ON THE PERIOD OF A ROD VIBRATING IN A LIQUID.

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AS early as 1786 Buat announced in his " Principes d'Hydraulique" that the period of a pendulum was greater when vibrating in a fluid than when vibrating in a vacuum, not only because of the loss in weight due to buoyancy but also because of the mass of fluid which must be considered as participating in the motion of the pendulum.   The latter is loaded by, or drags with it, a certain amount of the fluid.   Buat determined this added mass for the case of a sphere and for bodies of other forms.   Little attention was paid to this work until Bessel[1] about forty years later, in determining the length of the seconds pendulum, concluded from theoretical reasons that it was necessary to take into account the inertia of the air as well as its weight.   When the dimensions of the bob were small in comparison with the length of the suspending wire, Bessel represented the apparent increase in the weight of the pendulum by a constant, k, times the mass of the displaced fluid. For a sphere 2 in. in diameter vibrating in water he found for k the value 0.9459 ; later he changed this to 0.956.

Sabine[2] tried to find the effect of air on a pendulum by making it vibrate in air and then in a vacuum.   He found that the old correction for buoyancy should be multiplied by a factor m, whose value was 1.655, in order to obtain the total correction.

In 1832 Baily[3] published the results of some similar experiments.   Four spheres of about 1.5 in. in diameter gave for m the value 1.84 ; and three spheres of about 2 in. in diameter gave 1.748. For spheres of about the same size Bessel had found the value 1.956.   For small cylinders Baily found the value of m to increase regularly with decrease of diameter, but according to no apparent law.

[1] Abh. d. Akad. d. Wiss., Berlin, 1826 ; Coll. de Mem. Soc. Fr. de Phys., 5, 1.

[2] Phil. Trans., 1829 ; Coll. de Mem. Soc. Fr. de Phys., 5, 134.

[3] Phil. Trans., 1832, p. 399 ; Coll. de Mem. Soc. Fr. de Phys., 5, 185.

Poisson[1] determined analytically the value of *m* for spheres and found it to be 1.5 ; which agrees very well with the results of Buat's experiments made fifty years earlier.

In 1843 Stokes[2] found from theoretical considerations that for a long cylinder oscillating in a direction perpendicular to its length, the effect of the inertia of the fluid was the same as if a mass equal to that of the displaced fluid were distributed along the axis of the cylinder.   This agrees very well with the experimental results obtained by Baily for a long tube 1.5 in. in diameter.

Up to this time no law was known connecting the variation of *m* with the dimensions of the pendulum, and the only property of the fluid taken into consideration was its density.   In 1848 Stokes[3] treated the problem of a body oscillating in an infinite liquid medium. He dealt with the cases of a disk rotating about its axis, of a sphere oscillating in the direction of a diameter, and of an infinitely long cylinder vibrating in a direction perpendicular to its length.   He considered the effect of the viscosity as well as of the density of the fluid.   This paper is of especial interest here because the experiments about to be described approximate more nearly to this problem than to any other that has been treated mathematically. The main results of this important piece of work will be referred to later. .

In 1880 Montigny[4] published an account of some experiments which he had made in 1859 in order to find the effect of the density and of the compressibility of various liquids on the pitch of bells. (He intended these experiments to be preliminary only and to repeat them with electrically driven forks.   He evidently did not do so, although in a second paper[5] he describes his method of maintaining forks in vibration in a liquid, and states that the pitch is lower the denser the liquid.)   He found the pitch of the bell first in air, and then when submerged mouth upward in the liquid; also when in air and filled to the brim with the liquid ;   and finally when

[1] Mem. de l'Acad. de Paris, 11, 1831.

[2] Trans. Camb. Phil. Soc., 8, 105, 1843 ; B. A. Report, 1848.

[3] Trans. Camb. Phil. Soc., 9, pt. 2, p. 35 ;   Coll. de Mem. Soc. Fr. de Phys., 5, 277.

[4] Bull. de l'Acad. de Belg., [2], 50, 159, 1880.

[5] Bull. de l'Acad. de Belg., [2], 50, 300, 1880.

empty and immersed to the brim in the liquid. The pitch was the same for the last two cases, and the lowering of pitch was less than when the bell was entirely submerged; from this he concluded that the lowering was due to that part of the liquid which was in contact with the bell. The lowering was found to depend also on the shape and material of the bell. The pitch was determined by comparing the note by ear with that of a monochord. He found that the lowering of pitch was greater the denser the liquid and the graver the pitch. He gives the following intervals of lowering[1] for four of the bells used.

