MEASUREMENT OF THE INTERNAL RESISTANCE OF GALVANIC CELLS.

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IN the following paper it is proposed to set forth a new method of measuring electrolytic resistance, and to show that, in the case of certain galvanic cells, the *true* internal resistance is not a function of the current passing through the cell.

It is often questioned whether or not a galvanic cell has a fixed resistance, even at constant temperature; and many measurements have been made[1] which have been interpreted to show that the internal resistance decreases as the current passing through the cell increases. Where these measurements were made by steady-current methods, it has been impossibe to separate changes in electromotive force from changes in resistance, and in these cases the experimental results may be explained by assuming changes in electromotive force fully as well as by assuming changes in resistance. Where alternating currents have been used either the reristance has remained constant, or the variations have been much less pronounced than those shown by steady currents. Using an alternating current, F. Kohlrausch[2] found a value for the internal resistance which remained constant while the resistance in the bridge and the intensity of the current sent through the primary of the induction coil were varied between wide limits. Even with alternating currents Uppenborn[3] and Greef[4] found the resistance to depend on the current. Later, however, Haagn,[5] by a modification of Kohlrausch's method, found the internal resistance of galvanic cells to be independent of the current. The method here em-

[1] Streintz, Wied. Ann., 49, p. 571, 1893. Carhart, Physical Review, II., p. 392, 1895. Richarz, Wied. Ann., 47, p. 567, 1892.

[2] Pogg. Ann., Jubeland, p. 220, 1874. Pogg. Ann., 154, p. 1, 1875. Wied. Ann., 6, p. 1, 1878. Wied. Ann., 11, p. 653, 1880.

[3] Electrotech. Ztschr., 1891, p. 157.

[4] Greef Dissert. Marburg, 1895.

[5] Zeitschr. für Phys. Chem., 23, p. 97, 1897.

ployed, which is a bridge method, differs from the preceding in that the resistance and the capacity of the cell are separately, but in the final adjustment, simultaneously balanced, the self-induction being reduced to a negligible quantity. While attention is here directed to measurements of resistance, it may be noted that the method gives also the capacity of the cell.

The Method.

The method here used is a modification of Kohlrausch's method, and is shown schematically in Fig. 1. In the arms *AC* and *BC* of a wheatstone bridge, the resistance r_1 and r_2 are in series with the condensers c_1 and c_2, respectively. *AB* is the bridge wire ; *I*, an induction coil ; and *T* the telephone used in obtaining a balance. For the sake of convenience, this arrangement shall be referred to as the "capacity bridge." a and b are the segments into which the slider, *N*, divides the bridge wire *AB*. When the induction coil is running and the system so adjusted that there is silence in the telephone, the double relation,

$$\frac{r_1}{r_2} = \frac{a}{b} = \frac{c_2}{c_1} \tag{A}$$

is satisfied. If r_1 is the resistance to be measured, the capacities c_1 and c_2 may be fixed, and r_2, a, and b varied to satisfy the above relation. r_1 is then given by the equation

$$r_1 = \frac{a}{b} . r_2 . \tag{B}$$

This method was chosen because by it the internal resistance of a galvanic cell may be measured when no steady current is flowing through the cell as well as when the cell is sending a current through any desired resistance.[1] Thus if a battery is inserted between *A* and c_1, no steady current can flow because of the condenser c_1, and hence, by means of an alternating current and telephone, the resistance may be obtained when the battery is delivering no current. The battery may then be shunted by a resistance *S*, the combined

[1] The method employed by Haagn, though different from the one here used, also afforded this advantage.

resistance of the battery and shunt measured, and, by the law of shunts, the internal resistance of the battery calculated. A second reason for the choice of this method will be shown farther on in this paper.

The induction coil *I* was a small one, such as in ordinarily used with a Kohlrausch bridge. The resistance of the primary coil was 2 ohms and that of the secondary about 30 ohms. The primary was interrupted by the device known as Neeft's hammer. The length of the spring carrying the hammer could be varied, its greatest length being twice its shortest. In order that the noise from the interrupter might not disturb the observer when using the telephone, the induction coil was placed at some distance from the remainder of the apparatus, in a room by itself.

The bridge wire *AB* (Fig. 1) was a meter long, and, on different occasions, had a resistance of from 3 to 5 ohms.

In the first series of experiments, the condensers c_1 and c_2 were made of paper and their capacities were varied from 10 to 40 microfarads. In a later series of experiments, mica condensers of 30 microfarads each were used.

Several telephones were always at hand, and that one was chosen which seemed best suited to the conditions of the experiment under consideration. The one most used was a Swiss telephone of about 80 ohms resistance. Sometimes, however, an American telephone, having a resistance of about 1 ohm, was used.

The method of constructing the variable resistance, r_2, may be readily explained by reference to Fig. 1. Between *C* and *E* is a gap, *g*, into which a resistance coil of any desired size may be inserted. *EGHF* is a manganine wire, about two meters in length and of about 1.25 ohms resistance, fastened at *E* and *F*, and stretched over the two posts *G* and *H*. *MM'* is a trough carrying a drop of mercury connecting the two parts, *EG* and *HF*, of the wire. The resistance in *g*, together with the resistance of *EM'* and *MF*, makes

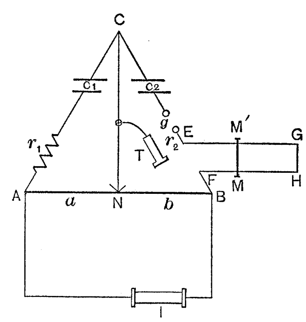

Fig. 1.

the resistance r_2. r_2 could be varied, therefore, by changing the resistance in g and by sliding MM' along the wire. By amalgamating the wire before each experiment, an excellent contact between the mercury and the wire was secured.

