial conditions in $\mathcal{C}$, which we compute in Sec. II.

The reader should also bear in mind that because of the scale separation of the evolution in the hierarchical region $\mathcal{B}$, there is an additional source of phase-space mixing: While the outer body moves along its two-body orbit about the inner binary, the mean anomaly of the latter evolves much more rapidly, and its value when the third star returns depends sensitively on its initial value, which implies that it is effectively quite random. Therefore, as the number of close approaches increases, σ should resemble Eq. (2) more closely—this fact is attested to by the simulations of Ref. [26]. Indeed, if the above arguments apply only approximately—so that, in a single close approach, mixing is only efficient in a fraction ν of the total volume of $\mathcal{C}$—then the phase-space distribution still tends to a fully mixed one, as ν^n (exponentially in the number n of close approaches). It may be possible that a close triple approach following a hierarchical phase ends more quickly than is required for the system to mix chaotically in $\mathcal{C}$ [25]. In that case, this phase mixing implies that the cross section is still mixed.

APPENDIX C: APPROXIMATION ERROR INDUCED BY THE MODEL

As the solution described in the previous sections is a statistical approximation to a deterministic, albeit chaotic, problem, it would be useful to be able to constrain the uncertainty due to the approximation above. We do so by evaluating f_{bin} for different values of β and comparing the resulting plots. An analogous check may be performed for η. [47] While this test is not an ideal uncertainty estimate, it is at least a test of the robustness of the model; we rely on the fact that, in many situations, the error induced by varying the parameters of a model is of a size similar to the one made by using that model as an approximation.

Figure 8 shows the marginal energy distribution, and Fig. 9 shows contour plots of the joint distribution of E_{bin} and $S = |\mathbf{S}|$; both figures indicate a weak dependence on β and therefore a high accuracy of the approximation model made here.

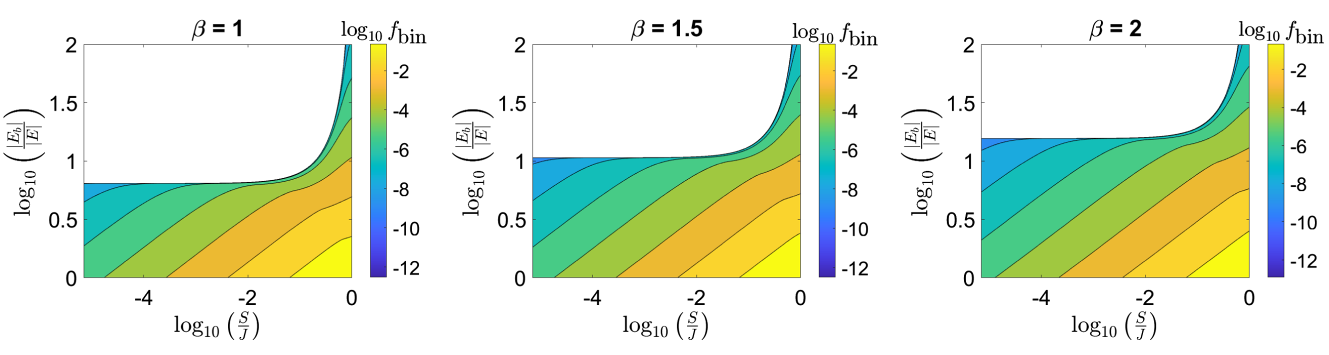

FIG. 9. Plots of Eq. (34) for $m_1 = m_2 = m_3$ for different values of β: $\beta = 1$ (left), $\beta = 1.5$ (center), and $\beta = 2$ (right). One can see that except for at energies where f_{bin} is minuscule anyway, the three plots agree well with each other.

APPENDIX D: INCLINATION

A yet-more-refined version of R may be obtained by not approximating the Legendre polynomial by unity. Instead, we use the singly averaged correction to the Hamiltonian, averaged over the inner binary's orbit, which yields a quadrupole term proportional to (see, e.g., Ref. [6] for orbit-averaging procedures)

$$\mathcal{H}_{\text{quad}} = -\frac{GM_2}{8}\frac{a_b^2}{r_s^3}\{-3(\cos^2 i(e_b^2-1)+4e_b^2+1)\cos(2\theta_b^s+\phi_s)+3\cos^2 i(e_b^2-1)-6e_b^2+1\}, \tag{D1}$$

where ϕ_s is the true anomaly of the outer orbit, and i is the mutual inclination between the two orbits,

$$i = i_s + i_{\text{bin}} = \arccos\left(\frac{J_z - S_z}{|\mathbf{J}-\mathbf{S}|}\right) + \arccos\left(\frac{S_z}{S}\right), \tag{D2}$$

where $S = J_b$ and $S_z = J_a$ are the magnitude and $\hat{\mathbf{z}}$ component of the binary angular momentum, respectively, and $\hat{\mathbf{z}}$ is parallel to $\mathbf{J}$. We also shorten the subscript bin to b for brevity.

