THE PENETRATION OF TOTALLY REFLECTED LIGHT INTO THE RARER MEDIUM.

By Elmer E. Hall.

THE fact that total reflection does not take place at the geometrical boundary of two media was discovered and experimentally investigated by Newton.[1] According to Newton the path of the ray during total reflection was a parabola,[2] the vertex being within the rarer medium. The problem was both experimentally and theoretically studied by Fresnel,[3] and more or less treated by Verdet,[4] Young,[5] Huygens,[6] Biot,[7] Babinet,[8] Billet,[9] Stokes,[10] and others. The first reliable quantitative work was published by G. Quincke[11] in 1866.

The purpose of the present paper is to investigate quantitatively the penetration in thin films, from the first boundary of which "total reflection" takes place, and to extend the study qualitatively to the case in which the second or rarer medium is no longer thin.

Introduction.

Many of the phenomena in the case of thin films may be easily obtained qualitatively. If two glass prisms be pressed together, and if the thickness of the air film between them be sufficiently small, total reflection no longer takes place in the air-layer between

[1] Newton, Opticks, Book II., Obs. 1, 2, 8; Book III., qu. 29.

[2] Newton, Principia, Phil. Nat., Book I., Prop. 96.

[3] Ann. d. Chem. et Phys., t. 23, 1823, p. 130; t. 29, 1825, p. 175; t. 46, 1831, p. 225; also Fresnel, Oeuvres, t. I., pp. 447, 453, 767.

[4] Verdet, Oeuvres, t. VI., pp. 430 and 563.

[5] Young's Lectures on Nat. Phil., 1807, I., p. 461, II., p. 623; also Phil. Trans., 12, 1801.

[6] Huygens, Traité da la Lumiere, 1690, Leide, p. 38.

[7] Biot, Traité de Physique, 1816, t. III., pp. 276, 290.

[8] Comptes Rendus, t. 8, 1839, p. 709.

[9] Ann. de Chemie, t. 64, 1862, p. 410.

[10] Phil. Trans., 8, 1849, p. 642.

[11] Pogg. Ann., Band 127, 1866, pp. 1 and 199.

the two prisms, and light is transmitted at all angles of incidence. That the two prisms are not in actual contact may be seen from the interference bands obtained when the angle of incidence is less than the critical angle. If a sunbeam, whose intensity has been reduced by a ground-glass screen, fall upon the air layer at an angle greater than the critical angle a portion of the light will be transmitted if the prisms be sufficiently close together. If the back prism now be pulled slowly away until transmission no longer occurs, and then the ground-glass screen be removed, allowing the full intensity of the sunlight to fall upon the prisms, transmission of a portion of the light again occurs, showing that an increase in intensity increases the depth to which the vibrations in the air layer extend. On separating the prisms the light becomes red before vanishing, showing that the longer wave-lengths affect the air layer to the greater depth.

The phenomena are best seen when one of the prisms has a convex surface. This gives an elliptical spot of transmitted light whose major diameter changes with the angle of incidence, the intensity, the wave-length, and, if the incident light be plane polarized, with the plane of polarization. In fact, the more important conclusions of Quincke can be verified qualitatively without difficulty. These conclusions may be summarized as follows.

1. The diameter of the transmitted elliptical spot of light, and therefore the penetration, increases with an increase in the intensity of the incident light.

2. In the neighborhood of the critical angle the color bands for a white light source are lost in dark bands. The diameter of the transmitted spot increases rapidly with increasing angle of incidence, reaching a maximum just as the dark bands disappear. The penetration then at first rapidly and later more slowly decreases with increased angle of incidence.

3. The penetration increases with the wave-length of the incident light.

4. As the angle of incidence is increased from the critical angle the penetration is at first greater for light polarized perpendicular to the plane of incidence, and then greater for light polarized parallel to the plane of incidence.

5. The penetration increases for an increase in the index of refraction of the film.

6. If the incident light be plane polarized the light which penetrates the film and passes through the second prism as well as that which is "totally reflected" is elliptically polarized.

Before obtaining a prism with the proper convex surface the writer employed the following method to obtain quantitative results. A thin layer of silver was deposited on one edge of the plane hypotenuse surface of a rectangular prism. This surface was placed in contact with the hypotenuse surface of a similar prism and the two prisms were placed in a brass clamp with edges vertical. The clamp was provided with set screws at top and bottom so that adjustment could be made to give interference bands parallel to the edge of the prism and extending from the silver nearly to the other side of the prism. The bands disappeared for an angle of incidence equal to or greater than the critical angle, and light was transmitted where before were interference bands. The nearer the part through which transmission occurred approached the silvered edge the greater was the penetration. The distance between the prisms at any point could be obtained from the interference bands. The method gives good qualitative results, but is not so well adapted for quantitative work as the method using a convex surface.

Apparatus and Method.

Two rectangular prisms, one of flint and one of crown glass, were ground by John A. Brashear for the University of California, where the writer began the investigation. The side face were square, being 25 mm. on each edge, and approximately plane. The hypotenuse surfaces were convex, being ground accurately spherical with a radius of curvature of about eight meters. These prisms were loaned the writer by the Physics Department of the University of California and the investigations continued at Cornell University.

A 90° prism with a good plane hypotenuse surface was used with one of the prisms having the hypotenuse surface slightly convex. The two prisms with their hypotenuse surfaces together were placed in a brass clamp. The clamp was provided with two set screws so that the pressure and position of contact could be regulated easily. The clamp containing the prisms *P* (see Fig. 1), was placed on the table of a spectrometer provided with leveling screws so that the prism edges could be set vertically.

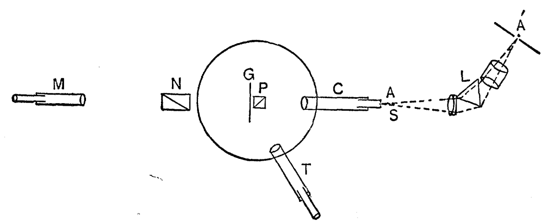

Fig. 1.

The light was incident on the prism whose hypotenuse surface was plane. From two to three millimeters back of the prisms was a ground-glass screen *G*, and back of this a Nicol prism *N*. The vertical diameter of the transmitted spot of light was measured by means of a telescope *M*, provided with a micrometer eyepiece. The values of the micrometer readings were obtained by replacing the ground-glass screen by a glass scale ruled to half millimeters. The magnification was such that one complete revolution of the micrometer screw corresponded to a distance of a little less than half a millimeter. The collimator *C* was focused for parallel rays. The focal length of the collimator lens was 34 cm., and the slit width in most of the work was 0.9 mm. The source was a single-jet acetylene burner placed at *A* for a white light source and at *A'* for an approximately monochromatic source, *L* being a lens and prism system to throw a pure spectrum on the slit. The spectrometer circle was graduated to 5 minutes of arc with a vernier reading to 5 seconds. Normal incidence was obtained by the usual spectrometer method, the cross hairs of the telescope *T* being set on a silk fiber stretched vertically over the center of the slit. The angle of incidence on the prism face could then be determined to within a few seconds of arc.

The thickness of the layer between the prisms was obtained by measuring the diameter of the interference rings for several angles of incidence less than the critical angle. The source for this purpose was a sodium flame s placed in front of the slit. The thickness d of the layer could be calculated from the formula

$$d = \frac{n\lambda}{4 \cos r}$$

where r is the angle of refraction and λ the wave-length within the film, and n takes the values 1, 3, 5, etc., for dark rings in the transmitted system. A calibration curve was plotted with diameter of rings as ordinates and thickness of film as abscissas. From this curve the thickness of the film penetrated could be obtained from the diameter of the transmitted spot of light. An air film was generally used in the calibration. The prisms were put together with enough pressure to make them in actual contact in the central part of the hypotenuse face. This was necessary since the introduction of a liquid film by placing a drop of liquid on the top of the prisms has a tendency to pull the prisms closer together. A new curve was plotted each time the prisms were put together.

The angle between the prism face on which the light was incident and the hypotenuse face was 45° 2′ 32″ for the crown-glass prism, and 44° 56′ 20″ for the flint-glass prism. The index of refraction, for sodium light, of the crown-glass prism having a convex hypotenuse surface was 1.5164, and for the similar flint-glass prism the index was 1.6166.

In working with films whose index of refraction differed little from the glass, as in the case of benzole and crown glass, an arrangement of the prisms as shown in Fig. 2 was used. Light was incident on prism A. B was the prism with a convex surface, C was stuck to B with Canada balsam to prevent total reflection at the side surface.

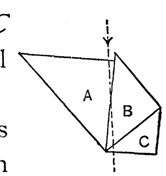

Fig. 2.

Great care was taken to have the glass surfaces clean. They were washed in nitric acid, in a solution of caustic potash in alcohol, and in distilled water. Water marks, when present, were removed by gently rubbing the surfaces with the optical paper furnished by Bausch and Lomb. Dust particles were removed with a camel's-hair brush. The room was kept partially darkened and the eye was rested every few minutes throughout a set of measurements.

Observations with White Light.