	Sol$_4$	Ut$_5$	Sol$_5$	Ut$_6$
When filled with water . .	1.204	1.125	1.127	1.118
When submerged in water .	1.411	1.327	1.286	1.261

Auerbach,[2] in a paper written in 1878, attempted to deduce a simple theoretical relation between the pitch of a tuning fork in air and that in a liquid as a consequence of the different way in which kinetic energy is dissipated in the two cases. The pitch, he contends, always depends on the square root of the coefficient of elasticity, and this can have two entirely different values according as the vibration takes place isentropically or isothermally. The former he considers to be the case for air and the latter for liquids. He then takes the coefficients of elasticity in the two cases to be as the ratio of the specific heats, 1.4 : 1, and the interval of lowering of pitch to be as $\sqrt{1.4}$: 1; that is, as 1.18 : 1. He tried four forks in air and in water, observing the pitch by ear, and obtained the following intervals :

Frequency of fork in air,	132	264	396	528
Interval of lowering,	1.11	1.12	1.13	1.15

He concluded from these experiments that the ideal interval, 1.18, would be approached more and more nearly the higher the pitch. He adds that experiments with other liquids show that the density and viscosity of the liquid have no influence on the lowering of pitch.

[1] The ratio of the pitch in air to that in the liquid is commonly called the interval of lowering.
[2] Wied. Ann., 3, 157, 1878.

Kolácek,[1] referring to this work by Auerbach, suggests that an explanation of this lowering of pitch can be given from simple mechanical principles, and that the effect of the liquid is mainly to load the fork by a certain quantity which is the product of the density, ρ, of the liquid and a factor, c, depending on the size and shape of the vibrating body. Accordingly he finds the interval of lowering to be $\frac{T'}{T} = \sqrt{1 + c\rho}$. Taking the mean, 1.13, of the intervals found by Auerbach, he determined the value for c to be 0.277. Using this value of c it follows that Auerbach's forks should give $\frac{T'}{T}$ equal to 2.18 when made to vibrate in mercury, and 1.21 when in sulphuric acid. On plunging a 435 fork into various liquids and noting its changed pitch by an ear comparison with a monochord, Kolácek found the value of $\frac{T'}{T}$ to be 2.1 for mercury, 1.21 for sulphuric acid, and about the same for ether and alcohol as for water. He proceeds to treat analytically the simple cases of the radial dilatational and linear translational oscillations of a sphere in a liquid, and finds expressions for $\frac{T'}{T}$ in the two cases. When applied to a solid iron sphere vibrating in water these expressions give for $\frac{T'}{T}$ the values 1.03 and 1.17 respectively. Neither of these cases is that of a tuning fork, but they give limiting values for it and the mean of the two agrees roughly with Auerbach's results. In a later paper Kolácek[2] gives a more rigorous hydrodynamical discussion of the general problem of the vibration of a solid in a liquid.

In 1882 Auerbach[3] experimented with glass cylinders filled with liquid. Assuming that the effect of the liquid was to load the cylinder with a mass, c, depending on the nature of the liquid and on the shape and dimensions of the cylinder, he deduced the equation, $\frac{T'}{T} = \sqrt{\frac{m + c}{m}}$, for the lowering of pitch, where m is the mass of the cylinder. He obtained the following results: (1) the interval was

[1] Wied. Ann., 7, 23, 1879.
[2] Sitzb. math.-naturw. Cl. Wien, 87, Abth. 2, 1147, 1883.
[3] Wied. Ann., 17, 964, 1882.

smaller the higher the pitch ; (2) the interval was independent of the
height of the cylinder ; (3) the interval was greater in proportion to
the narrowness of the cylinder ; (4) using different liquids whose
densities varied from 0.729 to 1.364, and in addition mercury, he
found that as a first approximation the specific lowering of pitch
(the ratio of the lowering for any liquid to that for water) depended
on the density only and increased with it, but more slowly, and as
a second approximation that the specific lowering increased as the
compressibility of the liquid decreased.   He determined the pitch
by Kolácek's method.

Miss L. R. Laird,[1] working in this laboratory, made some ex-
periments on the change in the period of a pianoforte wire 0.446
mm. in diameter and 107 cm. long, vibrating in the following liquids :
water, solutions of potassium carbonate, and mercury.   The wire
was made to vibrate by means of an electro-magnet and its record
was taken on a revolving drum along with that of a standard tun-
ing-fork.   The results for the lowering of pitch agreed very well
with the values obtained analytically by Stokes in his 1848 paper
for the case of an infinite cylinder.   The change of pitch gave the
following intervals : 1.06 for water ; 1.13 for potassium carbonate of
density 1.47 ; 1.095 for potassium carbonate of density 1.22 ; 1.72 for
mercury.   In the case of water the observed values agreed still more
closely with numbers calculated on the simple supposition that the
wire was loaded with the mass of the displaced liquid, which was
the conclusion arrived at by Stokes in his 1843 paper.   Some ex-
periments on a similar wire 0.933 mm. in diameter and 37 cm. long
gave the interval 1.065 for water, 1.18 for oil of density about 0.9,
and 1.12 for a sodium nitrate solution of density 1.38.   It will be
noticed that the lowering for water was the same for both wires.