The resistance to be measured was inserted in the arm AC between A and c_1, and its value calculated equation from (B). In order to determine the resistance r_2, it was thrown from the arm of the capacity bridge into an arm of a wheatstone bridge, where it was compared with a standard resistance by means of a steady current and galvanometer. Fig. 2 shows the connections of the two bridges. G represents the galvanometer and E the cell used with the second bridge. The transfer of r_2, from one bridge to the other was effected by means of a two-way

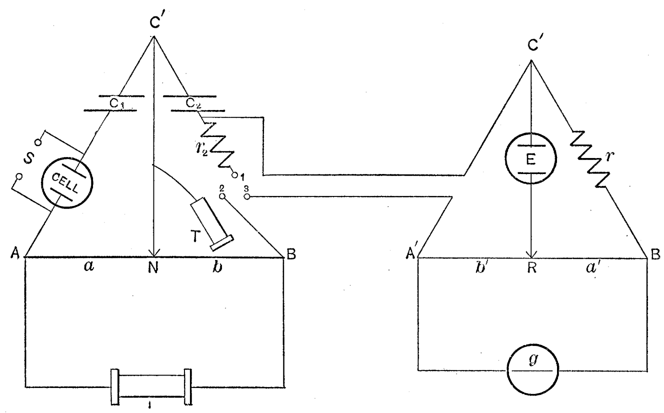

Fig. 2.

switch, which consists of three mercury wells and a heavy copper strap. When wells 1 and 2 were connected by the strap, r_2 was in the capacity bridge; and when the strap connected wells 1 and 3, r_2 was in the second or steady-current bridge. This second bridge was made entirely of copper, in order to reduce the thermo-electric effects to a minimum; but as r_2 was manganine, not copper, and thermo-electric currents might therefore be expected, the galvanometer was connected to the two ends of the bridge wire, its circuit being always closed, while the battery was placed across the bridge, its circuit being open, except for short intervals of time.

In this way, all trouble from thermoelectric effects was avoided.

It is not a difficult operation to satisfy the double relation

$$\frac{r_1}{r_2}=\frac{a}{b}=\frac{c_2}{c_1}. \tag{A}$$

One has merely to move the slider, N, along the bridge wire until a minimum is found, then vary r_2 so as to improve the minimum; then again move the slider, and so on until silence is obtained. Silence having been obtained, r_2 is put into the copper bridge and compared with a standard resistance. If r is the resistance against which r_2 is balanced in the copper bridge, a' and b' the segments of the bridge wire, then

$$r_2=\frac{b'}{a'}\cdot r$$

and, by equation (B),

$$r_1=\frac{b'}{a'}\cdot r\cdot\frac{a}{b} \tag{C}$$

a and b being the segments of the wire on the capacity bridge.

When metallic resistances, free from self-induction, are used in the bridge, perfect silence is obtainable in the telephone. This is not always the case when a liquid resistance is inserted in one arm of the bridge. When polarization takes place in the liquid cell, a disturbing effect is produced. This effect we shall refer to as the capacity effect of the cell, for it has been shown [1] that a polarized cell may be regarded as a resistance in series with a capacity. When such an effect exists, in the arrangement of the bridge here used, it is combined with the capacity of the condenser with which it is in series, and causes no error in the determination of resistance. If a current of a single frequency were used in working the bridge, it would always be possible to obtain silence in the telephone, but when the current is made up of vibrations of different frequencies, only a minimum sound is obtainable, because the capacity effect of a cell is different for different frequencies and it is impossible to balance for all frequencies at once.[2] Suppose the relations (A) are satisfied for metallic resistance r_1 and r_2. If any other metallic re-

[1] Varley, Phil. Mag., 4, 41, p. 310, 1871. Kohlrausch and Holborn, "Leitvermögen der Elektrolyte," p. 67.

[2] M. Wien, Wied. Ann., 42, 593, 1891; 47, 626, 1892; 58, 37, 1896; 59, 267, 1896.

ristance r_1' is substituted for r_1, silence may again be obtained by varying the resistance r_2, leaving the slider, N, unmoved, for c_1 and c_2 remain unchanged and

$$\frac{a}{b} = \frac{c_2}{c_1}.$$

Now suppose the resistance r_1' to be a liquid resistance in which polarization takes place. The capacity of the arm AC will no longer be c_1, but c_1', a combination of c_1 and the capacity effect of the cell. It will not now be possible to obtain a minimum by varying r_2 only, but the slider N must also be moved, for the segments a and b of the bridge wire must satisfy the relation

$$\frac{a}{b} = \frac{c_2}{c_1'}.$$

Even in case the capacity effect of the cell causes a considerable displacement of the position of the slider corresponding to a minimum, the resistance of the cell may be accurately measured, for a minimum is determined by the double relation

$$\frac{a}{b} = \frac{c_2}{c_1'} = \frac{r_1'}{r_2},$$

and r'_1 is given by the equation

$$r_1' = \frac{a}{b} \cdot r_2.$$

In this respect, the form of the bridge under consideration affords an advantage over the Kohlrausch bridge in which a displacement of the minimum causes an error in the determination of resistance. This advantage formed the second reason for the development of the method.

