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ABSOLUTE FORMULÆ FOR THE MUTUAL INDUCTANCE OF COAXIAL SOLENOIDS.

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1. Two absolute formulæ for the mutual inductance of coaxial solenoids have recently appeared, one by Cohen,[1] expressed in terms of elliptic integrals of Legendre and a second by Nagaoka[2] involving the $\wp$-function and σ-function of Weierstrass. The computation of any given case of mutual inductance by Cohen's formula necessitates the use of Legendre's tables of elliptic integrals, while a computation by Nagaoka's formula does not depend on the use of these tables, but on the introduction of the ϑ-functions of Jacobi.

A third formula, due to Kirchhoff, was published for the first time by Coffin.[3] This formula, which is of the same type as that of Cohen, was found, however, to be incorrect.

It is the object of this paper to derive a general expression from which the formulæ of Cohen and Nagaoka, as well as a correct form of Kirchhoff's formula may be obtained.

The mutual inductance of any two circuits is given by the expression

$$M = \int\int \frac{ds\,ds_1}{r} \cos \epsilon. \tag{1}$$

[1]L. Cohen, An Exact Formula for the Mutual Inductance of Coaxial Solenoids, Bull. Bur. Stand., 3, p. 295, 1907.

[2]H. Nagaoka, Note on the Mutual Inductance of Coaxial Coils, Mathematico-Physical Soc., Tokyo, Proc., 4, p. 192, 1907. Also l. c., p. 279, 1908.

[3]J. G. Coffin, Construction and Calculation of Absolute Standards of Inductance, Bull. Bur. Stand., 2, p. 125, 1906.

Applying this expression to two coaxial solenoids, it is easy to deduce

(2) $$M=4\pi nn' [I_1-I_2-I_3+I_4],$$ [1]

where n and n' are the respective numbers of turns per unit of length of the two coils, and the quantities in the brackets are integrals of the form

(3) $$I = a_1^2a^2\int_0^\pi \frac{\sin^2\psi d\psi}{a_1^2+a^2-2a_1a\cos\psi}\sqrt{a_1^2+a^2+c^2-2a_1a\cos\psi}.$$

In this a_1 and a are the respective radii of the coils and c has the following values:

$$c=d+l+l_1 \text{ for } I=I_1,$$
$$c=d+l_1-l \text{ for } I=I_2,$$
$$c=d+l-l_1 \text{ for } I=I_3,$$
$$c=d-l-l_1 \text{ for } I=I_4,$$

where d is the distance between the centers of the coils and $2l_1$ and $2l$ are their respective lengths.

To evaluate the above integral, let

$$\cos\psi = x,$$

then

(4) $$I = \frac{\beta^{\frac{3}{2}}}{4}\int_{+1}^{-1}\frac{(x-1)(x+1)\left(x-\frac{\gamma}{\beta}\right)}{x-\frac{\alpha}{\beta}}\frac{dx}{\sqrt{(x-1)(x+1)\left(x-\frac{\gamma}{\beta}\right)}},$$

where

(5) $$\alpha=a_1^2+a^2, \quad \beta=2a_1a, \quad \gamma=a_1^2+a^2+c^2.$$

In order to reduce this integral to the normal form of Weierstrass, we let

(6) $$x-\gamma/\beta=m(s-e_1), \quad x-1=m(s-e_2), \quad x+1=m(s-e_3)$$

and determine the quantities e_1, e_2 and e_3 in such a way that their sum is zero.

m is a parameter, which can be determined by imposing upon it any arbitrary condition.

[1]See L. Cohen, l.c., p. 297.

From (4) and (6) we obtain

$$(7)\quad I = \frac{(m\beta)^{\frac{3}{2}}}{8}\int_{e_2}^{e_3} \frac{4(s-e_1)(s-e_2)(s-e_3)}{(s-e_3)-\frac{1}{m}\left(\frac{\alpha}{\beta}+1\right)} \frac{ds}{\sqrt{4(s-e_1)(s-e_2)(s-e_3)}}.$$

Now let

$$\wp u = \wp(u \mid \omega_1, \omega_3)^1$$

be the elliptic function which is determined by e_1, e_2 and e_3 and put

$$(8)\qquad s = \wp u \quad \text{and} \quad \wp w = e_3 + \frac{1}{m}\left(\frac{\alpha}{\beta}+1\right).$$

We shall assume that ω_1 and ω_3/i are real and positive quantities. We have then

$$du = \frac{ds}{\sqrt{4(s-e_1)(s-e_2)(s-e_3)}}$$

and

$$4(s-e_1)(s-e_2)(s-e_3) = 4\wp^3 u - g_2\wp u - g_3 = (\wp' u)^2,$$

where

$$(9)\qquad g_2 = 12e_3^2 - 4(e_1-e_3)(e_2-e_3), \quad g_3 = 4e_1e_2e_3.$$

On substituting in (7), we get

$$I = \frac{(m\beta)^{\frac{3}{2}}}{8}\int_{\omega_2}^{\omega_3} \frac{4\wp^3 u - g_2\wp u - g_3}{\wp u - \wp w}\,du.$$

Expanding the integrand of this expression and remembering that

$$(\wp' w)^2 = 4\wp^3 w - g_2\wp w - g_3,$$

we obtain

$$I = \frac{(m\beta)^{\frac{3}{2}}}{8}\int_{\omega_2}^{\omega_3} \left\{ 4\wp^2 u + 4\wp w\cdot\wp u + 4\wp^2 w - g_2 + \frac{(\wp' w)^2}{\wp u - \wp w} \right\} du.$$

Integrating term for term, we have

$$(10)\quad I = \frac{(m\beta)^{\frac{3}{2}}}{2}\left[\wp w\cdot\eta_1 + \left(\frac{g_2}{6} - \wp^2 w\right)\omega_1 + \frac{\wp' w}{2}\left\{\eta_1 w - \omega_1 \frac{\sigma'}{\sigma}(w) + n\pi i\right\}\right],$$

where n is an integer due to the many-valuedness of a logarithm. A method for determining n will be given later.

