Adiabatic invariance of spin-orbit motion in accelerators

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It has been predicted and found experimentally that the polarization direction of particles on the closed orbit of a circular accelerator can be manipulated, without a noticeable reduction of polarization, by means of a slow variation of magnetic fields. This feature has been used to avoid imperfection resonances where the spin precession frequency is close to a multiple of the circulation frequency. As a first step we show that this property is related to an adiabatic invariant of spin motion. The proof is relatively simple since it involves only two frequencies, the spin-rotation frequency and the particle's rotation frequency on the closed orbit. The invariant spin field (ISF) describes a periodic polarization state of a beam's phase-space distribution. This ISF leads to a very useful parametrization of coupled spin and orbit dynamics. We prove that this ISF gives rise to an adiabatic invariant of spin-orbit motion. This proof is much more complicated since the orbital frequencies are involved. Because of this adiabatic invariance, a beam's spin field follows slow changes of the accelerator's ISF that can occur during a slow acceleration cycle. This feature is essential when high-order spin-orbit resonances are crossed, since it allows polarization that has been reduced at the resonance condition to be recovered, to a large degree, after the resonances have been crossed.

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I. INTRODUCTION

In order to maximize the number of collisions of particles inside experimental detectors of a storage ring system, one tries to maximize the total number of particles in the "bunches" and minimize the emittances, so that the particle distribution across phase space is narrow and the phase-space density is high. In addition, if the beam is spin polarized, high polarization is needed and it should be relatively stable over time. But to obtain high polarization levels at useful energies, particles must first be accelerated from low energy while retaining most of their initial polarization. It is for this reason that the concept of adiabatic invariance is important for spin-orbit motion in accelerators.

The spin $\vec{S}$ of a particle moving through the magnetic fields of a circular accelerator rotates according to the Thomas-BMT equation [1,2] along its phase-space trajectory $\vec{z}(\theta)$; i.e., $\dot{\vec{S}} = \vec{\Omega}(\vec{z}(\theta), \theta) \times \vec{S}$, where $\vec{\Omega}(\vec{z}, \theta)$ is 2π periodic in the azimuth θ, and the dot denotes the derivative with respect to θ. After a particle has traveled one turn along the closed orbit, from θ_0 to $\theta_0 + 2\pi$, the spin rotates around some unit rotation axis $\vec{n}_0(\theta_0)$ by a rotation angle $2\pi\nu_0$, where ν_0 is the so-called closed-orbit spin tune. In a flat accelerator without field errors, the closed orbit is in a horizontal plane and passes only through vertical fields. Thus $\vec{n}_0$ is vertical and independent of θ_0 and $\nu_0 = G\gamma$ (with anomalous gyromagnetic factor G and relativistic factor γ), which causes the number of spin rotations to increase with energy. When ν_0 is close to an integer, a case referred to as an imperfection resonance, the rotation matrix is close to the identity and spin directions hardly change from turn to turn. If field errors are now introduced, they can dominate the rotation direction and can rotate spins away from the vertical. Therefore when ν_0 crosses an integer value during acceleration, the rotation vector $\vec{n}_0$ can change significantly. When the spin rotation is much faster than this rotation vector change, spins which are nearly parallel to $\vec{n}_0$ are dragged along with the evolving $\vec{n}_0$ [3]. To illustrate this fact, one can imagine that $\vec{n}_0$ changes away from the spin sometimes and toward the spin at other times while the spin rotates around the slowly changing $\vec{n}_0$. Because of this rapid rotation, both cases occur frequently and the total effect averages out. This causes the spin to follow the slow change of $\vec{n}_0$, and the projection of spin on $\vec{n}_0$ hardly changes. It has therefore been conjectured that $\vec{S}(\theta) \cdot \vec{n}_0(\theta)$ is an adiabatic invariant. In [4,5] a proof of this statement was discussed under a restrictive assumption using a two-frequency averaging theorem of Neistadt (also often spelled Neishtadt). In this paper, we remove the restriction, give more detailed mathematical arguments, and discuss the underlying physics to which our results apply.

