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ON THE MAGNETIC SHIELDING EFFECT OF TRI-LAMELLAR SPHERICAL AND CYLINDRICAL SHELLS.

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THE subject of magnetic shielding has already received considerable attention, and, at the present day with the spread of the trolley system and the consequent increase of disturbing earth currents, the subject often demands attention from those having to do with suspended magnetic systems of any sort.

For a long time it was the custom to use extremely thick single iron shells as a protecting device. A single shell is in a measure effective, but in most cases one-tenth of the shielding material used could, if properly distributed, be made to produce many times the shielding actually produced by the usual distribution.

The advantage to be gained by using concentric shells separated by unmagnetic inter-spaces has been shown by Professor Rücker and by Professor du Bois. The former has given a general solution of the problem of multi-lamellar shielding with spherical shells; the results of this mathematical paper[1] are somewhat difficult to apply in actual shielding practice partly because there is involved the necessity of constant interpretation of the not very explicit equations used in the spherical harmonic analysis. Furthermore the practical advantages of cylindrical shells are much greater than those of spherical shells.

[1] Rücker, Phil. Mag. (5), 37, p. 95, 1894.

(Convenient cylindrical shells may be made by rolling up thin sheet iron strips of proper width.) Professor du Bois[1] has treated the case of bi-lamellar shielding both for spherical and cylindrical shells, and has, moreover, applied the results of his theoretical and experimental investigations in the construction of multi-lamellar ironclad galvanometers.[2] Experience showed that in cases of considerable disturbance it was desirable to use three shields instead of two as a protecting device.

Acting upon the suggestion of Professor du Bois the present writer has deduced the general explicit formulæ giving the shielding effect of tri-lamellar spherical and cylindrical shells under conditions to be mentioned later and in such mathematical form as to be readily interpreted. These general formulæ have then been applied in the graphical discussion of a particular problem well adapted to show the great advantage to be gained by introducing suitable air gaps within the shielding material.

In virtue of the formal analogy in the treatment of the spherical and cylindrical problems the corresponding equations for the two cases are given together at each step in the development.

We shall suppose a uniform magnetic field of strength H_e to be impressed upon the system of shells. We shall deal with the case where the space *within* the shells is to be shielded against H_e. Further we shall suppose the axes of the cylinders to be perpendicular to the lines of force; that the spherical and cylindrical shells are concentrically arranged; that the permeability is constant and equal for all shells. This last assumption involves, of course, a restriction in the values of H_e to those not exceeding, say .01 C.G.S. units, producing small variations in the magnetic condition of the shells superimposed upon their intrinsic magnetization which may be zero or have a finite residual value. It is true that this limitation invalidates the discussion for strong fields; but when we remember that the trouble with galvanometers arises from disturbing fields of the order of magnitude mentioned we may rest assured that our formulæ are capable of practical application in the designing of shielding apparatus.

[1] du Bois, Wied. Ann., 63, p. 348, 1897, and 65, p. 1, 1898. The Electrician, Vol. 40, 1898.

[2] du Bois and Rubens, Verhandl. physik. Gesellsch. Berlin, 17, p. 100, 1898.

Now, suppose a uniform disturbing field H_e to be impressed upon our tri-lamellar shielding systems, Fig. 1 representing both the spherical and cylindrical systems. The letters r_1, R_1, r_2, R_2, r_3, R_3,

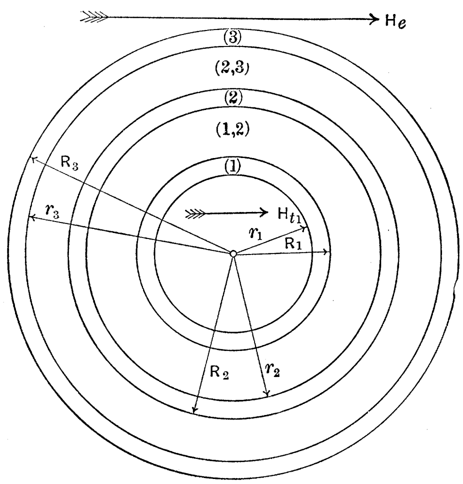

Fig. 1.

represent the inner and outer radii of the various shells; the small letters always referring to the inner radii and the large to the outer. We shall find a notation similar to that introduced by Professor du Bois in his treatment of the bi-lamellar problem to be convenient. It is given below in tabular form:

Spheres.		Cylinders.	
$p_1 = \frac{r_1^3}{R_1^3}$	$m_1 = 1 - \frac{r_1^3}{R_1^3}$	$q_1 = \frac{r_1^2}{R_1^2}$	$n_1 = 1 - \frac{r_1^2}{R_1^2}$
$p_2 = \frac{r_2^3}{R_2^3}$	$m_2 = 1 - \frac{r_2^3}{R_2^3}$	$q_2 = \frac{r_2^2}{R_2^2}$	$n_2 = 1 - \frac{r_2^2}{R_2^2}$
$p_3 = \frac{r_3^3}{R_3^3}$	$m_3 = 1 - \frac{r_3^3}{R_3^3}$	$q_3 = \frac{r_3^2}{R_3^2}$	$n_3 = 1 - \frac{r_3^2}{R_3^2}$
$p_{12} = \frac{R_1^3}{r_2^3}$	$m_{12} = 1 - \frac{R_1^3}{r_2^3}$	$q_{12} = \frac{R_1^2}{r_2^2}$	$n_{12} = 1 - \frac{R_1^2}{r_2^2}$
$p_{23} = \frac{R_2^3}{r_3^3}$	$m_{23} = 1 - \frac{R_2^3}{r_3^3}$	$q_{23} = \frac{R_2^2}{r_3^2}$	$n_{23} = 1 - \frac{R_2^2}{r_3^2}$

We shall consider the cylinders as infinitely long; but experience has shown that the equations to be derived hold in the central portion of cylinders whose length is not less than three or four times the radius.