[1]The notation employed in this paper is that of the "Formeln und Lehrsaetze zum Gebrauche der elliptischen Functionen. Nach Vorlesungen und Aufzeichnung des Herrn K. Weierstrass, bearbeitet und herausgegeben von H. A. Schwarz."

Equation (10) enables us to evaluate the integrals I_1, I_2, I_3 and I_4 in all cases. In order to do this, it is only necessary to express the quantities in brackets in terms of the constants of the problem and to choose such a form for the parameter m as will facilitate the computation as much as possible.

2. As the number n will be seen to depend on the value of w which satisfies the known values of $\wp w$ and $\wp' w$, it is advantageous at this point to evaluate these quantities.

From (6) we find

$$\text{(11)} \quad \begin{aligned} &e_1 - e_3 = \frac{1}{m}\left(1 + \frac{\gamma}{\beta}\right), \quad e_1 - e_2 = \frac{1}{m}\left(\frac{\gamma}{\beta} - 1\right), \quad e_2 - e_3 = \frac{2}{m}, \\ &e_1 = \frac{1}{m}\frac{2}{3}\frac{\gamma}{\beta}, \quad e_2 = \frac{1}{m}\left(1 - \frac{\gamma}{3\beta}\right), \quad e_3 = -\frac{1}{m}\left(1 + \frac{\gamma}{3\beta}\right). \end{aligned}$$

From (5), (8) and (11) we get

$$\text{(12)} \qquad \wp w = \frac{1}{m}\left\{\frac{3\alpha - \gamma}{3\beta}\right\} = \frac{1}{m}\frac{2(a_1^2 + a^2) - c^2}{6a_1 a}$$

and

$$\text{(13)} \qquad \left\{ \begin{aligned} \wp w - e_1 &= \frac{1}{m}\left\{\frac{\alpha - \gamma}{\beta}\right\} = \frac{-c^2}{2ma_1 a}, \\ \wp w - e_2 &= \frac{1}{m}\left\{\frac{\alpha - \beta}{\beta}\right\} = \frac{(a_1 - a)^2}{2ma_1 a}, \\ \wp w - e_3 &= \frac{1}{m}\left\{\frac{\alpha + \beta}{\beta}\right\} = \frac{(a_1 + a)^2}{2ma_1 a}. \end{aligned} \right.$$

Since

$$(\wp' w)^2 = 4(\wp w - e_1)(\wp w - e_2)(\wp w - e_3),$$

we have from (5) and (13)

$$\text{(14)} \qquad \frac{\wp' w}{2} = \pm\sqrt{\frac{(\alpha^2 - \beta^2)(\alpha - \gamma)}{m^3\beta^3}} = \pm\frac{i(a_1^2 - a^2)c}{(m\beta)^{\frac{3}{2}}}.$$

In this equation we assume that $a_1 > a$ and that m and c are real and positive, *i. e.*, the absolute value of c is always to be employed.

It is easily seen that the value of $\wp w$ is such that

$$e_1 > \wp w > e_2,$$

for we have from (5), (8) and (11)

$$e_1 = \frac{1}{3m\beta} 2\gamma = \frac{1}{3m\beta} 2(a_1^2 + a^2 + c^2),$$

$$\wp w = \frac{1}{3m\beta}(3\alpha - \gamma) = \frac{1}{3m\beta}\Big[2(a_1^2 + a^2) - c^2 \Big],$$

$$e_2 = \frac{1}{3m\beta}(3\beta - \gamma) = \frac{1}{3m\beta}\Big[2(a_1^2 + a^2) - \Big\{ 3(a_1 - a)^2 + c^2 \Big\} \Big].$$

Since $\wp w$ is real and comprised between the values e_1 and e_2, w must have either the form

$$w = (2p+1)\,\omega_1 + (2q+t)\omega_3 \tag{15}$$

or

$$w = (2p+1)\omega_1 + (2q+1+t)\omega_3, \tag{16}$$

where p and q are any positive or negative integers, including zero, and t is a real positive quantity less than unity. If w is of the first form, $\wp' w$ is purely imaginary and positive. If w has a value corresponding to the second form, $\wp' w$ is purely imaginary and negative.

3. The integer n, which enters into the expression for I, can be determined for any given value of w as follows:

We have

$$-\int_{\omega_2}^{\omega_3} \frac{\wp' w}{\wp u - \wp w} du = \Big[\log \frac{\sigma(u+w)}{\sigma(u-w)} - 2u\frac{\sigma'}{\sigma}(w) \Big]_{\omega_2}^{\omega_3}$$

$$= 2\eta_1 w - 2\omega_1 \frac{\sigma'}{\sigma}(w) + 2n\pi i.$$

If in the u-plane we start with any given value of w and let it vary along a straight line until the nearest value of w having the form

$$w_0 = (2p+1)\omega_1 + 2q\omega_3$$

is reached, the values of the definite integral and of the expression

$$2\eta_1 w - 2\omega_1 \frac{\sigma'}{\sigma}(w)$$

can only change continuously; the number n therefore must remain unchanged during this variation of w. When w has the value w_0, the definite integral equals zero, since $\wp' w$ is then zero, and

$$2\eta_1 w_0 - 2\omega_1 \frac{\sigma'}{\sigma}(w_0) = 2q\pi i \cdot$$

We have therefore

$$n = -q.$$

4. By means of the addition theorem of the function $\frac{\sigma'}{\sigma}(u)$, equation (10) can be transformed into another form from which Cohen's expression for I can be derived.