The following experiments were made in continuation of those
of Miss Laird, and were intended to separate, if possible, the effects
of the density and the viscosity of the liquid.   The change in pitch
of a tuning fork is of most direct interest, and, as the nearest simple
approach to this, a rod clamped at one end was studied.   As has
been stated, this problem is a very rough approximation to the case
of the infinite cylinder discussed by Stokes.   The present problem

[1] PHYS. REV., 7, 102, 1898.

differs from his in that the rod is of a different cross section and has edges, which greatly changes the stream lines ; that it has not the same amplitude at all points, and is of finite length. Nevertheless, a consideration of Stokes' work will be of great service, and will throw at least some light upon the way in which the constants of the liquid must enter.

If we suppose that a mass m has a simple oscillatory motion as a whole in the x direction, and is in a vacuum, its equation of motion is

(1)
$$m\frac{d^2x}{dt^2} + px = 0 ;$$

and its period

(2)
$$T = 2\pi\sqrt{\frac{m}{p}}.$$

Stokes found that the effect of a liquid medium is to introduce two terms into the above equation of motion. The first of these is of the nature of an acceleration, the second of a velocity. The former is equivalent to an increase of the mass of the vibrating body by a certain fraction of the mass of the displaced fluid ; m becomes $m + km'$ where m' is the mass of the displaced fluid. This is the same as saying with Buat that the body drags along with it a definite quantity of the liquid. The quantity k will evidently be determined by the form of the stream lines and must accordingly be a function of the shape and size of the cross-section of the vibrating body, of the density of the liquid, and of the coherence of its separate layers, or of its viscosity; and this Stokes found to be the case. The second of the terms which he found to enter into the equation of motion is a retarding force proportional to the velocity. The form of this term can be assigned from general considerations ; for the energy absorbed by the friction will be determined by the amount of the fluid displaced, and will depend not only on the velocity but also on the number of times its direction is changed. Stokes found it to be of the form $k'm'n$, where n is the pitch of the body in vacuum, and k' a quantity depending on the dimensions of the body and on the density and viscosity of the liquid. For the infinitely long cylinder he found, when $\dfrac{2}{a}\sqrt{\dfrac{1}{\pi n}\cdot\dfrac{\mu}{\rho}}$ was small,

(3) $\qquad k = 1 + \dfrac{2}{a}\sqrt{\dfrac{1}{\pi n}\cdot\dfrac{\mu}{\rho}},$ and $\quad k' = \dfrac{2}{a}\sqrt{\dfrac{1}{\pi n}\cdot\dfrac{\mu}{\rho}},$

and hence $k = 1 + k'$ where a is the radius of the cylinder.

The quantity $\sqrt{\dfrac{1}{\pi n}\cdot\dfrac{\mu}{\rho}}$ enters also into the values of k and k' found for the disk and the sphere.

Our equation of motion (1) can now be written

(4) $\qquad (m + km')\dfrac{d^2x}{dt^2} + k'm'n\dfrac{dx}{dt} + px = 0.$

The well-known solution of this equation gives for the period T' the value

$$T' = 2\pi\left[\dfrac{p}{m + km'} - \tfrac{1}{4}\left(\dfrac{k'm'n}{m + km'}\right)^2\right]^{-\frac{1}{2}}$$

$$= 2\pi\left(\dfrac{m + km'}{p}\right)^{\frac{1}{2}}\left[1 + \tfrac{1}{8}\dfrac{k'^2m'^2n^2}{p(m + km')}\right]$$

approximately, assuming k'^2 to be small. This can be put in the form

$$T' = 2\pi\left(\dfrac{m}{p}\right)^{\frac{1}{2}}\left[1 + k\dfrac{\rho}{\Delta}\right]^{\frac{1}{2}}\left[1 + \tfrac{1}{8}k'^2\left(\dfrac{\rho}{\Delta}\right)^2\dfrac{1}{1 + k\dfrac{\rho}{\Delta}}\right]$$

where Δ is the density of the body. Using equation (2) this becomes

$$\dfrac{T'}{T} = \left(1 + k\dfrac{\rho}{\Delta}\right)^{\frac{1}{2}}\left[1 + \tfrac{1}{8}k'^2\left(\dfrac{\rho}{\Delta}\right)^2\dfrac{1}{1 + k\dfrac{\rho}{\Delta}}\right]$$

