Student understanding of time dependence in quantum mechanics

Paul J. Emigh, Gina Passante, and Peter S. Shaffer[*]

Department of Physics, University of Washington, Seattle, Washington 98195-1560, USA
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[This paper is part of the Focused Collection on Upper Division Physics Courses.] The time evolution of quantum states is arguably one of the more difficult ideas in quantum mechanics. In this article, we report on results from an investigation of student understanding of this topic after lecture instruction. We demonstrate specific problems that students have in applying time dependence to quantum systems and in recognizing the key role of the energy eigenbasis in determining the time dependence of wave functions. Through analysis of student responses to a set of four interrelated tasks, we categorize some of the difficulties that underlie common errors. The conceptual and reasoning difficulties that have been identified are illustrated through student responses to four sets of questions administered at different points in a junior-level course on quantum mechanics. Evidence is also given that the problems persist throughout undergraduate instruction and into the graduate level.

I. INTRODUCTION

There is an increasing body of research on the teaching and learning of quantum mechanics, a topic that is abstract, mathematical, and often counterintuitive [1–4]. The findings indicate that student conceptual and reasoning difficulties are widespread and independent of textbook, population, and instructor, a situation similar to that at the introductory level [5]. Efforts have been made to design instructional strategies to address some of the issues that have been identified [6–12], but much more needs to be done.

A fundamental concept in quantum mechanics is the time evolution of quantum states. It is necessary for interpreting results from laboratory techniques such as NMR, for explaining new physics like neutrino oscillations, and for understanding the relationship between classical and quantum mechanics. Yet, there is evidence that many students fail to recognize the importance of time dependence in quantum mechanics [13]. Even for simple systems, students often have difficulty in identifying the correct time dependence not only for wave functions but also for probabilities and probability densities [14]. Many are unable to determine the source of the time dependence, and some believe that probabilities or expectation values never depend on time.

Although some student difficulties related to time dependence have been previously identified [2,15,16], this paper details a systematic and in-depth investigation into student ideas about time evolution in quantum mechanics. We describe some of the overall problems that students have in determining and generalizing the time dependence of quantum states, together with the prevalence in various contexts at different points during instruction [17]. In particular, we focus on patterns in student reasoning after lecture instruction on time dependence and related topics.

In this investigation, we use the framework for research and curriculum development that characterizes the work of the Physics Education Group at the University of Washington (UW) [18]. Student responses to carefully designed questions (both in written form and administered during interviews) are examined to identify common incorrect answers or explanations that might suggest incorrect beliefs or lines of reasoning. Multiple versions of each question are used to assess the extent to which the ideas are strongly held or persist throughout instruction. Strategies for improving instruction are then developed and tested. They are deemed effective if, afterward, students can apply the appropriate concepts properly in contexts that differ from those they have previously seen. The investigation into student understanding, the development of curriculum, and the testing of instructional materials are done together in an iterative process with each component informing the other. This article focuses on the first part of this process—identification of common difficulties and of ways to elicit them.

II. CONTEXT FOR RESEARCH

This investigation has been carried out over a period of many years (2003–2015). The data come primarily from a two-quarter, junior-level quantum mechanics sequence at UW. Supporting results are also reported from sophomore- and graduate-level courses. In the junior-level sequence, between 50 and 90 students are typically enrolled in the first-quarter course each year. About half of the students continue to the second quarter. The textbook used is Griffiths' *Introduction to Quantum Mechanics* [19].

[*]Corresponding author.
shafferp@uw.edu

In addition to data from UW, some of our results are drawn from junior-level courses at other universities [20]. The textbook and content have been essentially the same across all the courses in this study, but the courses have been taught by different instructors and using different lecturing styles. Some have included tools for interactive engagement, such as clickers. The results have been similar for all of the courses included in this study [21].

Each course also utilized *Tutorials in Physics: Quantum Mechanics* [22]. These are modeled after the tutorials that our group has developed to supplement instruction in introductory physics [23]. Each tutorial is preceded by a tutorial "pretest." These pretests are given before tutorial instruction but after lecture instruction on the relevant material. Each pretest consists of one or more tasks involving a series of questions. Students are asked to explain their answers to each question. The pretests are timed and have been given both on paper during class time and online outside of class [24]. The concepts and reasoning addressed in the tutorials are assessed through tutorial post-test questions on course examinations.

Since the focus of this paper is on student understanding after lecture instruction, most of the findings discussed in this paper are based on student responses to tasks given as tutorial pretests. In those places where data are drawn from exams administered after relevant tutorial instruction, it is explicitly noted. We also report results from individual student interviews conducted to probe various aspects of student reasoning in greater detail.