Now suppose the total resultant field in the direction of H_e within the innermost shell to be denoted by H_{i1}. The ratio of H_e to H_{i1} may be taken as a measure of the effectiveness of the shielding device. Denote this ratio by g then $g = \frac{H_e}{H_{i1}}$. $g = \infty$ would, of course, mean perfect shielding.

Now it becomes necessary to find an equation expressing the relation existing among the various quantities entering into the problem, namely, "the shielding ratio" g, the permeability μ and the geometrical data given by the p's and m's in the case of the spherical shells, the q's and n's in the case of the cylindrical shells.

FUNDAMENTAL EQUATIONS.

Since Fig. 1 represents a plan section of our double system of shells, the axis of the cylindrical system will be perpendicular to the plane of the paper. The direction of the impressed field is indicated by the arrows. There will be produced upon each of the six surfaces of separation between iron and air surface distributions of "free magnetism." In the theory of such distributions it is shown that a distribution upon a spherical or cylindrical surface produces within the surface, when the impressed force is uniform, a uniform magnetic field which is coincident with the impressed field but acting in the opposite direction and of different intensity. If S denote any one of the surfaces of the spherical system and H_s the field produced within S by the distribution upon S, the theory shows that at a point without S, P say, the radial force produced by the distribution upon S is $-2\frac{r^3}{R^3}H_s \cos\vartheta$, where r is the radius of S and R the distance from the center of S to the point P and the tangential force at P (perpendicular to the radial force) is $-\frac{r^3}{R^3}H_s \sin\vartheta$, where ϑ is the angle between the radius to P and the impressed field. With cylindrical shells, for any surface S and a point P without S we should have the corresponding forces $-\frac{r^2}{R^2}H_s \cos\vartheta$, and

$-\frac{r^2}{R^2} H_s \sin \vartheta$. Now considering the surface S we may think of the resultant force within it to be made up of two parts one of which is due to causes external and the other to causes internal. Call the first H_t and the second H_i. Starting in each case with the inner surface we may write down the following equations, applying to both spherical and cylindrical systems.

$$R \begin{cases} H_{t1} = H_{t2} + H_{i1} = H_e + H_{i6} + H_{i5} + H_{i4} + H_{i3} + H_{i2} + H_{i1} \\ H_{t2} = H_{t3} + H_{i2} = H_e + H_{i6} + H_{i5} + H_{i4} + H_{i3} + H_{i2} \\ H_{t3} = H_{t4} + H_{i3} = H_e + H_{i6} + H_{i5} + H_{i4} + H_{i3} \\ H_{t4} = H_{t5} + H_{i4} = H_e + H_{i6} + H_{i5} + H_{i4} \\ H_{t5} = H_{t6} + H_{i5} = H_e + H_{i6} + H_{i5} \\ H_{t6} = H_{t7} + H_{i6} = H_e + H_{i6} \end{cases}$$

From what has been said above and from Fig 1 the meaning of the notation is clear.

We do not require to know the exact distribution of the lines of force throughout the shells but the quantities H_{i6}, H_{i5}, H_{i4}, H_{i3}, H_{i2}, H_{i1} must be such as to satisfy the two surface conditions namely: that at surfaces of separation of iron from air the tangential components of the magnetic force must be equal in the two media and that the normal component of the magnetic induction must be continuous. The first condition, from what has been already said, may be seen to be always satisfied whatever the values of $H_{i6} \cdots H_{i1}$. But in order that the condition of continuity in the normal component of the magnetic induction be satisfied we must have the following equations also satisfied. In these equations μ represents the permeability of the iron. The equations pertaining to the spherical shells are given first. In both sets of equations cos θ occurs as a common factor in both members of each equation but is not written down.

$$
S\left\{
\begin{aligned}
&H_{t1} = \mu\left(H_{t2} - 2H_{i1}\right) \\
&\mu\left(H_{t2} - 2\frac{r_1^3}{R_1^3}H_{i1}\right) = H_{t3} - 2H_{i2} - 2\frac{r_1^3}{R_1^3}H_{i1} \\
&H_{t3} - 2\frac{R_1^3}{r_2^3}H_{i2} - 2\frac{r_1^3}{r_2^3}H_{i1} = \mu\left(H_{t4} - 2H_{i3} - 2\frac{R_1^3}{r_2^3}H_{i2} - 2\frac{r_1^3}{r_2^3}H_{i1}\right) \\
&\mu\left(H_{t4} - 2\frac{r_2^3}{R_2^3}H_{i3} - 2\frac{R_1^3}{R_2^3}H_{i2} - 2\frac{r_1^3}{R_2^3}H_{i1}\right) = H_{t5} - 2H_{i4} - 2\frac{r_2^3}{R_2^3}H_{i3} - 2\frac{R_1^3}{R_2^3}H_{i2} - 2\frac{r_1^3}{R_2^3}H_{i1} \\
&H_{t5} - 2\frac{R_2^3}{r_3^3}H_{i4} - 2\frac{r_2^3}{r_3^3}H_{i3} - 2\frac{R_1^3}{r_3^3}H_{i2} - 2\frac{r_1^3}{r_3^3}H_{i1} = \mu\left(H_{t6} - 2H_{i5} - 2\frac{R_2^3}{r_3^3}H_{i4} - 2\frac{r_2^3}{r_3^3}H_{i3} - 2\frac{R_1^3}{r_3^3}H_{i2} - 2\frac{r_1^3}{r_3^3}H_{i1}\right) \\
&\mu\left(H_{t6} - 2\frac{r_3^3}{R_3^3}H_{i5} - 2\frac{R_2^3}{R_3^3}H_{i4} - 2\frac{r_2^3}{R_3^3}H_{i3} - 2\frac{R_1^3}{R_3^3}H_{i2} - 2\frac{r_1^3}{R_3^3}H_{i1}\right) \\
&= H_e - 2H_{i6} - 2\frac{r_3^3}{R_3^3}H_{i5} - 2\frac{R_2^3}{R_3^3}H_{i4} - 2\frac{r_2^3}{R_3^3}H_{i3} - 2\frac{R_1^3}{R_3^3}H_{i2} - 2\frac{r_1^3}{R_3^3}H_{i1}
\end{aligned}
\right.
$$