Using a white light source the results for a number of substances are plotted in Fig. 3. Angles of incidence on the film are plotted

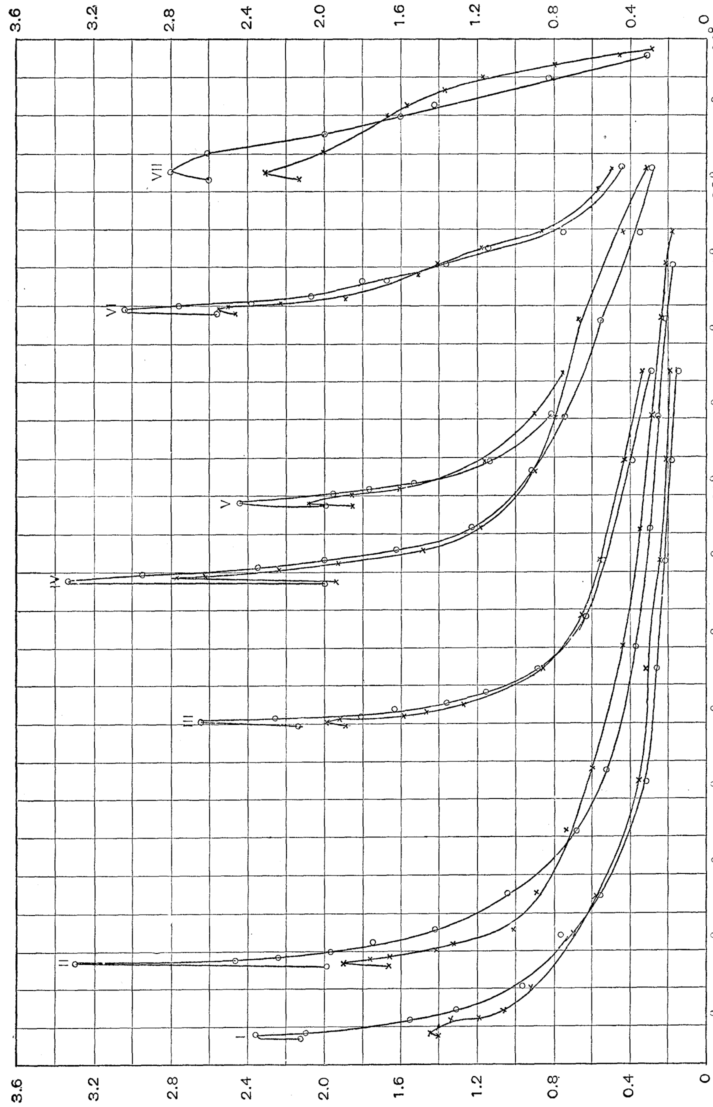

Fig. 3.

I., Flint-glass, air, flint-glass; II., Crown-glass, air, crown-glass; III., Flint-glass, water, flint-glass; IV., Crown-glass, water, crown-glass; V., Flint-glass, benzole, flint-glass; VI., Crown-glass, turpentine, crown-glass; VII., Crown-glass, benzole, crown-glass.

as abscissas, and the thicknesses of the film, through which light was transmitted to the second prism, expressed in thousandths of millimeters are plotted as ordinates. The curves have the same form for all substances used, and if produced they would intersect at the point whose coördinates are 90° and 0°. When the light was polarized perpendicular to the plane of incidence the observed points are indicated by small circles, and when the light was polarized in the plane of incidence the observed points are indicated by small crosses. In the case of such substances as air, water, and benzole the index of refraction obtained with a hollow prism agreed with the value obtained from the angle of incidence for which the bands disappeared and the diameter of the transmitted spot of light was a maximum. In the case of benzine and turpentine such was not the case. For turpentine the index of refraction by the prism method was 1.4671, and by calculation from the incidence angle for which sodium bands disappeared the index was 1.4690, the result varying slightly for different films. This disagreement may be due to the fact that turpentine is not a single chemical compound, and because of greater surface tension, or greater adhesion for the glass, that portion having the higher index of refraction is drawn between the prisms to form the film.

For angles of incidence equal to the critical angle the edges of the transmitted spot of light become diffuse, and it becomes difficult to determine just where the edge is. In most of the curves in Fig. 3 the maximum penetration given corresponds to angles of incidence a few minutes greater than the critical angle. That portion of each curve before the maximum is reached gives the diameter of the transmitted spot within the first dark ring.

Monochromatic Source.

In order to get measurements which might be compared with theory an approximately monochromatic source of fairly constant intensity was necessary. This was secured by allowing the desired portion of the spectrum of the acetylene source to fall upon the collimator slit. The lenses used were achromatic. The one between the prism and the collimator was of the same focal length as the collimator lens, thus minimizing the error due to a wide slit.

In the following tables each result is the average of from 10 to 30 readings, not differing widely among themselves. The latter i gives the angle of incidence on the face of the first prism, φ the angle of incidence on the film, d and d/λ give, respectively, the penetrations in the film expressed in thousandths of millimeters and in wave-lengths within the film of the light used. The symbols ⊥ and // denote that the plane of polarization is perpendicular and parallel respectively to the plane of incidence. (Fresnel interpretation.) μ_1 is the index of refraction of the first prism, μ_2 of the film, and μ the relative index between the film and the first prism. The critical angle is designated by c. The columns marked T and D will be explained later. The first five tables are for the combination crown-glass, air, crown-glass.

Table I.

Crown-glass, air, crown-glass. Red light, $\lambda = .618\,\mu$ *to* $.658\,\mu$. $\mu_1 = \mu = 1.5137$. $c = 41°\ 21'$.

i	ϕ	d		d/λ		T		D	
		//	⊥	//	⊥	//	⊤	Obs.	Cal.
5°45′	41°15′	0.78	1.55	1.20	2.43				
5 30	41 25	.98	2.15	1.54	3.33	2.68	4.43	1.79	2.54
5 15	41 35	.97	1.77	1.52	2.78	2.26	3.49	1.26	1.33
5	41 44	.90	1.55	1.41	2.43	2.03	2.97	1.02	1.00
3	43 4	.85	1.06	1.33	1.67	1.39	1.77	.34	.38
0	45 3	.71	.82	1.11	1.29	1.09	1.26	.18	.18
5	48 21	.55	.60	.86	.94	.87	.92	.08	.05
10	51 38	.49	.46	.77	.72	.76	.74	−.05	−.05
20	58 6	.39	.35	.61	.55	.60	.54	−.06	−.06
30	64 21	.30	.26	.47	.40	.50	.42	−.07	−.08

Table II.

Yellow light, $\lambda = .571\,\mu$ *to* $.608\,\mu$. $\mu = 1.5151$. $c = 41°18'$.

i	ϕ	d		d/λ		T		D	
		//	⊥	//	⊥	//	⊥	Obs.	Cal.
5°45′	41°15′	1.	1.71	1.70	2.90				
5 30	41 25	1.05	2.11	1.78	3.58	2.15	3.77	1.80	1.93
5 15	41 35	1.03	1.74	1.75	2.95	1.82	3.01	1.20	1.22
5	41 44	.99	1.54	1.68	2.61	1.78	2.63	.93	.93
3	43 4	.80	.98	1.35	1.66	1.26	1.63	.31	.37
0	45 3	.61	.71	1.03	1.20	.99	1.18	.17	.18
5	48 20	.48	.52	.82	.88	.81	.86	.06	.05
10	51 37	.41	.40	.69	.68	.70	.69	−.01	−.01
20	58 5	.34	.31	.58	.53	.56	.50	−.05	−.06
30	64 19	.27	.22	.46	.37	.47	.41	−.09	−.08

TABLE III.

Green light, $\lambda = .520\,\mu$ *to* $.550\,\mu$. $\mu = 1.5175$. $c = 41°12'$.

i	ϕ	d		d/λ		T		D	
		//	⊥	//	⊥	//	⊥	Obs.	Cal.
5°45′	41°16′	0.92	1.73	1.72	3.24	2.30	4.68	1.52	3.95
5 30	41 25	.90	1.55	1.68	2.90	1.97	3.25	1.22	1.36
5 15	41 35	.89	1.46	1.66	2.73	1.78	2.76	1.07	1.01
5	41 44	.86	1.19	1.60	2.22	1.66	2.43	.62	.66
3	43 4	.66	.85	1.23	1.59	1.22	1.58	.36	.38
0	45 3	.54	.62	1.	1.16	.98	1.16	.16	.17
5	48 20	.45	.47	.84	.88	.79	.85	.04	.05
10	51 37	.37	.37	.68	.68	.68	.68	.00	−.00
20	58 4	.30	.27	.56	.51	.55	.49	−.05	−.06
30	64 17	.24	.20	.45	.37	.46	.38	−.08	−.08

TABLE IV.

Violet, $= .441\,\mu$ *to* $.466\,\mu$. $\mu = 1.5233$. $c = 41°2'$.

i	ϕ	d		d/λ		T		D	
		//	⊥	//	⊥	//	⊥	Obs.	Cal.
5°45′	41°16′	0.61	1.23	1.35	2.71	1.22	2.24	1.36	1.34
5 30	41 26	.58	1.00	1.28	2.21	1.16	1.99	.93	.99
5 15	41 36	.54	.92	1.19	2.03	1.11	1.82	.84	.81
5	41 45	.49	.78	1.08	1.72	1.08	1.71	.64	.61
3	43 5	.38	.56	.84	1.23	.88	1.20	.39	.33
0	45 3	.30	.38	.66	.84	.74	.91	.18	.16

With the yellow light, of the same wave-length as in Table II., on the slit, a yellow glass, which was found by the usual photometric method to absorb approximately one-half the light of this wave-length, was placed in front of the collimator slit and the following results obtained.

TABLE V.

