Coherent synchrotron radiation for laminar flows

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We investigate the effect of shear in the flow of charged particle equilibria that are unstable to the coherent synchrotron radiation (CSR) instability. Shear may act to quench this instability because it acts to limit the size of the region with a fixed phase relation between emitters. The results are important for the understanding of astrophysical sources of coherent radiation where shear in the flow is likely.

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I. INTRODUCTION

When the wavelength of synchrotron radiation emitted by a bunch of relativistic particles is comparable to the size of the bunch, the particles may radiate coherently. The coherent emission produces significantly greater power than the incoherent emission. Coherent synchrotron emission is seen from different astrophysical sources but most notably from the rotationally powered radio pulsars (e.g., [1]). In particle accelerators the coherent synchrotron emission is usually undesirable because it causes very rapid energy loss. In this context we would also like to mention efforts to study coherent curvature radiation as a pulsar emission mechanism in the laboratory [2].

Several effects may act to stabilize a system unstable to the coherent synchrotron radiation (CSR) instability. In [3] the authors analyzed the influence of a small energy spread in a beam of charged particles in approximately circular motion. A distribution function with a single value of the canonical angular momentum was considered. The radial width of the beam was given by the amplitude of the betatron oscillations which is nonzero for a nonzero energy spread. It was shown that the decoherence introduced by the betatron oscillations leads to a characteristic frequency spectrum, whereas the dependence on the Lorentz factor and the number density remains unaffected.

In this paper we relax the assumption of zero spread in the canonical angular momentum P_ϕ of the equilibrium distribution. In general, this leads to shear in that the average angular velocity becomes dependent on the radius of the orbit. In practice, the requirement of a small spread in the canonical angular momentum may be harder to satisfy than a small energy spread. Shear is expected in astrophysical sources of coherent emission [4].

Shear itself can be the cause of instabilities, for example, the "diocotron instability" [5], but this is not the focus of the present paper. In the case of CSR it is reasonable to expect that the shear acts to stabilize the CSR instability due to particles with different radii "slipping away." Using a linear perturbation analysis of the fluid equations for a laminar Brillioun flow, we show that even with a spread of the canonical angular momentum $\Delta P_\phi > 0$, previous results for equilibria with constant P_ϕ can be recovered treating the plasma as a relativistic cold fluid. Computer simulations of CSR emission in Brillioun flows [6] are compatible with this picture. With the cold fluid approximation adopted here, the stability depends on the number density and the angular velocity which depends on the radius. The azimuthal velocity is approximately equal to the speed of light.

The problem of CSR emission has been investigated by several authors. Goldreich and Keeley [7] considered the stability of a charge distribution whose motion is confined to a thin ring with the particle motion being one-dimensional. This calculation led to confusion as to how the proposed CSR mechanism works in detail (cf. [8]).

As proposed in [3], the CSR instability is related to the classical negative mass instability [9–12] in the sense that an increase in particle energy leads to a decrease in its angular velocity. While the classical negative mass instability is caused by the Coulomb part of the electromagnetic potential, the CSR instability is caused by the radiation field. The negative mass effect is not immediately apparent in a one-dimensional treatment based on conservation of energy and charge. Heifets and Stupakov [13] effectively built in the negative mass effect by hand having a constrained radius and particle energy.

The treatments [3,7,13] give the growth rate in the absence of an energy spread. However, considering a nonzero energy spread requires a truly two-dimensional model. Larroche and Pellat investigated the effect of steep boundaries in the particle distribution function [14].

Section II describes the assumed models, and Sec. III approximate solutions of the equations. Section IV derives a dispersion relation for the considered perturbation. Section V discusses the results.

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II. THEORY

We consider two cases illustrated in Fig. 1. The first case (a) corresponds to a thin layer of relativistic electrons gyrating in a uniform external magnetic field, an "E-layer." The second case (b) corresponds to the electrons moving almost parallel to a toroidal magnetic field.