III. TASKS USED TO PROBE STUDENT REASONING

A variety of questions has been used in this investigation. Four representative tasks are discussed in this paper. The first two are based on physical contexts normally introduced near the beginning of instruction on quantum mechanics. They require students to reason about time dependence, in particular for spatial probability densities and probabilities of energy measurements for a variety of initial states. The other two tasks probe similar ideas for more complicated systems. One of these involves degenerate energy eigenvalues, the other, a time-dependent Hamiltonian. The four tasks are shown in Figs. 1(a)–1(d).
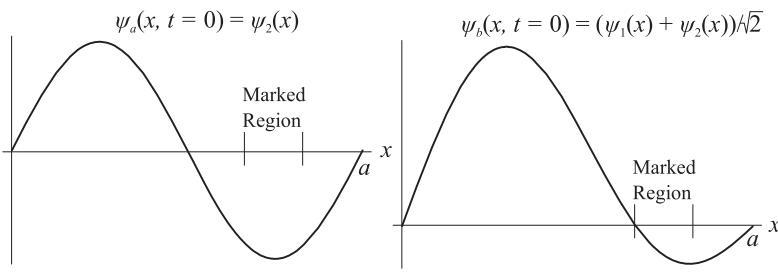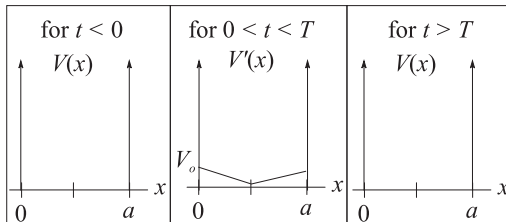

(a) Task 1

A system consists of a particle in an infinite square well of width a. Two possible wave functions for the system at time $t = 0$ s are shown at right. Both wave functions are entirely real at the instant shown.

Does the probability of finding the particle within the marked region depend on time? Explain your reasoning.

$\psi_a(x, t = 0) = \psi_2(x)$

$\psi_b(x, t = 0) = (\psi_1(x) + \psi_2(x))/\sqrt{2}$

Marked Region

Marked Region

(b) Task 2

Assume a particle in a QM harmonic oscillator is prepared in the initial state:

$$\psi = \frac{i}{\sqrt{3}}\psi_0 - \frac{\sqrt{2}}{\sqrt{3}}\psi_1$$

Consider an energy measurement made on this system. Are there times when the probability of measuring E_1 is zero and the probability of measuring E_0 is one?

(c) Task 3

Consider the following wave functions for a particle in the three-dimensional isotropic harmonic oscillator potential:

$$\Psi_A(x, y, z, t = 0) = N(\psi_1(x)\psi_0(y)\psi_0(z) + \psi_0(x)\psi_0(y)\psi_1(z))$$

$$\Psi_B(x, y, z, t = 0) = N(\psi_1(x)\psi_0(y)\psi_0(z) + \psi_1(x)\psi_0(y)\psi_1(z))$$

For both states, answer the following two questions:
1. Does the state change as time evolves? Explain.
2. Does the probability density change as time evolves? Explain.

(d) Task 4

A time-dependent perturbation to the infinite square well potential is illustrated below. At a time $t_- \ll 0$ the total energy of the system is measured and found to be the ground state energy of the unperturbed well.

for $t < 0$	for $0 < t < T$	for $t > T$
$V(x)$	$V'(x)$	$V(x)$

1. Describe the possible values that could result from an energy measurement during each interval.
2. Does the spatial probability distribution change with time during each interval?

FIG. 1. Example versions of four tasks that were administered over the course of many years in a junior-level quantum mechanics course. (a) Task 1 and (b) task 2 were each administered to more than 400 students. (c) Task 3 was given to more than 250 students, and (d) task 4 was given to more than 200 students.

TABLE I.   Percentage of junior-level students who gave correct answers (with correct reasoning) for each of the four research tasks (see Fig. 1). Results are shown for the individual parts of each task and for each task as a whole. Data are aggregated across all courses in which each task was administered after relevant lecture instruction but before tutorial instruction.

Question	Task 1 ($N = 416$)			Task 2 ($N = 439$)	Task 3 ($N = 285$)			Task 4 ($N = 215$)			
	ψ_a	ψ_b	Both		Ψ_A	Ψ_B	Both	Before	During	After	All
Correct with reasoning	35%	20%	20%	5%	20%	50%	20%	50%	20%	20%	10%

All of the tasks are intended to probe the specific approaches that students use when answering questions about time dependence. In each task, students are given one or more quantum states for a single instant in time (e.g., for $t = 0$). The form of the wave function for other times is not given explicitly. Multiple versions of each task have been used. They have been continually modified to probe different aspects of student understanding and to determine whether or not the results are due to a particular phrasing, representation, or physical context. The versions discussed below are representative, as are the student responses used to illustrate common lines of reasoning. Some of the tasks have been given to more than 400 students in junior-level courses at multiple universities. In addition, we discuss results from a version of task 1 given in a sophomore-level course and from versions of tasks 1–3 given in a graduate-level course.

Table I shows the percentage of junior-level students who answered correctly and provided correct explanations for each task. The criteria used for an explanation to be considered correct are discussed below with the description of the corresponding task. The results are aggregated over several different years, instructors, and universities [25], as well as over different versions of the research tasks. During this study, the percentage of the students who have answered a given task correctly has typically been similar [26]. The table includes only results from courses in which the research tasks were administered after lecture instruction and before students had worked through any tutorials on the relevant material.

Task 1.—On task 1 [Fig. 1(a)], students consider two different initial states for a particle in the infinite square well potential: an energy eigenstate and a superposition of two energy eigenstates. In each case, they are asked whether or not the probability of finding the particle within a specified region of the well depends on time. In some versions, students are asked to rank the probabilities of finding the particle within this region at specific times.