ϕ	d		d/λ		D		ϕ	d		d/λ		D	
	//	⊥	//	⊥	Obs.	Cal.		//	⊥	//	⊥	Obs.	Cal.
41°15′	0.83	1.33	1.40	2.26			48°21′	0.54	0.64	0.91	1.08	0.17	0.18
41 25	.84	1.67	1.42	2.84	1.42	1.93	57 37	.46	.49	.78	.83	.05	.05
41 35	.85	1.51	1.44	2.57	1.13	1.22	58 5	.38+	.38−	.65	.64	−.01	−.01
43 4	.81	1.34	1.37	2.28	.91	.93	64 19	.28	.24	.48	.41	−.07	−.06
45 3	.65	.90	1.10	1.52	.42	.37							

By comparing Table V. with Table II. where approximately twice the intensity was used, it appears that doubling the intensity increases the penetration in the ratio 1 : 1.18, the ratio being the same for either plane of polarization. This ratio is not constant however but varies with the magnitude of the original intensity.

No simple relation appears which would give the way in which the penetration varies with the intensity or with the wave-length of the incident light. Quincke states that the penetration varies almost, but not exactly, as the wave-length. This can be accepted only as a very rough approximation.

In working with white light it was noticed that the penetration for glycerine was approximately the same as for turpentine. Since the indices of refraction of these substances are approximately equal it would indicate, other conditions being the same, that the penetration is a function of the index of refraction alone. This point was tested by preparing a solution of common salt in water of such density as to give the same index of refraction as wood alcohol. Yellow light of the same wave-length as before was used.

TABLE VI.

Crown-glass, Wood-alcohol, Crown-glass. $\mu_2 = 1.3388$, $\mu = 1.1317$. $c = 62°5'$.						Crown-glass, Salt Solution, Crown-glass. $\mu_2 = 1.3388$. $\mu = 1.1317$. $c = 62°5'$.					
i	ϕ	d		d/λ		d		d/λ		D	
		//	⊥	//	⊥	//	⊥	//	⊥	Obs.	Cal.
26°30′	62°13′	1.60	1.99	3.64	4.52	1.64	2.03	3.71	4.58	0.87	0.87
26 45	62 21	1.63	1.92	3.69	4.35	1.62	1.93	3.67	4.37	.70	.59
27	62 30	1.57	1.78	3.55	4.04	1.56	1.80	3.54	4.07	.53	.47
28	63 7	1.28	1.40	2.90	3.18	1.27	1.38	2.88	3.13	.25	.27
30	64 20	1.04	1.10	2.36	2.49	1.02	1.08	2.31	2.45	.14	.15
40	70 10	.63	.64	1.42	1.45	.60	.61	1.36	1.38	.02	.02
50	75 27	.48	.46	1.08	1.04	.43	.40	.97	.91	−.06	−.01

The differences in the above table between the values for alcohol and the salt solution are easily within the range of experimental error. Since substances of such different chemical nature give the same values, the conclusion seems justifiable that the penetration is a function of the index of refraction alone, other conditions being the same.

TABLE VII.

Flint-glass, Air, Flint-glass. Red light, $\lambda = .618\ \mu$ *to* $.658\ \mu$. $\mu_1 = \mu = 1.6450$. $c = 37°26'$.

i	φ	d		d/λ		T		D	
		//	⊥	//	⊥	//	⊥	Obs.	Cal.
12°30′	37°23′	0.75	1.51	1.18	2.36				
12 15	37 32	.82	2.01	1.30	3.16	1.66	3.49	1.86	1.92
12	37 41	.78	1.66	1.22	2.60	1.51	2.78	1.38	1.45
10	38 53	.72	1.02	1.13	1.60	1.10	1.57	.47	.48
5	41 54	.52	.65	.81	1.03	.80	.95	.22	.23
0	44 56	.44	.47	.69	.74	.68	.72	.05	.04
10	51 0	.34	.32	.53	.50	.53	.49	−.03	−.04
20	56 56	.29	.23	.45	.36	.45	.38	−.09	−.07
30	62 38	.25	.20	.39	.31	.38	.29	−.08	−.08
40	67 56	.20	.15	.32	.23	.31	.22	−.09	−.09

An interesting case is where the first and third media are different. Red light of the same wave-length as before was used.

TABLE VIII.

Crown-glass, Air, Flint-glass. $\mu = 1.5137$. $c = 41°21'$.

i	φ	d		d/λ	
		//	⊥	//	⊥
5°45′	41°15′	0.84	1.61	1.31	2.52
5 30	41 25	.86	2.07	1.35	3.25
5 15	41 35	.84	1.64	1.32	2.57
5	41 44	.82	1.38	1.28	2.16
3	43 4	.67	.94	1.05	1.47
0	45 3	.61	.67	.96	1.05
10	51 38	.44	.40	.69	.63
20	58 6	.33	.28	.52	.44

TABLE IX.

Flint-glass, Air, Crown-glass. $\mu = 1.6450$. $c = 37°26'$.

i	φ	d		d/λ	
		//	⊥	//	⊥
12°30′	37°23′	0.86	1.66	1.35	2.60
12 15	37 32	.89	2.09	1.39	3.28
12	37 41	.88	1.71	1.37	2.68
10	38 53	.70	1.10	1.10	1.73
5	41 54	.58	.65	.91	1.02
0	44 56	.51	.49	.80	.77
10	51 0	.38	.33	.60	.52

Plotting these results, Fig. 4, it appears that the curves for crown glass as first medium are approximately the same as when flint glass is the first medium excepting that they are displaced nearly four degrees to the right, that is, displaced by a distance equal to the difference between the critical angles. It will be seen from the tables that for values near the critical angles, the penetration when flint glass is first medium is slightly greater than when the crown glass is the first medium. This difference, however, is within experimental error. Furthermore since the angles of incidence do not correspond, and also since different pairs of prisms are used in the two cases no conclusion can be drawn. To test this point more fully the flint glass prism with a plane hypotenuse was used with the crown-glass prism having the hypotenuse surface spherical. In the

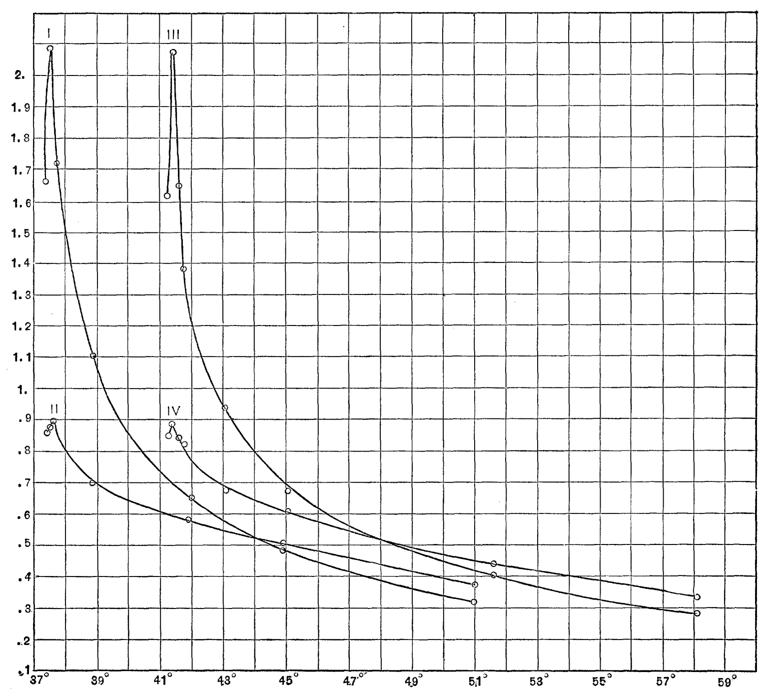

Fig. 4.

I., Flint-glass, air, crown-glass, light polarized perpendicular to plane of incidence; II., Flint-glass, air, crown-glass, light polarized in the plane of incidence; III., Crown-glass, air, flint-glass, light polarized perpendicular to plane of incidence; IV., Crown-glass, air, flint-glass, light polarized in the plane of incidence.

crown-glass prism the angle made by the side on which the light was incident with a plane tangent at the center of the convex surface was 45°. The results are given in Table X. Yellow light was used. The angles of incidence on the film, φ for the direction crown-glass, air, flint-glass and θ for the reverse direction, are made to correspond from the relation $\sin\varphi = \frac{\mu_C}{\mu_F}\sin\theta$.

The same calibration curve was used for both sets of measure-

TABLE X.

Crown-glass, Air, Flint-glass. $\mu = 1.5164$.						Flint-glass, Air, Crown-glass. $\mu = 1.6488$.					
i_1	ϕ	d		d/λ		i_3	θ	d		d/λ	
		//	⊥	//	⊥			//	⊥	//	⊥
5°	41°42′	0.96	1.40	1.62	2.38	11°58′30″	37°42′45″	0.98	1.45	1.66	2.46
3	43 1 20″	.87	.94	1.48	1.59	10 1	38 53 10	.90	.97	1.53	1.64
0	45	.75	.76	1.27	1.29	7 14 10	40 33 30	.77	.80	1.30	1.36
5	48 18	.60	.57	1.02	.97	2 36 20	43 21 30	.63	.58	1.07	.98

ments. The measurements are smaller throughout for the case where crown glass is the first medium. A water film also gave results slightly greater for the case where flint glass was the first medium. Although the differences are small and probably not in excess of experimental error, the consistency of the differences in both the case of a water and an air film make it seem highly probable that the penetration is greater for the direction flint glass film crown glass than for the reverse direction. This conclusion agrees with Quincke[1] although the difference is much less than his results show. The method he employed, which assumed that the prisms were always just in contact, is open to too great error to have any value on this point.