A. Case a

We consider a laminar Brillioun-type equilibrium of a long, non-neutral, cylindrical relativistic electron (or positron) layer in a uniform external magnetic field $\mathbf{B}_e = B_e\hat{\mathbf{z}}$, where we use a nonrotating cylindrical (r, ϕ, z) coordinate system. The electron velocity is $\mathbf{v} = v_\phi(r)\hat{\phi} = r\Omega(r)\hat{\phi}$. The self-magnetic field is in the z-direction while the self-electric field is in the r-direction. The radial force balance of the equilibrium is

$$-\gamma\Omega^2 r = \frac{q}{m_e}(E + vB), \tag{1}$$

where $\gamma = (1 - v_\phi^2)^{-1/2}$ is the Lorentz factor with velocities measured in units of the speed of light, $B = B_e + B_s$ is the total (self plus external) axial magnetic field, E is the total (= self) radial electric field, and q and m_e are the particle charge and rest mass. We have

$$\frac{1}{r}\frac{d(rE)}{dr} = 4\pi\rho_e, \qquad \frac{dB}{dr} = -4\pi\rho_e v, \tag{2}$$

where $\rho_e(r)$ is the charge density of the electron layer.

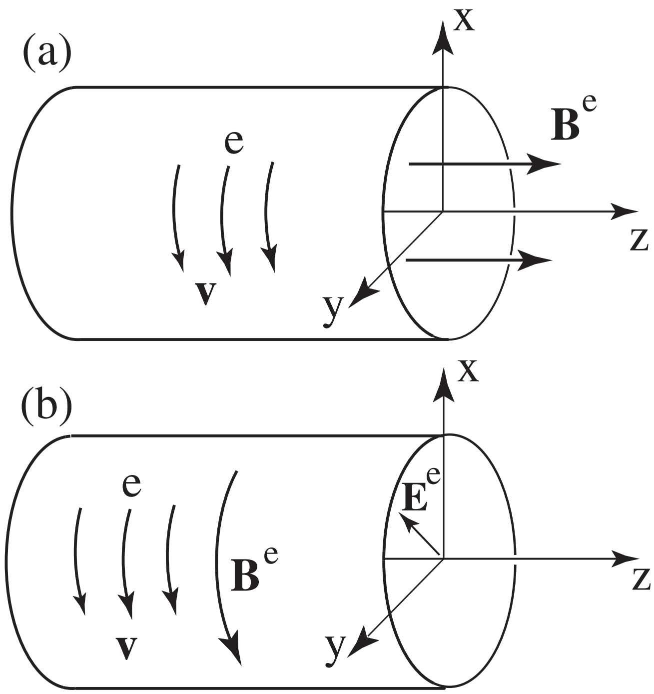

FIG. 1. Panel (a) shows the geometry of a relativistic E-layer for the case of a uniform external axial magnetic field. Panel (b) shows the second case of a relativistic E-layer in an external toroidal magnetic field and an external radial electric field.

We consider weak layers in the sense that the "field reversal" parameter

$$\zeta \equiv -\frac{4\pi}{B_e}\int_{r_1}^{r_2} dr\rho_e v \tag{3}$$

is small compared with unity, $\zeta^2 \ll 1$. Under this condition, Eq. (1) gives $\Omega = -qB_e/(m_e\gamma)$. Here, we have assumed that the layer exists between r_1 and r_2. We also consider that the Lorentz factor is appreciably larger than unity in the sense that $\gamma^2 \gg 1$. Furthermore, we consider radially thin layers

$$\delta \equiv \frac{\Delta r}{r_0} \ll 1. \tag{4}$$

We consider general electromagnetic perturbations of the electron layer with the perturbations proportional to

$$f_\alpha(r)\exp(im\phi - i\omega t), \tag{5}$$

where $\alpha = 1, 2, \ldots$ for the different scalar quantities, $m =$ integer, and ω the angular frequency of the perturbation. Thus the perturbations give rise to field components δE_r, δE_ϕ, and δB_z. The perturbed equation of motion is

$$\left(\frac{\partial}{\partial t} + \mathbf{v}\cdot\nabla\right)(\mathbf{v}\delta\gamma + \gamma\delta\mathbf{v}) + \delta\mathbf{v}\cdot\nabla(\gamma\mathbf{v}) = \frac{q}{m_e}(\delta\mathbf{E} + \mathbf{v}\times\delta\mathbf{B} + \delta\mathbf{v}\times\mathbf{B}), \tag{6}$$