Results and Discussions.

Experiments with Paper Condensers.—The cell to be measured was always inserted in the arm AC so that its resistance corresponded to that which has been called r_1. Care was taken to make the resistance of the connecting wires in the arms AC and BC as small as possible. In order to determine whether or not these wires caused serious error in the measurement of resistance, standard coils were measured. The results showed that resistance varying from one to three ohms could be measured with an accuracy of a few tenths of one per cent. The results given here are all above one ohm, and are probably accurate to within .5 per cent. Allowance was made for the resistances of the end connections of the bridge wires. In the copper bridge, in which the wire was of very low resistance, the end connection at A' (Fig. 2) was equivalent to the resistance of .7 cm. of the bridge wire, while that at B' was equivalent to .8 cm. On the capacity bridge, the connection resistance at A was equivalent to .19 cm. of the bridge wire, and that at B to .33 cm. In the experiments with mica condensers, the connection resistances were so reduced as to be negligible.

The first cell measured was a modified Daniel cell, composed of copper in sulphate of copper and zinc in sulphate of zinc. The copper electrode and the sulphate of copper were placed in a large glass jar, and into the solution of copper sulphate was sunk a porous cup containing the zinc and the zinc sulphate. The cell is shown in the adjoining figure, and will be referred to by its commercial name, "Excello Cell." In the tables of observations, given below, T represents the temperature of the cell, as given by a thermometer, the bulb of which was placed in the porous cup; N and R denote the positions of the sliders on the capacity and copper bridges, respectively; S denotes the resistance by which the cell was shunted, and b is the value found for the resistance of the cell. When no shunt is used, the symbol ∞ is placed in column S.

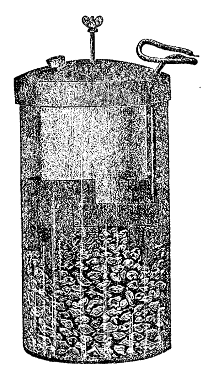

Fig. 3.

It is noticeable that the setting of the slider was practically the same for all these measurements. This shows that if the capacities c_1 and c_2 of the paper condensers changed at all during the experiment, the changes were such that the ratio c_2/c_1 remained constant.

Next the cell was shunted by resistances varying from 100 to 3 ohms, and its resistance measured.

In these experiments, the alternating current in the wires leading from the induction coil was less than .005 ampère, of which only a small part went through the cell. Thus the current used in

$C_1 = C_2 = 20$ *microfarads.* *August* 2, 1900.

T	*N*	*R*	*S*	*b*
25.6	51.20	38.40	100	1.700
25.6	51.20	37.98	∞	1.704
25.6	51.20	38.27	100	1.708
25.6	51.20	38.10	∞	1.693
25.6	51.20	38.47	100	1.698
25.6	51.20	38.03	∞	1.698
25.6	51.20	38.40	100	1.700
25.6	51.20	38.00	∞	1.700
25.5	51.20	38.41	100	1.700
25.5	51.20	38.02	∞	1.698
25.5	51.20	38.41	100	1.700
25.5	51.20	37.94	∞	1.704
25.5	51.20	38.40	100	1.700
25.4	51.20	37.98	∞	1.704
25.4	51.20	38.44	100	1.698
25.4	51.20	37.97	∞	1.704
25.4	51.20	38.44	100	1.698
25.4	51.21	37.88	∞	1.709
25.4	51.21	38.42	100	1.700
25.4	51.22	38.00	∞	1.700

Mean of open circuit resistance = 1.701 ohms. Mean of closed circuit resistance = 1.700 ohms.

$C_1 = C_2 = 20$ *microfarads.* *August* 3, 1900.

T	*N*	*R*	*S*	*b*
23.7	51.20	39.57	100	1.613
23.7	51.20	39.16	∞	1.611
23.7	51.20	39.85	50	1.616
23.7	51.20	39.15	∞	1.611
23.7	51.20	40.10	40	1.616
23.7	51.20	39.20	∞	1.608
23.7	51.20	40.52	30	1.604
23.7	51.20	39.18	∞	1.610
23.7	51.20	41.16	20	1.603
23.7	51.20	39.15	∞	1.612
23.7	51.20	42.81	10	1.611
23.7	51.20	39.21	∞	1.607
23.7	51.10	49.52	3	1.604
23.7	51.20	39.25	∞	1.605

measuring the resistance was not merely alternating, but also extremely small in comparison with the steady current.