One way to answer this question is to determine whether or not the probability density, which is given by the modulus (or complex) square of the wave function, depends on time. Since students are given the wave function at $t = 0$, one step in the reasoning is to recognize that each term in the wave function has a time-dependent phase that depends on the associated energy eigenvalue. This phase cancels for the particle in the energy eigenstate, resulting in a constant probability density, which is the hallmark of a *stationary*

state. The components of the wave function for the super-position state, however, have unequal phases. Thus, for this state, there are time-dependent cross terms in the square of the modulus. Students were considered to have correct reasoning for this task if they identified a connection between time dependence and energy eigenstates (e.g., if students simply stated that the first state is a stationary state and the second is not).

Task 1 has been given to 416 students at three institutions since 2007. The questions have proved challenging for most students. The percentage of correct answers (with correct reasoning) has been about 35% for ψ_a (the energy eigenstate) and about 20% for ψ_b (the superposition state), with about 20% correct for both.

A version of task 1 involving only the superposition state has also been asked in a graduate course on quantum mechanics at UW. It was given at the start of the course before any instruction. About two-thirds of the students ($N = 31$) answered correctly with correct reasoning.

Task 2.—On task 2 [Fig. 1(b)], students are presented with an initial state written as a superposition of two energy eigenstates in an harmonic oscillator potential. They are asked if there are times when the probability of measuring the energy associated with one of the eigenstates is zero. Some variants of this task directly asked whether or not the probabilities of the energy measurements depend on time.

Since the potential is constant in time and the state is expressed in the energy eigenbasis, the probability of measuring a given energy depends only on the square of the coefficient of the corresponding energy eigenstate. The time-dependent phase of each coefficient (not given to the students) squares to 1, so the probability does not depend on time. Answers were considered as demonstrating correct reasoning if students related the coefficients in the wave function to the probability for measuring the energy.

About half of the students answered correctly ($N = 439$). However, most of these students simply asserted that the result does not depend on time. Only about 5% of the students also gave correct explanations [27]. Incoming graduate students at UW performed better: about 70% answered correctly, although only 15% provided correct explanations ($N = 31$).

Task 3.—On task 3 [Fig. 1(c)], students are given two initial states for a three-dimensional harmonic oscillator, which has degenerate energy eigenstates. The two states (Ψ_A and Ψ_B) are each superpositions of two eigenstates that have the same or different energies, respectively. Students

are also given the functional form of the potential, the energy eigenvalues, and the energy eigenfunctions (not shown). Students are asked whether or not each state changes with time and whether or not each probability density changes with time.

The criteria we used to assess reasoning on this task are very similar to the criteria for task 1. Each term in each state has a time-dependent phase (not given to the students) that depends on the corresponding energy eigenvalue of that term; therefore, each state has time dependence. However, the degenerate energy eigenvalues for Ψ_A result in a single time-dependent phase for the entire wave function. Thus, the probability density has no time dependence. The degeneracy therefore results in a time-independent probability density for the first state, even though it is written as a superposition of energy eigenstates—in fact, this state is also an energy eigenstate (or stationary state) because of this. The same is not true for Ψ_B, which has distinct energy eigenvalues, so the phases do not cancel and the probability density depends on time.

In the junior-level course ($N = 285$), about 20% of the students answered the entire task correctly (with correct reasoning). Relatively few students gave correct answers without correct reasoning. About 20% answered correctly for Ψ_A alone and about 50% answered correctly for Ψ_B alone [28]. Students who did not answer correctly for either state typically gave a time evolution that was incorrect for both the state and its associated probability density. Overall, these results suggest that even near the end of the course, many students struggle with the underlying ideas of time dependence. Even graduate students have difficulty with this task. When a version of task 3 was given at the start of the third quarter of a graduate course on quantum mechanics, only about 60% of the students ($N = 19$) answered the entire task correctly with correct reasoning.

We have asked similar tasks involving other physical systems that have degenerate states (e.g., systems with angular momentum or identical particles). The results have been similar across the different contexts.

Task 4.—The context for task 4 [Fig. 1(d)] is time-dependent perturbation theory. Students consider a particle that is initially in the ground state of the infinite square well. A perturbation is applied to the potential and then removed. (Both changes are made instantaneously.) Students are asked whether or not the probability density depends on time for three separate time intervals: *before*, *during*, and *after* the applied perturbation. On some versions of this question, students are also asked whether or not the probabilities of energy measurements depend on time for the same three intervals.

The probability density for the first interval (before the perturbation) can be determined in exactly the same manner as task 1: the particle is in an energy eigenstate, so the probability density will not depend on time. The wave function itself is unchanged immediately after the potential

changes. Therefore, it does not correspond to an energy eigenfunction of the perturbed system. As a result, the probability density changes in time during the interval in which the potential is perturbed. Finally, the wave function is also continuous in time at the instant the perturbation is removed, but it has changed, so it no longer corresponds to an energy eigenstate of the unperturbed potential. Thus, the probability density also changes in time after the perturbation has been removed.

A total of 215 juniors near the end of their second quarter of introductory quantum mechanics at UW have responded to this task. About 50%, 20%, and 20% of the students correctly identified and explained whether or not the probability density would depend on time before, during, and after the perturbation, respectively. About 10% of the students answered correctly for all three intervals [29]. As with the other tasks discussed above, different variants of task 4 have been asked in different years [30].

