EXPERIMENTS ON RESONANCE IN WIRELESS TELEGRAPH CIRCUITS. PART V.[1]

XIII. THE ELECTRICAL OSCILLATIONS IN CONNECTED SYSTEMS OF CIRCUITS.

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AS a contribution toward the study of the nature of the oscillation at the sending station used in wireless telegraphy, the following account is given of a series of experiments on the wave-lengths obtained when two condenser circuits inductively or directly connected together are set into oscillation.

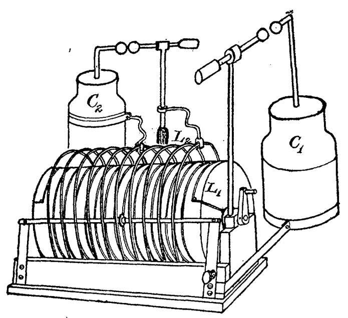
The problem here undertaken is the measurement of the wave-length of each of two condenser circuits with each circuit standing alone, and then a measurement of the resultant wave-lengths when the two circuits are coupled together after the manner of the electromagnetically coupled and the direct coupled sending stations of wireless telegraphy. This problem has already had ample theoretical study, but I am not aware of its having been given extended experimental examination. So the present experiments were undertaken, in part, as a test of the applicability of the theory, and, in part, for the benefit to be derived from an experimental study ot specific cases.

The dimensions of the condensers and inductances were chosen to have such values as would bring the wave-lengths within the range ordinarily used in wireless telegraphy (200 meters to 2,000 meters).

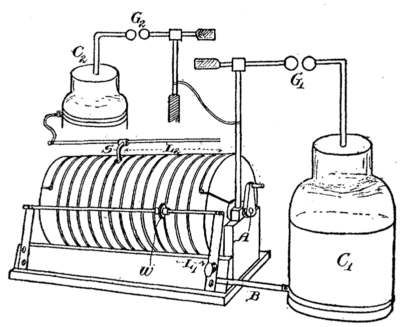
Forms of Circuit Studied. — The two types of circuits in which the oscillations were studied are shown in Fig. 49 and Fig. 50. In the *Electromagnetically Connected System* shown in Fig. 49 two condensers C_1 and C_2 are connected to two coils L_1 and L_2 which

[1] The previous parts were published in the PHYSICAL REVIEW, as follows: Part I., Vol. 19, p. 196, 1904; part II., Vol. 20, p. 220, 1905; part III., Vol. 21, p. 367, 1905; part IV., Vol. 22, p. 159, 1906.

are inductively related but insulated from each other. The number of active turns of wire on each of the coils may be varied. Each of the condenser circuits is provided with a spark-gap, so that either circuit, when connected to a step-up transformer, may be used as the discharge circuit. The other circuit may then be looked upon as a secondary circuit. When the spark-gap of the secondary is opened too wide to permit the passage of a spark, or better when the secondary is removed, the period of oscillation is the period of the primary alone. When, on the other hand, the secondary is left in place and the spark-gap of the secondary is closed, the



INDUCTIVE
Fig. 49.



DIRECT
Fig. 50.

oscillations of the discharge circuit are modified by the presence of the closed secondary. It is the purpose of the experiment to measure the wave-lengths produced under these several conditions and to compare the measured values with values computed from theoretical considerations.

In the *Direct Coupled System*, represented in Fig. 50, which was also studied, the transformer of the inductive coupling is replaced by an auto-transformer; that is the two condensers C_1 and C_2 are made to discharge through parts of the same coil. In this case also, both the inductances L_1 and L_2 can be varied independently by the motion of the contacts W and S. Also both the condenser circuits are provided with spark-gaps so that either circuit may be caused to oscillate alone or to constitute the discharge circuit in a connected system with closed secondary.

These two forms of circuits, Figs. 49 and 50, are derived from the ordinary wireless telegraph circuits by replacing the antenna and ground of the wireless telegraph station by the two coatings of a condenser respectively. The circuits were thus simplified for the purpose of the present experiments because in the simplified form the theoretical formulas ought to apply exactly. In a subsequent investigation it is proposed to study also the less simple cases.

Dimensions of the Inductances.—The coils employed in the apparatus shown in Figs. 49 and 50 had the following dimensions.

Coil.	No. Turns.	Diam. Wire.	Coil Diam.	Pitch.
Outer of Fig. 49	24	.208 cm.	18 cm.	.81 cm.
Inner of Fig. 49	51.5	.208	13	.42
Coil of Fig. 50	51.5	.208	13	.42

The inductances of various numbers of turns of these several coils was measured on a Rayleigh's bridge, and these values are recorded in the tables containing the wave-length measurements given below.

Measurement of Wave-Length. — The measurement of wave-length was made by a resonance method, in which an auxiliary circuit consisting of a variable condenser in series with an inductance was attuned to resonance with the oscillations to be determined. The adjustable auxiliary circuit is called the *wave-meter circuit.*

A wave measuring apparatus based on this principle was described by Drude in 1902 in a paper entitled "Resonanz-methode zur Bestimmung der oscillatorischen Condensatorentladung."[1] In Drude's apparatus the wave-meter circuit consisted of a condenser connected in series with a rectangular loop of wire, whose inductance could be varied by a bridge across the loop. The position of the bridge when the wave-meter circuit was in resonance with the oscillation to be determined gave the wave-length in the oscillation circuit, which could be read off on a scale attached to the wave-meter circuit. Drude ascertained when the wave-meter circuit was in resonance with the oscillation circuit by observing the glow in a Geissler tube attached to one plate of the condenser. The glow is a maximum at resonance.

Several commercial wave-meters based on the principle of reso-

[1] P. Drude, Ann. d. Phys., 9, p. 611, 1902.

nance have been constructed, prominent among which are the wave-meter of Doenitz and the wave-meter of Fleming.

In both of these wave-meters, as in the apparatus of Drude, the essential parts are a condenser and inductance in series, one or both of which may be varied in a practically continuous manner, some means of ascertaining when the wave-meter circuit is in resonance with the oscillations to be determined and a scale from which the wave-length corresponding to the resonant adjustment may be read.

In the Doenitz wave-meter[1] the adjustment of the wave-meter circuit is made chiefly by the variation of the capacity of the condenser, and the resonant condition is ascertained by observing the deflection of a Riess hot-wire air thermometer.

In Fleming's apparatus,[2] both the capacity and the inductance are varied in a practically continuous manner, and the resonant condition is determined by observing the glow in a Geissler tube.

On account of the feebleness of some of the maxima to be investigated in the present experiments it seemed doubtful if either of these wave-measuring instruments is sufficiently sensitive for the measurements required, so that an apparatus was constructed for the purpose.

The Wave-Meter. — One form given to the wave-metric apparatus is shown in Fig. 51. C is a variable air condenser of two

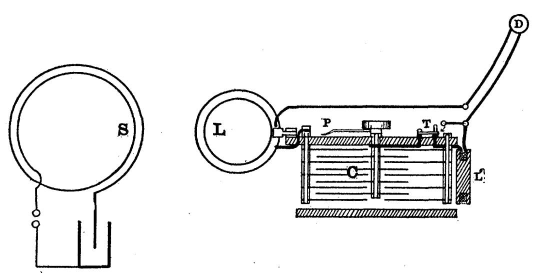


Fig. 51.

sets of semi-circular plates, one of which may be rotated to vary the capacity. The position or the movable plates is given by the reading of a pointer P passing over a semi-circular scale. This

[1] Doenitz, E. T. Z., 24, pp. 920–925, 1903.