We have

$$\frac{\sigma'}{\sigma}(u - v) = \frac{\sigma'}{\sigma}(u) - \frac{\sigma'}{\sigma}(v) + \frac{1}{2}\frac{\wp' u + \wp' v}{\wp u - \wp v}.$$

If we let

$$u = w \quad \text{and} \quad v = \omega_\lambda,$$

we obtain

$$\frac{\sigma'}{\sigma}(w - \omega_\lambda) = \frac{\sigma'}{\sigma}(w) - \eta_\lambda + \frac{1}{2}\frac{\wp' w}{\wp w - e_\lambda}, \tag{17}$$

where

$$\lambda = 1, 2 \text{ or } 3.$$

By means of (17) and the relation

$$\eta_1\omega_3 - \omega_1\eta_3 = \tfrac{1}{2}\pi i, \tag{18}$$

we can easily find three new expressions for I, but, for reasons which will appear later, we shall only consider the case $\lambda = 3$.

If $\lambda = 3$, we have

$$\eta_1 w - \omega_1 \frac{\sigma'}{\sigma}(w) = \eta_1(w - \omega_3) - \omega_1 \frac{\sigma'}{\sigma}(w - \omega_3) + \frac{\omega_1}{2}\frac{\wp' w}{\wp w - e_3} + \tfrac{1}{2}\pi i. \tag{19}$$

Substituting (19) in (10), we get

$$\begin{aligned} I = \frac{(m\beta)^{\frac{3}{2}}}{2}\Bigg[\eta_1 \wp w &+ \left\{\frac{g_2}{6} - \wp^2 w + \left(\frac{\wp' w}{2}\right)^2 \frac{1}{\wp w - e_3}\right\}\omega_1 \\ &+ \frac{\wp' w}{2}\left\{\eta_1(w - \omega_3) - \omega_1 \frac{\sigma'}{\sigma}(w - \omega_3) + (n + \tfrac{1}{2})\pi i\right\}\Bigg]. \end{aligned} \tag{20}$$

This equation leads to Cohen's formula, while from (10) Kirchhoff's formula can be deduced.

If in equation (10) we put

$$w=\omega_1+w_1 i,$$

where w_1 fulfils the condition

$$0<w_1<\omega_3/i,$$

then w is of the form (15), the value of n is zero and $\wp' w$ *is positive.*[1]

Since we also have

$$\frac{\sigma'}{\sigma}(\omega_1+w_1 i)=\eta_1+\frac{\sigma_1'}{\sigma_1}(w_1 i),$$

the expression (10) becomes

$$(21)\quad I=\frac{(m\beta)^{\frac{3}{2}}}{2}\Bigg[\eta_1\wp w+\left\{\frac{g_2}{6}-\wp^2 w\right\}\omega_1 + \frac{\wp' w}{2}\left\{\eta_1 w_1 i-\omega_1\frac{\sigma_1'}{\sigma_1}(w_1 i)\right\}\Bigg].$$

If in (20) we let w have such a value that

$$w-\omega_3=\omega_1+w_2 i,$$

or

$$w=\omega_2+w_2 i,$$

where

$$0<w_2<\omega_3/i,$$

then w is of the form (16), n is -1 and $\wp' w$ is negative. Since we also have

$$\frac{\sigma'}{\sigma}(w-\omega_3)=\eta_1+\frac{\sigma_1'}{\sigma_1}(w_2 i)$$

equation (20) becomes

$$(22)\quad I=\frac{(m\beta)^{\frac{3}{2}}}{2}\Bigg[\eta_1\wp w+\left\{\frac{g_2}{6}-\wp^2 w+\left(\frac{\wp' w}{2}\right)^2\frac{1}{\wp w-e_3}\right\}\omega_1 + \frac{\wp' w}{2}\left\{\eta_1 w_2 i-\omega_1\frac{\sigma_1'}{\sigma_1}(w_2 i)-\frac{1}{2}\pi i\right\}\Bigg].$$

5. In order to transform (21) and (22) into the corresponding formulæ of Kirchhoff and Cohen, we must find the relations between

[1]Nagaoka has given $\wp' w$ the wrong sign in this case; however, in his computation $\wp' w$ has the positive sign.

the integrals of Legendre and the quantities which enter into these equations.

If in the incomplete integral of the first kind to the modulus k

$$F(\varphi, k') = \int_0^{\phi} \frac{d\varphi}{\sqrt{1 - k'^2 \sin^2 \varphi}},$$

we let

$$\varphi = 90 - \psi,$$

we get

$$F(\varphi, k') = \frac{1}{k} \int_{\pi/2-\phi}^{\pi/2} \frac{d\psi}{\sqrt{1 + \left(\frac{k'}{k}\right)^2 \sin^2 \psi}}.$$

where

$$k^2 + k'^2 = 1.$$

Changing the variable again, by letting

$$t = \sin^2 \psi$$

we obtain

$$F(\varphi, k') = \frac{i}{k'} \int_{\cos^2\phi}^{1} \frac{dt}{\sqrt{4t(t-1)\left(t + \frac{k^2}{k'^2}\right)}}. \tag{23}$$

To transform this expression into the normal form of Weierstrass, let

$$t + k^2/k'^2 = m'(s - e_3), \quad t = m'(s - e_2), \quad t - 1 = m'(s - e_1), \tag{24}$$
$$e_1 + e_2 + e_3 = 0,$$

m' being a parameter. From these equations we find

$$e_1 - e_2 = \frac{1}{m'}, \quad e_1 - e_3 = \frac{1}{m'k'^2}, \quad e_2 - e_3 = \frac{k^2}{m'k'^2}, \tag{25}$$
$$k^2 = \frac{e_2 - e_3}{e_1 - e_3}, \quad k'^2 = \frac{e_1 - e_2}{e_1 - e_3}$$

From (23), (24) and the second equation of (25) we have

$$F(\varphi, k') = i\sqrt{e_1 - e_3} \int_s^{e_1} \frac{ds}{\sqrt{4(s - e_1)(s - e_2)(s - e_3)}}, \tag{26}$$

where the value of the lower limit is easily determined from the second equation of (24).

We have

$$s = \frac{t}{m'} + e_2 = \frac{\cos^2\varphi}{m'} + e_2 = \cos^2\varphi(e_1 - e_2) + e_2. \tag{27}$$

It is evident that for real values of φ, s is always real and comprised between e_2 and e_1.