IV. SPECIFIC DIFFICULTIES ELICITED BY RESEARCH TASKS

Student responses to each task were examined by several members of our group to attempt to identify each student's underlying reasoning. The questions on each task were analyzed both individually and together with all other questions on the same task. When many responses were compared, the patterns that emerged suggested some common approaches to the tasks as a whole. In many cases, interactions with students in class or in individual interviews provided supporting evidence for the interpretations. In what follows, incorrect lines of reasoning that were common among many students are termed *difficulties*. The most common and persistent of these are described below, together with representative student statements. Some of these difficulties have been documented elsewhere (see, for example, Refs. [6,14]). They are included below if we have identified them as arising in response to different kinds of questions than previously asked or in a broader variety of contexts than has been previously reported.

The specific difficulties have been grouped into four categories that can be considered to represent similar types of errors. Note that some responses do not fit into a single category. While other categorization schemes are possible, we have found these categories to be useful in communicating the specific difficulties to other instructors teaching quantum mechanics and for guiding the design of curriculum to address the issues [22].

We also report the percentage of student responses that are consistent with each difficulty (aggregated across all courses) and the tasks on which they have been identified. The data in this section are taken mostly from tasks administered after lecture instruction in junior-level courses (other sources of data are noted explicitly). In some cases, which are noted when relevant, we found that the percentages changed somewhat from year to year or from version

to version. These variations may be due to differences in the lectures or a result of small changes to the questions. We regard the exact percentages reported below as less important than the presence and persistence of the individual student difficulties.

Some of the difficulties we have identified occurred at only the 5% level on individual questions. They are included here if they occurred consistently across multiple contexts. Moreover, we have observed that in some cases students made multiple errors on a given question or task. Thus, some difficulties may be masked by others. In addition, some student answers (ranging between 5% and 20% on various questions) were too brief for us to be able to characterize their reasoning. As a result, the given percentages represent a lower bound on the prevalence of specific difficulties.

A. Tendency to confuse the time dependence of different quantum mechanical quantities

In many of their responses, students attributed the time dependence of one quantity to that of another. In some cases, they treated the presence (or absence) of time dependence for one quantity as implying the presence (or absence) of time dependence for another quantity. Often, they seemed to believe that the time evolution of both quantities must be exactly the same. This tendency was evident for quantities such as the wave function, the probability density, the probabilities of individual energy measurements, and the potential. It was especially common in student responses for the more advanced contexts of tasks 3 and 4, which suggests that even students who have completed most of their instruction in quantum mechanics may confuse the time dependence of various quantities.

1. Confusion between the time dependence of wave functions and probability densities

On task 1 ($N = 416$), about half of the students correctly stated that the probability of finding the particle in a given region does not depend on time for an energy eigenstate. However, between 5% and 20% of the students seemed to assign the time dependence of the wave function to the probability density (i.e., without referencing the modulus square).

The wave function is time independent. Thus, its probability density does not change. If the wave function is time dependent, then [its] probability density would change in time too. (task 1)

This student has clearly connected the time dependence of the two quantities. Many students obtained the correct answer because they incorrectly treated the wave function as being time independent (see also Sec. IV B 1). Students often demonstrated reasoning of this sort even when they were able to express the relationships mathematically (e.g., by writing that the probability density is the modulus square of the wave function).

About 5% of the students stated explicitly that the wave function itself does not depend on time.

This is a stationary state so the wave function will not evolve with time. (task 1)

The reasoning above suggests that this student thinks the term *stationary state* means that the wave function, rather than the probability density, has no time dependence. This error was more common for energy eigenstates than for superpositions of energy eigenstates. Some students also seemed to believe that everything about stationary states is time independent.

On task 3 ($N = 285$), about 25% of the students seemed to be using the time dependence of the probability density to determine whether or not the state itself is time dependent. On one version of task 4 ($N = 34$), about 25% of the students stated that the probability density would (or would not) depend on time because the wave function did (or did not) depend on time [31]. The presence of these errors in the more advanced contexts of tasks 3 and 4 suggests that they are particularly resistant to instruction, since they seem to persist throughout all of undergraduate quantum mechanics.

2. Confusion between the time dependence of probabilities of energy measurements and other quantities

Some students incorrectly stated that the probability of measuring a given energy depends on time. They related the time dependence of the probability to the time dependence of other quantities. About 10% of the responses to task 2 ($N = 439$) involved this line of reasoning.

It [the energy probability] would change because the wave function would change. (task 2)

It [the energy probability] depends on the probability density. If it's time independent then no, if time dependent then yes. (task 2)

The first student infers a time dependence for the probability of an energy measurement based on the fact that the wave function changes in time. The second student associates the probability of measuring a given energy to the time dependence of the *probability density*. However, the probabilities for energy measurements do not depend on time, regardless of whether or not the wave function or the probability density has a time dependence [32]. In some cases (e.g., for stationary states) this incorrect line of reasoning led students to give the correct answer.

Confusion about the time dependence of the probabilities for energy measurements was also common among graduate students. Even after two quarters of graduate instruction, 40% ($N = 19$) gave answers to a version of task 3 consistent with the line of reasoning discussed above.