where the deltas indicate perturbation quantities. This equation can be simplified to give

$$\begin{bmatrix} -i\gamma\Delta\omega & -\gamma\Omega(1+\gamma^2) - \frac{q}{m_e}B \\ \gamma\Omega + (\gamma\Omega r)' + \frac{q}{m_e}B & -i\gamma^3\Delta\omega \end{bmatrix}\begin{bmatrix} \delta v_r \\ \delta v_\phi \end{bmatrix} = \frac{q}{m_e}\begin{bmatrix} \delta E_r + v\delta B_z \\ \delta E_\phi \end{bmatrix}, \tag{7}$$

where the prime denotes a derivative with respect to r, and

$$\Delta\omega(r) \equiv \omega - m\Omega(r)$$

is the Doppler shifted frequency seen by a particle rotating at Ω. We also define the dimensionless quantity

$$\Delta\tilde{\omega} \equiv \frac{\Delta\omega}{m\Omega} \tag{8}$$

which will turn out to be useful later.

Using the equilibrium equation (1) and the condition $\zeta^2 \ll 1$, the matrix in Eq. (7) is approximately

$$\mathcal{D} = \begin{bmatrix} -i\gamma\Delta\omega & -\gamma^3\Omega \\ \gamma^3(\Omega r)' & -i\gamma^3\Delta\omega \end{bmatrix}, \tag{9}$$

We have used the fact that $(\gamma\Omega r)' = \gamma^3(\Omega r)'$. For $\zeta^2 \ll 1$ we have $(\Omega r)' = \Omega/\gamma^2$ and $\Omega' = -v^2\Omega/r$. In the absence of shear the latter quantity would be zero. Consequently,

$$\det(\mathcal{D}) = \gamma^4(\Omega^2 - \Delta\omega^2). \tag{10}$$

Inverting Eq. (7) gives

$$\delta v_r = \frac{q\gamma^3}{m_e \det(\mathcal{D})}[-i\Delta\omega(\delta E_r + v\delta B_z) + \Omega\delta E_\phi], \quad (11)$$

and

$$\delta v_\phi = \frac{q\gamma}{m_e \det(\mathcal{D})}[-i\Delta\omega\delta E_\phi - \Omega(\delta E_r + v\delta B_z)]. \quad (12)$$

Here, $q\rho_e\gamma^3\Omega/[m_e \det(\mathcal{D})]$ has the role of the distribution function of angular momentum [15].

B. Case b

Here we consider an equilibrium with the same number density and velocity profile as before but with different external fields as shown in Fig. 1(b). Instead of an external magnetic field in the z-direction, we consider an equilibrium with an azimuthal magnetic field acting as a guiding field and a radial electric field. The latter is included in the equilibrium condition and therefore does not enter the linearized Euler equation. B^e_ϕ would only enter if we considered motion in the axial direction and nonzero axial wave numbers.

Thus, we obtain the matrix $\mathcal{D}$ again without the B_0 terms, i.e., for $\gamma \gg 1$

$$\mathcal{D} = \begin{bmatrix} -i\gamma\Delta\omega & -\gamma^3\Omega \\ 2\gamma\Omega & -i\gamma^3\Delta\omega \end{bmatrix}, \quad (13)$$

with

$$\det(\mathcal{D}) = \gamma^4(2\Omega^2 - \Delta\omega^2). \quad (14)$$

We obtain

$$\delta v_r = \frac{q\gamma^3}{m_e \det(\mathcal{D})}[-i\Delta\omega(\delta E_r + v\delta B_z) + \Omega\delta E_\phi], \quad (15)$$

and

$$\delta v_\phi = \frac{q\gamma}{m_e \det(\mathcal{D})}[-i\Delta\omega\delta E_\phi - 2\Omega(\delta E_r + v\delta B_z)]. \quad (16)$$

Such a configuration is a more realistic possibility for the magnetosphere of a radio pulsar. The conclusions for the two configurations do not differ significantly, and we will proceed analyzing case a.