If k' is at all small we may neglect the second factor, especially for liquids of small density; in this case the interval of lowering of the pitch becomes

(5) $\qquad \dfrac{T'}{T} = \sqrt{1 + k\dfrac{\rho}{\Delta}}.$

This is equivalent to neglecting in our equation of motion the term which depends on the velocity. The following experiments were

made to test this equation and to find the form and value of the quantity *k*. One could approximate more closely to Stokes' theoretical problem than was done in the present investigation by using rods of circular cross-section; but the mechanical difficulties of maintaining such rods in vibration in a plane are too great, and those used had a rectangular cross-section, and so corresponded to one-half of a tuning-fork. Although in the theoretical discussion *T* is the period in a vacuum, it is a sufficiently good approximation for this investigation to take it as the period in air.

The rod to be experimented on was held in a heavy clamp screwed to the inside of a wooden box, which held, when required, the liquid under consideration. The rod was maintained in vibration by an electro-magnet which could be suspended at any height from a movable bar across the top of the box. The circuit was completed through the rod and a piece of platinum soldered to its free end dipping into a mercury cup fitted with a plunger and projecting up from the bottom of the box. When the rod was of brass a thin piece of soft iron 9 mm. wide and 32 mm. long was soldered to it just beneath the electro-magnet. The vibrations were recorded on a revolving drum, upon which a marker also recorded seconds as given by a standard clock. The pitch of a given length of the rod used was found first with the vibrations taking place in air; then without disturbing anything the box was filled with the liquid under consideration, and the lowering of pitch calculated. The box was large in order that the nearness of its walls might not influence appreciably the pitch of the rod. Three boxes were used, as follows :

	With Water.	With Oil.	With Sodium Nitrate Solution.
Internal length.	57 cm.	55 cm.	55 cm.
" breadth	26 "	21 "	25 "
" depth	30 "	22 "	22 "

To show that the size of the box is of great importance it may be stated that the pitch of the rod *A* (described below) in water was increased nearly five per cent. when it was changed from the large box to one whose dimensions were 52 x 4.5 x 4.5 cm. In order to

prevent as far as possible the absorption of energy by the box, etc., the clamp was made of a mass of metal weighing fifteen pounds; the whole apparatus was placed on a stone pier; and the box, the cross-bar supporting the electro-magnet, and the projecting end of the rod, if any, were heavily weighted down with masses of iron. Unless this were done the pitch was sensibly affected.

The duration of contact of the platinum wire with the mercury, the distance of the electro-magnet and piece of soft iron from the clamp, the shape of the electro-magnet and its height above the rod, and the strength of the driving current were all found to affect the pitch of the rod. The time of contact used was always the shortest that would maintain the rod in vibration. The distance of the electro-magnet from the clamp affected the pitch in the following way : the shorter the distance the greater was the interval of lowering, no matter what was the vibrating length of the rod. Table I. gives a specimen of the readings taken for the purpose of studying this question; they were made on the rod *A* with half of its length (21.1 cm.) vibrating.

TABLE I.

Distance of Electro-magnet from Clamp.	Frequency in Air.	Frequency in Water.	Height of Electro-magnet above Soft Iron.	Interval of Lowering.	Average Interval.
2.1 cm.	44.67	38.45	3 mm.	1.162	
" "	44.63	38.40	3 "	1.162	1.162
" "	44.62	38.37	3 "	1.163	
11.0 "	38.89	33.63	8 "	1.153	
" "	39.09	33.72	8 "	1.156	1.154
" "	39.09	33.91	8 "	1.153	
17.5 "	36.92	32.21	8 "	1.146	
" "	36.95	32.22	8 "	1.147	1.147
19.5 "	36.64	32.09	8 "	1.142	
" "	36.63	32.08	8 "	1.142	1.142

The final average intervals, found as above, for various lengths of the rod were :

| Average distance of electro-magnet, | 6 cm. | 11 cm. | 19 cm. |
| Average interval, | 1.160 | 1.153 | 1.148 |