The following table contains the results obtained on LeClanche, Daniell, and Excello cells :

	$c_1 = c_2 = 20$ m.f.			$c_1' = c_2 = 40$ m.f.		
	Excello Cell No. 1.	Excello Cell No. 2.	Excello Cell No. 3.	LeClanche Cell.	Daniell Cell No. 1.	Daniell Cell No. 2.
∞	1.603			1.584	1.613	1.184
100	1.602	1.325	1.092	1.540		
50	1.603					
40	1.603		1.072	1.575		
30	1.603		1.095	1.561	1.611	
20	1.604		1.095	1.551	1.620	
10	1.600	1.321	1.096			1.181
3	1.603		1.110			

Figures for Excello and LeClanche cells represent single observations, while those for the Daniell cells are means of a number of observations taken in succession. It was impossible to obtain a reliable minimum when low shunts were used on the LeClanche cells. In all these experiments, the minimum was sharp and its position was readily found. In the case of the Excello cells, it was found that if the zinc electrode was kept clean and well amalgamated the minimum was not only sharp, but symmetrical, as is the case when metallic resistances are used. Moreover, when the zinc electrode of these cells was clean, the position of the minimum was practically the same as for metallic resistances. This shows that there was no disturbing effect due to polarization. In the case of the LeClanche and Daniell cells, the minimum was displaced from 1 to 2 mm. from that given by metallic resistances. Under these circumstances, of course, the effect being different for different frequencies, the minimum was not symmetrical.

Experiments with Mica Condensers.

The mica condensers used in this series of experiments had a capacity of 30 microfarads each. The resistance of the wires connecting the various sections was very small and as nearly as possible the same in both condensers. Before beginning these experiments, the two bridges were thoroughly overhauled and the resistances of the end connections of the bridge wires were made negligible. Measurements on known metallic resistance, made to test the accuracy of the bridge, gave results correct to one-twenty-fifth per cent. In the experiments with this apparatus, the cells were measured first with no shunt and then with a shunt of 1 ohm. In order to keep the condition of the cells as steady as possible, they were shunted by a resistance of 10 ohms during the time between the different observations.

The table below contains the results obtained from five different cells. The zinc electrode of Daniell cell No. 1 consisted of a rod about three inches long and three-sixteenths of an inch in diameter. The zinc electrode of Daniell cell No. 2 was in the shape of a rectangle about 10 centimeters long and 2 centimeters wide. The electrode in the Excello cells, as well as those of the Daniell cells given in the first table, were of the size ordinarily used in these cells.

S.	Excello Cell No. 1.	Excello Cell No. 2.	Daniell Cell No. 1.	Daniell Cell No. 2.	Columbia Dry Cell.
∞	.6133	.5838	2.650	2.044	.1151
1	.6123	.5845	2.651	2.027	.1151
∞	.6138	.5843	2.656	2.036	.1157
1	.6117	.5855	2.652	2.052	.1151
∞	.6112	.5863	2.659	2.036	.1147
1	.6117	.5863	2.651	2.027	.1158

The results given above show that the resistance was the same whether the cell was in open or closed circuit, within high limits of accuracy. However, it is to be noted that the condition of a cell, of course, depends upon its previous history, among other things upon any prolonged current that may have been flowing, insofar as this actually alters the cell. What is here shown is that the resistance of a cell is not a function of the current strength at the time, a result contrary to the conclusions of most other observers.

ON THE THEORY OF THE BRIDGE.

Thus far we have considered only the apparatus used and the results obtained in actual experiment. It will now be of interest to consider the relations between the current in the telephone and the magnitude of the various quantities used in the bridge, for from these relations we may see what conditions are necessary for accurate work. Our arrangement of the bridge is represented in Fig. 1. In the arms AC and BC the resistance r_1 and r_2 are in series with the capacities c_1 and c_2, respectively. T denotes the telephone, and I the induction coil used with the bridge. Let r_3 and r_4 be the resistances of the segments AN and NB of the bridge wire AB. For the sake of clearness, the various quantities will be considered under separate heads.

Capacity.—In order to form an idea of how much capacity must be used to render accurate measurements of resistance possible, it is necessary to write down the expression for the current in the telephone. For the sake of simplicity, assume the resistance and self-induction of the main branch, AIB, to be so large that changes in the capacities c_1 and c_2 do not materially alter the current supplied to the bridge. If the current in the main branch is assumed to be

$$\cos(nt) = \tfrac{1}{2}\left({}^{int}e + e^{-int}\right),$$

the current in the telephone at any time may be found by taking one-half the sum of the currents in the telephone when currents e^{int} and e^{-int} are separately taken as the currents in the main branch.

If there is a current e^{int} in the main branch, the current in the telephone is given by

$$Z' = e^{int}\frac{a_2a_3 - a_1a_4}{(a_2 + a_4)(a_1 + a_3) + a(a_1 + a_2 + a_3 + a_4)} \qquad (1)$$

where

$$a_1 = r_1 + \frac{1}{inc_1},$$

$$a_2 = r_2 + \frac{1}{inc_2},$$

$$a_3 = r_3,$$

$$a_4 = r_4,$$

and

$$a = r + inL,$$

r being the resistance, and L the inductance of the telephone. Putting these values for the a's in equation (1), we obtain,