III. APPROXIMATE SOLUTION

Using Eqs. (11) and (12), the linearized continuity equation gives

$$\begin{aligned} i\Delta\omega\delta\rho = &+\frac{im}{r}\left[\frac{q\gamma\rho_0}{m_e \det(\mathcal{D})}[-i\Delta\omega\delta E_\phi - \Omega(\delta E_r + v_\phi\delta B_z)]\right] \\ &+ \frac{1}{r}\frac{\partial}{\partial r}\left[r\frac{q\gamma^3\rho_0}{m_e \det(\mathcal{D})}[-i\Delta\omega(\delta E_r + v_\phi\delta B_z) + \Omega\delta E_\phi]\right], \end{aligned} \quad (17)$$

where we used $\delta J_r = \rho_0\delta v_r$ and $\delta J_\phi = v_\phi\delta\rho + \rho_0\delta v_\phi$. We consider conditions where $\delta E_r + v_\phi\delta B_z$ terms can be neglected. The sufficient conditions are

$$\left|\frac{\Delta\omega}{\Omega}\right| \le 1, \quad (18)$$

and

$$|\delta E_r + v\delta B_z| \ll \left|\frac{\Delta\omega}{\Omega}\right| |\delta E_\phi|. \quad (19)$$

We estimate the relative magnitude of the field components using the Maxwell equations. From Faraday's law we obtain the relation

$$D_r(\delta E_\phi) - ik_\phi\delta E_r = i\omega\delta B_z, \quad (20)$$

where $D_r(\ldots) \equiv r^{-1}\partial/\partial r(r\ldots) \approx ik_r$ assuming the radial dependence $\exp(ik_r r)$ for the perturbed quantities as well as thin layers with $\delta \ll 1$. We have

$$k_r\delta E_\phi = k_\phi(\delta E_r + v_\phi\delta B_z) + k_\phi v_\phi\Delta\tilde{\omega}\delta B_z, \quad (21)$$

with $\delta B_z = D_r\delta A_\phi$, $\delta A_\phi = v_\phi\delta\Phi$, and $\delta E_\phi = -imr^{-1}\delta\Phi + i\omega\delta A_\phi$:

$$\begin{aligned} \delta E_r + v_\phi\delta B_z &= \frac{k_r}{k_\phi}\delta E_\phi - v_\phi^3\Delta\tilde{\omega}k_r\frac{\delta E_\phi}{\Delta\omega - \omega/\gamma^2} \\ &\approx \frac{\bar{k}_r}{m}\delta E_\phi. \end{aligned} \quad (22)$$

Inequality (19) turns into

$$\bar{k}_r \ll m\frac{\Delta\omega}{\omega} = m^2\Delta\tilde{\omega}, \quad (23)$$

where $\bar{k}_r \equiv k_r r_0$. We will always assume radial wave numbers k_r that are sufficiently small such that the latter condition is met. A consequence of inequality (19) is that

$$\delta v_\phi = \gamma^{-2}\frac{\Delta\omega}{\Omega}\delta v_r. \quad (24)$$

Making use of approximation (19), Eq. (11) can be entirely written in terms of δE_ϕ. Note that the resonant term due to $\det(\mathcal{D})$ (the Lindblad resonance) is canceled in the limit $\delta v_\phi \longrightarrow 0$. In general, we will have to invoke the assumptions (18) and (19), though. Neglecting the small δv_ϕ term, the linearized continuity equation gives

$$\frac{1}{r}\frac{\partial}{\partial r}(r\delta J_r) = i\Delta\omega\delta\rho. \tag{25}$$

Thus,

$$r\delta\rho = \frac{-i}{\Delta\omega}\frac{\partial}{\partial r}\left(\frac{qr\rho_0}{m_e\gamma\Omega}\delta E_\phi\right). \tag{26}$$

Integrating over dr and integrating by parts gives

$$\int rdr\delta\rho = -\int dr\frac{im\Omega/r}{(\Delta\omega)^2}\frac{qr\rho_0}{m_e\gamma\Omega}\delta E_\phi. \tag{27}$$

In the Lorentz gauge

$$\begin{gathered}\delta E_\phi = -\frac{im}{r}\delta\Phi + i\omega\delta A_\phi \\ \delta E_r = -\frac{\partial}{\partial r}\delta\Phi + i\omega\delta A_r \qquad \delta B_z = \frac{1}{r}\frac{\partial}{\partial r}(r\delta A_\phi).\end{gathered} \tag{28}$$