The following response illustrates another way in which students applied the time dependence of some quantity inappropriately to that of energy measurements:

A linear combo of stationary states is not stationary. The system will oscillate around E_0 and E_1. (task 2)

This student appears to believe that the probability of measuring a given energy changes with time, alternating between being greater for one of the eigenstates in the wave function and then, at later times, greater for the other. This behavior could correctly describe the real (or imaginary) part of the wave function, which oscillates in the real and imaginary planes, or the probability density, which has a peak that moves back and forth. However, it is incorrect when applied to energy probabilities, which are constant in time.

3. Confusion between the time dependence of the potential and other quantities

On task 4 ($N = 215$), some students used the time dependence of the potential to argue about the time dependence of other quantities. On this task, students are asked to consider the effect of a perturbation on the time dependence of the probability density for a particle.

If the perturbation is time dependent, then yes [the probability density depends on time]. (task 4)

Although this student arrived at the correct answer, the reasoning is not based on the relevant aspects of the quantum formalism. Approximately 20%–40% of the students gave answers to task 4 consistent with this response. This type of reasoning was evident on other questions as well, such as task 3, although only at about the 5% level.

B. Failure to ascribe the correct time-dependent phases to the wave function

Many students had difficulty in determining, applying, or interpreting the phases that arise from the time-dependent Schrödinger equation. Each term in a wave function that is written in the energy eigenbasis is associated with a phase that depends both on time and on the energy of the associated eigenstate. Students often failed to apply this formalism to the wave function. Some treated the wave function as not having a time-dependent phase; others included only a single time-dependent phase for the entire wave function. Still others incorrectly associated phases with *distinct* time dependence to terms that have the *same* energy (e.g., when there was degeneracy).

1. Belief that the wave function is time independent

In the research tasks, the wave functions were usually given to students, either mathematically or graphically, for a particular instant in time. The functional form of the time dependence was not given. On some versions of tasks 1 and 2 (not shown), we asked students explicitly about the time dependence of the wave function prior to asking about probabilities of energy or position measurements. In these cases, some students responded that there is no time dependence or that they had not been given enough information to find it. The following response is illustrative:

The graph does not provide any information about the time dependence of the wave function. (task 1)

This student did not recognize that, in the absence of external factors, the wave function at an initial time determines the wave function for all times. On each of the four tasks, about 5% gave similar responses, regardless of whether the question asked about the wave function, the probability density, or both. These students tended to focus on the fact that no explicit time dependence was given for the wave function. For example, one student answered task 2 by saying, "No, there is no time in the equation." Although this student correctly stated that the probability for an energy measurement does not change with time, the reasoning is incorrect.

Some students attempted to give a mathematical basis for why they believed that the wave function does not depend on time.

[Both wave functions] satisfy the time-independent Schrödinger equation so ψ_1 and ψ_2 do not have time dependence. (task 2)

This student is correct that the solutions to the time-independent Schrödinger equation do not depend on time. However, the solutions appear as terms in the wave function with a time-dependent phase. Like the students above, this student does not seem to recognize that the wave function at an instant does not provide a complete description of the state. When reasoning about the time dependence of nonstationary states, this belief has led some students to think that no states have time dependence.

Some of the student responses discussed in Sec. IV A also reflect the idea that a wave function consisting of a single eigenstate is time independent. These students often argued about the time evolution of the state based on the fact that the probability density does not depend on time (see Sec. IV A 1).

We have probed student thinking about the time evolution of states in greater detail during interviews. The results suggest that the errors are not superficial. For example, we conducted a series of interviews in the context of perturbation theory (similar to task 4) [4]. Even those students who recognized that there is a time dependence associated

with an energy eigenstate of the infinite square well often did not do so for an eigenstate of an unspecified potential. This finding suggests that many students do not have a general understanding of the time evolution of quantum states.

2. Tendency to treat all wave functions as having a single phase

Perhaps the most prominent error documented by prior research on student understanding of quantum mechanics is a tendency of students to associate a *single* time-dependent phase with the entire wave function, rather than to associate an individual phase with each term [14]. The equation below is a student's response to the superposition state question from task 1. Note that the student wrote a wave function, even though the question asked about the probability density:

$$\sqrt{\frac{1}{2}}e^{-iEt/\hbar}(\psi_1 + \psi_2) \quad \text{(task 1)}$$

The time dependence expressed by this student would be correct for an energy eigenfunction, but not for a superposition of energy eigenfunctions with different energies. Moreover, the student has not identified which energy should be used, but has simply written a generic "E." Other students wrote "E_n" without specifying the value of n that should be used [33].

Some students correctly identified the different phase factors for each term in the wave function but failed to recognize how the different phases impact the probability density:

> While it is true that the general wave function is of the form $\sqrt{1/2}\varphi_1 e^{-iE_1 t/\hbar} + \sqrt{1/2}\varphi_2 e^{-iE_2 t/\hbar}$, again the function we're interested in is $P(x) = |\varphi|^2$ which loses its t-dependence. (task 1)

Although the wave function is correct, the student claims that the overall phase vanishes for the probability density. This statement is consistent with what we have observed when working with students in class. Many simply state that "time drops out" or "probability is squared and [the] time component won't matter," independent of whether or not the initial state is an energy eigenstate or not. Between 10% and 20% of the responses to task 1 at the junior level were similar ($N = 416$).