$$Z' = \frac{e^{int}(A + iB)}{1 - C + iD} \qquad (2)$$

where

$$A = n^2 c_1 c_2 (r_1 r_4 - r_2 r_3),$$

$$B = n(r_3 c_1 - r_4 c_2),$$

$$C = n^2 k_1 k_2 c_1 c_2 + n^2 r c_1 c_2 (k_1 + k_2) + n^2 L(c_1 + c_2),$$

$$D = n[K_2 c_2 + k_1 c_1 + r(c_1 + c_2) - n^2 L(k_1 + k_2) c_1 c_2],$$

$$k_1 = r_1 + r_3$$

and

$$k_2 = r_2 + r_4.$$

Multiplying the numerator and denominator of (2) by $(1 - C) - iD$, we obtain,

$$Z = e^{int} \frac{(A + Bi)[(1 - C) - iD]}{(1 - C)^2 + D^2}$$

or

$$Z' = e^{int}(A'' + iB'') \tag{3}$$

where

$$A'' = \frac{A(1 - C) + BD}{(1 - C)^2 + D^2} \tag{4}$$

and

$$B'' = \frac{B(1 - C) - AD}{(1 - C)^2 + D^2} \tag{5}$$

If e^{-int} is taken as the current in the main branch, the current in the telephone is given by

$$Z'' = e^{-int}(A'' - iB''),$$

and the current in the telephone corresponding to a current cos (nt), in the main branch, is

$$Z = \tfrac{1}{2}(Z' + Z'')$$

$$= A'' \frac{e^{int} + e^{-int}}{2} + B'' \frac{e^{int} - e^{-int}}{2} i$$

$$= A'' \cos(nt) - B'' \sin(nt)$$

whence

$$Z = \sqrt{A^{2''} + B^{2''}} \cos(nt - \varphi) \tag{6}$$

where

$$\tan \varphi = -\frac{B''}{A''}.$$

When $r_1/r_2 = r_3/r_4$, we have $A = 0$, and when $c_1/c_2 = r_4/r_3$, $B = 0$. When both A and B are zero, A'' and B'' are zero; and, therefore, by (6), $Z = 0$. Now if it is desired to measure resistances there must be a disturbance in the telephone whenever $r_1/r_2 \neq r_3/r_4$, even if $c_1/c_2 = r_4/r_3$; that is, even if $B = 0$. It is, therefore, desirable to study the relations that must exist between the resistances, capacity, and inductance in order that the current in the telephone may be detected as long as the resistances do not satisfy the condition $r_1/r_2 = r_3/r_4$. The work is much simplified, while its usefulness is not at all impaired, by the assumption $B = 0$. If $B = 0$,

$$A'' = \frac{A(1 - C)}{(1 - C)^2 + D^2},$$

and

$$B'' = \frac{-AD}{(1 - C)^2 + D^2},$$

hence, if the maximum value of Z is called $\bar{Z}$, we may write

$$\bar{Z} = \sqrt{\left[\frac{A(1 - C)}{(1 - C)^2 + D^2}\right]^2 + \left[\frac{-AD}{(1 - C)^2 + D^2}\right]^2},$$

or

$$\bar{Z} = A[(1 - C)^2 + D^2]^{-\frac{1}{2}}. \qquad (7)$$

Now

$$A = n^2 c_1 c_2 (r_1 r_4 - r_2 r_3);$$

therefore, A increases as the product $c_1 c_2$, and, if C and D are small in comparison with unity as is the case when $c_1 c_2$ are small, $\bar{Z}$ will increase as the product $c_1 c_2$. In this case evidently $\bar{Z}$ may be increased by increasing r_3 and r_4, *i. e.*, by increasing the resistance of the bridge wire. Let us now assume that $c_1 = c_2$, as was the case in all the experiments made with this arrangement of the bridge, and write

$$P = \frac{1 - C}{c_1^2},$$

and

$$Q = \frac{D}{c_1^2},$$

or

$$P = \frac{1}{c_1^2} - \left[n_2 k_1 k_2 + n^2 r(k_1 + k_2) + \frac{2n^2 L}{c_1}\right] \qquad (8)$$

and

$$Q = n\left[\frac{k_1 + k_2 + 2r}{c_1} - n^2L(k_1 + k_2)\right]. \tag{9}$$

Then equation (7) becomes

$$\bar{Z} = \frac{n^2c_1^{\ 2}}{c_1^{\ 2}}(r_1r_4 - r_2r_3)\left[P^2 + Q^2\right]^{-\frac{1}{2}},$$

or

$$\bar{Z} = n^2(r_1r_4 - r_2r_3)\left[P^2 + Q^2\right]^{-\frac{1}{2}}. \tag{10}$$

Consider c_1 the only variable in the right hand side of equation (10). Z is largest for that value of c_1 which makes $[P^2 + Q^2]^{\frac{1}{2}}$ smallest, or for that value which makes

$$D_{c_1}P \cdot P + D_{c_1}Q \cdot Q = 0 \tag{11}$$

Putting in (11) the values of P and Q given by (8) and (9), we obtain,

$$ac_1^{\ 3} - a_1c_1^{\ 2} + a_2c_1 - a_3 = 0 \tag{12}$$

where

$$a = n^4L\,(k_1^{\ 2} + k_2^{\ 2})$$
$$a_1 = n^2[k_1^{\ 2} + k_2^{\ 2} + 2r\,(k_1 + k_2) + 4r^2] + 4n^4L_2$$
$$a_2 = 6n^2L$$

and

$$a_3 = 2.$$

It is evident that equation (12) always has at least one real, positive root. Hence for any given resistances and inductance there is always at least one value for c_1 that makes $\bar{Z}$ a maximum.