By comparing Tables VIII. with I., and IX. with VII., it will be noticed that the values obtained for crown-glass, air, flint-glass or flint-glass, air, crown-glass lie between those obtained for crown-glass, air, crown-glass and flint-glass, air, flint-glass. The same relation is brought out in the following tables, where yellow light of the same wave-length as before was used.

TABLE XI.

Flint-glass, Benzole, Flint-glass. $\mu_1 = 1.6488$. $\mu_2 = 1.4998$. $\mu = 1.0994$. $c = 65°27'$.					
i	ϕ	d		d/λ	
		//	⊥	//	⊥
35°30′	65°34′	1.61	2.21	4.10	5.63
36	65 49	1.48	1.82	3.77	4.64
38	66 52	1.03	1.24	2.62	3.16
40	67 53	.89	.99	2.26	2.52
45	70 20	.62	.62	1.78	1.58

TABLE XII.

Flint-glass, Benzole, Crown-glass. $\mu = 1.0994$. $c = 65°27'$.				
ϕ	d		d/λ	
	//	⊥	//	⊥
65°34′	2.13	2.50	5.42	6.25
65 49	1.82	2.03	4.64	5.16
66 52	1.10	1.28	2.80	3.26

[1] Loc. cit., p. 16.

TABLE XIII.

Crown-glass, Benzole, Crown-glass.

$\mu_1 = 1.5151$. $\mu_2 = 1.4998$. $\mu = 1.0102$. $c = 81°51'$.

i	ϕ	d		d/λ		T		D	
		//	⊥	//	⊥	//	⊥	Obs.	Cal.
12°	82°6′	2.68	2.73	6.83	6.95	6.70	6.81	0.12	0.10
11 30′	82 25	2.29	2.40	5.83	6.11	5.56	5.63	.28	.06
11	82 45	2.16	2.22	5.50	5.65	4.87	4.92	.15	.04
10	83 24	1.65	1.72	4.20	4.38	4.06	4.09	.18	.02
9	84 5	1.40	1.44	3.56	3.67	3.55	3.56	.11	.01
7	85 22	1.12	1.11	2.85	2.83	2.87	2.86	−.02	−.01−
5	86 41	.90	.88	2.32	2.30	2.32	2.30	−.02	−.01+

The measurements are exceedingly hard to obtain when the relative index of refraction is as small as in the case of crown glass and benzole. The percentage of error is therefore high, individual measurements varying from 10 to 15 per cent. from the average. The arrangement of the prisms was the same as shown in Fig. 2, the "right angle" of the prism on which the light was incident being 89° 59′ 5″. The intensity would be slightly greater for Table XIII. than for Tables XI. and XII., but allowing for this, the penetration is still greater for crown glass-benzole-crown glass than for flint glass-benzole-crown glass and this in turn is greater than for flint glass-benzole-flint glass. It follows from this that as the index of refraction of the third medium increases, the distance to which the light vibration penetrates the film decreases; and as the index of refraction of the third medium decreases this distance increases. It was pointed out before that as the relative index of refraction between the first and second mediums decrease the penetration increases. The same relation therefore exists between the second and third mediums as between the first and second.

It would be interesting to know to what extent the penetration depends upon the layer of carbon dioxide, water vapor, etc., condensed upon the surface of the glass. There seems to be no method of getting rid of this layer except by heating to redness for a considerable length of time, a method hardly practical in this case.

For an air film measurements taken after the prisms had been in a dry vacuum for two days agreed with measurements taken after the prisms had been for nearly the same length of time in moist carbon dioxide under a pressure slightly greater than atmospheric pressure. It is doubtful whether any change in the layer of condensed gases was produced by this treatment.

Strongly Absorbing Medium.

The penetration was next examined in the film of a substance showing anomalous dispersion and hence having a strong absorp-

Table XIV.

Flint-glass, Fuchsine, Flint-glass.

Red $\lambda = .620\mu$ to $.658\mu$ $\mu_1 = 1.6450$. $\mu_2 = 1.389$. $\mu = 1.184$. $c = 57°36'$.						Yellow $\lambda = .579\mu$ to $.612\mu$. $\mu_1 = 1.6488$. $\mu_2 = 1.412$. $\mu = 1.167$. $c = 58°55'$.					
i	ϕ	d		d/λ		i	ϕ	d		d/λ	
		//	⊥	//	⊥			//	⊥	//	⊥
21°	57°31′	1.17	1.28	2.54	2.78	23°	58°39′	1.37	1.85	3.26	4.40
21 15′	57 40	1.33	1.66	2.89	3.61	23 30′	58 56	1.39	2.18	3.31	5.18
21 30	57 49	1.18	1.61	2.56	3.50	24	59 13	1.28	1.66	3.04	3.95
22	58 6	1.10	1.42	2.39	3.09	25	59 47	1.17	1.25	2.78	2.97
23	58 41	.98	1.28	2.13	2.78	30	62 36	.87	.86	2.06	2.04
24	59 15	.91	1.06	1.98	2.30	35	65 18	.68	.65	1.61	1.54
25	59 50	.88	.99	1.91	2.15	40	67 53	.55	.51	1.30	1.21
30	62 38	.72	.77	1.56	1.67	45	70 20	.45	.43	1.07	1.02
35	65 21	.63	.60	1.37	1.31						
40	67 56	.50	.48	1.09	1.04						
45	70 24	.43	.41	.94	.89						

Green $\lambda = .513\mu$ to $.548\mu$. $\mu_1 = 1.655$. $\mu_2 = 1.383$. $\mu = 1.197$. $c = 56°41'$.						Blue $\lambda = .440\mu$ to $.458\mu$. $\mu_1 = 1.664$. $\mu_2 = 1.372$. $\mu = 1.212$ $c = 55°32'$.					
i	ϕ	d		d/λ		i	ϕ	d		d/λ	
		//	⊥	//	⊥			//	⊥	//	⊥
20°	33° 1′	1.80	1.85	4.70	4.83	21°	57°23′	0.40	0.67	1.22	2.23
0	44 56	1.31	1.41	3.42	3.68	21 30′	57 40	.37	.54	1.13	1.67
10	50 49	.97	1.05	2.53	2.74	22	57 57	.36	.46	1.10	1.41
18	55 42	.65	.73	1.70	1.90	23	58 31	.35	.41	1.07	1.25
20	56 52	.51	.55	1.33	1.43	24	59 5	.33	.39	1.01	1.19
21	57 27	.53	.51	1.38	1.33	25	59 39	.34	.38	1.04	1.16
22	58 1	.49	.51	1.28	1.33	30	62 25	.28	.30	.86	.92
23	58 36	.47	.50	1.23	1.30	35	65 6	.25	.24	.76	.73
24	59 10	.46	.48	1.20	1.25	40	67 37	.23	.22	.70	.67
25	59 44	.43	.44	1.12	1.15						
30	62 31	.39	.39	1.02	1.02						
35	65 13	.35	.34	.91	.89						
40	67 48	.33	.32	.86	.84						
45	70 14	.28	.27	.73	.71						

tion band. A 7 per cent. solution of fuchsine in ethyl alcohol was used. The fuchsine was manufactured by the Heller and Merz Co.

The indices of refraction were obtained with a prism of the liquid, the prism angle varying from a few degrees for the red to two minutes for the green. For red and yellow light the slit width was 0.75 mm., and for the green and blue light the slit width was 1.2 mm.

These values do not differ in their qualitative relations from those where the film is not strongly absorbent. This is true even for the green where the measurements were made for the region including the absorption band. The values are numerically smaller than those obtained with an alcohol film, as might be expected. This is especially so in the case for the green light. It will also be noticed that for the green light the diameter of the transmitted spot of light does not decrease as the angle of incidence decreases from the critical angle. This is due to the fact that no interference bands appear in the transmitted light, on account of the strong absorption, so that what is measured in this case is not the diameter of the inner spot of light but the diameter corresponding to the total effect of all the transmitted light. In the reflected system one interference band was distinctly visible, which disappeared as the critical angle was reached.

Penetration with two Media.

It has been shown that as the optical density of the third medium decreases the penetration increases. If it be allowable to extend this reasoning, it would appear that if the third medium became optically homogeneous with the second, the penetration would be greater than for any case in which the third medium was denser than the second. This is equivalent to saying that the penetration is greater where the second medium is thick, and hence only two media are concerned, than it is when the second medium is a thin film enclosed between denser substances.

To show that penetration does occur where only two media are involved the following method was employed. Sensitized gelatine was prepared from the formula given by Valenta for the Lippman process of color photography. The silver bromide grains were exceedingly fine and the emulsion fairly transparent. One side of a 60° flint-glass prism was covered with a heavy coat of emulsion. When the coat was nearly dry another was added, thus giving a layer of gelatine of great thickness as compared with the penetration phenomena. A beam of sunlight, nearly one centimeter in diameter was introduced into a dark room and allowed to fall upon the prism so that the light passing through the prism would be incident on the emulsion at an angle greater than the critical angle. The prism was in a small black box; the totally reflected light passing out of a small opening, made for the purpose, was absorbed by the black walls of the room. An exposure of five minutes gave, after development, a sharply defined black spot of the size of the incident beam. The film was transferred from the prism to a paraffine block. The darkened spot was on the side of the film next the prism and other parts of the film gave no evidence of exposure. The depth of this darkening was about 0.005 mm. This was obtained by examining a cross-section of the film under a high-power microscope. The fact that the spot was sharply defined shows that the effect was not due to light diffused from the first surface of the prism. That the effect is not due to the light diffused from the second surface is shown by the very small depth to which the darkening extends. As a further proof on the last point the prism was cleaned and placed in the same position as when the exposure was made. A glass plate coated with emulsion was placed about one fourth of a millimeter back of the prism surface from which the light was totally reflected. An exposure of the same length as before failed to show the slightest effect on the sensitive plate. The darkening obtained cannot be due to the light diffused from the surfaces of the prism.