[2] J. A. Fleming, Phil. Mag., 9, p. 758, 1905. See also Principles of Electric Wave Telegraphy, London, 1906.

scale is divided in circular degrees, and is also calibrated in wave-lengths by the method described below. In series with the variable condenser is a loop of inductance L (8 turns 13.5 cm. in diameter) and the high-frequency dynamometer with variable sensitiveness D described in Parts I. and II. The loop L, is the "receiving" loop to be acted on inductively by the oscillations to be measured, and the dynamometer is for the purpose of determining when the wave-meter circuit is in resonance with the required oscillations.

The wave-meter circuit is also provided with a switch T, by means of which a second inductance L', may be thrown in or out of the circuit. For short waves, between 200 meters and 650 meters, L' may be thrown out, while for waves between 550 meters and 1,800 meters, L' may be thrown in. This gives an easy method of extending the range of the instrument, which must, of course, bear a double calibration giving two wave-length scales, one to be used with L' in, the other with L' out.[1]

The manner of using the apparatus is as follows: If, for example, it was desired to determine the wave-length of the oscillation in the circuit S, Fig. 51, the loop L was placed in such a position that the currents in S acted inductively on the loop L, and gave a deflection of the dynamometer. By observing the deflection of the dynamometer for different adjustments of the variable capacity C, the value of the capacity that put the wave-meter circuit in resonance with the required oscillations could be determined, and from the calibration on the scale of the capacity, the wave-length of the required oscillation could be read off directly in meters.

The apparatus in this form may be extremely sensitive, or its sensitiveness may be decreased to any degree by one of two methods: first, by decreasing the sensitiveness of the dynamometer by drawing the coil of the dynamometer away from the suspended disc, or second, by rotating the receiving loop L about a horizontal axis so that the inductive action upon it is diminished.

Only one precaution is necessary in the use of such a resonance method of determining wave-lengths, namely, care must be taken

[1] Dœnitz's wave-meter has a triple range, attained by replacing the loop L by loops of higher or lower inductance.

that the inductive coupling between the wave-meter circuit and the discharge circuit be so " loose " that the reaction of the wave-meter circuit on the oscillation circuit is negligible. On account of the high sensitiveness of the present apparatus, there was never any temptation to bring the wave-meter circuit too near to the oscillation circuit.

In some of the experiments described below the dynamometer of the apparatus above described was replaced by other instruments for detecting the resonant condition, but the description of the other instruments is for the present deferred.

Experiment XIX.: Calibration of the Wave-Meter. — The wave-meter was calibrated by an experimental method as follows: A glass-plate condenser in series with a variable inductance and spark-gap was set up in such a position that the spark across the spark-gap could be photographed with the aid of a revolving mirror after the manner of Feddersen's original experiment. The speed of revolution of the mirror was kept constant by the aid of a stroboscope [1] operated by a tuning fork of known period. From the speed of revolution of the mirror and the distance apart of the impressions on the photographic plate the time of one oscillation was determined. This time multiplied by the velocity of light gave the wave-length of the oscillation. The wave-meter circuit was then brought up to within 20 or 30 cm. of the oscillation circuit, adjusted to resonance with it, and the resonant point of the scale of the wave-meter was marked with the wave-length determined by the photograph. In this way a large number of points on the wave-meter scale were determined, and from these points, with the aid of a curve, the wave-length corresponding to any value of the wave-meter scale could be obtained. It was found convenient to write these wave-lengths upon the scale of the wave-meter and to make subsequent readings directly in wave-lengths.

To show the probable accuracy of this method of calibrating the wave-meter some of the data from the measurements of the photographs are given in Table XVI.

In these experiments the speed of the mirror was 41.3 revolu-

[1] G. W. Pierce, On the Cooper-Hewitt Mercury Interrupter, Proc. Am. Acad., XXXIX., No. 18, Feb., 1904.

TABLE XVI.

Photographic Determination of Wave-Length. Capacity Four Boxes of Flat Plates.

No. Turns Inductance.	.5	2.5	4.5	6.5	8.5	10.5	13.5
Inductance in 10^{-5} Henrys.	.50	1.12	2.20	3.43	5.05	6.66	9.38
Distance in cm. between oscillations on plate.	.1096	.164	.227	.301	.340	.404	.483
	.1080	.165	.224	.290	.352	.402	.481
	.1094	.164	.230	.290	.343	.403	.488
	.1103	.160	.238	.292	.343	.404	.484
	.1094		.230	.292	.342		.475
	.1091		.229		.350		
	.1108		.230		.353		
	.1097				.342		
					.346		
Average.	.1095	.163	.230	.293	.344	.403	.482
Wave-Length Meters.	416	630	875	1115	1320	1530	1835
Mean Error per cent.	.5	1.0	1.0	1.0	1.0	.2	.7

tions per second. The distance from the mirror to the sensitive plate was 153 cm., so that the image of the spark travelled across the photographic plate with a speed of $41.3 \times 4\pi \times 153$ cm./sec. $=$ $.792 \times 10^6$ mm./sec. Multiplying the reciprocal of this quantity by the velocity of light in meters/sec., we have

1 mm. between oscillations $=$ 380 meters wave-length.

In determining the distance between oscillations on the photographic plate the length of a whole series of oscillations was measured and divided by the number of oscillations. The table shows that this part of the work was done with a mean error of about 1 per cent.

Besides the values given in the table, the wave-lengths 463 meters and 350 meters were also determined photographically.

Experiment XX.: Extending the Calibration to Shorter Wave-Lengths. — With the particular apparatus at my disposal it was difficult to photograph oscillations giving rise to wave-lengths shorter than 350 meters, so that wave-lengths below this value were obtained by partitioning the condenser. This was done as follows: By the aid of the photographs the wave meter has already been calibrated from 350 meters to 1,835 meters. Using such values of the inductance in the discharge circuit as will bring the wave-length always within this range the following table (Table XVII.) was obtained by discharging each of four condensers and then the four together through various inductances.

TABLE XVII.

Wave-Lengths with Different Condensers Separate and Together in Discharge Circuit.

Turns of Inductance.	Wave-Length in Meters Given with Condenser.				
	1.	2.	3.	4.	All Together.
4.5	430	415	450	440	875
6.5	555	533	575	570	1115
8.5	675	640	700	685	1320
10.5	790	750	815	800	1530
12.5	895	850	905	870	1780
13.5	950	895	970	935	1835

If now we divide the wave-lengths in columns marked Condenser 1, 2, 3, 4 respectively by the wave-length in the last column of Table XVII., we obtain the ratio of the wave-lengths with the separate condensers in the discharge circuit to the wave-length with the four condensers together in parallel in the discharge circuit. These relative values are compiled in Table XVIII.

TABLE XVIII.

Wave-Lengths with Separate Condensers Relative to Wave-Length with Aggregate.