The tendency to treat the time dependence of all wave functions as consistent with the presence of only a single phase has been very persistent. We have observed this difficulty in response to questions asked on exams in multiple courses. Even after students had worked through tutorials designed to address this difficulty, the problem still arose when the context of the exam question was different from that of the tutorials.

For example, on a final exam in a sophomore-level quantum mechanics course, students were given a version of task 1 that explicitly gave the time-dependent form of the wave function. (In task 1, we typically gave the wave function for only a single instant in time.) The students were asked about the time dependence of the probability of a particular outcome of a position measurement. About 25% of the students ($N = 223$) gave answers consistent with treating the wave function as having a single time-dependent phase. This finding suggests that the underlying problem goes beyond a failure to recognize that the wave function has time dependence. Some students appear to have a strongly held belief that the time dependence vanishes in the probability density, even when they are given the information necessary to perform a calculation that contradicts this belief.

We have also found that the tendency to treat the wave function as having a single phase (whether or not students explicitly write it with a single phase) is particularly prevalent when the initial state is not given in terms of the energy eigenbasis. For example, on several exams in the junior-level course, we have asked students about the time dependence of both wave functions and probabilities for an initial state that was not written explicitly as a sum of energy eigenstates. The contexts for these questions have included the infinite square well, the harmonic oscillator, and spin. The number of students in these classes has ranged from 40 to 70. Between 10% and 25% of the responses to each exam question were consistent with treating the wave function as having a single, time-dependent phase.

3. Tendency to treat every superposition as having multiple distinct phases

In responding to the questions in the research tasks about the time dependence of a wave function or another quantity, even students who answered correctly did not usually discuss the phases of the individual terms. Many appeared to use a rule that any superposition state has a time-dependent probability density. This reasoning gives the correct answer for cases with distinct energy eigenvalues. However, it does not work for task 3, which involves a superposition of states with degenerate energies. On this task, about 30% of the students ($N = 285$) gave reasoning consistent with this idea.

> It is a linear combo so time dependence does not cancel out. (task 3)

This student responded in the same way to *both* of the questions on task 3, regardless of whether or not the energies were degenerate. Many students gave identical responses for both states.

The tendency to treat all superpositions as having distinct phases has arisen in student answers to several variants of task 3. For example, on some versions, students were shown four different superpositions of two eigenstates of the hydrogen atom potential. Two of the superpositions had degenerate energies and two had nondegenerate energies. About 30% of junior-level students ($N = 42$) answered that the probability density for all four states depends on time. The result was the same on similar tasks in the context of identical particles, spin, or angular momentum. About 10% of graduate students ($N = 19$) also made similar errors, even after almost a full year of advanced instruction on quantum mechanics.

C. Tendency to misinterpret the mathematical formalism used for time dependence in quantum mechanics

In this investigation, we have found that many students seem to have difficulty in determining the effect that the time-dependent phases has on the evolution of different physical quantities. Even students who are able to determine the correct time dependence of a state often misinterpret the mathematics in one or more ways. Some confuse the behavior of traveling waves and standing waves, others have difficulty in distinguishing between real and imaginary exponentials or between coordinate space and Hilbert space. We have grouped these problems together because they have in common a strong mathematical component.

1. Tendency to treat wave functions for bound systems as traveling waves

Students in quantum mechanics classes have typically studied waves in a variety of contexts (e.g., in introductory and upper-division courses on classical mechanics and electromagnetism). We have found that some students seem to confuse the properties of two common kinds of waves: traveling waves and standing waves.

The explanation below was given in response to a version of task 1. The student was shown a graph of the wave function at an initial time, $t = 0$, and asked how the graph would change in time for a single point labeled x_1:

The graph of [the] wave function shows that it is a function of $\sin(t)$ and at $t = 0$, $x_1 = 3\pi/2$. So when the wave function move[s] to the right $\pi/2$ second later, the zero point on graph hit[s] x_1. (task 1)

This student appears to be treating the wave function as a traveling wave that moves from left to right, rather than as a standing wave. On task 1 ($N = 416$), about 10% of the students gave similar responses. This tendency has been most prominent for tasks that include a visual representation of the wave function.

2. Tendency to treat time-dependent phase factors as decaying exponentials

The time-dependent phase factors in a superposition of energy eigenstates are imaginary exponentials. One of their properties is that the overall magnitude does not change; it is always equal to 1. Previous research has shown that many students treat these factors as rising or falling exponentials [6]. This tendency appears in about 10% of the responses to task 1 ($N = 416$), for which the students were required to determine and describe the time dependence of probability densities.

Since the wave equation will gain a $e^{-E_2t/\hbar}$ term to represent its evolution as time goes on, the probability of finding the particle in the marked area will decrease […] since the square of its wave equation will decrease as well. (task 1)

This student has omitted the imaginary unit in the exponent, which is not uncommon [13]. However, even students who wrote the term correctly often gave answers consistent with treating the term as a rising or falling exponential.

It should be noted that falling exponentials are sometimes described as *decaying* exponentials. In addition to the mathematical error in the response above, some students may interpret "decay" as referring to a change in which the energy of a particle decreases (see Sec. IV D 2).