If we now let

$$s = \wp u = \wp(u \mid \omega_1,\ \omega_3)$$

be the elliptic function belonging to the quantities e_1, e_2 and e_3, as determined above, we have

$$du = \frac{ds}{\sqrt{4(s-e_1)(s-e_2)(s-e_3)}},$$

and

$$F(\varphi, k') = i\sqrt{e_1 - e_3}\int_u^{\omega_1} du = i\sqrt{e_1 - e_3}(\omega_1 - u), \tag{28}$$

the lower limit u being determined by (27) for any given value of φ.

Since $s = \wp u$ is real and comprised between e_1 and e_2, we may put

$$u = \omega_1 + u_1 i,$$

$$0 < u_1 < \omega_3/i,$$

so that (28) becomes

$$F(\varphi, k') = u_1\sqrt{e_1 - e_3}. \tag{29}$$

If φ equals $\pi/2$, $F(\varphi, k')$ becomes a complete integral to the modulus k', *i. e.*,

$$F(k') = \frac{\omega_3}{i}\sqrt{e_1 - e_3}.$$

In the same manner it can be shown that the incomplete integral of the second kind to the modulus k'

$$E(\varphi, k') = \int_0^{\phi}\sqrt{1 - k'^2\sin^2\varphi}\, d\varphi,$$

reduces to

$$E(\varphi, k') = \frac{i}{\sqrt{e_1 - e_3}}\left[e_3(u - \omega_1) + \frac{\sigma'}{\sigma}(u) - \eta_1\right],$$

$$= \frac{i}{\sqrt{e_1 - e_3}}\left[\frac{\sigma_1'}{\sigma_1}(u_1 i) + e_3 u_1 i\right]. \tag{30}$$

From this we find for the complete integral to the modulus k'

$$E(k') = \frac{i}{\sqrt{e_1 - e_3}} [\eta_3 + e_3\omega_3].$$

The expressions for the complete integrals of the first and second kind to the modulus k may be deduced in the same manner, as that in which the corresponding ones to the modulus k' were obtained. We have

$$F(k) = \omega_1\sqrt{e_1 - e_3}, \tag{31}$$

$$E(k) = \frac{1}{\sqrt{e_1 - e_3}} \left[\eta_1 + e_1\omega_1 \right]. \tag{32}$$

6. If in equation (21) we now express w_1, $\frac{\sigma_1'}{\sigma_1}(w_1 i)$, ω_1 and η_1 in terms of elliptic integrals of Legendre and g_2, $\wp w$, $\wp' w$, e_1 and e_3 in terms of the constants of the problem, we obtain the correct form of Kirchhoff's formula.

If after having fixed the parameter m, we assign to e_1, e_2 and e_3 in (25) the values given in (11), the moduli k and k' and the parameter m' will be definitely determined. If we also let

$$u = w = \omega_1 + w_1 i, \quad s = \wp w,$$

the value of the amplitude φ of the incomplete integrals can be determined from (27).

From (29), (30), (31) and (32) the following relations can be easily be deduced:

$$\tag{33} \left\{ \begin{aligned} w_1 &= \frac{1}{\sqrt{e_1 - e_3}} F(\varphi, k') \\ -\frac{\sigma_1'}{\sigma_1}(w_1 i) &= i\sqrt{e_1 - e_3} E(\varphi, k') + \frac{ie_3}{\sqrt{e_1 - e_3}} F(\varphi, k'), \\ \eta_1 &= \sqrt{e_1 - e_3} \left[E(k) - \frac{e_1}{e_1 - e_3} F(k) \right], \\ \omega_1 &= \frac{1}{\sqrt{e_1 - e_3}} F(k). \end{aligned} \right.$$

Substituting these values in (21), we get

$$(34)\quad \begin{aligned} I = \frac{(m\beta)^{\frac{3}{2}}}{2}\Bigg[&\sqrt{e_1-e_3}\cdot \wp w\cdot E(k) + \frac{1}{\sqrt{e_1-e_3}}\left\{\frac{g_2}{6} - \wp w(\wp w+e_1)\right\}F(k) \\ &- \frac{i\wp' w}{2}\{F(k)[F(\varphi,k')-E(\varphi,k')]-E(k)F(\varphi,k')\}\Bigg]. \end{aligned}$$

The value of $g_2/6$ in this equation can be obtained from (9), which, by substituting from (11) and (5), becomes

$$(35)\quad \frac{g_2}{6} = \frac{2}{9m^2}\left[\frac{\gamma^2}{\beta^2}+3\right] = \frac{2}{9m^2}\left[\left(\frac{a_1^2+a^2+c^2}{2a_1a}\right)^2+3\right].$$

From (5), (11), (12) and (35) we easily find

$$(36)\quad \begin{aligned} \wp w\sqrt{e_1-e_3} &= \frac{1}{(m\beta)^{\frac{3}{2}}}\left[\frac{3\alpha-\gamma}{3}\right]\sqrt{\gamma+\beta} \\ &= \frac{1}{(m\beta)^{\frac{3}{2}}}\,\frac{2(a_1^2+a^2)-c^2}{3}\sqrt{(a_1+a)^2+c^2} \end{aligned}$$

and

$$(37)\quad \begin{aligned} \left\{\frac{g_2}{6} - \wp w(\wp w+e_1)\right\}\frac{1}{\sqrt{e_1-e_3}} &= \frac{1}{(m\beta)^{\frac{3}{2}}}\left[\frac{\gamma^2+2\beta^2-3\alpha^2}{3\sqrt{\gamma+\beta}}\right] \\ &= \frac{1}{(m\beta)^{\frac{3}{2}}}\left[\frac{c^4+2c^2(a_1^2+a^2)-2(a_1^2-a^2)^2}{3\sqrt{(a_1+a)^2+c^2}}\right]. \end{aligned}$$

Substituting (14), taken with the positive sign, (36) and (37) in (34), we obtain

$$(38)\quad \begin{aligned} I = \tfrac{1}{2}\Bigg[&\frac{2(a_1^2+a^2)-c^2}{3}\sqrt{(a_1+a)^2+c^2}\,E(k) \\ &+ \frac{c^4+2c^2(a_1^2+a^2)-2(a_1^2-a^2)^2}{3\sqrt{(a_1+a)^2+c^2}}F(k) \\ &+ (a_1^2-a^2)c\{F(k)[F(\varphi,k')-E(\varphi,k')]-E(k)F(\varphi,k')\}\Bigg], \end{aligned}$$

which is the correct value of I for Kirchhoff's formula.