We now introduce the notion of spin fields in a precise way. The coupled spin-orbit equations may be written

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$$\dot{\vec{S}} = \mathbf{A}(\theta, \vec{z})\vec{S}, \qquad \vec{S}(\theta_0) = \vec{S}_0, \tag{1}$$

$$\dot{\vec{z}} = \vec{F}(\vec{z}, \theta), \qquad \vec{z}(\theta_0) = \vec{z}_0, \tag{2}$$

where $\vec{z} \in \mathbb{R}^6$ and $\mathbf{A}$ is skew-symmetric with $A_{12} = \Omega_3$, $A_{13} = -\Omega_2$, $A_{23} = \Omega_1$, and $\mathbf{A}(\cdot, \vec{z})$ is 2π periodic. Denoting the general solution of Eq. (2) by the transport map $\vec{z} = \vec{M}(\theta; \theta_0, \vec{z}_0)$ allows us to write Eq. (1) as $\dot{\vec{S}} = \mathbf{A}(\theta, \vec{M}(\theta; \theta_0, \vec{z}_0))\vec{S}$. If we let $\mathbf{R}(\theta; \theta_0, \vec{z}_0)$ be the associated principal solution matrix, i.e., the matrix solution of

$$D_1\mathbf{R}(\theta; \theta_0, \vec{z}_0) = \mathbf{A}(\theta, \vec{M}(\theta; \theta_0, \vec{z}_0))\mathbf{R}(\theta; \theta_0, \vec{z}_0), \tag{3}$$

$$\mathbf{R}\,(\theta_0; \theta_0, \vec{z}_0) = I, \tag{4}$$

then the solution of the system comprising Eqs. (1) and (2) may be written

$$\vec{S}(\theta) = \mathbf{R}(\theta; \theta_0, z_0)\vec{S}_0, \qquad \vec{z} = \vec{M}(\theta; \theta_0, \vec{z}_0). \tag{5}$$

We use the notation D_k to denote the derivative with respect to the kth argument, be it scalar or vector.

A spin field $\vec{f}(\vec{z}, \theta)$ describes the polarization direction for each phase-space point of a beam and has $||\vec{f}|| = 1$. Now consider $\vec{f}(\vec{z}_0, \theta_0)$; this spin vector starts at θ_0 and becomes $\mathbf{R}(\theta; \theta_0, \vec{z}_0)\vec{f}(\vec{z}_0, \theta_0)$ at θ, but the particle is now at $\vec{z} = \vec{M}(\theta; \theta_0, \vec{z}_0)$, so $\vec{f}(\vec{M}(\theta; \theta_0, \vec{z}_0), \theta) = \mathbf{R}(\theta; \theta_0, \vec{z}_0)\vec{f}(\vec{z}_0, \theta_0)$ and the basic law for the evolution of spin fields is

$$\vec{f}(\vec{z}, \theta) = \mathbf{R}(\theta; \theta_0, \vec{M}(\theta_0; \theta, \vec{z}))\vec{f}(\vec{M}(\theta_0; \theta, \vec{z}), \theta_0). \tag{6}$$

If $\vec{f}(\vec{z}, \theta)$ is such that $\vec{f}(\vec{z}, \theta + 2\pi) = \vec{f}(\vec{z}, \theta)$, then we call $\vec{f}$ an invariant spin field (ISF) and denote it by $\vec{n}(\vec{z}, \theta)$. In a slight abuse of notation we define $\vec{n}(\vec{z}) := \vec{n}(\vec{z}, \theta_0)$, $\vec{M}(\vec{z}_0) := \vec{M}(\theta_0 + 2\pi; \theta_0, \vec{z}_0)$, and $\mathbf{R}(\vec{z}_0) := \mathbf{R}(\theta_0 + 2\pi; \theta_0, \vec{z}_0)$, so that the basic equation for the ISF at θ_0 is

$$\vec{n}(\vec{M}(\vec{z})) = \mathbf{R}(\vec{z})\vec{n}(\vec{z}). \tag{7}$$

General conditions on $\vec{M}$ and $\mathbf{R}$ for the existence of $\vec{n}$ is an unsolved and difficult mathematical problem, although some progress has been made [6,7]. However, there is good experimental evidence that—at the very least—an approximate $\vec{n}$ exists in real machines.