The emulsion may be regarded as gelatine holding in suspension grains of silver bromide. The average diameter of these grains, as obtained with a high-power microscope with a micrometer eyepiece, was less than a wave-length of light. Since only a portion of the surface of the grain could be in contact with the glass, it would not seem probable that the effect is due to those grains which happen to be in actual contact with the glass. That the effect is not due to this cause is clearly shown by the following procedure. One surface of the prism was wet with a very dilute solution of collodion in ether, thus a very thin film of collodion was obtained. The prism was cooled and the emulsion applied to the collodion surface. An exposure, as above, was made, total reflection taking place at the boundary between the glass and the collodion. A dark spot was again obtained on the sensitized gelatine. The film was removed from the prism, under water, the collodion coming off with the gelatine. The collodion film was partially separated from the gelatine, while under water, that portion of the film over the darkened spot was clear, hence no grains of silver bromide had penetrated it. Where the collodion film became several wave-lengths thick no effect was produced in the sensitized gelatine.

The photographic effect, then, must be due to the energy which has passed into the second medium.

Voigt[1] has given a method for showing, without the help of a third body, that in total reflection the vibrations enter the second medium. From a rectangular prism *efg*, Fig. 5, a wedge *afb* is cut. The surfaces *ba* and *ae* are carefully ground, the edge *a* being cylindrical. If light be allowed to fall nearly normally upon *gb*, so as to be totally reflected from the surfaces *ba* and *ae*, and then if one looks at *a* from some point in *oh*, the edge *a* appears as a streak of light. The intensity of this light decreases rapidly as the angle of incidence is increased, and other characteristics peculiar to penetration phenomena are noted.

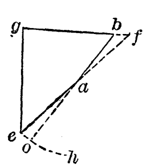

Fig. 5.

In support of the experiment Voigt advances theoretical considerations which have been called in question by Ketteler.[2]

Elliptical Polarization.

The two prisms were clamped together as they were when the penetration distances were measured. The transmitted and reflected light was then examined with a Babinet compensator and the following conclusions drawn. The incident light was polarized in the azimuth 45°.

[1] Wied. Ann., 67, 1899, p. 185.
[2] Wied. Ann., 67, 1899, p. 879. For Voigt's reply see Wied. Ann., 68, p. 135.

1. If the incident parallel light be plane-polarized in any azimuth, both the transmitted and "totally reflected" light will be elliptically polarized.

2. The phase difference, in both the transmitted and reflected light, is independent of the intensity.

3. The phase difference decreases as the index of refraction of the film increases.

4. The phase difference is the same whether the order be flint glass-air-crown glass or crown glass-air-flint glass.

5. The phase difference in the transmitted system increases with an increase in the thickness of the layer penetrated.

6. As the angle of incidence increases, the phase difference increases from zero at the critical angle to a maximum, and then decreases to zero for an angle of incidence of 90°. The maximum value occurs at the angle of incidence for which the distance of penetration is independent of the plane of polarization. That is, at the angle of incidence corresponding to the intersection of the two curves representing the penetration as given in Fig. 3.

6. The direction of the axes of polarization in the transmitted light are parallel and perpendicular to the plane of polarization of the incident light.

The above conclusions are to be considered simply as very close approximations.

Theoretical.

We think of light energy as being propagated by means of ether waves, whatever may be the substance through which propagation takes place. That there is a discontinuity in the condition of the ether at the geometrical boundary between two bodies, or between a body and free space, is hardly conceivable. Therefore whatever may be the difference in the behavior of the ether in two different media, the change in this behavior as one passes from one medium to the other must be a gradual, that is, a continuous change. We should therefore expect, in the case of total reflection, some such phenomena as have been experimentally observed.

For the case where a thin film is enclosed between two denser media Voigt[1] has given a very complete theory. Voigt's theory

[1] Nachrichten v. d. Koniglichen Gesellschaft d. Wissenschaft Göttingen, 1884, p. 49.

has for its starting point the equations of reflection and refraction as deduced from the Neumann and MacCullagh theory, combined with the principle "that in the boundary between two different media a certain work between the ponderable and ether particles must be done by the active vibrating force."[1] Voigt shows the equations he derives to be in qualitative accord with the experimental results of Quincke, with one exception. Quincke states that the penetration is greater when the light passes through the prisms in the direction flint glass-air-crown glass than when the order is reversed. According to Voigt's theory the penetration should be the same in the two cases.

Charles Fabry,[2] starting with the equations for interference in thin films, has also developed equations for the case where the first and third media are the same.

Drude, in Winkelmann's Handbuck der Physik,[3] gives a short discussion of total reflection in thin films, the first and third media being the same.

The electromagnetic theory of light affords a method of deriving equations which agree, both qualitatively and quantitatively, with experimental results. In the following development the symbols used are in accord with those used in Drude's Lehrbuch der Optik. Let the three media be designated 1, 2, 3, with dielectric constants ε_1, ε_2, ε_3, respectively. The index of refraction from 1 to 2 is then given by the equation $n_1 = \frac{\sqrt{\varepsilon_2}}{\sqrt{\varepsilon_1}}$, and from 2 to 3 by the equation $n_2 = \frac{\sqrt{\varepsilon_3}}{\sqrt{\varepsilon_2}}$. We have then $n_1 < 1 < n_2$. Let the plane of incidence be the plane of the paper, or the XZ plane; the XY plane being the boundary between media 1 and 2, and the plane $z = d$ the boundary between media 2 and 3. X, Y, Z are the components of electric force parallel respectively to the axes x, y, z; α, β, γ, are the corresponding components of the magnetic force. Let the incident wave be a plane parallel wave whose angle of incidence on the boundary 1–2 is φ, the angle of refraction in 2 and 3 being ψ and θ respectively. E, R and D are, respectively, the electric forces in the incident, re-

[1] G. Kirchhoff, Berlin Acad., 1876, p. 75.
[2] Compte Rendus, t. 120, 1895, p. 314.
[3] B. II., 1, p. 780.

flected, and transmitted wave. The electric force will be considered as broken up into two components, one perpendicular and the other parallel to the plane of incidence, distinguishing between them by the subscripts s and p respectively. For the case where the electric force is perpendicular to the plane of incidence we have the real parts of the following expressions.

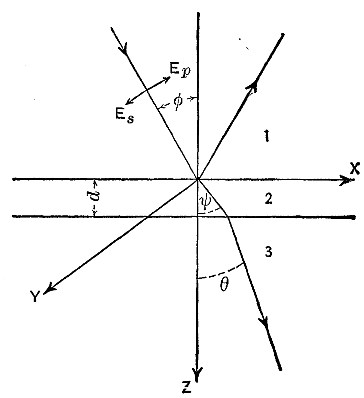

Fig. 6.

For the incident wave in the first medium,[1]

$$(A)\quad \begin{aligned} Y_e &= E_s e^{i\frac{2\pi}{T}\left(t-\frac{x\sin\phi+z\cos\phi}{v_1}\right)} \\ a_s &= -E_s \cos\varphi \sqrt{\varepsilon_1}\, e^{i\frac{2\pi}{T}\left(t-\frac{x\sin\phi+z\cos\phi}{v_1}\right)} \\ \gamma_e &= E_s \sin\varphi \sqrt{\varepsilon_1}\, e^{i\frac{2\pi}{T}\left(t-\frac{x\sin\phi+z\cos\phi}{v_1}\right)}. \end{aligned}$$

For the reflected wave in the first medium,

$$(B)\quad \begin{aligned} Y_r &= R_s e^{i\frac{2\pi}{T}\left(t-\frac{x\sin\phi-z\cos\phi}{v_1}\right)} \\ a_r &= R_s \cos\varphi \sqrt{\varepsilon_1}\, e^{i\frac{2\pi}{T}\left(t-\frac{x\sin\phi-z\cos\phi}{v_1}\right)} \\ \gamma_r &= R_s \sin\varphi \sqrt{\varepsilon_1}\, e^{i\frac{2\pi}{T}\left(t-\frac{x\sin\phi-z\cos\phi}{v_1}\right)}. \end{aligned}$$

For the wave moving in the second medium so that z is increasing.[2]

$$(C)\quad \begin{aligned} Y' &= D_s' e^{i\frac{2\pi}{T}\left(t-\frac{x\sin\psi+z\cos\psi}{v_2}\right)} \\ a' &= -D_s' \cos\psi \sqrt{\varepsilon_2}\, e^{i\frac{2\pi}{T}\left(t-\frac{x\sin\psi+z\cos\psi}{v_2}\right)} \\ \gamma' &= D_s' \sin\psi \sqrt{\varepsilon_2}\, e^{i\frac{2\pi}{T}\left(t-\frac{x\sin\psi+z\cos\psi}{v_2}\right)}. \end{aligned}$$

For the wave moving in the second medium so that z is decreasing,

[1] Drude, Lehrbuch der Optik, p. 258.