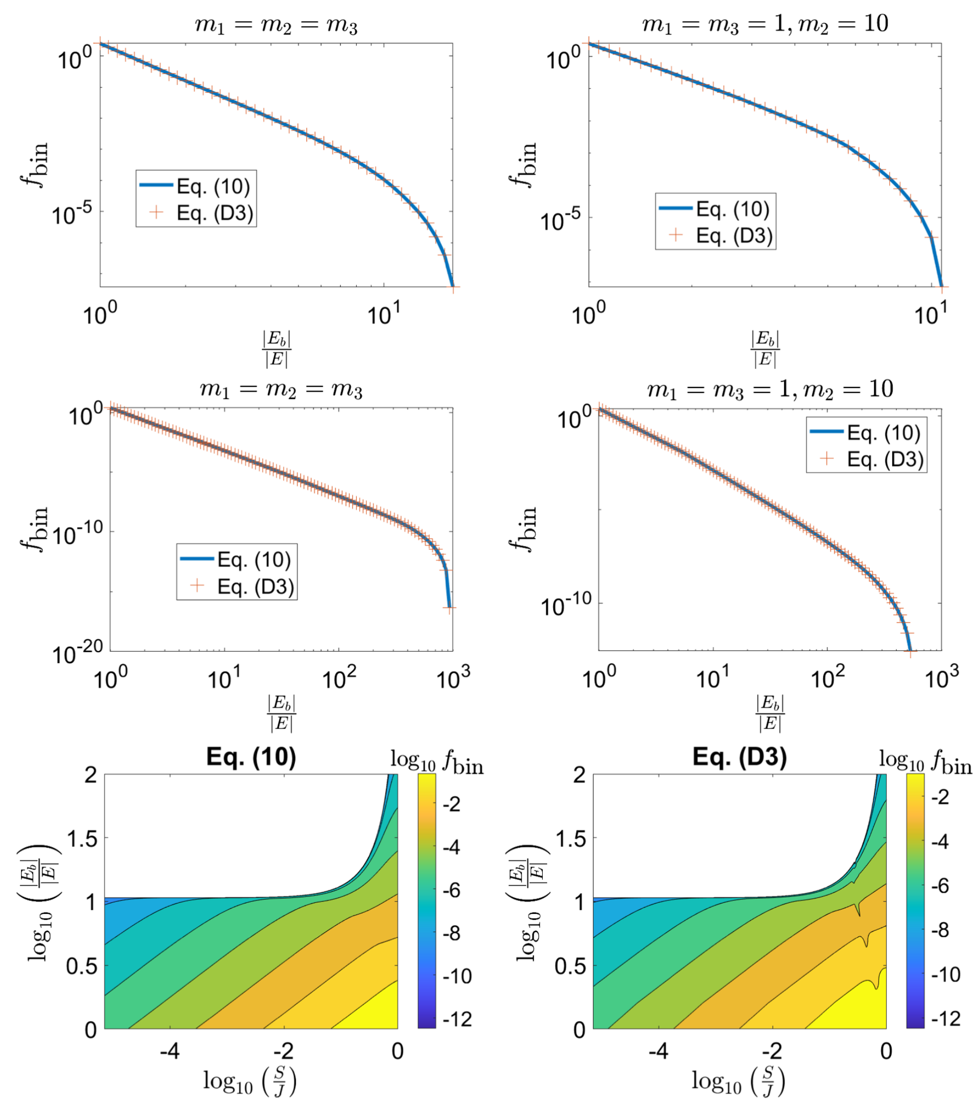

FIG. 10. Comparison between f_{bin} calculated with R defined by Eq. (10) and by Eq. (D3). The first two rows show the marginal energy distribution, and the last line shows that joint energy-spin distribution [left is for Eq. (10); right is for Eq. (D3)]. The top row displays f_{bin} for a large value of J, while the second is for a low value. For the bottom row, we use the same value of J as for the top row, and equal masses. All rows have $\beta = 1.5$.