The corresponding set of equations for the cylinders is

$$T\left\{\begin{aligned}
&H_{t1} = \mu(H_{t2} - H_{i1})\\
&\mu\left(H_{t2} - \frac{r_1^2}{R_1^2}H_{i1}\right) = H_{t3} - H_{i2} - \frac{r_1^2}{R_1^2}H_{i1}\\
&H_{t3} - \frac{R_1^2}{r_2^2}H_{i2} - \frac{r_1^2}{r_2^2}H_{i1} = \mu\left(H_{t4} - H_{i3} - \frac{R_1^2}{r_2^2}H_{i2} - \frac{r_1^2}{r_2^2}H_{i1}\right)\\
&\mu\left(H_{t4} - \frac{r_2^2}{R_2^2}H_{i3} - \frac{R_1^2}{R_2^2}H_{i2} - \frac{r_1^2}{R_2^2}H_{i1}\right) = H_{t5} - H_{i4} - \frac{r_2^2}{R_2^2}H_{i3} - \frac{R_1^2}{R_2^2}H_{i2} - \frac{r_1^2}{R_2^2}H_{i1}\\
&H_{t5} - \frac{R_2^2}{r_3^2}H_{i4} - \frac{r_2^2}{r_3^2}H_{i3} - \frac{R_1^2}{r_3^2}H_{i2} - \frac{r_1^2}{r_3^2}H_{i1} = \mu\left(H_{t6} - H_{i5} - \frac{R_2^2}{r_3^2}H_{i4} - \frac{r_2^2}{r_3^2}H_{i3} - \frac{R_1^2}{r_3^2}H_{i2} - \frac{r_1^2}{r_3^2}H_{i1}\right)\\
&\mu\left(H_{t6} - \frac{r_3^2}{R_3^2}H_{i5} - \frac{R_2^2}{R_3^2}H_{i4} - \frac{r_2^2}{R_3^2}H_{i3} - \frac{R_1^2}{R_3^2}H_{i2} - \frac{r_1^2}{R_3^2}H_{i1}\right)\\
&= H_e - H_{i6} - \frac{r_3^2}{R_3^2}H_{i5} - \frac{R_2^2}{R_3^2}H_{i4} - \frac{r_2}{R_3^2}H_{i3} - \frac{R_1}{R_3^2}H_{i2} - \frac{r_1^2}{R_3^2}H_{i1}
\end{aligned}\right.$$

TREATMENT OF FUNDAMENTAL EQUATIONS.

From the set of equations (S) we may eliminate the H_t's with the aid of the set (R) giving for the spherical shells after combination and rearrangement of terms the following set:

$$(\mu-1)H_{i6}+(\mu-1)H_{i5}+(\mu-1)H_{i4}+(\mu-1)H_{i3}+(\mu-1)H_{i2}$$
$$-(2\mu+1)H_{i1}=-(\mu-1)H_e$$
$$(\mu-1)H_{i6}+(\mu-1)H_{i5}+(\mu-1)H_{i4}+(\mu-1)H_{i3}+(\mu+2)H_{i2}$$
$$-2\frac{r_1^3}{R_1^3}(\mu-1)H_{i1}=-(\mu-1)H_e$$
$$(\mu-1)H_{i6}+(\mu-1)H_{i5}+(\mu-1)H_{i4}-(2\mu+1)H_{i3}-\frac{2R_1^3}{r_2^3}(\mu-1)H_{i2}$$
$$-2\frac{r_1^3}{r_2^3}(\mu-1)H_{i1}=-(\mu-1)H_e$$
$$(\mu-1)H_{i6}+(\mu-1)H_{i5}+(\mu+2)H_{i4}-2\frac{r_2^3}{R_2^3}(\mu-1)H_{i3}-2\frac{R_1^3}{R_2^3}(\mu-1)H_{i2}$$
$$-2\frac{r_1^3}{R_2^3}(\mu-1)H_{i1}=-(\mu-1)H_e$$
$$(\mu-1)H_{i6}-(2\mu+1)H_{i5}-2\frac{R_2^3}{r_3^3}(\mu-1)H_{i4}-2\frac{r_2^3}{r_3^3}(\mu-1)H_{i3}-2\frac{R_1^3}{r_3^3}(\mu-1)H_{i2}$$
$$-2\frac{r_1^3}{r_3^3}(\mu-1)H_{i1}=-(\mu-1)H_e$$
$$(\mu+2)H_{i6}-2\frac{r_3^3}{R_3^3}(\mu-1)H_{i5}-2\frac{R_2^3}{R_3^3}(\mu-1)H_{i4}-2\frac{r_2^3}{R_3^3}(\mu-1)H_{i3}-2\frac{R_1^3}{R_3^3}(\mu-1)H_{i2}$$
$$-2\frac{r_1^3}{R_3^3}(\mu-1)H_{i1}=-(\mu-1)H_e$$