[2] Equations C and D are summations. They take account of the multiple reflections, the phase differences introduced being included in the amplitudes D_s' and D_s'' which are to be regarded as complex.

$$Y'' = D_s''e^{i\frac{2\pi}{T}\left(t-\frac{x\sin\psi - z\cos\psi}{v_2}\right)}$$

$$(D) \qquad \alpha'' = D_s''\cos\psi\sqrt{\varepsilon_2}\,e^{i\frac{2\pi}{T}\left(t-\frac{x\sin\psi - z\cos\psi}{v_2}\right)}$$

$$\gamma'' = D_s''\sin\psi\sqrt{\varepsilon_2}\,e^{i\frac{2\pi}{T}\left(t-\frac{x\sin\psi - z\cos\psi}{v_2}\right)}.$$

For the wave moving in the third medium,

$$Y_3 = D_s e^{i\frac{2\pi}{T}\left(t-\frac{x\sin\theta + z\cos\theta}{v_3}\right)},$$

$$(E) \qquad \alpha_3 = -D_s\cos\theta\sqrt{\varepsilon_3}\,e^{i\frac{2\pi}{T}\left(t-\frac{x\sin\theta + z\cos\theta}{v_3}\right)},$$

$$\gamma_3 = D_s\sin\theta\sqrt{\varepsilon_3}\,e^{i\frac{2\pi}{T}\left(t-\frac{x\sin\theta + z\cos\theta}{v_3}\right)}.$$

The boundary conditions are :

(I) $Y_e + Y_r = Y' + Y''$, $\alpha_e + \alpha_r = \alpha' + \alpha''$, $\gamma_e + \gamma_r = \gamma' + \gamma''$ for $z = 0$,

(II) $Y' + Y'' = Y_3$, $\quad \alpha' + \alpha'' = \alpha_3$, $\quad \gamma' + \gamma'' = \gamma_3$ $\quad$ for $z = d$.

Substituting in these equations of boundary condition the value of Y_e, Y_r, etc., as obtained from equations A to E inclusive we have, from I,

$$(1) \qquad E_s + R_s = D_s' + D_s'',$$

$$(2) \qquad E_s - R_s = n_1(D_s' - D_s'')\frac{\cos\psi}{\cos\varphi},$$

and from (II.), letting

$$u = i\frac{2\pi d}{\lambda_2}\cos\psi \quad \text{and} \quad q = \frac{2\pi d}{\lambda_3}\cos\theta,$$

$$(3) \qquad D_s'e^{-u} + D_s''e^{u} = D_s e^{-iq},$$

$$(4) \qquad D_s'e^{-u} - D_s''e^{u} = D_s e^{-iq}\cdot n_2\frac{\cos\theta}{\cos\psi}.$$

When φ becomes greater than the critical angle ψ becomes imaginary. Under these conditions we may write, since $\sin\psi = \sin\varphi/n_1$

$$(5) \qquad \cos\psi = -i\sqrt{\frac{\sin^2\varphi}{n_1^2} - 1}.^1$$

[1] Cos ψ is negative imaginary, since otherwise the amplitude of vibration would increase with an increase in the distance of penetration, as will be seen from later equations

If we introduce the relation (5) in our equations the amplitude D_s becomes complex. The physical meaning of a complex amplitude is that a phase change has been introduced.[1] We may write the relation then $D_s = D_s e^{i\delta}$ where D is real and δ is the phase difference between the transmitted and incident light. u also takes a new value and is no longer imaginary.

$$u = i\frac{2\pi d}{\lambda_2}\left(-i\sqrt{\frac{\sin^2\varphi}{n_1^2} - 1}\right) = \frac{2\pi}{n_1}\frac{d}{\lambda_2}\sqrt{\sin^2\varphi - n_1^2}.$$

Eliminating D_s', D_s'', and R_s from equations 1, 2, 3, and 4, and introducing the relation (5) we have

$$(6)\quad D_s e^{i\delta} e^{-iq} = E_s \cdot \frac{4i\cos\varphi\sqrt{\sin^2\varphi - n_1^2}}{(e^u + e^{-u})(\cos\varphi + n_1 n_2\cos\theta)\, i\sqrt{\sin^2\varphi - n_0^2} + (e^u - e^{-u})[\sin^2\varphi - n_1^2 - n_1 n_2 \cos\varphi\cos\theta]^2}.$$

Multiplying this equation by its conjugate imaginary gives

$$(7)\quad D_s^2 = E_s^2 \cdot \frac{4\cos^2\varphi(\sin^2\varphi - n_1^2)}{(1 - n_1^2)(\sin^2\varphi + n_1^2 n_2^2\cos^2\theta - n_1^2)\sinh^2 u + (\cos\varphi + n_1 n_2\cos\theta)(\sin^2\varphi - n_1^2)}.$$

A similar proceeding gives for the square of the amplitude in the third medium when the incident light is polarized so that the electric force is in the plane of incidence,

$$(8)\ D_p^2 = E_p^2 \frac{4n_1^2\cos^2\varphi(\sin^2\varphi - n_1^2)}{(1 - n_1^2)(\sin^2\varphi - n_1^2\cos^2\varphi)[n_2^2(\sin^2\varphi - n_1^2) + n_1^2\cos^2\theta]\sinh^2 u + n_1^2(n_1 n_2\cos\varphi + \cos\theta)^2(\sin^2\varphi - n_1^2)}.$$

The numerator of either (7) or (8) is always small. If d, which enters the expression for u. exceeds a very small value, the denominator becomes practically infinite, and hence there is no vibration in the third medium.

If the third medium is the same as the first $\theta = \varphi$ and $n_2 = 1/n_1$. Equations (7) and (8) then become

$$(9)\qquad \frac{D_s^2}{E_s^2} = \frac{4\cos^2\varphi(\sin^2\varphi - n_1^2)}{(1 - n_1^2)^2\sinh^2 u + 4\cos^2\varphi(\sin^2\varphi - n_1^2)}$$

$$(10)\qquad \frac{D_p^2}{E_p^2} = \frac{4n_1^4\cos^2\varphi(\sin^2\varphi - n_1^2)}{(1 - n_1^2)^2(\sin^2\varphi - n_1^2\cos^2\varphi)^2\sinh^2 u + 4n_1^4\cos^2\varphi(\sin^2\varphi - n_1^2)}.$$

[1] Lehrbuch der Optik, p. 268.

It remains to be shown that these equations are in complete accord with experiment. The conditions of the experiment are such that the intensity incident on the surface of the first prism is a constant, independent of the plane of polarization and the angle of incidence. Also the intensity observed in the transmitted light is not the intensity in the third medium, but the intensity after the light has left the third medium. Denoting the amplitudes in air by brackets we have

$$E_s^2 = (E_s)^2 \cdot s_1, \quad E_p^2 = (E_p)^2 \cdot p_1, \quad (\mathrm{D}_s)^2 = \mathrm{D}_s^2 \cdot s_3, \quad (\mathrm{D}_p)^2 = \mathrm{D}_p^2 \cdot p_3$$

where

$$s_1 = \frac{\sin 2i \sin 2r}{\sin^2 (i + r)}, \quad p_1 = \frac{\sin 2i \sin 2r}{\sin^2 (i + r) \cos^2 (i - r)}$$

and similarly for s_3 and p_3 at the surface between the third medium and air.

1. There is nothing periodic about equations (7) and (8) and they therefore show that there can be no interference rings for angles of incidence greater than the critical angle.

2. If we divide both numerator and denominator of the second member of equations (7) and (8) by $\sin^2 \varphi - n_1^2$ we shall have as a factor of one term in the denominator,

$$\frac{\sinh^2 \left\{ \frac{2\pi}{n_1} \frac{d}{\lambda} \sqrt{\sin^2 \varphi - n_1^2} \right\}}{\sin^2 \varphi - n_1^2}.$$

The value of this expression is a minimum when $\sin \varphi = n_1$, that is, at the critical angle and increases as φ increases. It follows therefore that the amplitude, and hence the depth of penetration in the second medium, is a maximum at the critical angle, and decreases as the angle of incidence increases.

3. θ decreases when n_2 increases. It follows then from (7) and (8) that as the index of refraction of the third medium increases the amplitude, and hence the penetration in the second medium, decreases and vice versa.

4. The wave-length, λ, occurs directly in the equations and also indirectly in the indices of refraction. The penetration decreases with decreasing wave-length.

5. We have the relation $\sin\varphi = n_1 n_2 \sin\theta$. Inserting this condition, and also introducing the corrective factors s_1 and s_3, equation (7) may be reduced to

$$(11)\quad \frac{(D_s)^2}{(E_s)^2} = s_1 s_3 \frac{4\cos^2\varphi\left(\frac{\sin^2\varphi}{n_1^2} - 1\right)}{(1 - n_1^2)(n_2^2 - 1)\sinh^2 u + (\cos\varphi - n_1 n_2 \cos\theta)^2\left(\frac{\sin^2\varphi}{n_1^2} - 1\right)}.$$

If now the beam of light be reversed in direction and become incident at the angle θ on what has been designated the third medium, the factors s_1 and s_3 and also φ and θ will be interchanged. n_1 must now be replaced by $1/n_2$ and n_2 by $1/n_1$. Inserting these changes in (11) we have for the equation expressing the condition where light passes from the third to the first medium,

$$(12)\quad \frac{(D_s)^2}{(E_s)_2} = s_3 s_1 \frac{4 n_1^2 n_2^2 \cos^2\theta\,(n_2 \sin\theta - 1)}{(1 - n_1^2)(n_2^2 - 1)\sinh^2 u' + (n_1 n_2 \cos\theta + \cos\varphi)^2 (n_2^2 \sin^2\theta - 1)}.$$

The denominators of (11) and (12) are equal since

$$n_2^2 \sin^2\theta = \frac{\sin^2\varphi}{n_1^2}.$$

which relation makes $u = u'$. The numerators become the same if $\varphi = \theta$, that is, if the third and first media have the same optical density.