Turns Inductance.	Condenser.			
	1. λ_1/λ	2. λ_2/λ	3. λ_3/λ	4. λ_4/λ
4.5	.492	.474	.514	.502
6.5	.492	.477	.515	.511
8.5	.502	.477	.521	.510
10.5	.517	.490	.532	.522
12.5	.501	.477	.508	.490
13.5	.517	.488	.528	.508
Mean	.507	.480	.520	.507
Mean error	2 per cent.	1 per cent.	1.4 per cent.	1.4 per cent.

The ratio in each column of Table XVIII. is approximately constant whatever the inductance in the discharge circuit. On the assumption that these ratios are also constant when still smaller inductances are used, as they should be from Thomson's formula, we can extend the calibration to the necessary shorter waves. For example, the four condensers discharging together through 2.5 turns of inductance gave wave-lengths 630 meters; condenser 1 discharging through the same inductance should give $.507 \times 630 = 320$ meters. Condensers 2, 3 and 4 respectively give wave-lengths 303, 327 and 320 meters, each of which was used to determine a point on the wave-meter scale. In a similar manner the wave-lengths 200, 211, 217, 245, 259, 266, etc., were found and used in the calibration of the scale of the wave-meter.

Experiment XXI.: Resonance Curves Taken in Calibration of Wave-Meter. — In the preceding paragraphs, it was shown how certain standard discharge circuits were made up, and their wave-lengths determined by revolving mirror photographs of the spark. These standard circuits were used in the calibration of the circuit of the wave-meter. The curves of Fig. 52 show how the standard

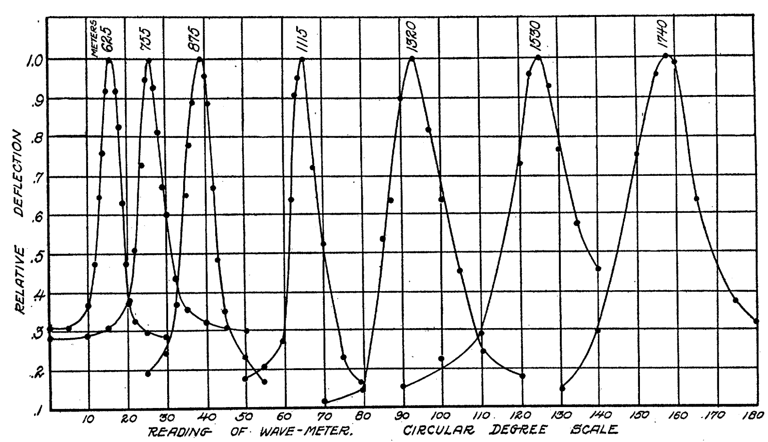


Fig. 52.

wave-lengths were transferred to the wave-meter scale. Here it should be recalled that there are two scales on the wave-meter, one for use when the inductance L' of Fig. 51 was thrown into the wave-meter circuit thus lengthening the period of the wave-meter, the other when the inductance L' was thrown out. The curves of Fig. 52 were taken with the lengthening coil in. For example, with the discharge circuit arranged to give out wave-length 625 meters, the first curve at the left was constructed by plotting the deflections of the dynamometer against the readings of the circular degree scale attached to the variable condenser of the wave-meter. It is seen that resonance is obtained when the condenser is set at 17 degrees of the circular scale. Since the wave-meter circuit was at a distance sufficiently great from the discharge circuit so that the reaction of the wave-meter on the discharge had no appreciable effect in modifying its period, the period of the wave-meter when set at 17 degrees is the period corresponding to the wave-length 625 meters. In a similar manner from the other curves of the series the number of circular degrees corresponding to the other known wave-lengths was obtained. It is interesting to note that while the resonance curves widen as the wave-length is increased the interval on the scale comprehended within a given number of meters also widens, so that the percentage accuracy of

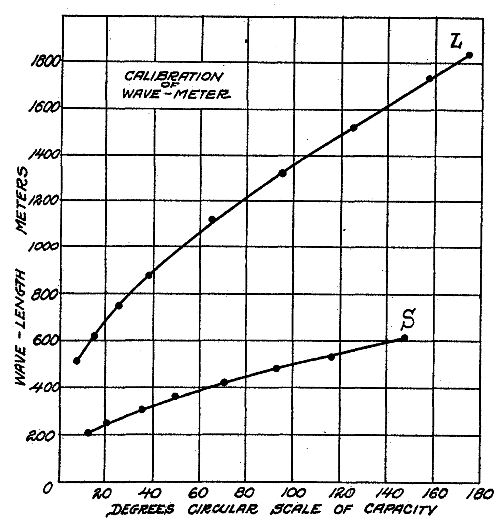


Fig. 53.

the settings of the wave-meter is about the same throughout the scale. An idea of the accuracy of the settings will be had from an examination of the results obtained in the applications that come later in the paper.

If we plot the wave-length at the vertices of Fig. 52 against the reasonant value in circular degrees on the scale of the condenser of the wave-meter, the curve marked L in Fig. 53 is obtained. In like manner when the switch on the wave-meter is thrown so as to cut out the lengthening coil L', and the discharge circuit is made to produce the known shorter waves, the curve marked S in Fig. 53 is obtained. From these curves respectively two scales were constructed and attached to the wave-meter condenser so that the readings could be made directly in wave-lengths.

Experiment XXII.: Use of the Wave-Meter in the Determination of the Capacity of the Discharge Condenser. — Fleming has pointed out the utility of the waver-meter in the determination of the capacity of a condenser. His method consists of discharging the condenser across a spark-gap through a known inductance and measuring the wave-length by the wave-meter and calculating the capacity from the formula

$$\lambda = v \cdot 2\pi\sqrt{LC}.$$

This method was utilized in the determination of the capacity of the several condensers used in the present experiments. To show the probable accuracy of the method, the following table (Table XIX.) is given showing the results obtained for Leyden jar No. 45 used in the later experiments. The values of the inductances were measured on a Rayleigh's bridge.

TABLE XIX.

Determination of Capacity by the Wave-Meter. Leyden Jar No. 45.

Inductance in Discharge Circuit in Henrys.	Wave-Length in Meters.	Capacity in Farads Computed by Thomson's Formula.
3.10×10^{-5}	690	$.00432 \times 10^{-6}$
4.90	865	.00432
6.61	1005	.00430
8.35	1130	.00432
10.0	1235	.00430
12.0	1345	.00427
14.05	1450	.00418
16.1	1560	.00423
		Mean $.00428 \times 10^{-6} \pm 1$ per cent.

The mean error in the measurement of the capacity recorded in Table XIX. is about 1 per cent. This is perhaps better than we ought to expect of the method because in the calculation of the capacity the error in the measurement of the wave-length has been doubled.

Turning our attention now to the main problems of the investigation let us study the oscillations in, first, The Electromagnetically Connected System of Circuits and, second, The Direct Connected System of Circuits.

XIV. Wave-Metrical Study of the Oscillations in the Electromagnetically Connected System of Circuits.

This form of oscillation system is represented in Fig. 49. The theory[1] of this system has been the subject of numerous mathematical researches, so that only so much of the theory will be given here as is necessary in order to enable us to examine the results of the experiments in the light of the theory.

Brief Sketch of the Theory of the Electromagnetically Connected System. — Let the secondary circuit C_2L_2, Fig. 54 be closed, while the primary circuit C_1L_1 is charged and allowed to discharge across a spark-gap.