Let us now study the variations of $\bar{Z}$ with c_1 for the special case in which $c_1 = c_2$, $r_3 = r_4 = 1$, $r_1 = 1 + x$, $r_2 = 1$, $r = 1$, $L = \frac{1}{2}\,10^{-4}$, and $n = 10^3$, these values being chosen because they are of the same order of magnitude as those that were used occasionally in actual experiment. x is here the amount by which r_1 deviates from satisfying the relation $r_1/r_2 = r_3/r$. For this case,

$$k_1 = 2 + x;$$
$$k_2 = 2;$$
$$A = 10^6c_1^{\ 2}x;$$
$$C = 10^6(8 + 3x)c_1^{\ 2} + c_1 10^2$$

and

$$D = 10^3\,[6c_1 + xc_1 - 50\,(4 + x)\,c_1^{\ 2}].$$

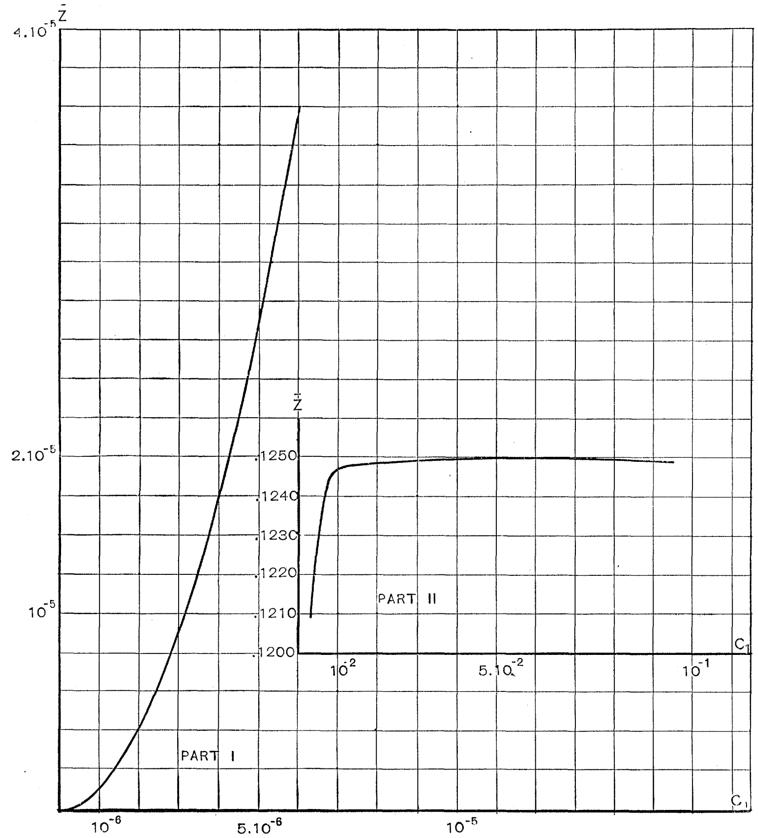

Fig. 4.

If x is small in comparison with unity, we may write,

$$C = 8 \cdot 10^6 \cdot c_1^2 + c_1 \cdot 10^2,$$

and

$$D = 10^3 \, [6c_1 - 2 \cdot 10^2 \cdot c_1^2].$$

This makes, by (8) and (9),

$$P = \frac{1}{c_1^2} - 10^2 \left[10^4 \cdot 8 + \frac{1}{c_1} \right]$$

and

$$Q = 10^8 \left[\frac{6}{c_1} - 2 \cdot 10^2 \right].$$

By equation (10),

$$\bar{Z} = 10^6 x [P^2 + Q^2]^{-\frac{1}{2}}. \tag{13}$$

To find the value of c_1 for which $\bar{Z}$ is a maximum, we substitute for P and Q in the equation,

$$D_{c_1}P \cdot P + D_{c_1}Q \cdot Q = 0.$$

This gives,

$$4 \cdot 10^8 c_1 - 2001 \cdot 10^4 c_1^2 + 3 \cdot 10^2 c_1 - 2 = 0. \tag{14}$$

By Sturm's theorem, equation (14) is seen to have but one real root; this root lies between $c_1 = 10^{-1}$ and $c_1 = 10^{-2}$. Hence there is but one value of c_1 which renders $\bar{Z}$ a maximum. The relation between c_1 and $\bar{Z}$ is shown by Fig. 4. This curve is given in two parts, each part having a scale of its own. Part I. shows the rapid increase of $\bar{Z}$ with c_1, when c_1 is small. Part II. shows the maximum value of $\bar{Z}$, which corresponds very nearly to $5 \cdot 10^{-2}$ farads. Between Parts I. and II. a portion of the curve, having no peculiar points, is omitted. The table below gives a few values of c_1, together with the corresponding values of $\bar{Z}$.

c_1	$\bar{Z}$	c_1	$\bar{Z}$
0	0	10^{-2}	$.1248x$
10^{-6}	$10^{-6}x$	2.10^{-2}	$.1249x$
2.10^{-5}	$4.10^{-6}x$	5.10^{-2}	$.1250x$
10^{-5}	$10^{-4}x$	10^{-1}	$.1249x$
10^{-4}	$9.2x10^{-3}$	1	$.1249x$
10^{-3}	$.109x$	∞	$.1249x$

The table shows that $\bar{Z}$ at the maximum is only slightly greater than the limit approached as c_1 increases without limit; that is, $\bar{Z}$ is never much larger than it would be if resistances only were used in the bridge. The value of c_1 corresponding to a maximum of $\bar{Z}$ is, in the case under consideration, exceedingly large. It is, therefore, fortunate that accurate work may be done long before the best condition is attained. With a bridge wire of two or three ohms resistance, and with capacities of from twenty to forty microfarads, a resistance as low as half an ohm may be measured with an accuracy of .5 per cent.