The interval, therefore, diminishes as the driving force is placed nearer to the free end of the rod. In order to get results undisturbed by this possible source of variation, the electro-magnet was

placed at a constant distance from the clamp throughout the whole series of experiments, the piece of soft iron being taken off the rod and soldered on in a new place whenever the vibrating length of the rod was changed. The distance from the clamp chosen was such that with the shortest length of rod used the soft iron was at the tip end. For the rod *A* this length was 10.55 cm. and for the other rods it was 21.1 cm. A set of observations was taken to find also the effect of the height of the electro-magnet above the rod. The electro-magnet first used was of the *U* shape. Its cores did not project beyond the bobbins, whose ends were each about 2 sq. in. in area. The effect of the height was not very noticeable for air, but for water a change of height from 7 to 23 mm. made the rod vibrate more slowly by more than 1 per cent. There was no doubt that with this electro-magnet near the rod, the coils acted partly as a wall, and they were replaced by a single coil whose core projected 15 mm. and was 6 mm. in diameter. With this electro-magnet it was found that a diminution of height increased the pitch, no matter how far the electro-magnet was from the clamp, but in no case by so much as 1 per cent. and usually by much less. It was evident, therefore, that the height must be kept constant throughout the experiments; but as the height of the electro-magnet was increased, the current also had to be increased, and it seemed desirable to keep the current also constant. The current strength chosen was the smallest that would drive the shortest length of rod used, and the height chosen was the greatest possible for that current and length.

Five rods were used during the investigation. Four of them, called *A, B, C* and *D*, were of brass, and one, *E*, of steel. *B* had approximately twice the thickness of *A* and the same width; *C* twice the width and the same thickness; and *D* twice the width and twice the thickness. *E* had approximately the same dimensions as *A*. The dimensions and densities of the rods were:

	A	*B*	*C*	*D*	*E*
Width. . . .	0.94 cm.	0.97 cm.	1.91 cm.	1.95 cm.	0.97 cm.
Thickness. .	0.32 "	0.65 "	0.34 "	0.67 "	0.33 "
Density. . .	8.56	8.56	8.70	8.56	7.79

The liquids used were water, cotton-seed oil, and a solution of sodium nitrate saturated at 15° C. The first two had approximately the same density but very different viscosities, while the first and last had approximately the same viscosity but quite different densities. The viscosities of the oil and the solution of sodium nitrate were found by the method used by Gartenmeister.[1] The temperature throughout was kept as nearly as possible at 17°.6 C. At this temperature the liquids have the following constants :

	Water.	Cotton-Seed Oil.	Sodium Nitrate Solution.
Density, ρ	1.000	0.921	1.355
Viscosity, μ	0.011	0.781	0.030

The tables which follow give the results obtained. All of the separate readings are not given, but the average frequency in air and in the liquid for each length, and the corresponding interval of lowering. In no case were less than two distinct sets of observations made on each length of the rod used. Ordinarily 1,000 vibrations for each observation were counted on the drum and the corresponding number of clock seconds, and the pitch calculated therefrom.

TABLE II.

Rod A. Dimensions, 42.2 × 0.94 × 0.32 *cm. In water.*

Vibrating Length.	Frequency in Air.	Number of Readings.	Frequency in Water.	Number of Readings.	Interval of Lowering.
Full length.	10.33	4	9.04	3	1.148
¾ "	18.16	2	15.78	2	1.151
⅝ "	25.58	2	22.27	2	1.149
½ "	39.10	2	33.95	2	1.152
⅜ "	66.73	2	58.09	3	1.149
¼ "	136.44	4	119.45	6	1.143
Average Lowering.					1.149

From this table it is seen that for a range of about 3 ½ octaves the lowering is independent of the pitch within the limits of accuracy

[1] Zeitschr. f. physikal. Chem., **6**, 524, 1890.

attained.   The same deduction can be made from the numbers in each of the other tables.   The lowering here is a little less than a minor whole tone plus a small semitone $\left(\dfrac{10}{9} \times \dfrac{25}{24} = 1.157\right)$.

TABLE III.

Rod E.   Dimensions, 42.2 × 0.97 × 0.33 *cm.   In water.*

Vibrating Length.	Frequency in Air.	Number of Readings.	Frequency in Water.	Number of Readings.	Interval of Lowering.
¾ length.	24.86	2	21.48	2	1.157
⅝ "	35.62	2	30.71	2	1.159
½ "	53.54	2	46.20	2	1.158
Average Interval.					1.158

The lowering here again is independent of the pitch.   The interval is exactly a minor whole tone plus a small semitone, and differs by less than 1 per cent. from the interval for the brass rod of approximately the same dimensions.   It seems safe to conclude that no great importance is to be attached to the materials used for the rods, at least for such materials as are ordinarily used for tuning forks, bells and kindred musical instruments.

TABLE IV.

Rod B.   Dimensions, 42.0 × 0.97 × 0.65 *cm.   In water.*

Vibrating Length.	Frequency in Air.	Number of Readings.	Frequency in Water.	Number of Readings.	Interval of Lowering.
Full length.	19.80	2	18.29	2	1.083
¾ "	34.56	2	31.65	2	1.094
⅝ "	47.50	2	43.60	2	1.089
½ "	72.14	3	66.51	3	1.085
Average Interval.					1.088

The lowering is nearly a semitone plus a comma, or a great limma ($\frac{16}{15} \times \frac{81}{80} = 1.080$), and is not far from being one-half of that for rod *A* of one-half the thickness.