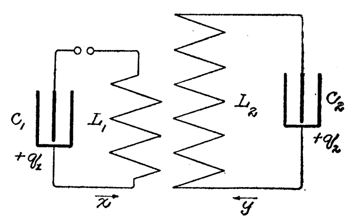


Fig. 54.

Let q_1 be the quantity of positive electricity on the outer plate of the condenser C_1 at any time t during the discharge; let x be the current in the primary circuit at that time; let q_2 be the quantity of positive electricity on the outer coating of C_2, y the current in the secondary circuit: then taking the electromotive force around each of the circuits we have

$$(1)\qquad L_1\frac{dx}{dt} + R_1x + M\frac{dy}{dt} = \frac{q_1}{C_1},$$

$$(2)\qquad L_2\frac{dy}{dt} + R_2y + M\frac{dx}{dt} = \frac{q_2}{C_2},$$

in which L_1 and R_1 are the self-inductance and resistance of the

[1] Lord Rayleigh, Theory of Sound; J. v. Geitler, Sitz. d. k. Akad. d. Wiss. z. Wien, February and October, 1905; B. Galitzine, Petersb. Ber., May and June, 1895; V. Bjerknes, W. A., 55, p. 120, 1895; Oberbeck, Wied. Ann., 55, p. 623, 1895; Domalip and Koláček, W. A., 57, p. 731, 1896; M. Wien, W. A., 61, p. 151, 1897, and D. A., 8 p. 686, 1902; Compare also Webster, Theory of Electricity and Magnetism, p. 499, 1897, and Fleming, The Principles of Electric Wave Telegraphy, p. 209, 1906.

primary, L_2 and R_2 the self-inductance and resistance of the secondary, and M the mutual inductance between the two circuits.

Also

$$x = -\frac{dq_1}{dt}; \; y = \frac{dq_2}{dt}. \tag{3}$$

Differentiating equations (1) and (2) and substituting from equation (3), we have

$$L_1 \frac{d^2x}{dt^2} + R_1 \frac{dx}{dt} + M \frac{d^2y}{dt^2} + \frac{x}{C_1} = 0, \tag{4}$$

$$L_2 \frac{d^2y}{dt^2} + R_2 \frac{dy}{dt} + M \frac{d^2x}{dt^2} + \frac{y}{C_2} = 0. \tag{5}$$

If we eliminate y from these two equations we have

$$\begin{aligned}(L_2L_1 - M^2)\frac{d^4x}{dt^4} + (R_1L_2 + R_2L_1)\frac{d^3x}{dt^3} + \left(R_1R_2 + \frac{L_1}{C_2} + \frac{L_2}{C_1}\right)\frac{d^2x}{dt^2} \\ + \left(\frac{R_1}{C_2} + \frac{R_2}{C_1}\right)\frac{dx}{dt} + \frac{1}{C_1C_2} = 0.\end{aligned} \tag{6}$$

If we eliminate x we get the same equation with x replaced by y.

The equation (6) solved for period of oscillation, on the assumption that the resistance terms have no effect on the period, give the result that the current in the primary oscillates with double periodicity, the current in the secondary also oscillates with double periodicity, and the two periods of the primary are the same as the two periods of the secondary.

The two periods obtained by the solution of equation (6) are

$$T_1' = \sqrt{\frac{T_1^2 + T_2^2 + \sqrt{(T_1^2 - T_2^2)^2 + 4\tau^2 T_1^2 T_2^2}}{2}}, \tag{7}$$

$$T_2' = \sqrt{\frac{T_1^2 + T_2^2 - \sqrt{(T_1^2 - T_2^2)^2 + 4\tau^2 T_1^2 T_2^2}}{2}}, \tag{8}$$

in which

T_1 = period of the primary when standing alone,

T_2 = period of the secondary when standing alone, and

$$\tau^2 = \frac{M^2}{L_1L_2}. \tag{9}$$

τ is called the "coefficient of coupling."

If equations (7) and (8) are each multiplied by the velocity of light, we have, remembering that $vT=$ wave length,

$$(10)\qquad \lambda_1' = \sqrt{\frac{\lambda_1^2+\lambda_2^2+\sqrt{(\lambda_1^2-\lambda_2^2)^2+4\tau^2\lambda_1^2\lambda_2^2}}{2}},$$

$$(11)\qquad \lambda_2' = \sqrt{\frac{\lambda_1^2+\lambda_2^2-\sqrt{(\lambda_1^2-\lambda_2^2)^2+4\tau^2\lambda_1^2\lambda_2^2}}{2}}.$$

In these equations

$\lambda_1 =$ the natural wave-length of the primary alone,

$\lambda_2 =$ the natural wave length of the secondary alone,

$\tau\ =$ the coefficient of coupling, and

λ_1' and λ_2' are the resultant wave lengths in both primary and secondary when the circuits are coupled together.

Measurement of Wave-Lengths Produced by the Electro-magnetically Connected System. — With the aid of the apparatus described above the oscillations in the electromagnetically connected system, Fig. 49, were studied, by measuring the wave-length of the primary circuit and of the secondary circuit when each stands alone and then the wave-length in each circuit when they are coupled together. The results obtained are compared with computations by the aid of the equations (10) and (11). For the computations we need also to know the self inductance of each circuit and their mutual inductance. These quantities were measured on a Rayleigh's bridge.

Experiment XXIII.: E.M.C. System. $L_2 = 24$ Turns of Outer Coil. $\lambda_2 = 1,060$ Meters. — The results obtained in this experiment are plotted in the curves of Fig. 55. The method of taking the observations is as follows: First the condenser $C_2(=.00482$ m.f.) was connected about 24 turns of the outer coil (Fig. 49) and was provided with a spark-gap. In this position, with the inner coil thrown out of circuit by disconnecting both plates of its condenser, the wave-length λ_2 was found to be 1,060 meters. Next, with the secondary condenser disconnected, the wave-length of the primary (inner) circuit was determined with its condenser $C_1(=.00432$ m.f.) connected about 50 turns of the inner coil. This wave-length λ_1 was 1,560 meters. Next, with the primary left unaltered, the secondary was closed by attaching its condensers without spark-gap to the 24 turns of the outer coil. This is the case of the closed secondary, and when the discharge was established in the primary, the wave-lengths were found to be $\lambda_2' = 710$ meters and λ_1' too great for the wave-meter scale. The value $\lambda_2' = 710$ was plotted against $\lambda_1 = 1,560$, Fig 55. Now decreasing the primary inductance to 45 turns, the values $\lambda_1 = 1,460$, $\lambda_1' = 1,650$ and $\lambda_2' = 680$ were obtained, and the last two values were plotted against the value of λ_1, and so forth.

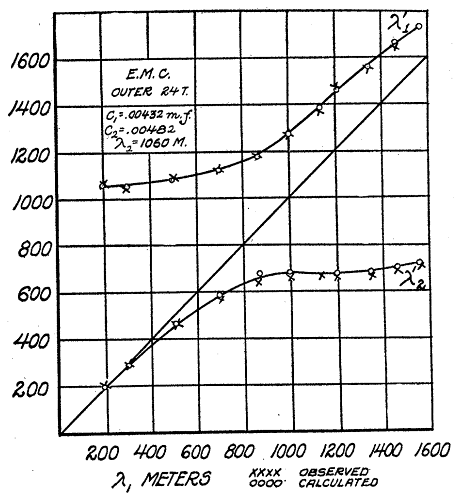


Fig. 55.