The corresponding set for the cylinders is

$$
\begin{aligned}
&(\mu-1)H_{i6}+(\mu-1)H_{i5}+(\mu-1)H_{i4}+(\mu-1)H_{i3}+(\mu-1)H_{i2}-(\mu+1)H_{i1}=-(\mu-1)H_e\\
&(\mu-1)H_{i6}+(\mu-1)H_{i5}+(\mu-1)H_{i4}+(\mu-1)H_{i3}+(\mu+1)H_{i2}-\frac{r_1^2}{R_1^2}(\mu-1)H_{i1}=-(\mu-1)H_e\\
&(\mu-1)H_{i6}+(\mu-1)H_{i5}+(\mu-1)H_{i4}-(\mu+1)H_{i3}-\frac{R_1^2}{r_2^2}(\mu-1)H_{i2}-\frac{r_1^2}{r_2^2}(\mu-1)H_{i1}=-(\mu-1)H_e\\
&(\mu-1)H_{i6}+(\mu-1)H_{i5}+(\mu+1)H_{i4}-\frac{r_2^2}{R_2^2}(\mu-1)H_{i3}-\frac{R_1^2}{R_2^2}(\mu-1)H_{i2}-\frac{r_1^2}{R_2^2}(\mu-1)H_{i1}=-(\mu-1)H_e\\
&(\mu-1)H_{i6}-(\mu+1)H_{i5}-\frac{R_2^2}{r_3^2}(\mu-1)H_{i4}-\frac{r_2^2}{r_3^2}(\mu-1)H_{i3}-\frac{R_1^2}{r_3^2}(\mu-1)H_{i2}-\frac{r_1^2}{r_3^2}(\mu-1)H_{i1}=-(\mu-1)H_e\\
&(\mu+1)H_{i6}-\frac{r_3^2}{R_3^2}(\mu-1)H_{i5}-\frac{R_2^2}{R_3^2}(\mu-1)H_{i4}-\frac{r_2^2}{R_3^2}(\mu-1)H_{i3}-\frac{R_1^2}{R_3^2}(\mu-1)H_{i2}-\frac{r_1^2}{R_3^2}(\mu-1)H_{i1}=-(\mu-1)H_e
\end{aligned}
$$

We have now two sets of six linear equations in the six quantities $H_{i1} \cdots H_{i6}$, one set corresponding to the spherical shells and the other to the cylindrical shells. We have sufficient equations in each case to enable us to evaluate $H_{i1} \cdots H_{i6}$ in terms of the geometrical data, the permeability and the impressed field.

It will be convenient to treat these equations by means of determinants, and the method used in treating the spherical problem only will be given; the cylindrical problem is subject to a treatment mathematically analogous to that indicated for the spherical problem.

Referring to the set of equations applying to spherical shells, let the determinant formed by taking in order the coefficients of the H's in the left-hand member of the first equation for the elements of its first row, the coefficients, in order, of the H's in the left-hand member of the second equation for the elements of its second row and so on be called Δ. Let the determinant which is equal to Δ except in the first column, where in place of the elements found in the first column is put in the place of each element the common right-hand member of the system of equations namely—$(\mu - 1)H_e$, be called $\Delta 1$. Let $\Delta 2$, $\Delta 3$, $\Delta 4$, $\Delta 5$ and $\Delta 6$ be defined in an analogous way. The determinants thus defined are seen to be of the sixth order.

We shall have then according to Leibnitz's rule for the solution of simultaneous linear equations the following values of $H_{i1} \cdots H_{i6}$ pertaining to the spherical shells

$$H_{i1} = \frac{\Delta 1}{\Delta}$$
$$\vdots \quad \vdots \quad \vdots$$
$$H_{i6} = \frac{\Delta 6}{\Delta}.$$

These are the values to be substituted in the equation

$$H_{t1} = H_e + H_{i6} + \cdots + H_{i1}$$

and it will be remembered H_{t1} stands for the field within the innermost spherical shell and H_e for the impressed field. From the method of formation of $\Delta 1 \cdots \Delta 6$ it will be seen that each of these determinants contains H_e as a factor, and the shielding ratio $g = \dfrac{H_e}{H_{t1}}$

is found from the equation just given. Substituting in this equation the values found for $H_{i1} \cdots H_{i6}$ we get

$$H_{t1} = H_e \left\{ 1 + \frac{1}{H_e}\left[\frac{\Delta 1 + \Delta 2 \cdots \Delta 6}{\Delta}\right] \right\}$$

and therefore

$$g = \frac{H_e}{H_{t1}} = \frac{He\Delta.}{\Delta H_e + \Delta 1 + \Delta 2 + \cdots + \Delta 6}.$$

This then expressed symbolically in determinantal form is the result sought. It remains now to expand the Δ's and reduce the somewhat complicated resultant expression. It will be sufficient to illustrate the method used in expanding the various determinants to consider one of them only, say Δ.

It is in the ordinary notation

$$\begin{vmatrix} (\mu-1), & (\mu-1), & (\mu-1), & (\mu-1), & (\mu-1), & -(2\mu+1) \\ (\mu-1), & (\mu-1), & (\mu-1), & (\mu-1), & (\mu+2), & -2(\mu-1)\frac{r_1^3}{R_1^3} \\ (\mu-1), & (\mu-1), & (\mu-1), & -(2\mu+1), & -2(\mu-1)\frac{R_1^3}{r_2^3}, & -2(\mu-1)\frac{r_1^3}{r_2^3} \\ (\mu-1), & (\mu-1), & (\mu+2), & -2(\mu-1)\frac{r_2^3}{R_2^3}, & -2(\mu-1)\frac{R_1^3}{R_2^3}, & -2(\mu-1)\frac{r_1^3}{R_2^3} \\ (\mu-1), & -(2\mu+1), & -2(\mu-1)\frac{R_2^3}{r_3^3}, & -2(\mu-1)\frac{r_2^3}{r_3^3}, & -2(\mu-1)\frac{R_1^3}{r_3^3}, & -2(\mu-1)\frac{r_1^3}{r_3^3} \\ (\mu+2), & -2(\mu-1)\frac{r_3^3}{R_3^3}, & -2(\mu-1)\frac{R_2^3}{R_3^3}, & -2(\mu-1)\frac{r_2^3}{R_3^3}, & -2(\mu-1)\frac{R_1^3}{R_3^3}, & -2(\mu-1)\frac{r_1^3}{R_3^3} \end{vmatrix}$$