The ISF was first introduced by Derbenev and Kondratenko [8] in the theory of radiative electron polarization and is often called the Derbenev-Kondratenko $\vec{n}$ axis. Note that $\vec{n}(\vec{z})$ is usually not an eigenvector of the spin transport matrix $\mathbf{R}(\vec{z})$ at a given phase-space point, since a particle's spin changes after one turn around the ring but the eigenvector does not change when it is transported by $\mathbf{R}(\vec{z})$. Since the spin vector of a particle at phase-space position $\vec{z}$ is transported by the same rotation matrix, the angle between $\vec{S}$ and $\vec{n}$ does not change and $I(\vec{S}, \vec{z}, \theta) = \vec{S} \cdot \vec{n}(\vec{z}, \theta)$ is an invariant of spin-orbit motion. Therefore spins rotate around $\vec{n}(\vec{z}(\theta), \theta)$ while they travel around an accelerator. Once orthogonal unit vectors $\vec{u}_1(\vec{z}, \theta)$ and $\vec{u}_2(\vec{z}, \theta)$ that are perpendicular to $\vec{n}(\vec{z}, \theta)$ and 2π periodic in θ have been defined, then the rotation rate around $\vec{n}$ can be defined. This rate is referred to as the spin tune ν and depends on the amplitude of a particle's oscillation around the closed orbit.

The guide fields in storage rings are produced by dipole and quadrupole magnets. The dipole fields constrain the particles to almost circular orbits and the quadrupole fields focus the beam, thus ensuring that particles do not drift too far from the central orbit. In these fields, spins precess according to the T-BMT equation.

In horizontally bending dipoles, spins precess only around the vertical field direction. The quadrupoles have vertical and horizontal fields which cause the spins to precess away from the vertical direction. The strength of the spin precession and the precession axis in machine magnets depend on the trajectory and the energy of the particle. Thus in one turn around the ring, the effective precession axis can deviate from the vertical and can strongly depend on the initial position of the particle in the 6-dimensional phase space. From this it is clear that if an invariant spin field $\vec{n}(\vec{z})$ exists, it can vary across the orbital phase space.

At very high energy, as, for example, in the Hadron Electron Ring Accelerator (HERA) proton ring [9–13], for particles with realistic phase-space amplitudes, $\vec{n}(\vec{z})$ may deviate by tens of degrees from the beam average $\langle\vec{n}\rangle$ at azimuth θ_0. Thus even if each point in phase space were 100% polarized parallel to $\vec{n}(\vec{z})$, the average beam polarization $|\langle\vec{n}\rangle|$ might be much smaller than 100%. This was first pointed out in [14] for the Superconducting Super Collider (SSC) and in [15] for HERA-p. Clearly it is very important to have accurate and efficient methods for calculating approximate $\vec{n}(\vec{z})$ and for ensuring that the spread of $\vec{n}(\vec{z})$ is as small as possible.

However, although it is straightforward to define $\vec{n}(\vec{z})$, it is not easy to calculate this spin field in general. Much effort has been expended in this direction [16–23], but mainly for electrons at energies up to 46 GeV. All algorithms developed before the polarized proton project at HERA-p rely on perturbation methods at some stage, and either do not go to high enough order [23–25], or have problems with convergence at high order and high proton energies [19,26].