The amplitude φ may be obtained from (27). We have

$$\cos^2\varphi = \frac{\wp w - e_2}{e_1 - e_2}$$

or

$$\sin^2\varphi = 1 - \frac{\wp w - e_2}{e_1 - e_2},$$

and by (5), (11) and (13) this becomes

$$\sin^2 \varphi = \frac{\gamma - \alpha}{\gamma - \beta} = \frac{c^2}{(a_1 - a)^2 + c^2}. \tag{39}$$

The moduli k and k' are obtained from (5), (11) and (25). We have

$$k^2 = \frac{4a_1a}{(a_1 + a)^2 + c^2}, \quad k'^2 = \frac{(a_1 - a)^2 + c^2}{(a_1 + a)^2 + c^2}. \tag{40}$$

7. If we now let

$$u = w - \omega_3 = \omega_1 + w_2 i,$$

where

$$0 < w_2 < \omega_3/i,$$

and substitute the corresponding values of (33) in (22), we obtain

$$\begin{aligned} I = \frac{(m\beta)^{\frac{3}{2}}}{2}\Bigg[& \sqrt{e_1 - e_3}\cdot \wp w \cdot E(k) + \frac{1}{\sqrt{e_1 - e_3}}\left\{ \frac{g_2}{6} - \wp w(\wp w + e_1) \right. \\ & \left. + \left(\frac{\wp' w}{2}\right)^2 \frac{1}{\wp w - e_3} \right\} F(k) \\ & - \frac{i\wp' w}{2}\left\{ F(k)[F(\varphi, k') - E(\varphi, k')] - E(k)F(\varphi, k') + \frac{\pi}{2} \right\}\Bigg]. \end{aligned} \tag{41}$$

In this equation $\wp' w/2$ is to be taken with its negative sign.

The only quantities in this equation, which have not been expressed in terms of the constants of the problem, are the coefficient of $F(k)$ of the second term in the square brackets and the amplitude φ.

From (5), (11), (13) and (14) we get

$$\begin{aligned} \frac{1}{\sqrt{e_1 - e_3}}\left(\frac{\wp' w}{2}\right)^2 \frac{1}{\wp w - e_3} &= \frac{1}{(m\beta)^{\frac{3}{2}}} \frac{(\alpha - \beta)(\alpha - \gamma)}{\sqrt{\gamma + \beta}} \\ &= -\frac{1}{(m\beta)^{\frac{3}{2}}} \frac{(a_1 - a)^2 c^2}{\sqrt{(a_1 + a)^2 + c^2}}. \end{aligned}$$

Adding this to (37) and reducing, we get

$$\begin{aligned} & \frac{1}{\sqrt{e_1 - e_3}}\left\{ \frac{g_2}{6} - \wp w(\wp w + e_1) + \left(\frac{\wp' w}{2}\right)^2 \frac{1}{\wp w - e_3} \right\} \\ & \qquad = \frac{1}{(m\beta)^{\frac{3}{2}}}\left[\frac{c^4 - (a_1^2 - 6a_1a + a^2)c^2 - 2(a_1^2 - a^2)^2}{3\sqrt{(a_1 + a)^2 + c^2}} \right]. \end{aligned} \tag{42}$$

Substituting (36), (42) and (14), the latter taken with the negative sign, in (41), we finally have

$$
\begin{aligned}
I = \frac{1}{2}\Bigg[&\frac{2(a_1^2+a^2)-c^2}{3}\sqrt{(a_1+a)^2+c^2}\,E(k)\\
(43)\qquad &+\frac{c^4-(a_1^2-6a_1a+a^2)c^2-2(a_1^2-a^2)^2}{3\sqrt{(a_1+a)^2+c^2}}F(k)\\
&-(a_1^2-a^2)c\left\{F(k)[F(\varphi,k')-E(\varphi,k')]-E(k)F(\varphi,k')+\frac{\pi}{2}\right\}\Bigg],
\end{aligned}
$$

which is the expression V in Cohen's formula.

It remains to find the amplitude φ for this expression. If in (27) we let

$$s=\wp(w-\omega_3),$$

and subtract e_3 from both sides of the equation, we obtain

$$\wp(w-\omega_3)-e_3=\cos^2\varphi(e_1-e_2)+(e_2-e_3)=(e_1-e_3)-(e_1-e_2)\sin^2\varphi.$$

Now

$$\wp(w-\omega_3)-e_3=\frac{(e_3-e_1)(e_3-e_2)}{\wp w-e_3},$$

and consequently

$$\sin^2\varphi=\frac{e_1-e_3}{e_1-e_2}\frac{\wp w-e_2}{\wp w-e_3},$$

which by (11), (13) and (5) becomes

$$(44)\qquad \sin^2\varphi=\frac{\gamma+\beta}{\gamma-\beta}\frac{\alpha-\beta}{\alpha+\beta}=\frac{(a_1^2-a^2)^2+c^2(a_1-a)^2}{(a_1^2-a^2)^2+c^2(a_1+a)^2}.$$

The values of k^2 and k'^2 are of course the same as those in Kirchhoff's formula.

8. By letting λ equal 1 and 2 successively in (17) two additional expressions for I can be derived, but the value of the amplitude φ becomes complex in these cases. Nothing new is obtained by letting

$$v=-\omega_\lambda,$$

instead of

$$v=\omega_\lambda,$$

as was done on page 622 in deriving Cohen's formula.