For $(\Delta r/r)^2 \ll 1$ we have the approximation $r^{-1}\partial[r(\partial\ldots/\partial r)] = \partial(\ldots)/\partial r$. Assuming the radial dependence $\delta\Phi \propto \exp(ik_r r)$, we obtain from the gauge condition

$$ik_r\delta A_r + \frac{im}{r}\delta A_\phi - i\omega\delta\Phi = 0 \tag{29}$$

and δA_z is negligible because of the symmetry of the problem. δA_r is nonzero since $\delta J_r = \rho_0\delta v_r + v_r\delta\rho \neq 0$ and $|\delta v_\phi| \sim \gamma^{-2}|\Delta\omega/\Omega||\delta v_r|$. δA_ϕ can be computed from the Green's function (cf. Appendix) to give $\delta A_\phi = v_\phi\delta\Phi$. Fortunately, we only need δE_ϕ. Note that

$$\delta E_\phi = -imr^{-1}\delta\Phi/\gamma^2 + O(\Delta\tilde{\omega}). \tag{30}$$

IV. DISPERSION RELATION

The derivation of the Green's function can be found in the Appendix:

$$\delta\Phi(r) = 2\pi^2 i\int_0^\infty r'dr'J_m(\omega r_<)H_m(\omega r_>)\delta\rho(r'). \tag{31}$$

The argument of the Bessel functions is assumed to be independent of r and r' in the important region $1-\delta < r/r_0 < 1+\delta$ which we will justify later. Thus,

$$\delta\Phi(r) = -\frac{2\pi^2 ie^2m^2}{H\gamma^2}J_mH_m\int_0^\infty dr'\frac{n(r')\delta\Phi(r')}{r'[\omega - m\Omega(r')]^2}, \tag{32}$$

where Eq. (26) has been used and $H \equiv m_e\gamma$. Since the right-hand side of the last equation is independent of r, $\Phi(r)$ has to be constant and we obtain with $\Omega(r) \approx 1/r$

$$1 = -\frac{2\pi^2 ie^2m^2}{H\gamma^2}J_mH_m\int_0^\infty dr'\frac{n(r')}{r'(\omega - m/r')^2}, \tag{33}$$

where the Bessel functions are evaluated at $m[1 - 1/(2\gamma^2)]$. The remaining integral can be evaluated if the Gaussian number density profile $n(r) = n_0\exp[-\delta r^2/2(\Delta r)^2]$ is replaced by a rectangle with width $2\Delta r$ and height $n_0\sqrt{\pi/2}$:

$$\begin{aligned}\int_0^\infty dr'\frac{n(r')}{r'(\omega - m/r')^2} &\approx \int_{r_0(1-\delta)}^{r_0(1+\delta)} dr' n_0\frac{\sqrt{\pi/2}}{r'(\omega - m/r')^2} = n_0\sqrt{\frac{\pi}{2}}\left[\frac{\ln(\omega r' - m)}{\omega^2} - \frac{m}{\omega^2(\omega r' - m)}\right]_{r_0(1-\delta)}^{r_0(1+\delta)} \\ &\approx n_0\sqrt{\frac{\pi}{2}}\frac{2r_0\delta}{\omega}\frac{m}{(\omega r_0 - m)^2 - \omega^2r_0^2\delta^2}.\end{aligned} \tag{34}$$

The logarithm can be neglected because we are interested in the resonant case. With $\zeta = 4\pi e^2n_0\sqrt{\pi/2}r_0^2\delta/m_e\gamma$ we obtain the dispersion relation

$$1 = -\pi\zeta Z\gamma^{-2}\frac{1}{(\Delta\tilde{\omega})^2 - \delta^2}, \tag{35}$$

where $Z = iJ_mH_m$. Thus,

$$\Delta\tilde{\omega} = \pm\sqrt{\delta^2 - \pi\zeta Z/\gamma^2}. \tag{36}$$

The Bessel functions can be expressed in terms of Airy functions for $m \gg 1$:

$$J_m(z) = 2^{1/3}m^{-1/3}\mathrm{Ai}(w) \tag{37}$$

$$Y_m(z) = -2^{1/3}m^{-1/3}\mathrm{Bi}(w), \tag{38}$$

where

$$\begin{aligned}w &\equiv -2^{1/3}m^{2/3}\left(\frac{z}{m} - 1\right) \\ &= \left(\frac{m}{2}\right)^{2/3}[\gamma^{-2} - 2\Delta\tilde{\omega} - 2(\xi - 1)]\end{aligned} \tag{39}$$

with $\xi \equiv r/r_0$ which at the outer edge is $\xi - 1 \sim \delta$. Assuming $m \ll 2\gamma^3 \equiv m_2$, $\gamma^{-2} \ll 1$, $|\Delta\tilde{\omega}| \ll \gamma^{-2}$ (assuming that the layer essentially exists in the region $1 - \delta < r/r_0 < 1 + \delta$), and $2^{1/3}\delta \ll m^{-2/3}$, we can consider Z to be a pure function of m since $|w|^2 \ll 1$:

$$Z \approx (0.347 + 0.200i)/m^{2/3}. \tag{40}$$

Therefore, the Bessel functions do not depend on r and r' anymore. For $|w| \gg 1$ the Airy functions can be approximated to give

$$Z \approx \frac{1}{2\pi\sqrt{w}}\left(\frac{2}{m}\right)^{2/3}, \tag{41}$$

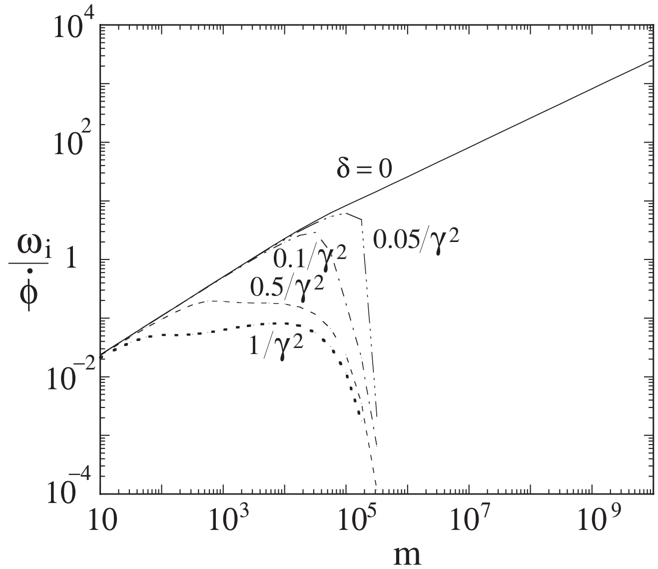

FIG. 2. Growth rates as a function of azimuthal mode m for $\zeta = 0.02$, $\gamma = 30$, and various energy spreads.

but we additionally have to demand $\delta \ll \gamma^{-2}$ in order for Z to remain independent of r and r'. For zero energy spread the condition $|\Delta\tilde{\omega}| \ll \gamma^{-2}$ is equivalent to $m \ll \gamma^3\zeta^{3/2} \equiv m_1$. As we will discuss in Sec. V this condition may be too strict.

Some results are plotted in Fig. 2. For zero energy spread we recover the usual scaling relation $\mathrm{Im}(\omega)/\Omega(r_0) \propto m^{2/3}$ for $m < 2\gamma^3$ [3,7,13] and $\mathrm{Im}(\omega)/\Omega(r_0) \propto m^{1/3}$ for $m > 2\gamma^3$ [3] with the $|w| \gg 1$ approximation of the Airy functions. The most dramatic consequence of a nonzero energy spread is the presence of a very sudden and steep cutoff. Pushing the cutoff to higher values of m requires increasingly small energy spreads. For large energy spreads the growth rate $\mathrm{Im}(\omega)/\Omega(r_0)$ scales as $m^{1/3}$ if the $|w| \ll 1$ approximation is used. Retaining the Airy function the growth rate grows more slowly. According to Fig. 2 the scaling relation for $\delta = 1/\gamma^2$ is in the order of $m^{1/4}$ before the cutoff is reached. This power law has been derived before in [3] where the decoherence is due to betatron oscillations instead of a nonzero spread in the average angular velocity. Note that these two results also agree with computer simulations that were carried out for Brillioun [6] flows.