If the third medium is optically denser than the first we have $\cos\varphi < n_1 n_2 \cos\theta$. It follows therefore that where the first and third media are different the penetration is greater when the direction is such that the light passes from the denser to the rarer medium. Equation (8) leads to the same result.

For the case of an air film between crown glass and flint glass a value for $(D_s)^2/(E_s)^2$ was assumed and the values for d computed. The results as calculated from equation (12) were only a few hundredths of a wave-length greater than those obtained from equation (11).

This conclusion is directly opposed to Voigt's theory. Voigt's theory leads to the conclusion that the phenomenon is completely reversible. The path through the prisms is clearly reversible, but it is by no means evident that the intensities are independent of the direction. Furthermore, experiment seems to confirm the conclusion that the penetration and hence the intensities are not independent of the direction. Voigt's equations, in the form in which they are given, are complicated when expressed in terms of the indices of refraction and the angles involved. They may however be reduced to the form of (7) and (8) and become identical with (7) and (8) if the second member be multiplied by $n_1 n_2 \cos\theta/\cos\varphi$. This factor reduces to unity when the first and third media are the same and hence for this special case Voigt's equations become identical with (9) and (10).

6. Assuming that the sensitiveness of the eye does not change throughout a series of readings, the conditions of experiment are such that the amplitudes at the edge of the transmitted spot of light are constant. They are the amplitudes which just fail to produce an effect on the retina. Since the intensity of the source is assumed constant, we have, for the edge of the transmitted spot of light for any set of readings

$$\frac{(D_s)^2}{(E_s)^2} = \frac{(D_p)^2}{(E_p)^2} = C,$$

where C is some constant.

Rewriting equations (9) and (10) inserting the factors s_1 and p_1,

$$(9') \quad C = s_1^2 \frac{4\cos^2\varphi(\sin^2\varphi - n_1^2)}{(1 - n_1^2)^2 \sinh^2 u_s + 4\cos^2\varphi(\sin^2\varphi - n_1^2)},$$

$$(10') \quad C = p_1^2 \frac{4n_1^4 \cos^2\varphi(\sin^2\varphi - n_1^2)}{(1 - n_1^2)^2(\sin^2\varphi - n_1^2\cos^2\varphi)^2 \sinh^2 u_p + 4n_1^4\cos^2\varphi(\sin^2\varphi - n_1^2)}.$$

It is to be remembered that equations (9′) and (10′) hold only for the edge of the transmitted spot of light. If we multiply both numerator and denominator of (9′) by n_1^4 and examine the denominators of (9′) and (10′) we have

$$n_1^4 > [\sin^2\varphi - n_1^2\cos^2\varphi]^2 \quad \text{for} \quad \sin\varphi = n_1,$$

and

$$n_1^4 < [\sin^2\varphi - n_1^2\cos^2\varphi]^2 \quad \text{for} \quad \sin\varphi = 1.$$

That is, the penetration for increasing angle of incidence is at first greater for light polarized perpendicular to the plane of incidence, and later greater for light polarized parallel to the plane of incidence. To find the condition that $d_s = d_p$ we have, since the second term in the denominator is small in comparison with the first and hence can be neglected,

$$p_1^2 n_1^4 = s_1^2(\sin^2\varphi - n_1^2\cos^2\varphi)^2$$

which becomes

$$\sin^2\varphi = \frac{n_1^2(s_1 + p_1)}{s_1(1 + n_1^2)}, \tag{13}$$

or considering simply the three media

$$\sin^2\varphi = \frac{2n_1^2}{1 + n_1^2}.$$

This formula is in close agreement with experiment. It gives the values of φ for the intersection of the curves in Fig. 3.

7. The theory of elliptical polarization has been fully developed.[1] It can be deduced from the equations given and from a similar pair expressing the conditions in the reflected system. It is interesting to note that if the incident light be polarized in the azimuth of 45°, the phase difference between the components parallel and perpendicular to the plane of incidence, in both the reflected and transmitted systems, attains a maximum value at the value of φ given by equation (13).

8. The second term in the denominator of either (9′) or (10′) is small in comparison with the first term, and may be neglected. Dividing equation (9′) by (10′) we have, after extracting the square root,

$$p_1 n_1^2 \sinh u_s = s_1(\sin^2\varphi - n_1^2\cos^2\varphi)\sinh u_p.$$

Since in general the values of n are not small, we may neglect e^{-u} in comparison with e^u and obtain, on substituting the values of u

$$\left(u = \frac{2\pi}{n_1}\cdot\frac{d}{\lambda}\sqrt{\sin^2\varphi - n_1^2}\right),$$

$$\frac{d_p}{\lambda_2} - \frac{d_s}{\lambda_2} = \frac{\log\text{nat}\left\{\sin^2\varphi\left(\frac{1 + n_1^2}{n_1^2}\right) - 1\right\}\frac{p}{s}}{\frac{2\pi}{n_1}\sqrt{\sin^2\varphi - n_1^2}}. \tag{14}$$

[1] Wüllner, Experimentalphysik, Vol. 4, p. 732.

This gives an approximate formula for testing experimental results. The formula is identical with that given by Voigt. In the experimental results in the columns marked D are given the values of $\frac{d_p}{\lambda_2} - \frac{d_s}{\lambda_2}$ as observed experimentally, and as calculated from equation (14). The agreement between the observed and calculated values is very close, excepting near the critical angle, where the approximations made are not justifiable. Near the critical angle $\sqrt{\sin^2\varphi - n_1^2}$ is small and hence e^{-u} cannot be neglected in comparison with e^u. The introduction of this approximation gives values higher than would otherwise be obtained. This fully explains the apparent discrepancy near the critical angles.

9. If the intensity of the incident light be increased to x times the original intensity, then we have, neglecting the second term of the denominator,

$$C = x s_1^2 \frac{4\cos^2\varphi(\sin^2\varphi - n_1^2)}{(1 - n_1^2)^2 \sinh^2 u_{s \cdot \kappa}}. \tag{15}$$

Dividing equation (9′) by (15) and neglecting e^{-u}, we have

$$\frac{d_{s\kappa}}{\lambda} - \frac{d_s}{\lambda} = \frac{\log \text{nat} \sqrt{x}}{\frac{2\pi}{n_1}\sqrt{\sin^2\varphi - n_1^2}}. \tag{16}$$

The values for the differences in penetration obtained from this formula agree rather poorly from those obtained experimentally in Tables II. and V. This is probably due to the fact that the ratio of the intensities was only approximately obtained.

10. If we assume some value for d the ratio $(D_s)^2/(E_s)^2$ can be determined. The results in the preceding tables were plotted and the value of d at the intersections of the curves, where $d_s = d_p$, was assumed to be correct. From this value of d the ratio of the intensities was computed. Knowing this ratio the value of d for any angle of incidence could be calculated. Equations (9′) and (10′) were used, hence no approximations introduced. The columns marked T in the preceding tables give the calculated results. The results are also plotted in Figs. 7 and 8, the dotted lines being the computed curves, the observed values being denoted by circles (o) and crosses (×), as in Fig. 3. In all cases, with the exceptions

perhaps of Tables I. and III., the agreement between the calculated and observed results is, considering the severity of the test, remarkably close. The agreement is equally good whether the film be air, salt solution, or benzole. Near the critical angle the observed values are all less than the calculated values with

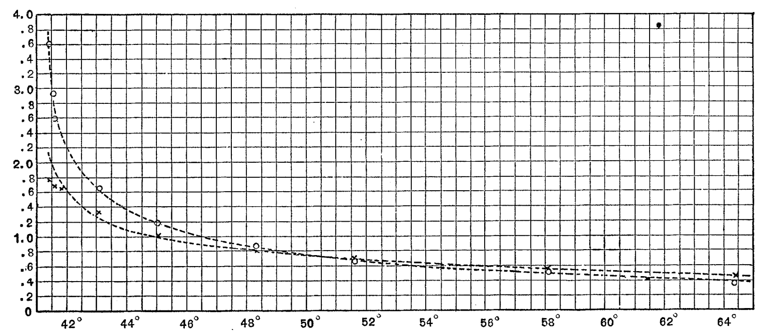

Fig. 7.

Crown glass air crown glass, Table II.

the exception of those for benzole. This is to be expected. Measurements near the critical angle are difficult to obtain, the transmitted spot of light spreads out and the edges become very diffuse. In making settings on this diffuse edge at a point where

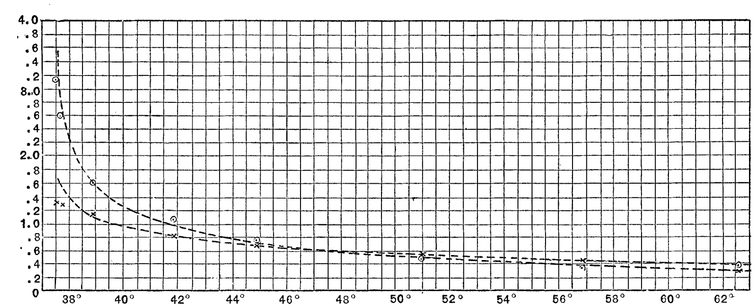

Fig. 8.

Flint-glass air flint glass, Table VII.

one is sure light can be seen, the tendency will be to underestimate rather than overestimate the diameter of the transmitted spot. Especially is this true since much more light is transmitted near the critical angle, and hence the eye rendered less sensitive to small intensities, than is transmitted for larger angles of incidence. In the neighborhood of a degree from the critical angle the edges of the transmitted spot become fairly sharp. Furthermore, in the calculations, the value of d assumed correct is taken from a region where, for the air films, d is changing slowly. A small change in d at this point introduces, in the calculated result, a comparatively large change near the critical angle.