The complete record of the observed and calculated values is given in Table XX. In the curves of Fig. 55 the *crosses* are the observed values and the *circles* are the corresponding calculated values. When the observed and calculated values fall together the point is indicated by a *combination* of cross and circle. The 45° line between the two curves may be looked upon as λ_1 plotted against itself, while a horizontal line across the figure at 1,060 meters (not shown) would represent the graph of λ_2. With this in mind it will be seen that the two derived wave-lengths λ_1' and λ_2' are asymptotic toward the origin to λ_2 and λ_1 respectively. The observed and the calculated values are in satisfactory agreement, and their departure one from the other may be due as much to the inaccuracies of the data for the calculated values as to the errors in wave-length measurements.

The formulas for the calculation of λ_1' and λ_2' are the formulas (10) and (11) of page 19, which involve merely the independent periods

Table XX.

Electromagnetically Connected System.

Primary capacity .00432 microfarad.

Primary inductance varied.

Secondary capacity .00482.

Secondary inductance 24 turns outer coil, $L_2 = 6.60 \times 10^{-5}$ Henry.

Wave-length of secondary $\lambda_2 = 1060$ meters.

Turns Primary.	L_1 Primary Inductance. Henry.	M Henry.	τ^2
50	15.85×10^{-5}	6.52×10^{-5}	.412
45	13.9	6.14	.421
40	11.8	5.80	.430
35	10.0	5.12	.397
30	8.20	4.45	.360
25	6.50	3.56	.295
20	4.82	2.70	.228
15	3.15	1.95	.183
10	1.72	1.20	.128
5	.69	.47	.048
3	.32	.23	.0277

Turns Primary.	λ_1 Meters.	Calculated.		Observed.	
		λ_1' Meters.	λ_2' Meters.	λ_1' Meters.	λ_2' Meters.
50	1560	1740	727		710
45	1460	1670	712	1650	685
40	1350	1567	686	1570	665
35	1230	1462	680	1480	660
30	1130	1390	660	1370	660
25	1000	1273	685	1280	660
20	870	1185	680	1185	630
15	700	1127	595	1125	565
10	510	1080	467	1090	460
5	300	1060	292	1040	285
3	210	1062	193	1075	210

of the two circuits and their coefficient of coupling. The latter quantity was obtained by the measurement on a Rayleigh's bridge of L_1, L_2 and M for each setting of the oscillation circuit. The values of these inductances and of the values of τ calculated from them is also included in Table XX.

The intensity of the various periods of the circuits under the different conditions of the experiment varies greatly. No attempt was made to record these intensities, since it was found that their relative values depended on the position of the receiving loop of the wave-meter circuit with respect to its relative distance from the primary and secondary circuits. It is proposed, however, to examine these intensities in a later investigation in which the wave-meter circuit is to be placed at a great distance from the discharge circuit.

Experiment XXIV.: E.M.C. System. $L_2 = 15$ *Turns of Outer Coils.* $\lambda_2 = 775$ *Meters.* In a similar manner an experiment was performed with λ_2 constantly equal to 775 meters, obtained by connecting C_2 about 15 turns of the outer coil. The results are plotted in Fig. 56, from which a comparison of the observed and

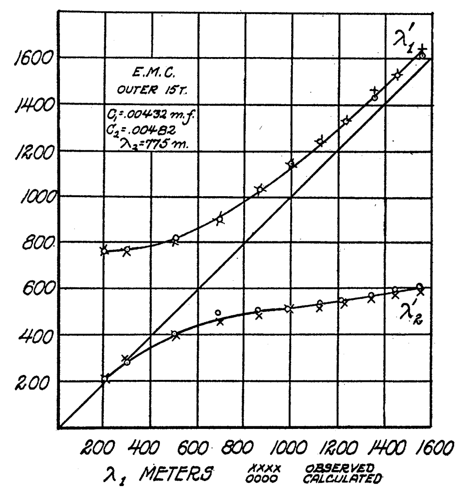


Fig. 56.

calculated values may be had. Here also the agreement is within the limit of error of the method. An abbreviated record of the experiment is given in Table XXI.

Experiment XXV.: E.M.C. System. Special Case $\lambda_2 = \lambda_1$. — A case of especial interest is the case in which the primary and secondary have the same independent periods. This is the case of so-called "resonance" between the two circuits. It is of interest because in this case the wave-length formulas (10) and (11) become greatly simplified, as may be seen by substituting $\lambda_2 = \lambda_1$ in these equations, which under this condition become

$$(\lambda_1')^2 = \lambda_1^2(1 + \tau) \tag{12}$$

$$(\lambda_2')^2 = \lambda_1^2(1 - \tau). \tag{13}$$

In the present experiment the two independent wave-lengths λ_1 and λ_2 were made equal, and the wave-lengths produced by the compound system were then measured and compared with calculations from the formulas (12) and (13). To make the two wave-lengths of the separate circuits the same, the wave-lengths obtained with the

Table XXI.

Electromagnetically Connected System.

Primary capacity .00432 microfarad.
Primary inductance varied.
Second capacity .00482.
Secondary inductance 15 turns outer coil, $L_2 = 3.53 \times 10^{-5}$ Henry.
Wave-length of secondary $\lambda_2 = 775$ meters.

Turns Primary.	τ^2	λ_2 Meters.	Calculated.		Observed.	
			λ_1'	λ_2'	λ_1'	λ_2'
50	.295	1560	1610	627	1650	600
45	.317	1460	1530	612	1540	580
40	.345	1350	1440	590	1460	560
35	.377	1230	1337	560	1345	550
30	.396	1130	1255	545	1240	525
25	.390	1000	1152	522	1150	510
20	.354	870	1040	518	1050	490
15	.305	700	918	491	900	465
10	.252	510	830	410	800	410
5	.107	300	780	281	760	300
3	.062	210	774	206	775	210

$\tau^2 = \left(\frac{M^2}{L_1 L_2}\right)$ was obtained by measurements of L_1, L_2 and M.

capacity C_1 discharging through various turns of the inner coil were measured, with the secondary removed, and the curve of wave-length against turns was plotted. In a similar manner the curve of wave-length against turns was obtained for the outer coil with the inner coil disconnected from C_1. From these two wave-length curves, the number of turns of the inner coil that with its condenser acting alone gave the same wave-length as a given number of turns of the outer coil with its condenser acting alone could be selected. The two circuits were now allowed to oscillate together. That is, the primary condenser C_1 was allowed to discharge through L_1 and the spark gap, while the gap in the secondary circuit was closed, so that oscillations were also set up in the secondary. Two wave-lengths λ_1' and λ_2' were obtained. The results are recorded in Table XXII., and plotted in the curves of Fig. 57.

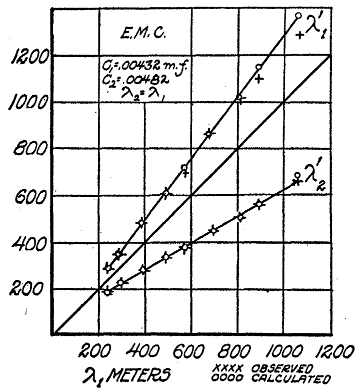


Fig. 57.