V. RESULTS AND DISCUSSION

In order for the growth rate to vanish the expression inside the root must be real and non-negative. Using our approximations for the Bessel functions it is seen immediately that the former condition is satisfied if $m > m_2$. This explains why the drop off occurs at $m = m_2$ regardless of the value of δ as long as the remaining real part is positive. This is the case if the field reversal parameter does not exceed the critical threshold

$$\zeta > m\gamma\delta^2. \tag{42}$$

If the last inequality is not satisfied, complex roots can only exist if Z is complex which is the case for $m < m_2$. If however this inequality is satisfied, the expression inside the root becomes negative for $m > m_2$ and an unstable solution exists, but the sharp cutoff is replaced by Eq. (42).

As discussed in the last section the azimuthal mode number m resulting in the largest growth rate is either given by $m = 2\gamma^3$ or $m = \zeta\gamma^{-1}\delta^{-2}$ whatever is greater. This guarantees that $(\Delta\tilde{\omega})^2$ is either still complex or real and negative with both leading to an unstable mode. For our allowed parameter range $2\gamma^3$ is typically larger. Thus,

$$\mathrm{Im}\,(\omega) \leq 2^{2/3}\dot{\phi}\gamma\sqrt{\zeta} \sim \gamma^{-1/2}\omega_p, \tag{43}$$

where ω_p is the nonrelativistic plasma frequency.

Finally, let us consider small values of the azimuthal mode number m. So far we have approximated the Bessel functions by Airy functions which makes them easier to compute especially for large orders. For small m this approximation cannot be justified and the Bessel functions have to be retained. In Fig. 3 we solved the dispersion relation in the small m regime. Also, their arguments depend on $\Delta\tilde{\omega}$ since $m < m_1$. An accurate calculation of the growth rates of modes with azimuthal mode numbers m is important for the determination of the total power radiated, because the latter decreases with increasing m. Even for $m = 1$, Eqs. (37) and (38) give excellent results, presumably because the argument of the Bessel functions is close to their order for $\gamma \gg 1$ and Z is independent of w if $|w|^2 = |(\frac{m}{2})^{2/3}[\gamma^{-2} - 2\Delta\tilde{\omega} - 2(\xi - 1)]|^2 \ll 1$. The latter condition may be satisfied even if $|\Delta\tilde{\omega}|$ violates our assumption $|\Delta\tilde{\omega}| \ll \gamma^{-2}$.

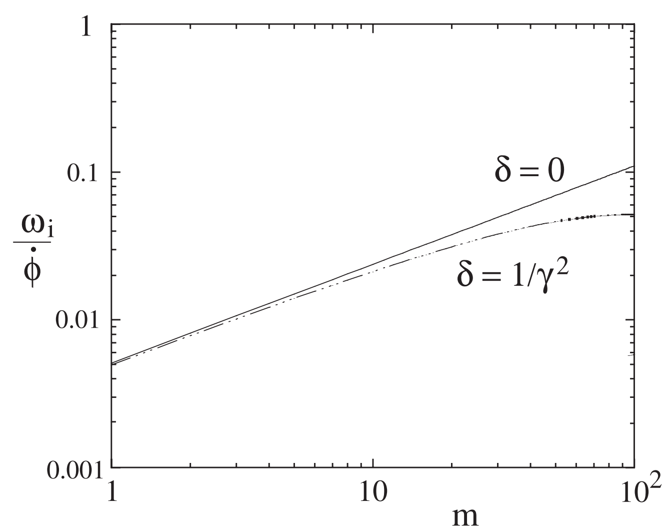

FIG. 3. Growth rates as a function of azimuthal mode m for $\zeta = 0.02$, $\gamma = 30$, and various energy spreads. The Bessel functions were retained in order to compute the growth rates in the low m regime correctly.

Looking at Eq. (34) we conclude that the quenching of the instability, i.e., the existence of the additional δ^2 term in the denominator, is due to both the nonzero thickness of the layer and $\Omega' \neq 0$. However, it is important to note that the instability relies on the negative mass effect and would not exist in the absence of shear in our model.