TABLE V.

Rod C.   Dimensions, 42.2 × 1.91 × 0.34 *cm.   In water.*

Vibrating Length.	Frequency in air.	Number of Readings.	Frequency in Water.	Number of Readings.	Interval of Lowering.
Full length.	10.12	2	7.95	2	1.273
¾ "	17.81	2	14.01	2	1.274
⅝ "	25.05	2	19.78	2	1.267
½ "	37.86	5	30.32	5	1.244
Average Interval.					1.265

The lowering is exactly a major third plus a comma ($\frac{5}{4} \times \frac{81}{80}$ = 1.265) and is nearly twice as much as that for rod A of one-half the width.

TABLE VI.

Rod D.   Dimensions, 42.0 × 1.95 × 0.67 *cm.   In water.*

Vibrating Length.	Frequency in Air.	Number of Readings.	Frequency in Water.	Number of Readings.	Interval of Lowering.
Full length.	19.89	3	17.19	3	1.157
¾ "	35.13	8	30.29	7	1.160
⅝ "	48.23	2	42.34	2	1.140
Average Interval.					1.152

The lowering here is a little less than a minor whole tone plus a small semitone, and is almost exactly the same as that for rod A of one-half the width and one-half the thickness, as we should expect from the results shown in Tables IV. and V.

GENERAL APPROXIMATE RESULTS.

1. The interval of lowering for a rod of given cross section is independent of the length.

2. The interval of lowering for a rod of given cross section is approximately the same for brass and steel and is probably independent of the material within the range of substances ordinarily used.

3. The interval of lowering for a rod of given width is approximately inversely proportional to the thickness.

4. The interval of lowering for a rod of given thickness is approximately directly proportional to the width. More definite relations will be given later.

The above experiments were then repeated with cotton-seed oil instead of water. This oil was used because it is inexpensive, and consequently can be used in large quantities, and has its coefficient of viscosity about seventy times that of water. With this liquid it was hoped that the importance of the viscosity in the lowering of pitch could be determined. Whereas in the case of water the contact breaker was under the liquid, it had to be outside for the oil and the nitrate solution. The platinum wire was replaced by a platinum-tipped thin steel wire which reached above the surface of the liquid where it was bent over and dipped into the mercury cup. No special difficulty was experienced in maintaining the rod in vibration in the viscous liquid. The results of these experiments were as follows :

TABLE VII.

In Cotton-seed Oil.

Rod.	Vibrating Length.	Frequency in Air.	Number of Readings.	Frequency in Oil.	Number of Readings.	Interval of Lowering.	Av. Interval of Lowering.
A	Full length.	10.00	10	8.67	10	1.140	
"	¾ "	17.32	2	14.81	2	1.169	
"	⅝ "	24.57	2	20.91	2	1.175	
"	½ "	37.15	4	31.90	4	1.164	1.162
B	Full length.	20.26	3	18.42	4	1.105	
"	¾ "	36.11	3	32.22	3	1.121	
"	⅝ "	49.67	4	44.72	3	1.111	
"	½ "	65.18	3	59.65	3	1.093	1.108
C	Full length.	10.21	4	8.13	4	1.256	
"	¾ "	18.38	3	14.33	3	1.283	
"	⅝ "	25.76	3	19.90	3	1.294	
"	½ "	38.60	2	30.58	2	1.262	1.272
D	Full length.	19.67	3	16.89	3	1.166	
"	¾ "	35.30	2	29.54	5	1.195	
"	⅝ "	48.43	3	41.44	3	1.169	1.176

We shall discuss these numbers more carefully later, but we can say at once in a general way that the effect of viscosity is not very marked. In order to throw still more light on the effect of viscosity as compared with that of density, the four brass rods were made to vibrate in a solution of sodium nitrate, which had a viscosity of only 0.0299, and so not very much greater than that of

water, and whose density was 1.355, about a third as much again as that of water. These determinations resulted as follows :

TABLE VIII.

In Sodium Nitrate Solution.