If in place of the telephone, we have a non-inductive conductor of resistance, r, across the bridge,

$$P = \frac{1}{c_1^2} - n^2 k_1 k_2 - n^2 r(k_1 + k_2),$$

and

$$Q = n[k_1 + k_2 + 2r]c_1^{-1}.$$

In this case,

$$Dc_1 P \cdot P + Dc_1 Q \cdot Q = 0,$$

becomes

$$c_1^2[4r^2n^2 + 2rn^2(k_1 + k_2) + n^2(k_1^2 + k_2^2)] + 2 = 0.$$

Since the roots of this equation are always imaginary, there is no value of c_1 for which $\bar{Z}$ is a maximum. In fact,

$$\bar{Z} = n^2(r_1 r_4 - r_3 r_2)\left[\frac{1}{c_1^4} + \frac{G}{c_1^2} + H\right]^{-\frac{1}{2}}$$

where

$$G = n^2(k_1^2 + k_2^2) + 4n^2r^2$$

and

$$H = n^4 k_1 k_2 + n^4 r^2(k_1 + k_2)^2 + 2n^4 r k_1 k_2(k_1 + k_2).$$

And it is evident that $\bar{Z}$ increases when c increases, as would naturally be expected.

Inductance.—That there is a value of the inductance, L, of the telephone which renders $\bar{Z}$ a maximum when the resistances and capacity in the bridge are fixed, may be seen by placing the derivative, with respect to L, of the right-hand side of the equation (10) equal to zero. This gives the equation,

$$D_L P \cdot P + D_L Q \cdot Q = 0,$$

an equation which is linear as regards L. Putting in the values of P and Q, we obtain,

$$L = \frac{c_1(k_1^2 + k_2^2) + \dfrac{2}{n^2 c_1^2}}{4 + n^2(k_1 + k_2)^2 c_1^2}. \tag{14}$$

From this equation, it is evident that the smaller c_1, the larger the inductance required to make $\bar{Z}$ a maximum. That the resistance of the telephone does not enter into this expression for the inductance is not surprising, for we have assumed that we could vary the inductance of the telephone without varying its resistance.

$$k_1 = k_2 = 2,$$

$$c_1 = 4 \cdot 10^{-5},$$

and

$$n = 10^3,$$

we find

$$L = .0124 \text{ henry.}$$

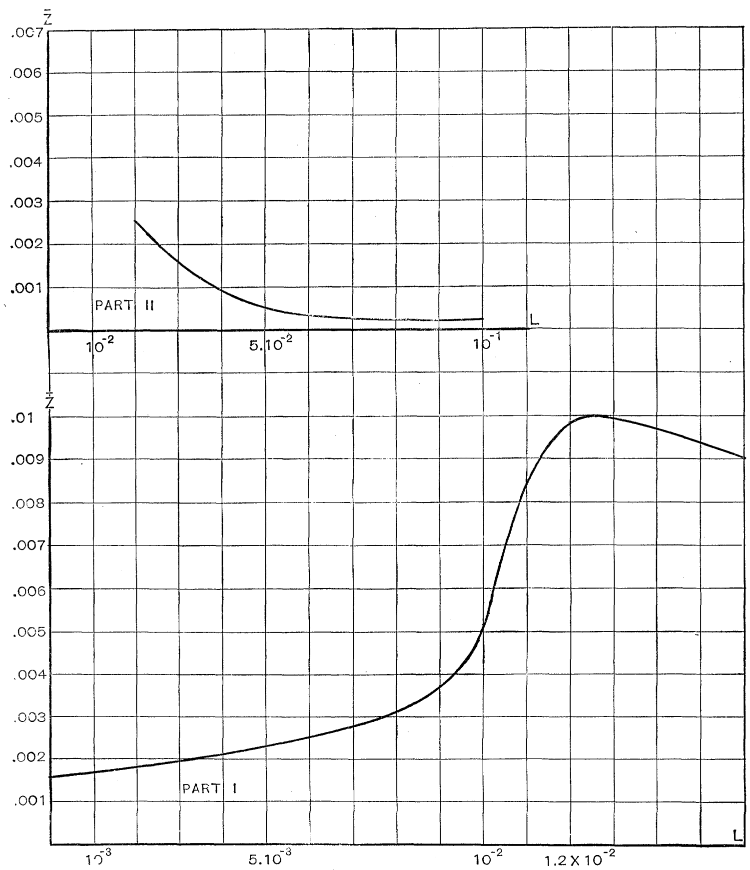

Fig. 5.