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APPENDIX A: GREEN'S FUNCTION

The Green's function for the potentials gives

$$\begin{aligned} \delta\Phi(\mathbf{r}, t) &= \int dt' d^3r' G(\mathbf{r} - \mathbf{r}', t - t')\delta\rho(\mathbf{r}', t'), \\ \delta\mathbf{A}(\mathbf{r}, t) &= \int dt' d^3r' G(\mathbf{r} - \mathbf{r}', t - t')\delta\mathbf{J}(\mathbf{r}', t'), \end{aligned} \tag{A1}$$

where

$$\begin{aligned} \left(\nabla^2 - \frac{\partial^2}{\partial t^2}\right)G(\mathbf{r}, t) &= -4\pi\delta(t)\delta(\mathbf{r}), \\ \tilde{G}(\mathbf{k}, \omega) &= \frac{4\pi}{\mathbf{k}^2 - \omega^2}, \\ G(\mathbf{r}, t) &= 4\pi \int_C d\omega \int d^3k \frac{\exp(i\mathbf{k}\cdot\mathbf{r} - i\omega t)}{\mathbf{k}^2 - \omega^2}, \end{aligned} \tag{A2}$$

where $\tilde{G}$ is the Fourier transform of the Green's function. The "C" on the integral indicates an ω-integration parallel to but above the real axis, $\mathrm{Im}(\omega) > 0$, so as to give the retarded Green's function.

Because of the assumed dependences of Eq. (5), we have for the electric potential

$$\begin{aligned} \delta\Phi_{\omega m k_z}(r) &= 2\int_0^\infty r'dr' \int_0^\infty \kappa d\kappa \int_0^{2\pi} d\alpha \delta\rho_{\omega m k_z}(r')[\cdots] \\ &= 4\pi \int_0^\infty r'dr' \int_0^\infty \kappa d\kappa \frac{J_m(\kappa r)J_m(\kappa r')}{\kappa^2 - (\omega^2 - k_z^2)} \\ &\quad \times \delta\rho_{\omega m k_z}(r'), \end{aligned} \tag{A3}$$

where

$$[\cdots] \equiv \frac{\exp(im\alpha)J_0\{\kappa[r^2 + (r')^2 - 2rr'\cos\alpha]^{1/2}\}}{\kappa^2 - (\omega^2 - k_z^2)},$$

where $\kappa^2 \equiv k_x^2 + k_y^2$. Because ω has a positive imaginary part, this solution corresponds to the retarded field. Also because $\mathrm{Im}(\omega) > 0$, the κ-integration can be done by a contour integration as discussed in [16] which gives

$$\delta\Phi_{\omega m k_z}(r) = 2\pi^2 i \int_0^\infty r'dr' J_m(kr_<)H_m^{(1)}(kr_>)\delta\rho_{\omega m k_z}(r'), \tag{A4}$$

where $k \equiv (\omega^2 - k_z^2)^{1/2}$, where $r_<$ $(r_>)$ is the lesser (greater) of (r, r'), and where $H_m^{(1)}(x) = J_m(x) + iY_m(x)$ is the Hankel function of the first kind. Because $(\delta A_\phi, \delta J_\phi) = -(\delta A_x, \delta J_x)\cos\phi + (\delta A_y, \delta J_y)\sin\phi$, we have instead of Eq. (A4),

$$\delta\Psi^{\omega m k_z}(r) = r\delta A_\phi^{\omega m k_z} = \pi^2 ri \int_0^\infty r'dr' \delta J_\phi^{\omega m k_z}(r')[\cdots], \tag{A5}$$

where

$$[\cdots] \equiv J_{m+1}(kr_<)H_{m+1}^{(1)}(kr_>) + J_{m-1}(kr_<)H_{m-1}^{(1)}(kr_>).$$

For $m \gg 1$, one can show that in this equation $[J_{m+1}H_{m+1}^{(1)} + J_{m-1}H_{m-1}^{(1)}]/2 = J_m H_m^{(1)}$ to a good approximation. Equations (A4) and (A5) are useful in subsequent calculations.

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