Rod.	Vibrating Length.	Frequency in Air.	Number of Readings.	Frequency in Nitrate Solution.	Number of Readings.	Interval of Lowering.	Av. Interval of Lowering.
A	Full length.	10.00	2	8.47	3	1.181	
"	¾ "	17.25	2	14.48	3	1.191	
"	⅝ "	24.70	2	20.46	3	1.207	
"	½ "	37.58	2	31.51	2	1.193	1.198
B	Full length.	20.35	2	18.10	2	1.123	
"	¾ "	35.75	2	31.84	2	1.123	
"	⅝ "	50.99	2	44.96	2	1.134	1.127
C	Full length.	10.21	2	7.61	2	1.343	
"	¾ "	18.51	2	13.76	2	1.347	
"	⅝ "	25.98	2	19.26	2	1.349	1.346
D	Full length.	19.97	2	16.57	2	1.206	
"	¾ "	35.30	3	29.51	3	1.196	
"	⅝ "	49.15	3	40.88	3	1.202	1.201

Comparing these numbers with those obtained for water it is seen that the lowering is increased very decidedly, and that the increase is not far from being directly proportional to the added density. It is evident therefore that the main factor in the lowering of pitch is the density of the medium, and that the effect of viscosity is relatively small.

Thus far the conclusions drawn from the investigation have been of an approximate and general nature; it remains to determine whether an expression for $\frac{T'}{T}$ can be found which will contain the results of every case investigated. Stokes' work can afford little direct help since he found the value of $\frac{T'}{T}$ to depend on n, as will be seen by reference to equation (3); whereas in the present investigation the interval of lowering is independent of the pitch within the limits of accuracy attained. Indirectly his work is of great service, and his results have been used already in the formation of the equation of motion (4). Since for the case he discussed, the value of k'

is considerably smaller than that of k, and since the results here ob-
tained for oil show that the change of pitch due to friction is rela-
tively small, it is allowable as a first approximation to neglect k',
and consider the lowering of pitch to be expressed by equation (5),

$$\frac{T'}{T} = \sqrt{1 + k\frac{\rho}{\varDelta}}.$$

The following table contains the values of k found from this equation.

TABLE IX.

Values for k.

Medium.	Vibrating Length.	Rod A.	Rod B.	Rod C.	Rod D.	Rod E.
Water.	Full length.	2.697	1.478	5.396	2.895	
	¾ "	2.776	1.682	5.418	2.954	2.641
	⅝ "	2.737	1.589	5.263	2.561	2.678
	½ "	2.796	1.515	4.761·		2.660
Average.		2.752	1.566	5.210	2.803	2.660
Oil.	Full length.	2.785	2.054	5.454	3.342	
	¾ "	3.407	2.385	6.101	3.978	
	⅝ "	3.537	2.178	6.369	3.407	
	½ "	3.298	1.809	5.596		
Average.		3.257	2.107	5.880	3.576	
$NaNO_3$	Full length.	2.483	1.642	5.152	2.858	
	¾ "	2.632	1.643	5.220	2.707	
	⅝ "	2.873	1.799	5.255	2.797	
	½ "	2.662				
Average.		2.663	1.698	5.209	2.787	

The quantity k can be a function of the constants of the liquid
and the dimensions of the rod. Stokes has shown that μ and ρ
enter always as a ratio, and further that in the equation for the
lowering of pitch they enter as $\sqrt{\frac{\mu}{\rho}}$; an attempt was accordingly
made to express k as a function of $\sqrt{\frac{\mu}{\rho}}$ and the cross section of the
rod (the length has been shown not to enter). This it has been
found can be done in several ways of about the same degree of
satisfactoriness. The following four equations give values for $\frac{T'}{T}$

with various forms of the quantity k; w denotes the width of the rod, and t the thickness.

$$(a) \qquad \frac{T'}{T} = \sqrt{1 + 2.63 \frac{w^{\frac{7}{8}}}{t^{\frac{3}{4}}} \left(1 + \frac{1}{4}\sqrt{\frac{\mu}{\rho}}\right)\frac{\rho}{\varDelta}}.$$

$$(b) \qquad \frac{T'}{T} = \sqrt{1 + 1.12 \frac{w}{t^{\frac{3}{4}}} \left(1 + \frac{1}{4}\sqrt{\frac{\mu}{\rho}}\right)\frac{\rho}{\varDelta}}.$$

$$(c) \qquad \frac{T'}{T} = \sqrt{1 + \frac{w}{t^{\frac{7}{8}}} \left(1 + \frac{1}{4}\sqrt{\frac{\mu}{\rho}}\right)\frac{\rho}{\varDelta}}.$$

$$(d) \qquad \frac{T'}{T} = \sqrt{1 + 0.91 \frac{w}{t} \left(1 + \frac{1}{4}\sqrt{\frac{\mu}{\rho}}\right)\frac{\rho}{\varDelta}}.$$

The last of these equations is preferred on account of its simplicity, but its concordance with the observed values is not so good as is the case with the first, where the greatest deviation is 1 part in 110. The observed average values for $\frac{T'}{T}$ and those calculated from equation (d) are given in Table X., along with their differences.

TABLE X.