This quadrupole term is to be evaluated at the value θ_c^s that corresponds to $r_s = R$ (i.e., at the value of ϕ_s that corresponds to it), and then equated with E and solved for R. This procedure gives an R that depends on the binary actions and on θ_b^s; but, as we already know that the dependence of $\theta_{\max}$ on the precise value of R is weak, we neglect this dependence on θ_b^s, by setting $\mathcal{H}_{\text{quad}}$ to its average value, when averaging it over θ_b^s, thereby allowing us to solve for R, explicitly:

$$R = \beta \min\left\{a_{\text{bin}}, \frac{a_{\text{bin}}^{2/3}}{2}\left(\frac{GM_2}{|E|}\right)^{1/3}|3(e_{\text{bin}}^2 - 1)\cos^2(i) - 6e_{\text{bin}}^2 + 1|^{1/3}\right\}. \tag{D3}$$

In Fig. 10, we display a comparison between f_{bin} for R given by Eq. (10), and the correction implied by using Eq. (D3) (for the unbound case) instead. The two definitions yield almost identical values of f_{bin}; therefore, we use the simpler Eq. (10) in the paper.

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nonhierarchical three-body problem, which can be continued analytically to the bound case. This solution contains an unknown "emissivity" multiplicative factor, which has not yet been computed and therefore does not constitute a closed-form solution.

[30] Strictly speaking, one needs to normalize this integral so that it has the correct dimensions. This can be done by working in the appropriate system of units—and by multiplying the integral by functions that are symmetric in all three stars—and is therefore unimportant for the analysis presented in this paper.

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[33] The reason it is not exact is that, when converting the integration over $E_{\rm bin}$ into an integration over $a_{\rm bin}$, one finds that there are now two parameters with dimension of length that can be used to rescale $a_{\rm bin}$ for the purposes of dimensional analysis: R and the original semimajor axis a_0. Therefore, there is an additional mass dependence hidden here, which Eq. (36) does not account for [but an integration of Eq. (48) over $E_{\rm bin}$ does].

[34] We remark that one cannot assume that $\mu_s v^2$ is much smaller than $E_{\rm bin}$; if we did, then thermal equilibrium would be precluded, as interactions between hard binaries and single stars are known to be a source of heat for globular clusters (see, e.g., Ref. [2]).

[35] Here, we sum up all orders, so this assumption is innocuous insofar as the general model described in this section is concerned. In Sec. VIII, we truncate the series at linear order.

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[37] D. C. Heggie, P. Hut, and S. L. W. McMillan, *Binary–Single-Star Scattering. VII. Hard Binary Exchange Cross Sections for Arbitrary Mass Ratios: Numerical Results and Semianalytic FITS*, Astrophys. J. **467**, 359 (1996).

[38] The cross sections were found by integrating the ejection probabilities *given* angular momentum J (and energy E) over the allowed values of the impact parameter b, using the law of total probability: Each value of b determines the angular momentum J and, thence, the ejection probabilities, which are then integrated to yield the cross section.

[39] W. H. Press and S. A. Teukolsky, *On Formation of Close Binaries by Two-Body Tidal Capture*, Astrophys. J. **213**, 183 (1977).

[40] This is justified by the fact that the amount of angular momentum that goes into the tidal excitations is about $\sqrt{r_*^3/Gm}\Delta E$ [41], which is very small in comparison with the initial orbital angular momentum of the binary, $\mu_{\rm bin}\sqrt{Gm_{\rm bin}a_0}$.

[41] C. S. Kochanek, *The Dynamical Evolution of Tidal Capture Binaries*, Astrophys. J. **385**, 604 (1992).

[42] A. C. Fabian, J. E. Pringle, and M. J. Rees, *Tidal Capture Formation of Binary Systems and X-Ray Sources in Globular Clusters*, Mon. Not. R. Astron. Soc. **172**, 15 (1975).

[43] P. Heinämäki, H. J. Lehto, M. J. Valtonen, and A. D. Chernin, *Chaos in Three-Body Dynamics: Kolmogorov—Sinai Entropy*, Mon. Not. R. Astron. Soc. **310**, 811 (1999).

[44] We assume that the initial longest distance between stars is $a_{\rm bin}$ and that the binary is hard.

[45] A. Lichtenberg and M. Lieberman, *Regular and Chaotic Dynamics* (Springer-Verlag, New York, 1992).

[46] S. Redner, *A Guide to First-Passage Processes* (Cambridge University Press, Cambridge, England, 2001), pp. x+312.

[47] The two are not independent, and in the plots, we have kept $\eta\beta$ constant when varying β; this affects only the low-energy cutoff of the bound case.

Correction: Equation (D3) contained an error and has been fixed.