TABLE XXII.

Electromagnetically Connected System. Special Case $\lambda_1 = \lambda_2$.

Turns Inner Coil.	Turns Outer.	$\lambda_1=\lambda_2$ Meters.	Observed.		Calculated.	
			λ_1'	λ_2'	λ_1'	λ_2'
27	24	1060	1290	655	1335	680
19.2	19	900	1095	555	1150	555
18	16	810	1025	503	1032	507
14.9	13	690	860	450	870	440
11.5	10	570	700	380	714	374
9.5	8	487	600	330	609	322
7.3	6	395	480	280	485	278
4.9	4	290	345	224	352	210
3.6	3	252	275	204	294	174

The values of the inductances, used in computations, were

L_1 Henry.	L_2 Henry.	M Henry.	τ^2
6.70×10^{-5}	6.95×10^{-5}	4.02×10^{-5}	.348
4.90	4.45	2.90	.386
3.87	4.10	2.30	.374
2.87	3.00	1.72	.343
1.95	2.00	1.14	.323
1.40	1.55	.83	.316
.92	1.10	.50	.253
.45	.64	.257	.227
.21	.40	.14	.230

In this case also the agreement is satisfactory with the exception of the two largest values of λ_1'. No significance is attributed to the departure of these two observations from the theoretical values, as no tendency to a departure of this character was noted in the more general cases described above. It may be that errors were made in reading the wave-meter for these values or, as is more probable, the selection of the turns to make λ_1 and λ_2 equal was erroneous.

The two curves of Fig. 57 are approximately straight lines, which do not pass through the origin. This is due to the particular manner in which τ happens to vary. Constant τ would make the lines pass through the origin.

From Experiments XXIII., XXIV., XXV., with this system of circuits, it is seen that the theoretical formulas give accurately values of the wave-lengths in the electromagnetically connected system, and that the values of the coefficient of coupling required in the equations are accurately enough given by measurements of self and mutual inductance with the comparatively slow frequencies (500 to 1,200 per second) used with an ordinary induction bridge. The theoretical and experimental investigations given below show that the same formulas apply also to the Direct Connected System of circuits.

XV. Wavemetrical Study of the Oscillation in the Direct Connected System of Circuits.

This form of oscillation system is represented in Fig. 50 and in diagram in Fig. 58. The experiments given below show that so far as concerns the wave-length of oscillations exactly the same formulas apply as have been found to be applicable to the electromagnetically connected system. The following sketch of the theory of this system of circuits leads also to this result.

Sketch of the Theory of the Direct Coupled System of Circuits. — In Fig. 58, let the circuit C_1L_1 be looked upon as the discharge circuit, and the circuit C_2L_2 be looked upon as a secondary circuit.

At any time t, let the charge on the outer plate of the condenser C_1 be q_1, let the current flowing from this plate at this time be i, and let the current in the coil L_1 be x. Let q_2 be the charge on the outer plate of C_2, and y the current flowing towards this plate: then

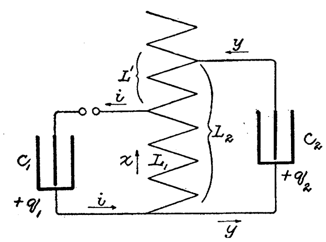


Fig. 58.

(14) $$i = x + y,$$

(15) $$q_1 = -\int idt,$$

$$= -\int (x+y)dt,$$

(16) $$q_2 = \int ydt.$$

Let L_1 be the self-inductance of the discharge circuit, L_2 the self-inductance of the secondary circuit, and L' the self-inductance in that part of L_2 that is not common to L_1, and let M' be the mutual inductance between L_1 and L'.

Let us suppose that the inductances are localized in the coils, and the capacities localized in the condensers.

If we neglect the resistances of the two circuits, which have but small influence on the period of the circuits, and take the electromotive force around the circuits C_1 L_1 and C_2 L_2 we have

(17) $$L_1\frac{dx}{dt} - M'\frac{dy}{dt} - \frac{q_1}{C_1} = 0,$$

(18) $$L_1\frac{dx}{dt} + M'\frac{dx}{dt} - L'\frac{dy}{dt} - M'\frac{dy}{dt} - \frac{q_2}{C_2} = 0.$$

Replacing q_1 and q_2 by their values from (15) and (16), we have

(19) $$L_1\frac{dx}{dt} - M'\frac{dy}{dt} + \frac{1}{C_1}\int(x+y)dt = 0,$$

(20) $$(M' + L_1)\frac{dx}{dt} - (L' + M')\frac{dy}{dt} - \frac{1}{C_2}\int ydt = 0.$$

In order to eliminate x from these two equations let us take $(M' + L_1) \times$ equation (19) and add it to $(- L_1) \times$ equation (20), giving

(21) $$(L'L_1 - M'^2)\frac{dy}{dt} + \frac{M' + L_1}{C_1}\int xdt + \left(\frac{M' + L_1}{C_1} + \frac{L_1}{C_2}\right)\int ydt = 0.$$

The second derivative of (21) with respect to t, added to $-1/C_1$ $\times$ equation (20), gives

(22) $$(L'L_1 - M'^2)\frac{d^3y}{dt^3} + \left(\frac{L_1}{C_2} + \frac{L' + M_1 + L_1}{C_1}\right)\frac{dy}{dt} + \frac{1}{C_1C_2}\int y dt = 0,$$

which differentiated again gives

(23) $$(L'L_1 - M'^2)\frac{d^4y}{dt^4} + \left(\frac{L_1}{C_2} + \frac{L' + 2M' + L_1}{C_1}\right)\frac{d^2y}{dt^2} + \frac{y}{C_1C_2} = 0.$$

This equation may be somewhat simplified by introducing the total inductance of the secondary, L_2, and by expressing the mutual induction in terms of the total mutual inductance between L_1 and L_2 instead of in terms of M' which is the mutual inductance between L_1 and L'. Let the mutual inductance between L_1 and L_2 be M, then by reference to the diagram, Fig. 58, it may be seen that

(24) $$M = M' + L_1$$

and

(25) $$L_2 = L_1 + L' + 2M'.$$

From these equations it follows that

(26) $$L' = L_2 - L_1 - 2M' = L_2 + L_1 - 2M,$$

and

(27) $$L'L_1 - M'^2 = L_1(L_2 + L_1 - 2M) - (M - L_1)^2 = L_1L_2 - M^2.$$

Equations (25) and (27) substituted in equation (23) gives

(28) $$(L_1L_2 - M^2)\frac{d^4y}{dt^4} + \left(\frac{L_1}{C_2} + \frac{L_2}{C_1}\right)\frac{d^2y}{dt^2} + \frac{y}{C_1C_2} = 0.$$

In like manner, if instead of eliminating x from equations (19) and (20), we eliminate y, we get

(29) $$(L_1L_2 - M^2)\frac{d^4x}{dt^4} + \left(\frac{L_1}{C_2} + \frac{L_2}{C_1}\right)\frac{d^2x}{dt^2} + \frac{x}{C_1C_2} = 0.$$

This equation is identical with the equation used for the determination of the period of oscillation with the electromagnetically connected system; namely equation (6) with the resistances pu- equal to zero. Whence it follows that the formulas for determining the wave-lengths of the direct coupled system are identical with those for the electro-magnetically coupled system, which are given as equations (10) and (11) on page 19. These formulas are found to apply to the experimental cases that follow.