Fig. 5 shows the relation between L and $\bar{Z}$ as calculated from equation (13) on the assumption that

$$r_2 = r_3 = r_4 = r = 1 \text{ ohm.}$$

$$r_1 = 1 + x,$$

$$n = 10^3,$$

and

$$c_1 = 40 \text{ microfarads.}$$

This curve is given in two parts, having different scales. Part I. contains the maximum and is of chief interest. There is a small and unimportant part of the curve omitted between Parts I. and II. The following are a few values of L, with the corresponding values of $\bar{Z}$.

L	$\bar{Z}$	L	$\bar{Z}$
0	$.00157x$	$1.24 . 10^{-2}$	$.0100x$
10^{-5}	$.00158x$	$2 . 10^{-2}$	$.00257x$
10^{-4}	$.00158x$	$5 . 10^{-2}$	$.000506x$
10^{-3}	$.00170x$	10^{-1}	$.000230x$
10^{-2}	$.00525x$	∞	$.0000$

Frequency.—In order to see if, by varying n, keeping the resistances and inductance constant, a value of n can be found for which $\bar{Z}$ is a maximum, it will be well to write equation (7) in the form,

$$\bar{Z} = c_1^2(r_1 r_4 - r_2 r_3)[P_1^2 + Q_1^2]^{-\frac{1}{2}} \tag{15}$$

where

$$P_1 = \frac{1}{n^2} - [k_1 k_2 c_1 + r c_1^2 (k_1 + k_2) + 2Lc_1],$$

and

$$Q = \frac{k_1 c_1 + k_2 c_1 + 2rc_1}{n} - nL(k_1 + k_2)c_1^2,$$

c_1 being taken equal to c_2, as before.

To make $\bar{Z}$ a maximum, we make

$$D_n P_1 \cdot P_1 + D_n Q_1 \cdot Q_1 = 0. \tag{16}$$

Substituting the values of P_1 and Q_1 in this equation, we obtain, after reduction,

$$n^6 L^2(k_1 + k_2)^2 c_1^4 + n^2 [4Lc_1 - 4r^2 c_1^2 - 2rc_1^2(k_1 + k_2) - k_1^2 c_1^2 - k_2^2 c_1^2] - 2 = 0.$$

Or, if we let $n_1 = n^2$

$$n_1^3 L^2(k_2 + k_1)^2 c_1^4 + n[4Lc_1 - 4r^2 c_1^2 - 2rc_1^2(k_1 + k_2) - k_1^2 c_1^2 - k_2^2 c_1^2] - 2 = 0. \tag{17}$$

If, for example, we take

$$c_1 = 3 \cdot 10^{-5},$$

$$L = \tfrac{3}{2}\ 10^{-4},$$

and

$$k_1 = k_2 = 2,$$

equation (17) reduces to

$$n_1^3 = \frac{10^{28}}{145\ 8},$$

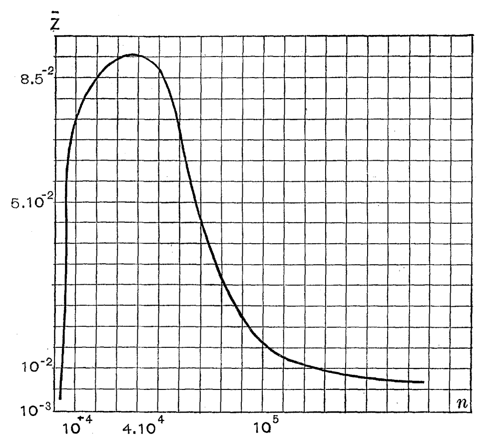

Fig. 6.

or

$$n_1^3 = \frac{10^{27}}{146};$$

hence

$$n_1 = \frac{10^9}{(146)^{\frac{1}{3}}}$$

and

$$n = \frac{10^{\frac{9}{2}}}{(146)^{\frac{1}{6}}},$$

$$n = 4 \cdot 10^4.$$

And the number of complete vibrations per second is given by $\frac{n}{2\pi}$, or $\frac{2.10^4}{\pi} = 6366$.

Fig. 6 shows the relation between $\bar{Z}$ and n as calculated from equation (13); it being assumed that $k_1 = k_2 = 2$,

$$c_1 = 3 \cdot 10^{-5},$$

and

$$L = \tfrac{3}{2}\, 10^{-4}.$$

The following table shows a few of the values of n, together with the corresponding values of $\bar{Z}$.

n	$\bar{Z}$	n	$\bar{Z}$
0	0	10^5	$1.6.10^{-2}x$
10^2	$9.10^{-6}x$	10^6	$1.7.10^{-3}x$
10^3	$9.10^{-4}x$	10^7	$1.7.10^{-4}x$
10^4	$7.10^{-2}x$	10^8	$1.7.10^{-5}x$
4.10^4	$8.6.10^{-2}x$	∞	0

It is to be noticed that $\bar{Z}$ increases rapidly at first as n increases, reaching a maximum when n is $4 \cdot 10^4$, and then decreases as n increases, approaching the value of zero as n increases without limit.

The above treatment shows how the accuracy of measurement depends upon the relations between inductance, capacity, resistance, and frequency of current used in the bridge.

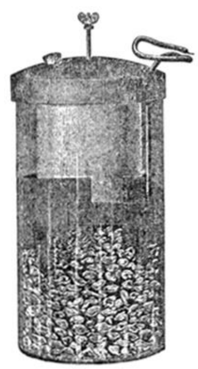

Fig. 3.