3. Tendency to misinterpret the real and imaginary components of the wave function

Complex numbers play a critical role in quantum mechanics, especially with respect to time dependence. Student responses on many of the tasks presented in this paper revealed fundamental errors in their reasoning about the real and imaginary parts. Figure 2 shows part of a handwritten response to a version of task 1. In this version, students were told to consider a particle in the first excited state of an infinite square well and asked about the probability of finding the particle in a given region at several times. This student correctly identified the time dependence of the wave function and recognized that the overall phase is given by 1, i, and -1 at times t_0, t_1, and t_2, respectively. However, the student appears to associate the

FIG. 2. A handwritten response to a version of task 1 that illustrates a common student difficulty in relating imaginary numbers to probabilities in quantum mechanics. The subscripts 0, 1, and 2 refer to instants in time.

imaginary wave function with a probability of zero, and is apparently uncertain about the probability when the wave function is negative.

The response below articulates the same idea for a student who is thinking about time evolution graphically:

> When the wave function rotates to being entirely in the imaginary axis the wave function will be equal to zero. (task 1)

In both of these responses, the students correctly identify a rotation of the wave function in the complex plane. However, both students seem to treat the imaginary part of the wave function as if it does not contribute to the probability density. This line of reasoning was evident in about 10% of the responses to both tasks 1 and 2.

In response to some versions of task 3, students made a different error involving imaginary numbers. In these versions, students were asked to consider several superposition states, with either degenerate or nondegenerate energies, some of which include a term that has an imaginary coefficient. Some students asserted that the wave function, the probability density, or the energy probabilities only change with time for the states that include an imaginary term:

> ψ^2 depends on time because it has a complex component. (task 3)

> The imaginary component rotates as time passes. (task 3)

These students seem to be associating a time dependence with the *presence* of imaginary numbers. We have also identified similar errors on questions about wave functions and complex numbers that do not ask about time dependence [17].

4. Confusion between coordinate and Hilbert spaces

On task 3 ($N = 285$), students are asked to consider a particle in an isotropic three-dimensional harmonic oscillator. Often, this is one of the first times that they have considered a potential that depends on more than one spatial dimension. One of the questions asked about the time evolution of a superposition state consisting of two eigenstates with degenerate energies. About 5% of the students seemed to treat the particle as if it were moving alternately in different spatial directions.

> [The state] will oscillate between x and z with y component constant. (task 3)

Students who give answers like this seem to be confusing the three spatial directions with the real and imaginary axes through which the wave function rotates in time. Similar difficulties involving student confusion of physical spaces

and Hilbert spaces have been previously documented [34,35]. It is also possible that this student views the particle as switching from one state to another, similar to the difficulty discussed in Sec. IV A 2.

D. Tendency to apply ideas about time evolution that lie outside the model for quantum mechanics

A number of errors that students made on the research tasks suggest that they were using ideas that are inconsistent with the basic model and formalism that are being presented in the course [36]. In some cases, these alternative ideas might be considered as arising from mathematical difficulties, but many seem to be conceptual in nature. We have grouped these together since they often appear to be an attempt by students to incorporate outside knowledge (e.g., electron transitions in an atom) into the model for quantum mechanics that they are being taught. We have also found this interpretation useful in guiding the design of curriculum to address the underlying problems.

1. Belief that the wave function will spread out over time

One of the systems typically described near the beginning of an introductory quantum mechanics course is that of a free particle. The general form of the wave function for a free particle is often termed a "wave packet," and the time dependence is frequently described as "spreading" from a narrow to a broad distribution in position space. Some students use similar language to describe the time dependence for other quantum systems.

For example, the student below describes how the probability density for a finite region in an infinite square well would change for the first excited state of the particle:

> Over time the function would need to spread out and become symmetric. (task 1)

This response is consistent with the behavior of a wave packet. The example below expresses the same idea in a different way:

> All regions have equal possibility because there is enough time for the wave to become undefined. (task 1)

Between 5% and 15% of the students gave responses to task 1 ($N = 416$) consistent with the idea that an energy eigenfunction would spread out over time.

In a variant of task 2, students were asked whether or not the probability of measuring the ground state energy would change with time for a state initially in an unequal superposition of the ground and first excited states. The following response to this task illustrates how reasoning about the spreading of the wave function has been applied to the probabilities of energy measurements:

Still [could measure] either E_1 or E_2 but [the probabilities] will disperse. (task 2)

This student seems to believe that the probabilities of each possible outcome, which were initially unequal, will change. Other students have stated that as time progresses it becomes possible to measure other energies, until all allowed energies are both possible and equally probable. (The response of the prior student, who stated that all positions eventually become equally probable, is consistent with the idea that the state eventually includes terms corresponding to all energy eigenvalues.) We have also seen evidence of this line of reasoning in other contexts, such as angular momentum.

2. Belief that the wave function will return to its initial state

On task 4 ($N = 215$), students are asked to consider the time dependence of a wave function in a potential that is perturbed. They are asked about several different intervals: *before*, *during*, and *after* the perturbation. The following response is for the interval after the perturbation is removed:

Now it will time evolve back to [the] original wave function. (task 4)

The idea that the wave function reverts to the form it had before the perturbation ("revival" of the wave function) is very common on this task. It is also common on questions that deal with successive measurements of a system or when students are asked about the long-term time dependence of a system, for example, on versions of task 2 that include questions about an energy measurement after an initial measurement of energy or position [17]. About 20% of student responses to all versions of task 4 are consistent with this idea (the percentage for individual quarters varied from as little as 5% to as high as 45% of the students).