Multiply the sixth row by R_3^3, the fifth by r_3^3, the fourth by R_2^3, the third by r_2^3, the second by R_1^3, the first by r_1^3. To keep the value of the determinant unchanged we must multiply the resulting form by $\frac{1}{R_3^3 \ldots r_1^3}$. Now subtract the fifth row from the sixth, the fourth from the fifth and so on down to the first row. These operations have not changed the value of the determinant. Now in the resultant form subtract the second column from the first, the third from the second and so on up to the sixth. Divide the sixth row of the resultant expression by R_3^3, the fifth by r_3^3 and so on down to the first row. To keep the value of the determinant unchanged multiply the resulting form by $R_3^3 \ldots r_1^3$. Employing the p and m notation explained on page 195 we may write the resulting form of the determinant Δ as follows :

$$\Delta = \begin{vmatrix} 0 & 0 & 0 & 0 & 3\mu & -(2\mu+1) \\ 0 & 0 & 0 & -3 & (\mu+2)m_1 & 3p_1 \\ 0 & 0 & 3\mu & -(2\mu+1)m_{12} & -3\mu p_{12} & 0 \\ 0 & -3 & (\mu+2)m_2 & 3p_2 & 0 & 0 \\ 3\mu & -(2\mu+1)m_{23} & -3\mu p_{23} & 0 & 0 & 0 \\ (\mu+2)m_3 & 3p_3 & 0 & 0 & 0 & 0 \end{vmatrix}$$

As the determinant now stands Laplace's Method of development will be found convenient. By this method the determinant Δ is at once seen to be equal an expression with two terms, each of which is the product of two determinants of the third order. Using the "criss-cross" method applicable to determinants of the third order the expansion of these determinants is easily obtained and thus the expansion of Δ. In a similar way the expansions of $\Delta 1 \ldots \Delta 6$ may be obtained.

We have now to substitute the values found for $\Delta, \Delta 1 \ldots \Delta 6$ in the expression

$$g = \frac{\Delta . H_e}{\Delta H_e + \Delta 1 + \Delta 2 + \ldots + \Delta 6}.$$

Assuming this to have been done and the proper simplifications and rearrangement of the terms to have been accomplished we get

$$(A) \qquad \begin{aligned} g - 1 = \tfrac{2}{9} \frac{(\mu - 1)^2}{\mu} \Big\{ & (1 - p_1 p_2 p_3) \\ & + \tfrac{1}{81} \frac{(2\mu + 1)^2 (\mu + 2)^2}{\mu^2} m_1 m_{12} m_2 m_{23} m_3 \\ & + \tfrac{1}{9} \frac{(2\mu + 1)(\mu + 2)}{\mu} [(m_1 m_3 + m_1 m_2 - m_1 m_2 m_3) m_{12} \\ & + (m_1 m_3 + m_2 m_3 - m_1 m_2 m_3) m_{23} - m_1 m_3 m_{12} m_{23}] \Big\}. \end{aligned}$$

A similar treatment of the fundamental equations for the cylindrical shells gives the corresponding formula

$$g - 1 = \tfrac{1}{4}\frac{(\mu - 1)^2}{\mu}\Big\{(1 - q_1q_2q_3) + \tfrac{1}{16}\frac{(\mu+1)^4}{\mu^2}\,n_1n_{12}n_2n_{23}n_3$$

$$(B) \qquad + \tfrac{1}{4}\frac{(\mu+1)^2}{\mu}\,[(n_1n_3 + n_1n_2 - n_1n_2n_3)\,n_{12}$$

$$+ (n_1n_3 + n_2n_3 - n_1n_2n_3)n_{23} - n_1n_3n_{12}n_{23}]\Big\}$$

Equations (A) and (B) are capable of transformation into the following forms:

$$g = \Big\{\Big(\tfrac{2}{9}\frac{(\mu-1)^2}{\mu}m_1 + 1\Big)\Big(\tfrac{2}{9}\frac{(\mu-1)^2}{\mu}m_2+1\Big)\Big(\tfrac{2}{9}\frac{(\mu-1)^2}{\mu}m_3+1\Big)\Big\}$$

$$+ \tfrac{2}{9}\frac{(\mu-1)^2}{\mu}\cdot\frac{(2\mu+1)(\mu+2)}{9\mu}\Big\{\frac{(2\mu+1)(\mu+2)}{9\mu}m_1m_2m_3p_{12}p_{23}$$

$$(A')$$

$$- \tfrac{2}{9}\frac{(\mu-1)^2}{\mu}(m_1m_2m_3p_{12} + m_1m_2m_3p_{23}) - (m_1m_2p_{12} + m_2m_3p_{23}$$

$$+ m_1m_3p_{12}p_{23})\Big\}.$$

$$g = \Big\{\Big(\tfrac{1}{4}\frac{(\mu-1)^2}{\mu}n_1+1\Big)\Big(\tfrac{1}{4}\frac{(\mu-1)^2}{\mu}n_2+1\Big)\Big(\tfrac{1}{4}\frac{(\mu-1)^2}{\mu}n_3+1\Big)\Big\}$$

$$(B') \quad + \tfrac{1}{16}\frac{(\mu^2-1)^2}{\mu^2}\Big\{\frac{(\mu+1)^2}{4\mu}n_1n_2n_3q_{12}q_{23}$$

$$-\tfrac{1}{4}\frac{(\mu-1)^2}{\mu}(n_1n_2n_3q_{12}+n_1n_2n_3q_{23})-(n_1n_2q_{12}+n_2n_3q_{23}+n_1n_3q_{12}q_{23})\Big\}.$$

A few experimental determinations of the shielding ratio of three cylindrical shells made of transformer iron have furnished a satisfactory verification of the formula (B').

Discussion of the Equations.