Experiments with the Direct Coupled System. — Several experiments were made with the direct connected system of circuits. In all these experiments the primary capacity was kept the same, $C_1 = .00432$ m.f. Two different values of the secondary capacity were used, $C_2 = .00178$ m.f. and $C_2 = .00445$ m.f. In each experiment the secondary inductance was set at some fixed value L_2, thus making λ_2 constant, and the primary inductance L_1 was varied from 50 turns to 3 turns, so that the two wave-lengths λ_1' and λ_2' derived when the secondary circuit is closed are functions of the variable primary wave-length λ_1.

Experiment XXVI.: D. C., $C_2 = .00178$, $L_2 = 50.5$ *Turns,* $\lambda_2 = 1,010$ *Meters.* — The apparatus for the experiments with the direct circuit is shown in Fig. 50. The steps of the experiment

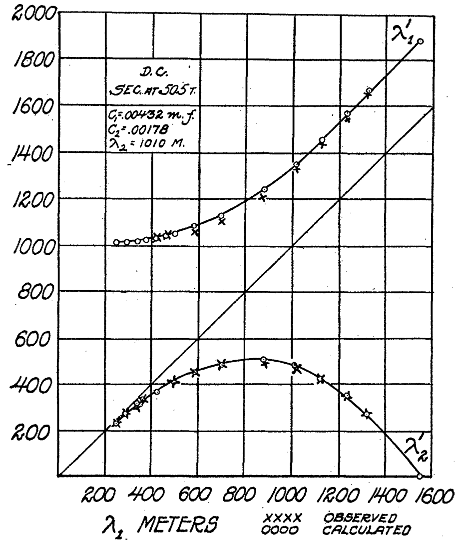


Fig. 59.

are similar to those with the other system of circuits. Table XXIII. contains a record of this experiment. The observed and calculated values of the wave-lengths in the compound oscillating system are plotted in Fig. 59. The formulas of calculation are the formulas (10) and (11), and the only difference between the calculations in the present case and those of the former case is in the method of obtaining M. In the previous case M had to be meas-

Table XXIII.

Direct Coupled System.

Primary capacity .00432 microfarad.
Primary inductance varied.
Secondary capacity .00178 microfarad.
Secondary inductance 50.5 turns of coil $= 16.0 \times 10^{-5}$ Henrys.
Wave-length of secondary 1010 meters.

Turns Primary.	L_1 Primary Inductance. Henry.	L'	M	τ^2
50	15.85×10^{-5}	$.05 \times 10^{-5}$	15.90×10^{-5}	1.00
45	13.9	.78	14.56	.95
40	11.8	1.85	12.97	.89
35	10.0	3.30	11.35	.805
30	8.20	5.00	9.60	.702
25	6.50	6.66	7.92	.601
20	4.82	8.40	6.21	.498
15	3.15	10.20	4.47	.396
12	2.30	11.30	3.50	.332
8	1.30	12.80	2.22	.237
7	1.05	13.25	1.89	.212
6	.88	13.63	1.62	.113
5	.69	14.00	1.35	.145
4	.50	14.40	1.05	.138
10	1.72	12.00	2.86	.296

Turns Primary.	λ_1 Meters.	Calculated.		Observed.	
		λ_1' Meters.	λ_2' Meters.	λ_1' Meters.	λ_2' Meters.
50		1870			
45	1450	1770	175		
40	1325	1665	272	1650	265
35	1235	1563	355	1540	342
30	1125	1450	427	1435	415
25	1020	1350	477	1325	465
20	880	1240	500	1200	485
15	695	1125	485	1100	485
12	585	1072	450	1050	450
8	425	1030	368	1000	360
7	375	1025	330		335
6	340	1018	317		300
5	290	1018	270		270
4	250	1015	234		235
10	500	1050	405		415

ured on an inductance bridge for every setting of the inductances in the experiments. With the direct coupled system the determination of M is somewhat simpler. It may be had by calculation from the calibration curve of the inductance coil used, for by equation (26) on page 93 we have

$$M = \frac{L_1 + L_2 - L'}{2}. \tag{30}$$

In this equation L_1 is the inductance of the primary, which may be obtained from the calibration curve of the coil and the number of turns of the primary; L_2 is the inductance of the secondary, had from the number of turns of the secondary; and L', see Fig. 58, is the inductance of that part of L_2 which is not common to L_1. This L' may be obtained as the inductance on the calibration curve belonging to the difference between the number of turns on L_2 and the number on L_1.

The value of M by the use of equation (30) found for each setting of the primary inductance is recorded in the fourth column of Table XXIII. From this value and the corresponding values of L_1 and L_2 a value of $\tau^2 = M^2/L_1L_2$ is obtained, and recorded in the fifth column. λ_1 was measured by the wave-meter for several values of the primary inductance. λ_2 was measured and left constant throughout the experiment. From this value of λ_2 and the several values of λ_1 and τ^2, the values of λ_1' and λ_2' were computed by the formulas (10) and (11). The corresponding values of λ_1' and λ_2' were also observed directly with the wave-meter. The observed values are represented by *crosses* in Fig. 59, and the calculated values by *circles*.

The agreement between the observed and calculated values is within the limit of accuracy of the measurement of the wave-lengths.

Experiment XXVII.: Same as XXVI. Except that $C_2 = .00445$ m.f., and $\lambda_2 = 1,575$ Meters. — The capacity about the secondary was replaced by a larger jar giving a longer secondary wave-length of 1,575 meters and observations like those of the preceding experiment were made. The results are plotted in Fig. 60. The curves in this case bear a marked resemblance to those of Fig. 59. The chief differences are that in the latter case with the longer secondary wave-length, λ_1' begins higher up, tangential to 1,575 meters instead of to 1,010 meters as in the preceding case; and the arch of the curve λ_2' is a little higher than in Fig. 59.