9. The values of g_2, η_1, ω_1 and w depend on the value of m. If, for example, we let

$$\frac{(m\beta)^{\frac{3}{2}}}{2} = 2a_1a,$$

so that

$$m = \left(\frac{2}{a_1a}\right)^{\frac{1}{3}},$$

and we then compute the above quantities, we obtain Nagaoka's values. His expression for I is obtained from (10) by giving m this value and letting n equal zero. A convenient value of m for computing I from (21) is the one for which $e_1 - e_3 = 1$. We shall give the special formulæ for this case later.

10. In addition to the expressions (11), (12), (13), (14), (35) and (40), two sets of formulæ are necessary for the computation of I from (21), one set for negative and another set for positive values of e_2. The series of the first set are more convergent when $e_2 < 0$, while those of the second set converge more rapidly when $e_2 > 0$.

From (5) and (11) we see that

$$e_2 \gtreqless 0 \quad \text{when} \quad 6a_1a \gtreqless a_1^2 + a^2 + c^2.$$

In the following formulæ all radicals are to be considered as positive quantities.

Formulæ for $e_2 < 0$.

Let

$$l = \frac{\sqrt[4]{e_1 - e_3} - \sqrt[4]{e_1 - e_2}}{\sqrt[4]{e_1 - e_3} + \sqrt[4]{e_1 - e_2}} = \frac{1 - \sqrt{k'}}{1 + \sqrt{k'}} \tag{45}$$

and

$$h = \tfrac{1}{2}l + 2(\tfrac{1}{2}l)^5 + 15(\tfrac{1}{2}l)^9 + 150(\tfrac{1}{2}l)^{13} + \cdots,$$

then ω_1 can be calculated from either of the two expressions

(46)
$$\omega_1 = \frac{2\pi}{(\sqrt[4]{e_1 - e_3} + \sqrt[4]{e_1 - e_2})^2}(1 + 2h^4 + 2h^{16} + \cdots)^2,$$

$$\omega_1 = \frac{\pi}{2\sqrt{e_1 - e_3}}(1 + 2h + 2h^4 + 2h^9 + \cdots)^2.$$

To determine η_1 we have

$$\eta_1\omega_1 = \frac{\pi^2}{12}\,\frac{1 - 3^3h^2 + 5^3h^6 - 7^3h^{12} + \cdots}{1 - 3h^2 + 5h^6 - 7h^{12} + \cdots}. \tag{47}$$

To find the value of w_1 which satisfies the values of $\wp w$ and $\wp' w$ given in (12) and (14), we write

$$\mathfrak{L}_{0,1} = (\tfrac{1}{2})^2 l^4 + \left(\frac{1\cdot 3}{2\cdot 4}\right)^2 l^8 + \left(\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\right)^2 l^{12} + \cdots$$

$$\mathfrak{L}_{0,2} = \left(\frac{1\cdot 3}{2\cdot 4}\right)^2 l^8 + \left(\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\right)^2 l^{12} + \cdots$$

$$\mathfrak{L}_{0,3} = \left(\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\right)^2 l^{12} + \cdots$$

and let

$$(48) \qquad \wp w = s, \quad \frac{\sqrt[4]{e_1 - e_3}\sqrt{s - e_2} - \sqrt[4]{e_1 - e_2}\sqrt{s - e_3}}{\sqrt[4]{e_1 - e_3}\sqrt{s - e_2} + \sqrt[4]{e_1 - e_2}\sqrt{s - e_3}} = lt;$$

then

$$(49) \qquad w_1 = \frac{\omega_1}{\pi}\log\text{nat}\,(\sqrt{t^2 - 1} - t) - \frac{2\sqrt{t^2 - 1}}{(\sqrt[4]{e_1 - e_3} + \sqrt[4]{e_1 - e_2})^2}\Big\{\mathfrak{L}_{0,1}t + \tfrac{2}{3}\mathfrak{L}_{0,2}t^3 + \frac{2\cdot 4}{3\cdot 5}\mathfrak{L}_{0,3}t^5 + \cdots\Big\}$$

Instead of (48) and (49) the following two formulæ may also be employed:

$$(50) \qquad \frac{\sqrt{e_1 - s} - \sqrt[4]{e_1 - e_3}\sqrt[4]{e_1 - e_2}}{\sqrt{e_1 - s} + \sqrt[4]{e_1 - e_3}\sqrt[4]{e_1 - e_2}} = lt',$$

$$(51) \qquad w_1 = \frac{\omega_1}{\pi}\log\text{nat}\,\frac{1}{h(\sqrt{t'^2 - 1} - t')} + \frac{2\sqrt{t'^2 - 1}}{(\sqrt[4]{e_1 - e_3} + \sqrt[4]{e_1 - e_2})^2}\Big\{\mathfrak{L}_{0,1}t' + \tfrac{2}{3}\mathfrak{L}_{0,2}t'^3 + \frac{2\cdot 4}{3\cdot 5}\mathfrak{L}_{0,3}t'^5 \cdots\Big\}.$$

If the absolute value of t is less than that of t', it will in general be better to use (49) to compute w_1, in the contrary case it is better to employ (51).

It can be shown that t and t' are always negative and that their absolute values are always greater than unity.