There appears therefore a close agreement, in every detail, both qualitatively and quantitatively, between the theory and the experimental results.

It is interesting to note that Righi[1] has measured the penetration in the case of electric waves using paraffine prisms. Unfortunately only one result is given, that for an angle of incidence of 45°, hence his results cannot be compared with theory.

Theory for Two Media.

If we introduce the condition that the third medium is the same as the second we have $\cos\theta = \cos\psi$ and $n_2 = 1$. Introducing these values in equation (6) and multiplying by the conjugate imaginary the equation reduces without approximation to

$$\frac{D_s^2}{E_s^2} = \frac{4\cos^2\varphi}{e^{2u}(1 - n_1^2)}. \tag{17}$$

Similarly for the case where the electric force is parallel to the plane of incidence,

$$\frac{D_p^2}{E_p^2} = \frac{4n_1^2\cos^2\varphi}{(1 - n_1^2)(\sin^2\varphi - n_1^2\cos^2\varphi)e^{2u}}. \tag{18}$$

u has the same meaning as before, d, which enters the expression u, now means the distance of any position under consideration from the boundary.

These equations give the same qualitative relations as hold for the case where the second medium is thin, and hence three media involved. The condition that $d_s = d_p$ is that

$$n_1^2 = \sin^2\varphi - n_1^2\cos^2\varphi \quad \text{or} \quad \sin^2\varphi = \frac{2n_1^2}{1 + n_1^2},$$

which is the same as before.

[1] Righi, L'Ottica delle Oscillazioni Elettriche, p. 158.

It follows from equations (17) and (18) that the planes of constant amplitude are parallel to the bounding surface, the magnitude of the amplitude decreasing rapidly as the distance from the geometrical surface increases.

To show how rapidly the intensity falls off as the distance from the boundary increases it is necessary simply to insert values for d/λ in the above equations. The following table gives the values for the ratio D_s^2/E_s^2 for different angles of incidence and for different distances from the geometrical boundary of crown glass of index of refraction 1.5151, and air as calculated from equation (18).

ϕ	d/λ								
	¼	½	1	2	3	4	5	10	20
41°25′	3.223	2.667	1.705	0.729	0.312	0.133	0.057	$.82\times10^{-3}$	$.17\times10^{-6}$
43 4	1.615	0.744	.125	.004	$.14\times10^{-3}$	$.46\times10^{-5}$	$.15\times10^{-6}$	$.15\times10^{-13}$	$.10\times10^{-28}$
48 20	.810	.209	.014	$.62\times10^{-4}$	$.28\times10^{-6}$	$.12\times10^{-8}$	$.55\times10^{-11}$	$.95\times10^{-23}$	$.29\times10^{-46}$
60	.131	.025	$.41\times10^{-4}$	$.95\times10^{-9}$	$.21\times10^{-13}$	$.51\times10^{-18}$	$.12\times10^{-24}$	$.77\times10^{-46}$	$.34\times10^{-92}$
75	.017	$.61\times10^{-3}$	$.80\times10^{-6}$	$.12\times10^{-11}$	$.24\times10^{-23}$	$.41\times10^{-28}$	$.70\times10^{-29}$	$.11\times10^{-57}$	$.24\times10^{-116}$

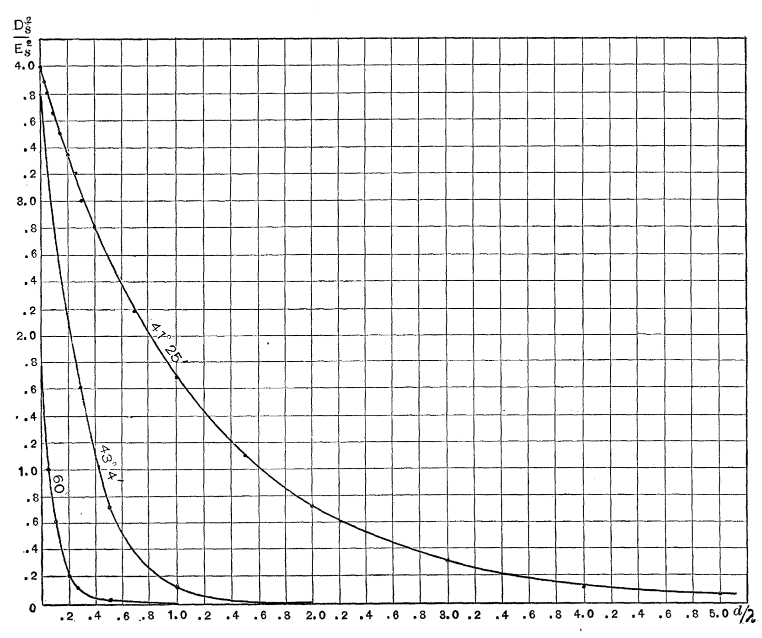

Fig. 9.

For the intensity used in the experimental work with thin films the ratio D/E was of the order of magnitude of .01 for the limit of visibility. Equation (19) would give higher values than are given in the above table for angle of incidence less than 51° 10′ (for this case of crown glass and air), and values less than the above table for angle of incidence greater than 51° 10′. Fig. 9 gives a plot of the above table.

At the critical angle where $\sin\varphi = n_1$, we have from (17) for $d = 0$, the relation $\frac{D_s}{E_s} = 2$. That this relation is true follows at once from the boundary condition derived from the principle of continuity, which is that $Y_e + Y_r = Y_2$. Both experiment and theory show that at the critical angle there is no phase difference. Since, in total reflection, the reflected energy is equal to the incident energy,[1] it follows that for an angle exceeding the critical angle by a differential amount $Y_e = Y_r$. Therefore $Y_2 = 2Y_e$.

The equation for the wave motion in the second medium for the electric force perpendicular to the plane of incidence is the real part of

$$(19) \qquad Y_2 = D_s e^{i\frac{2\pi}{T}\left(t - \frac{x\sin\psi + z\cos\psi}{v_2}\right)}.$$

putting in the relation (5) equation (19) becomes

$$(20) \qquad Y_2 = De^{i\delta}e^{-u}e^{i\frac{2\pi}{T}\left(t - \frac{x}{\frac{n_1}{\sin\phi}v_2}\right)}.$$

This equation represents a wave motion moving in the x direction, that is, along the boundary surface, with a velocity equal to $\frac{n_1}{\sin\varphi}\cdot v_2$. The velocity is therefore v_2 at the critical angle and decreases with an increase in the angle of incidence, reaching the minimum value of v_1 for grazing incidence. The direction of vibration of the electric force is still perpendicular to the plane of incidence and hence parallel to the boundary and transverse to the direction of propagation. The magnetic force is still perpendicular to the electric force but is not normal to the surface and hence not transverse to the direction of propagation. This must be so since a_2 is not zero.

[1] This is proved by electro-magnetic theory in Lehrbuch der Optik, p. 279.

At the boundary and for the critical angle equation (18) becomes $\frac{D_p}{E_p} = \frac{2}{n_1}$. Since n_1 is less than unity, at the critical angle the amplitude at the boundary is greater than twice the incident amplitude, and increases as n_1 decreases, that is, as the two media become more widely different. This is no contradiction to the statement that the penetration decreases as the media become more widely different for although the amplitude at the boundary may be greater, equation (18) shows that for any given angle of incidence, and given amplitude d decreases when n_1 decreases. The value $\frac{D_p}{E_p} = \frac{2}{n_1}$ for $z = 0$ at the critical angle can be derived directly from the equations of wave motion by applying the boundary condition.

Here, as in the previous case, the equation expressing the motion in the second medium may be put in a form similar to equation (20) showing the velocity of the wave along the boundary to be the same as given by equation (20). Since X_2 is not zero the amplitude is not normal to the surface, and not normal to the direction of propagation. The electric force in this case is parallel to the magnetic force in the previous case, where the electric force was perpendicular to the plane of incidence. We have then an ether wave in which the vibrations are not transverse. This is no contradiction to the Maxwell theory. The disturbance at any point is not due to the disturbance at some other point on the boundary but to the wave from the first medium incident at the point in question. This can be more clearly seen from Fig. 10. Let AB represent a particular phase of a portion of the inicdent wave. As the wave moves from B to O, the wave in the second medium corresponding to this particular phase will move from A to O. The motion at O is not due to the motion at A, but due directly to the incident wave from B. It can also be seen from the figure that the velocity along OA is that given by equation (20)

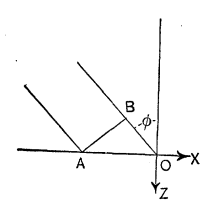

Fig. 10.

The above discussion has reference to an infinite wave, incident on an infinite plane. In the practical case, where neither the incident wave nor the bounding surface are infinite, the discussion would be practically unchanged for points a short distance away from the edge of the incident wave or the edge of the boundary. Just what takes place at the edge is difficult to determine. It is conceivable that here a disturbance may be set up which can propagate itself along the boundary. If so Voigt's experiment for showing penetration when only two media are involved would be justified.

There seems at present no method for experimentally testing the theory for two media in the case of light waves. It would seem feasible however to test the theory with short electric waves. This the writer hopes to do at some future time.

In conclusion the writer wishes to express his appreciation of the interest shown by Professor Slate, of the University of California, and by Professors Nichols and Merritt, and Dr. Shearer, of the Physics Department of Cornell University. The writer is also grateful to Professor Gage, of the Histology Department, for the use of microscopic facilities.

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