Rod.	Medium.	Water.	Oil.	NaNO$_3$.
	Obs.	1.149	1.162	1.198
A	Cal.	1.150	1.164	1.200
	Diff	+ .001	+ .002	+ .002
	Obs.	1.088	1.108	1.127
B	Cal.	1.078	1.086	1.106
	Diff.	− .010	− .022	− .021
	Obs.	1.265	1.272	1.346
C	Cal.	1.271	1.295	1.358
	Diff.	+ .006	+ .023	+ .012
	Obs.	1.152	1.176	1.201
D	Cal.	1.148	1.163	1.198
	Diff.	− .004	− .013	− .003
	Obs.	1.158		
E	Cal.	1.162		
	Diff.	+ .004		

It will be seen that the agreement is as close as that of the separate values of $\frac{T'}{T}$ in any of the tables II. to VIII.

The above equations (a), (b), (c) and (d) were found from a close consideration of Stokes' work. It is worthy of remark, however, that an entirely empirical formula can be found which will give an equally good agreement with the results. The following table contains the observed values of $\frac{T'}{T}$, and those calculated from the equation

$$(e) \qquad \frac{T'}{T} = 1 + 0.45 \frac{w^{\frac{1}{8}} \rho}{t} \frac{\rho}{\Delta} + 0.03 \, \mu.$$

TABLE XI.

Rod.	Medium.	Water.	Oil.	NaNO$_3$.
A	Obs.	1.149	1.162	1.198
	Cal.	1.157	1.168	1.212
	Diff.	+ .008	+ .006	+ .014
B	Obs.	1.088	1.108	1.127
	Cal.	1.079	1.096	1.107
	Diff.	− .009	− .012	− .020
C	Obs.	1.265	1.272	1.346
	Cal.	1.273	1.275	1.368
	Diff.	+ .008	+ .003	+ .022
D	Obs.	1.152	1.176	1.201
	Cal.	1.180	1.196	1.254
	Diff.	+ .028	+ .020	+ .053
E	Obs.	1.158		
	Cal.	1.155		
	Diff.	− .003		

In the work which has been done on this subject there is only one case where sufficient details are given to enable us to apply our equations to the values found for the lowering of pitch. Koláček[1] states that for a horseshoe magnet 340 mm. long, 12.5 mm. wide and 46 mm. thick vibrating in water the value of $\frac{T'}{T}$ was 1.03. Calculated from equations (d) and (e) the interval is 1.02.

[1] Loc. cit.

It was thought worth while to test equation (d) by applying it to the numbers found by Miss Laird, taking $\dfrac{w}{t}$ as equal to 1. It was found that the values thus determined for the interval of lowering in water agree with the observed values even better than do the values calculated on the simple supposition that the wire is loaded with the mass of the displaced fluid. For denser liquids, however, equation (d) does not agree so well as the equation of Stokes, since the lowering was found to depend on the pitch. The following table contains the frequencies in air and in liquid observed by Miss Laird, and the differences between the observed frequency in liquid and numbers calculated (1) from Stokes' equation, (2) from equation (d), and (3) on the assumption that the wire is loaded with the mass of the displaced liquid. These differences are called Δ_1, Δ_2 and Δ_3 respectively.

TABLE XII.

	Pitch in Air.	Pitch in Liquid.	Δ_1	Δ_2	Δ_3
Water.	73.8	70.1	− 2.9	− .4	− .6
	92.3	87.9	− 3.6	− .7	− 1.0
	105.5	99.4	− 2.8	+ .3	− .1
	113.4	107.7	− 3.8	− .5	− .9
	126.5	117.8	− 1.8	+ 1.7	+ 1.3
Solution of Pot. Carbonate.	73.8	65.1	− .8	+ 2.9	
	92.3	82.6	− 1.7	+ 2.4	
	105.5	93.9	− 1.2	+ 3.3	
	113.4	99.7	+ .1	+ 4.8	
	126.5	111.6	− .2	+ 5.2	
Mercury.	73.8	43.0	− 1.8	+ 2.8	
	92.3	53.3	− 1.3	+ 4.0	
	105.5	61.9	− 2.2	+ 3.5	
	113.4	65.8	− 1.5	+ 4.7	
	126.5	73.4	− 1.5	+ 5.2	

An attempt was made to see what would come from assuming that there was always attached symmetrically to the rod when in motion a constant mass of liquid of elliptical cross section. It will be seen from the way in which the quantity R was introduced that it is the ratio of the volume of this attached liquid to the volume of

the rod.   Using Table IX. it was found that if the major axis of the ellipse were 2 w and the minor axis $t + \frac{1}{2} w$, the periods calculated, on the assumption (used in this paper) that the lowering of pitch is due only to the added mass of liquid, agreed very well with the observed values.

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