Consider first equations (A) and (B). g is seen to be a function of the geometrical quantities represented by the p's and the m's, the q's and the n's and of the permeability μ. In both (A) and (B) the expression for g is of the third degree in μ. In both g equals unity when

$\mu = 1$ and ∞ when $\mu = \infty$. [These remarks apply also to equations (A') and (B').] If any one of the three shells be allowed to vanish we have the case of two shells and the form of the resulting equations is seen to be independent of the shell which we choose to make vanish. Suppose in the case of three spherical shells we make the outer one vanish by causing the outer radius to become equal to the inner radius. Equation (A) then takes the form

$$g - 1 = \frac{2}{9}\frac{(\mu - 1)^2}{\mu}\left\{(1 - p_1 p_2) + \frac{(2\mu + 1)(\mu + 2)}{9\mu} m_1 m_2 m_{12}\right\}.$$

This is identically the equation given for two spherical shells by Professor du Bois[1] and a similar reduction for the cylindrical case would give the corresponding formula for two cylindrical shells. If we suppose the two outer spherical shells to vanish we obtain the well-known formula expressing the shielding effect of a single shell

$$g - 1 = \frac{2}{9}\frac{(\mu - 1)^2}{\mu}(1 - p_1).$$

Referring to the equations (A), (B), (A'), (B'), it is to be noticed that the subscripts 1 and 3 may be throughout interchanged without changing the equations; that is to say the formulæ are symmetrical with respect to the inner and outer shells.

Of course (A') and (B') are capable of discussion in a manner similar to that given above for (A) and (B), difference of form in the results occurring. The forms (A') and (B') are particularly adapted to a form of the discussion to be given now.

Suppose the outer shell to have so far expanded that there is no "magnetic interference" between it and the middle shell ($p_{23} = 0$, $m_{23} = 1$) and so far as to be also consistent with the condition that the middle shell may so expand as to cause the mutual interference between it and the inner shell to vanish ($p_{12} = 0$, $m_{12} = 1$) and yet the mutual interference between the middle and outer shell to remain zero. In this assumed expansion the ratio of the radii of each shell is supposed to remain constant. This means, of course, that R_3, r_3, R_2, r_2, become very large. The field within the outer shell may be now considered to act directly upon the middle

[1] du Bois, Wied. Ann., 63, p. 353. 1898.

shell as an impressed field; denote it by H_1. Likewise the field within the middle shell may be considered to act directly upon the inner shell as an impressed field; denote it by H_2. Denote the field within the inner shell by H_1. We have now

$$g = \frac{H_e}{H_3} \cdot \frac{H_3}{H_2} \cdot \frac{H_2}{H_1} = g_3\, g_2\, g_1$$

and have so expressed what Professor du Bois designates as a multiplication of shielding ratios, g_3, g_2, g_1 representing the shielding effect of the outer, middle and inner shell respectively against the field impressed upon each of them. We have then

$$g_3 = \frac{H_e}{H_3} = \tfrac{2}{9}\frac{(\mu - 1)^2}{\mu} m_3 + 1,$$

$$g_2 = \frac{H_3}{H_2} = \tfrac{2}{9}\frac{(\mu - 1)^2}{\mu} m_2 + 1,$$

$$g_1 = \frac{H_2}{H_1} = \tfrac{2}{9}\frac{(\mu - 1)^2}{\mu} m_1 + 1,$$

for it will be remembered that the form of the expression giving the shielding effect of a single spherical shell is $g = \tfrac{2}{9}\frac{(\mu - 1)^2}{\mu} m + 1$. Now in the expansion of the shells mentioned above it was specified that the ratio of the inner to the outer radius should remain constant for each shell. This means p_1, p_2 and p_3 and therefore m_1, m_2 and m_3 must also remain constant. So we may write equations (A') and (B') as follows:

$$(A'') \quad g = g_1 g_2 g_3 + \tfrac{2}{9}\frac{(\mu - 1)^2}{\mu}\frac{(2\mu + 1)(\mu + 2)}{9\mu}\left\{\frac{(2\mu + 1)(\mu + 2)}{9\mu} m_1 m_2 m_3 p_{12} p_{23} - \tfrac{2}{9}\frac{(\mu - 1)^2}{\mu}(m_1 m_2 m_3 p_{12} + m_1 m_2 m_3 p_{23}) - (m_1 m_2 p_{12} + m_2 m_3 p_{23} + m_1 m_3 p_{12} p_{23})\right\}$$

$$(B'') \quad g = g_1 g_2 g_3 + \tfrac{1}{16}\frac{(\mu^2 - 1)^2}{\mu^2}\left\{\frac{(\mu + 1)^2}{4\mu} n_1 n_2 n_3 q_{12} q_{23} - \tfrac{1}{4}\frac{(\mu - 1)^2}{\mu}(n_1 n_2 n_3 q_{12} + n_1 n_2 n_3 q_{23}) - (n_1 n_2 q_{12} + n_2 n_3 q_{23} + n_2 n_3 q_{12} q_{23})\right\}$$

(B'') being derived for the cylindrical shells in a manner analogous to that used for (A''). A consideration of (A'') and (B'') shows that the combined effect of the terms following $g_1 g_2 g_3$, is strongly subtractive, that is the shielding is much greater when the shells are very far removed from one another and as we have seen the shielding ratio in this case is equal to $g_1 g_2 g_3$. We may think then of the product $g_1 g_2 g_3$ as representing an "ideal" shielding and the terms following this product as representing the departure from this ideal case caused by the shells mutually interfering.

If μ is large, say > 100, (A) and (B) may be simplified as follows

$$(A_1)\quad \begin{aligned} g - 1 = {} & \tfrac{2}{9}(\mu - 2)\{(1 - p_1 p_2 p_3) + \tfrac{4}{81}(\mu + 1)(\mu \\ & + 2) m_1 m_{12} m_2 m_{23} m_3 + \tfrac{2}{9}(\mu + 2)[(m_1 m_3 + m_1 m_2 \\ & - m_1 m_2 m_3) m_{12} + (m_1 m_3 + m_2 m_3 - m_1 m_2 m_3) m_{23} \\ & - m_1 m_3 m_{12} m_{23}]\} \end{aligned}$$

$$(B_1)\quad \begin{aligned} g - 1 = {} & \tfrac{1}{4}(\mu - 2)\{(1 - q_1 q_2 q_3) + \tfrac{1}{16}(\mu + 2)^2 n_1 n_{12} n_2 n_{23} n_3 \\ & + \tfrac{1}{4}(\mu + 2)[(n_1 n_3 + n_1 n_2 - n_1 n_2 n_3) n_{12} + (n_1 n_3 + n_2 n_3 \\ & - n_1 n_2 n_3) n_{23} - n_1 n_3 n_{12} n_{23}]\}. \end{aligned}$$

Similar approximate formulæ might be deduced from the equations (A'), (B'), (A''), (B'') and often prove useful.