If we now put

$$v_1 = \frac{w_1}{2\omega_1}, \quad z = e^{\pi v_1},$$

then

$$(52)\qquad \eta_1 w_1 i - \omega_1 \frac{\sigma_1'}{\sigma_1}(w_1 i) = \frac{\pi i}{2}\left\{\frac{(z - z^{-1}) + 3h^2(z^3 - z^{-3}) + 5h^6(z^5 - z^{-5})\cdots}{(z + z^{-1}) + h^2(z^3 + z^{-3}) + h^6(z^5 + z^{-5})\cdots}\right\}$$

Formulæ for $e_2 > 0$

$$(53)\qquad l_1 = \frac{\sqrt[4]{e_1 - e_3} - \sqrt[4]{e_2 - e_3}}{\sqrt[4]{e_1 - e_3} + \sqrt[4]{e_2 - e_3}} = \frac{1 - \sqrt{k}}{1 + \sqrt{k}},$$

$$h_1 = \tfrac{1}{2}l_1 + 2(\tfrac{1}{2}l_1)^5 + 15(\tfrac{1}{2}l_1)^9 + 150(\tfrac{1}{2}l_1)^{13} + \cdots,$$

$$\frac{\omega_3}{\pi i} = \frac{2}{(\sqrt[4]{e_1 - e_3} + \sqrt[4]{e_2 - e_3})^2}(1 + 2h_1^4 + 2h_1^{16} + \cdots)^2,$$

or

$$\frac{\omega_3}{\pi i} = \frac{1}{2\sqrt{e_1 - e_3}}(1 + 2h_1 + 2h_1^4 + 2h_1^9 + \cdots)^2,$$

$$(54)\qquad \omega_1 = \frac{\omega_3}{\pi i}\log\text{nat}\left(\frac{1}{h_1}\right),$$

$$(55)\qquad \eta_1 = \frac{\pi i}{2\omega_3}\left[1 - \tfrac{1}{6}\log\text{nat}\left(\frac{1}{h_1}\right)\cdot\frac{1 - 3^3h_1^2 + 5^3h_1^6 - 7^3h_1^{12} + \cdots}{1 - 3h_1^2 + 5h_1^6 - 7h_1^{12} + \cdots}\right].$$

The formulæ which are necessary to determine w_1 in this case are:

$$\mathfrak{L}_{1,1} = \left(\frac{1}{2}\right)^2 l_1^4 + \left(\frac{1\cdot 3}{2\cdot 4}\right)^2 l_1^8 + \left(\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\right)^2 l_1^{12} + \cdots,$$

$$\mathfrak{L}_{1,2} = \left(\frac{1\cdot 3}{2\cdot 4}\right)^2 l_1^8 + \left(\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\right)^2 l_1^{12} + \cdots.$$

$$\mathfrak{L}_{1,3} = \left(\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\right)^2 l_1^{12} + \cdots,$$

$$(56)\qquad \frac{\sqrt{s - e_3} - \sqrt[4]{e_1 - e_3}\,\sqrt[4]{e_2 - e_3}}{\sqrt{s - e_3} + \sqrt[4]{e_1 - e_3}\,\sqrt[4]{e_2 - e_3}} = l_1 t_1',$$

$$(57)\qquad \varphi = \text{arc tan}\frac{t_1'}{\sqrt{1 - t_1'^2}},\quad -\frac{\pi}{2} \leqq \varphi \leqq \frac{\pi}{2},$$

(58)
$$w_1 = \frac{\omega_3}{i}\left(\frac{1}{2} - \frac{\varphi}{\pi}\right) + \frac{2\sqrt{1 - t_1'^2}}{(\sqrt[4]{e_1 - e_3} + \sqrt[4]{e_2 - e_3})^2} \times \left\{ \mathfrak{L}_{1,\,1} t_1' + \tfrac{2}{3}\mathfrak{L}_{1,\,2} t_2'^3 + \frac{2\cdot 4}{3\cdot 5}\mathfrak{L}_{1,\,3} t_1'^5 + \cdots \right\}.$$

Now let

$$v = \frac{w_1 i}{2\omega_3},$$

then

(59)
$$\eta_1 w_1 i - \omega_1 \frac{\sigma_1'}{\sigma_1}(w_1 i) = i\pi v - \frac{\pi\omega_1}{\omega_3}\,\frac{2h_1 \sin 2v\pi - 4h_1^4 \sin 4v\pi + 6h_1^9 \sin 6v\pi - \cdots}{1 - 2h_1 \cos 2v\pi + 2h_1^4 \cos 4v\pi - 2h_1^9 \cos 6v\pi + \cdots}.$$

Formulæ corresponding to (48) and (49) could be given for this case, but the values $l_1 t_1$ and t_1 are complex and make the computation troublesome.

t_1' in the above formulæ may be either positive or negative; its absolute value is always less than unity.

In many cases l and l_1 are small quantities, so that the second terms of (49), (51) and (58) can be omitted. The approximate expressions for z and v are then quite simple.

We have from (49)

(60)
$$z^2 = \sqrt{t^2 - 1} - t,$$

from (51)

(61)
$$z^{-2} = h(\sqrt{t'^2 - 1} - t'),$$

and from (58)

(62)
$$v = \tfrac{1}{4} - \frac{\varphi}{2\pi}.$$

11. If we let

(63)
$$m = 1 + \frac{\gamma}{\beta} = \frac{(a_1 + a)^2 + c^2}{2a_1 a},$$

so that

$$e_1 - e_3 = 1, \quad e_2 - e_3 = k^2, \quad e_1 - e_2 = k'^2,$$

$$s - e_1 = \frac{-c^2}{(a_1 + a)^2 + c^2}, \quad s - e_2 = \frac{(a_1 - a)^2}{(a_1 + a)^2 + c^2}, \quad s - e_3 = \frac{(a_1 + a)^2}{(a_1 + a)^2 + c^2},$$

some of the formulæ of the last paragraph simplify somewhat. The other quantities which are necessary to calculate I from (21) have the following forms in this case:

$$m\beta = (a_1+a)^2+c^2, \tag{64}$$

$$s = \wp w = \frac{1}{3}\frac{2(a_1^2+a^2)-c^2}{(a_1+a)^2+c^2}, \tag{65}$$

$$\frac{\wp' w}{2} = \frac{i(a_1^2-a^2)c}{[(a_1+a)^2+c^2]^{\frac{3}{2}}}, \tag{66}$$

$$\frac{g_2}{6} = \frac{2}{9}\left[\left\{\frac{a_1^2+a^2+c^2}{(a_1+a)^2+c^2}\right\}^2 + 12\left\{\frac{a_1 a}{(a_1+a)^2+c^2}\right\}^2\right]. \tag{67}$$

Since $e_1 - e_3 = 1$, e_2 can be expressed in terms of m alone. We easily find

$$e_2 = \frac{1}{3}\left(\frac{4}{m} - 1\right),$$

from which we see that

$$e_2 \lesseqgtr 0 \quad \text{when} \quad m \gtreqless 4.$$

When $c=0$ and $a_1=a$ we have from (63)

$$m=2,$$

which is the minimum value of m. This is one of the values of m when the two coils coincide, so that we have self-inductance instead of mutual inductance. As a very convenient expression for computing the self inductance of a single layer coil has been given by Nagaoka,[1] it is not necessary to give formulæ for self inductance here.