In both experiments the curve λ_2' comes down to the horizontal axis in the neighborhood of λ_1 equal 1,575 meters. This point, which was calculated since the wave-meter scale does not extend below 200 meters, is the point of *perfect coupling* indicated by the theory. This value occurred in the experiments when the primary and secondary condensers were both connected about the *same inductance*, 50.5 turns of the coil, so that from the standpoint of the

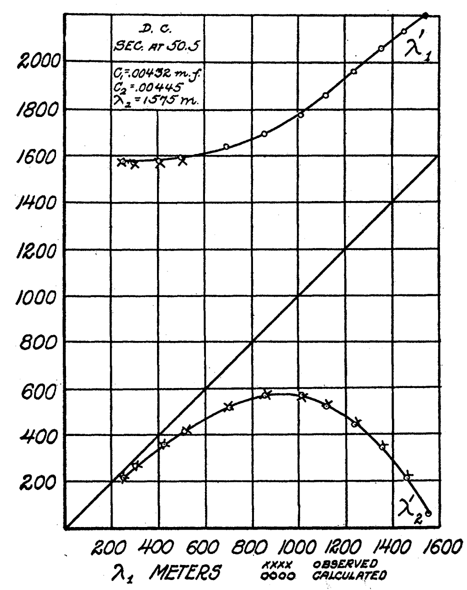


Fig. 60.

experiment the two condensers may be looked upon as discharging in parallel through the same inductance, and producing, therefore, only one wave of wave-length

$$\lambda_1' = 2\pi \cdot v \cdot \sqrt{L_1(C_1 + C_2)} = \sqrt{\lambda_1^2 + \lambda_2^2}.$$

This result is also obtainable from the theoretical equations (10) and (11)

$$\lambda_1' = \sqrt{\frac{\lambda_1^2 + \lambda_2^2 + \sqrt{(\lambda_1^2 - \lambda_2^2)^2 + 4\tau^2\lambda_1^2\lambda_2^2}}{2}},$$

$$\lambda_2' = \sqrt{\frac{\lambda_1^2 + \lambda_2^2 - \sqrt{(\lambda_1^2 - \lambda_2^2)^2 + 4\tau^2\lambda_1^2\lambda_2^2}}{2}}.$$

For when the primary and secondary condensers are connected about the same inductance

$$L_1 = L_2 = M, \text{ therefore}$$

$$\tau^2 = \frac{M^2}{L_1 L_2} = 1.$$

When τ is equal to unity *the coupling is said to be perfect* and the equations given above for the derived wave-lengths become

$$\lambda_1' = \sqrt{\lambda_1^2 + \lambda_2^2}\text{; and}$$

$$\lambda_2' = 0.$$

That is to say, the oscillation becomes single valued.

The case of perfect coupling was not observed in the experiments with the *electromagnetically coupled system*, because for perfect coupling the primary and secondary coils must have the same number of windings and the two coils must be so close together as to be practically coincident, conditions that could not be realized with the apparatus used in the experiments with the electromagnetic coupling

Other experiments made with the Direct Coupled System of circuits may be epitomized as follows.

	Experiment XXVIII.	Experiment XXIX.	Experiment XXX.
Secondary at	25.5 turns.	25.5 turns.	15.5 turns.
Secondary inductance L_2	6.7×10^{-5} Henry.	6.7×10^{-5} Henry.	3.33×10^{-5} Henry.
Secondary wave-length λ_2	645 meters.	1020 meters.	735 meters.
Secondary capacity C_2	.00178 m.f.	.00445 m.f.	.00445 m.f.
Primary capacity C_1	.00432 m.f.	.00432 m.f.	.00432 m.f.
Primary inductance L_1	varied.	varied.	varied.
Curves.	Fig. 61	Fig. 62	Fig. 63

In all of these experiments the agreement between the observed and calculated values is satisfactory. In the experiments that gave Fig. 61 and Fig. 62 the *point of perfect coupling* occurs at the same value of primary wave-length, about 1,010 meters. This is because the secondaries had the same number of turns in both cases. In Fig. 63 the point of perfect coupling occurs nearer the origin at wave-length $\lambda_1 = 650$ meters. In each of the figures, the curve of λ_2' after passing the zero point again rises toward the right giving again the two periods of oscillation. Lengthy comment on these curves is unnecessary, since on account of the agreement of the observations with the theory the complete trend of the curves can be obtained from the theoretical equations.

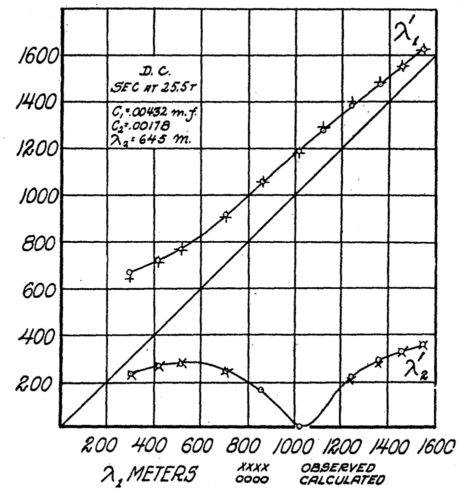


Fig. 61.

Experiment XXXI.: Special Case of the Direct Coupled System, in which the Primary and Secondary are Adjusted Separately to the same period. $\lambda_2 = \lambda_1$. — The observed and calculated values in this

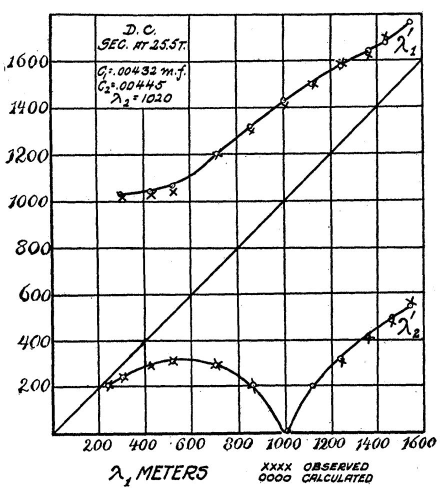


Fig. 62.

case are given in Fig. 64. The figure resembles the corresponding case with the electromagnetic connected system.

In conclusion it should be remarked that the experimental values of the wave-lengths for the entire set of experiments were taken before the calculations were made, and in no case were any of the experimental values redetermined. Each measurement of wave-length was made hastily and with a single setting of the wave-meter No effort was made to obtain high accuracy in the results, as might have been done by averaging several settings of the wave-meter. The results show that the theoretical formulas given above are suf-

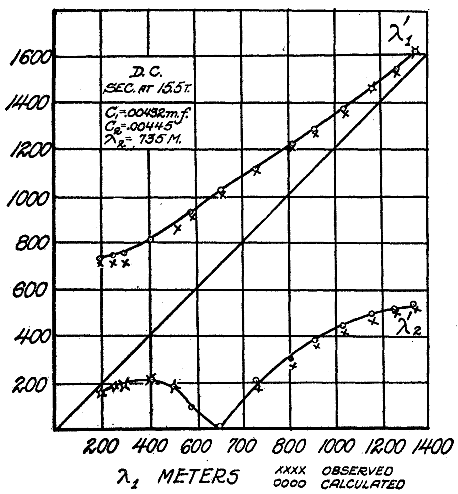


Fig. 63.

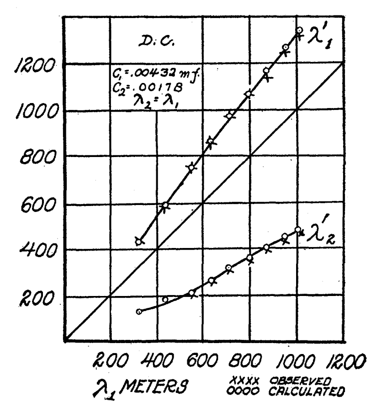


Fig. 64.

ficiently accurate to be used for computing the wave-lengths of the oscillation occurring in connected systems of condenser circuits. Also, on the other hand, the agreement between observed and calculated values show that the wave-meter used is correctly calibrated and is serviceable for the range of wave-lengths between 200 meters and 1,800 meters. It is proposed to utilize the apparatus in direct measurements with wireless telegraph circuits instead of the simpler condenser circuits of the present research.

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November 6, 1906.