Graphical Discussion of a Particular Problem.

Professor Rücker has discussed analytically the question of maximum shielding, with reference to the weight of material used, for several particular problems. Among other things he has shown that if the radius a of a spherical shielded space be fixed and the amount of shielding material given the latter is best employed in the form of a *single* shell until a limit t is reached defined by $\frac{t}{a} = \frac{3}{2\mu}$ (approximately) where t denotes the thickness. So for a comparatively small shielded space, since μ for the best material reaches the initial value 300, it is only for extremely thin shells that the shielding with a single shell is, strictly speaking, most economical as regards weight. He has shown further that with two and three spherical shells when the permeability is great the arrangement with regard to minimum weight of material is best when the radii of the successive bounding surfaces are in geometrical progression, the ratio of the innermost to the outermost radius being supposed given. Professor Rücker's discussion is limited to spherical shells.

Professor du Bois has discussed the bi-lamellar case for both spherical and cylindrical shells when the innermost radius is given, and has found the best arrangement to be that for which the radial ratio of the air-gap is 1.5538 for cylindrical shells and 1.3815 for spherical shells; strictly speaking these results only hold for two thin similar shells (*i. e.*, of equal radial ratios); but the best radial ratios of the air space, as calculated for other less simple cases in actual practice, never come out much different from the above values; hence for designing purposes it may be roughly fixed between 1.3 and 1.4 for spheres, and between 1.5 and 1.6 for cylinders.

The following graphical discussion is introduced in the hope of showing in a manner convincing and easy to understand the advantages of lamination of the shielding material and not with the intent to give the most general possible discussion of the equations used in plotting the curves. At the same time it is hoped that the general explicit equations given above and the following illustration of their applicability to the plotting of shielding curves may be of use to those who may have occasion to design shielding apparatus.

We shall discuss here the shielding of single, bi-lamellar and tri-lamellar spherical and cylindrical shells under the supposition that the permeability of the shielding material is, in all cases, 202, which is but about two-thirds of that of the best material hithero tested, and under other conditions to be mentioned later.

Referring to Fig. 2, the curves shown relate to single spherical and cylindrical shells. The abscissæ represent the radial ratios of the shells; if the radius of the shielded space should be made equal to unity ($r = 1$) then the abscissæ would represent the successive values of the outer radius of the shell. The ordinates of curves 1 and 2 represent the shielding ratio, g. Curves 1 and 2 show how the shielding increases as the thickness of the shells increases. At first the gradient of the cylindrical curve is not quite so sharp as that of the spherical; later as the thickness increases it becomes greater and does not reach its asymptotic value so soon as in the spherical curve. Curve 1 reaches very nearly its asymptotic value when its outer radius is say three times the inner. The corresponding curve 2 for the hollow cylinder has not quite reached this point even when the outer radius has become five times the inner. These

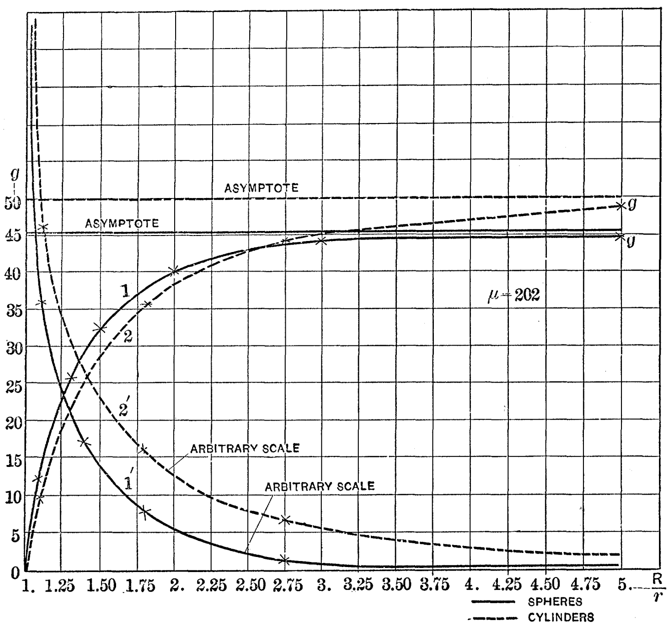

Fig. 2.

curves show clearly the futility of increasing the thickness of a single shielding shell beyond a certain point. Curves 1′ and 2′, corresponding to spherical and cylindrical shells respectively, represent, upon a scale quite arbitrary in each case as regards the ordinates, the way in which the shielding per unit weight of material varies as the thickness of the shells is increased.

Consider now Fig. 3. It refers to the case of two concentric spherical shells, also to that of two concentric cylindrical shells. If r and R denote the innermost and outermost radii respectively of either system of shells, a condition imposed is that $\frac{R}{r}$ shall always equal five. The other conditions are that the two shells must be equal in thickness and that $\mu = 202$. If the radius of the shielded space be taken equal to unity ($r = 1$) then $R = 5$. The figures given as the abscissæ in the present case and in the case of three shells are to be considered on the supposition that $r = 1$. The independent variable is the thickness of the air-gap. It may vary between the limits 0 and 4. The upper of the two rows of abscissæ represents the varying air-gap thickness and the lower the corresponding thickness of the shells. The ordinates represent the shielding ratio g. Curves 1 and 2 are without reference to the weight and refer to spherical and cylindrical shell respectively. Curves 1′ and 2′ represent upon scales quite arbitrary as regards the ordinates the way in which the shielding per unit weight varies as the air-gap is increased or decreased.