12. Cohen's as well as Kirchhoff's formula involves c^4, which is inconveniently large when one or both coils are long, or when there is a great distance between their centers. Kirchhoff's formula has the further disadvantage that, when c is large compared to (a_1+a), the arguments φ and θ of Legendre's tables are both near 90°, so that $E(\varphi, k')$ and especially $F(\varphi, k')$ cannot be determined with sufficient accuracy, unless terms of the fourth and even fifth order are taken into account in the interpolation.

[1]H. Nagaoka, Note on the Self-inductance of Solenoides, *l. c.*, p. 314, 1908.

13. The case computed by Cohen, where

$$d=0, \quad 2l=200, \quad 2l_1=20, \quad a=10, \quad a_1=15,$$

when calculated by Kirchhoff's formula, gave

$$M=4\pi nn'\,6212.9,$$

while formula (21), when (61) was used and m was so chosen that $e_1-e_3=1$, gave

$$M=4\pi nn'\,6213.77,$$

seven place logarithms having been employed. The values of M obtained by Cohen and Nagaoka were

$$M=4\pi nn'\,6213.4$$

and

$$M=4\pi nn'\,6213.51,$$

respectively.

A computation by the series given for $e_2>0$, which do not converge very rapidly for this case, gave

$$M=4\pi nn'\,6213.63.$$

The results of the computation by (21) and (61) are given on the following page.

	$a_1=15,\ a=10,\ c=110.$	$a_1=15,\ a=10,\ c=90.$
$\log \mathfrak{p}w = \log \frac{1}{3} \frac{2(a_1{}^2+a^2)-c^2}{(a_1+a)^2+c^2}$	$9_n.4770264$	$9_n.4542696$
$\frac{g_2}{6} = \frac{2}{9}\left[\left(\frac{a_1{}^2+a^2+c^2}{(a_1+a)^2+c^2}\right)^2 + 3\left(\frac{2a_1a}{(a_1+a)^2+c^2}\right)^2\right]$	0.2122382	0.2079914
$\log \frac{\mathfrak{p}'w}{2i} = \log \frac{(a_1{}^2-a^2)c}{[(a_1+a)^2+c^2]^{\frac{3}{2}}}$	7.9813160	8.1400044
$\log (m\beta)^{\frac{3}{2}} = \log [(a_1+a)^2+c^2]^{\frac{3}{2}}$	6.1569867	5.9111481
$k^2 = \frac{4a_1a}{(a_1+a)^2+c^2}$	0.0471513	0.0687679
$k'^2 = 1-k^2$	0.9528487	0.9312321
$\log l = \log \frac{1-\sqrt{k'}}{1+\sqrt{k'}}$	7.7808415	7.9496612
$\log h = \log \left(\frac{l}{2}+\cdots\right)$	7.4798115	7.6486312
$\log \omega_1 = \log \frac{2\pi}{(1+\sqrt{k'})^2}(1+\cdots)^2$	0.2013483	0.2038209
$\log \eta_1 = \log \frac{\pi^2}{12\omega_1}\frac{1-9h^2\cdots}{1-3h^2\cdots}$	9.7136753	9.7110910
$\sqrt{e_1-s} = \sqrt{\frac{c^2}{(a_1+a)^2+c^2}}$	0.9751330	0.9635180
$\log lt' = \log \frac{\sqrt{e_1-s}-\sqrt{k'}}{\sqrt{e_1-s}+\sqrt{k'}}$	7.8164572	7.9856940
$-t'$	1.085465	1.086508
$\log z = \log [h(\sqrt{t'^2-1}-t')]^{-\frac{1}{2}}$	1.1709451	1.0860006
$\log \left\{ \frac{(z-z^{-1})+3h^2(z^3-z^{-3})+\cdots}{(z+z^{-1})+h^2(z^3+z^{-3})+\cdots} \right\}$	9.9977951	9.9967327
$\log \left\{ \quad \right\} \frac{\mathfrak{p}'w}{2}\cdot\frac{\pi i}{2} = \log \frac{\mathfrak{p}'w}{2}\left\{\eta_1 w_1 i - \omega_1 \frac{\sigma_1'}{\sigma_1}(w_1 i)\right\}$	$8_n.1752310$	$8_n.3328570$
$\mathfrak{p}w\cdot\eta_1$	−0.1551321	−0.1463391
$\omega_1\left(\frac{g_2}{6}-\mathfrak{p}^2w\right)$	0.1943994	0.2030302
$\frac{\mathfrak{p}'w}{2}\left\{\eta_1 w_1 i - \omega_1 \frac{\sigma_1'}{\sigma_1}(w_1 i)\right\}$	−0.0149703	−0.0215207
Sum = Σ	0.0242970	0.0351704
$\log \Sigma$	8.3855527	8.5461773
$\log 2I = \log (m\beta)^{\frac{3}{2}}\cdot\Sigma$	4.5425394	4.4573254
$2I$	34877.02	28663.25

$$2(I_1 - I_2) = 2(I_4 - I_3) = 34877.02 - 28663.25 = 6213.77$$
$$M = 4\pi nn' 6213.77$$

Worcester Polytechnic Institute,
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