In some cases, we have asked a version of task 4 in which students consider a particle that is initially in an excited state, instead of being initially in the ground state. While some students still give answers consistent with revival, others explain that the particle will "decay" to the ground state. It is possible that student knowledge of atomic or radioactive decay may influence these answers, even though this is not consistent with the formalism they have learned. It is also possible that this idea is reinforced by the terminology of a decaying exponential discussed in Sec. IV C 2 [3].

V. CONCLUSION

This paper documents results from an investigation into student understanding of time dependence in quantum mechanics. The focus is on the extent to which students are able to recognize which quantities depend on time, how they depend on time, and the impact of the time dependence on real-world phenomena. The findings suggest that many students have not understood key aspects of the formalism that have been taught in the course. The persistence of specific difficulties throughout many different courses—taught by different professors, using a wide variety of lecture styles, over many years, and at several different institutions—indicates the extent to which they are strongly held and resistant to instruction. The identification of these difficulties is an important first step toward developing curriculum specifically tailored to address the most common student difficulties.

Some of the problems that we have identified are related to the mathematics underlying the physical concepts, for example, in interpreting the role of imaginary numbers. However, many are related directly to the physics, such as the finding that many students incorrectly relate the time dependence of different quantities. For example, some treat quantum states as if they do not depend on time and base this belief on the fact that the probabilities for stationary states are time independent. Others reason in reverse, assigning the time dependence of a quantum state to that of the associated probability density.

Perhaps most troubling is the failure of many students to recognize which phenomena can and cannot be accounted for by the model for quantum mechanics that they are studying. When asked about the time evolution of a system, many give an answer that is not prescribed by the Hamiltonian. Some treat quantum states as if they will "decay" to states with lower energy or as if the corresponding probability distributions will "spread out" over time. They seem not to understand critical elements of the model that is being presented (e.g., the basic assumptions that are included in the model), as well as how to use it to make predictions about the behavior of a system. This is particularly apparent in the relatively difficult context of time-dependent perturbation theory. In this application, we found that many students were unable to describe the time dependence of the wave function, the probability density, and the probabilities of energy measurements.

Most of the difficulties were more common among students in the junior-level course, but some occurred at all levels of instruction and persisted to the graduate level. In some cases, the graduate students were more likely to give correct answers; however, their explanations were often incomplete and contained conceptual and reasoning difficulties similar to those of students at the junior or sophomore levels. This finding suggests that the ideas are not easy, and are not addressed simply by additional instruction [37].

The findings from this research have guided our group in the design of *Tutorials* on quantum mechanics. We have been developing and testing instructional strategies for helping students deepen their understanding of this topic

while attempting to address the most common incorrect ideas. Overall, the results have been promising [38,39], but in the process we have found that certain errors seem to be very persistent. Small changes to the phrasing of a question or a change in context can elicit problems that had previously seemed to be addressed. Moreover, the underlying problems may appear to be resolved early in the course, but resurface when students study more complicated phenomena. The persistence of certain ideas suggests that the problems are deeply rooted and that more research needs to be done. There is a need for ongoing efforts that describe not only the specific problems that students encounter but that document instructional strategies that have proved effective at improving the learning and teaching of this difficult and abstract topic.

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[24] Students receive credit for completing the pretests; credit is not based on correctness. In our experience at the introductory level, most students take the pretests seriously and they provide a good gauge of student understanding. We have found the results to be similar when a given question

is asked either on a tutorial pretest or on a course exam when the corresponding tutorial is not used.

[25] A total of 115, 131, and 91 students from the University of Colorado, Boulder are included in the results for tasks 1, 2, and 3, respectively. A total of 5 and 9 students from Mount Allison University are included in the results for tasks 1 and 2.

[26] In most cases, the percentages are what would be expected statistically given the sizes of the different classes involved.

[27] On task 2 a substantial fraction of the students who responded correctly did not give complete reasoning. This was not the case on the other tasks. Most of these students did not give incorrect reasoning, but rather their explanations did not seem to be directly related to the time-dependent phases. Thus, it was difficult to determine whether or not they understood the underlying ideas.

[28] Note that the percentage of students who answered correctly with correct reasoning for a superposition state on task 3 ($\sim$50%) is greater than the percentage who correctly answered the corresponding part of task 1 ($\sim$20%). There is evidence that this difference might be a result of tutorial curriculum administered between the two tasks.

[29] Results from the administration of task 4, including a description of several follow-up interviews, are discussed in more detail in Ref. [4]. However, the percentages in this paper differ slightly due to the inclusion of additional data obtained in subsequent years.

[30] Task 4 has been given on paper and online. The online versions have tended to be multiple choice with explanations. We have noticed somewhat different patterns in the incorrect responses when the task was given online rather than handwritten.

[31] This was only present for the interval before the perturbation was turned on, and was not seen in the paper version of the task. We suspect this is because in the paper version students are aware that a perturbation will be added to the system. The students who took the online version did not receive this information until after they had answered the question for the "before" interval.

[32] Only an energy eigenstate with zero energy would have no time dependence.

[33] The tendency for students to give an answer in terms of a generic energy (e.g., E_n) has been identified previously. See, for example, Ref. [13].

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