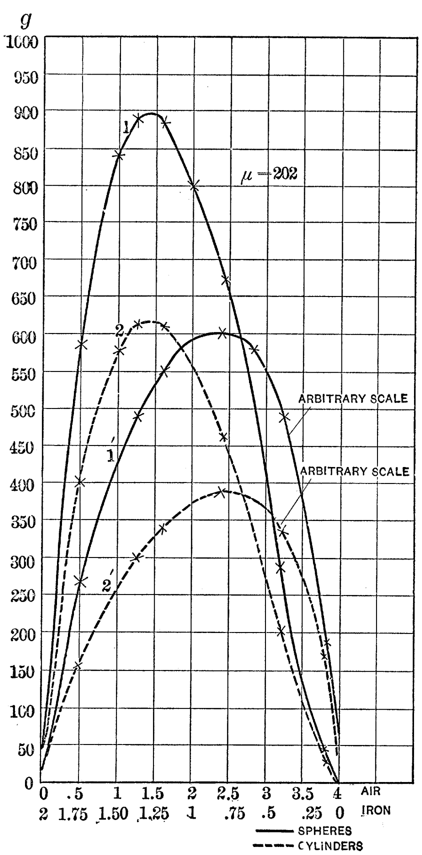

Fig. 3.

While graphically it is difficult to specify the exact position of the maximum points, yet it may be seen that the maxima for curves 1 and 2 occur for very nearly the same value of the air-gap distance and for that value for which the thickness of the shells and the airgap is the same. But the maxima for curves 1′ and 2′ occur much further to the right when the shells have become notably thinner; and approximately for the same value of the air-gap thickness and for that value which makes the air-gap thickness about three times the common thickness of the shells. By means of two shells arranged as specified above it would be possible, in the case of spherical shells, to reduce our impressed field to about $\frac{1}{900}$ of its original value within the space to be shielded. Had all of the available shielding space been filled up with iron we should have obtained but about $\frac{1}{20}$ as complete shielding and should have used about 1.3 as much iron, showing clearly the advantage of using air instead of iron in some parts of the shielding arrangement, and also how the addition of iron at the wrong place may be decidedly harmful.

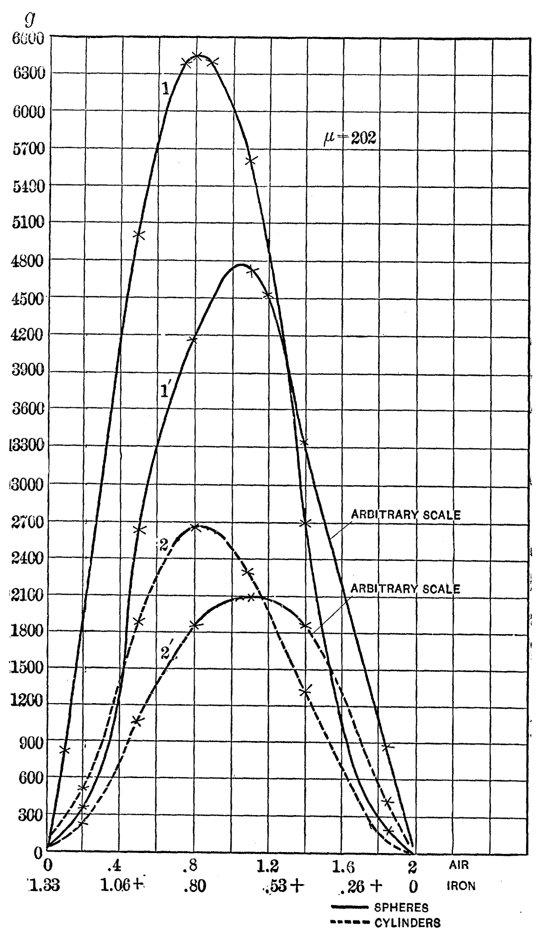

Fig. 4.

In Fig. 4 we have represented the case of three spherical shells and also of three cylindrical shells under conditions quite similar to those laid down for the bi-lamellar shells. The conditions are: that $\mu = 202$; that the shells have always a common thickness; that the two air-gaps have the same thickness; that $r = 1$, $R = 5$, r and R denoting the innermost and outermost radius respectively of the shells in both cases (spherical and cylindrical). Curves 1 and 2 refer, without reference to the weight, to the spherical and cylindrical shells respectively. Curves 1′ and 2′ are the corresponding curves representing the variation in the shielding per unit weight. The upper of the two rows of abscissæ represents the common thickness of the two air-gaps and it is the thickness which has been taken as the independent variable. The second row represents the corresponding common thickness of the three shells in both cases (spherical and cylindrical). Curves 1 and 2 reach a maximum very nearly simultaneously and for that value of the air-gap thickness at which very approximately, at any rate, the common thickness of the air-gaps equals that of the shells. Curves 1′ and 2′ whose ordinates are drawn upon independent arbitrary scales reach their maximum points nearly simultaneously and for a value of the air-gap thickness, which makes the thickness of each air-gap about twice the thickness of each shell. With three spherical shells as specified above it would be possible then to reduce an impressed field to about $\frac{1}{6400}$ of its original value within the shielded space. Had all of the available shielding space been filled up with iron we should have obtained but about $\frac{1}{150}$ as complete shielding and should have used about 1.5 as much iron. The advantages of an extended lamination of the shielding material are hereby made very obvious.

As was mentioned above, this graphic discussion is intended more as an illustrative problem than as one which would allow of the deduction of practical rules to cover the most probable conditions which might arise in practice.

In a joint paper by Professor du Bois and the writer the results of some experimental investigations with three cylindrical shells will be published.

I have to express my cordial thanks to Professor du Bois for his kind assistance during the development of the foregoing discussion.

BERLIN, April 15, 1899.