The system in Eq. (2) can be viewed as a Hamiltonian system, with Hamiltonian $H_{\text{tot}}(\vec{\phi}, \vec{I}, \theta) = H_0(\vec{I}) + \varepsilon H(\vec{\phi}, \vec{I}, \theta)$. The generalized positions are the components of $\vec{\phi}$ and the generalized momenta are those of $\vec{I}$, i.e. $\vec{z} = (\vec{\phi}, \vec{I})$. For $\varepsilon = 0$ the Hamiltonian is $H_0(\vec{I})$ and the momenta $\vec{I}$ are invariants of motion or action variables of this integrable system. For small ε the Hamiltonian is said to be nearly integrable. The perturbation $\varepsilon H(\vec{\phi}, \vec{I}, \theta)$ of the integrable Hamiltonian is chosen to have zero mean with respect to each component ϕ_i, and thus the motion is roughly quasiperiodic with tunes $Q_j(\vec{I}) = \partial_{I_j} H_0(\vec{I})$. Since the particles oscillate with the orbital tunes around the closed orbit, the spin-rotation vector $\vec{\Omega}(\vec{z}(\theta), \theta)$ is modulated by these frequencies. Therefore the spin motion can be strongly disturbed when the spin tune ν is in approximate resonance with the orbit tunes, i.e., when

$$\nu \approx j_0 P_s + j_1 Q_1 + j_2 Q_2 + j_3 Q_3, \qquad P_s, j_n, \in \mathbb{N}, \quad (8)$$

where P_s is the super period of the ring accelerator. These conditions are referred to as intrinsic resonances and can be of very high order [27,28], i.e. $\sum_{k=1}^{3} |j_k| \gg 1$. Therefore the ISF can change significantly while spin-orbit resonance conditions are crossed during beam acceleration. Similarly to $\vec{S} \cdot \vec{n}_0$, it can be conjectured that $\vec{S} \cdot \vec{n}(\vec{z}, \theta)$ is an adiabatic invariant. In [4] a proof of this conjecture was presented under a restrictive assumption using a general theorem of Neistadt. Here we remove the restriction and give a more complete argument. We are also working on a more direct proof which we hope will bypass the Neistadt theorem, give a stronger result, and provide more insight into the spin dynamics.

For the definition of adiabatic invariants we use [29], Sec. 8.1.

Definition: adiabatic invariants.—Consider $\frac{d}{d\theta}\vec{x} = \vec{g}(\vec{x}, \tau)$ with $\tau = \varepsilon\theta$ and $\vec{x} \in \mathbb{R}^n$ for a small parameter ε so that $\vec{g}$ is a slowly varying vector field. A function $\vec{A}(\vec{x}, \tau)$ is said to be an adiabatic invariant of this system if its variation on the interval $\theta \in [0, 1/\varepsilon]$ (i.e., $\tau \in [0, 1]$) is small together with ε, except perhaps for a set of initial conditions whose (Lebesgue) measure goes to zero with ε. That is, for "most" initial conditions the following limit of the supremum over the interval $[0, 1/\varepsilon]$ holds:

$$\limsup_{\varepsilon \to 0} {}_{\theta \in (0, 1/\varepsilon)} |\vec{A}(\vec{x}(\theta), \varepsilon\theta) - \vec{A}(\vec{x}(0), 0)| = 0. \quad (9)$$

In this paper we first prove the adiabatic invariance of $\vec{S} \cdot \vec{n}_0$ on the closed orbit. Certain properties of the ISF are then derived, and finally the adiabatic invariance of $\vec{S} \cdot \vec{n}(\vec{z}, \theta)$ is proved.

It is well known and it will be shown in Sec. IV C that the action variables of a Hamiltonian are adiabatic invariants. One could therefore try to find a Hamiltonian for spin-orbit motion for which $\vec{S} \cdot \vec{n}_0$ on the closed orbit is an action variable, or one for which $\vec{S} \cdot \vec{n}(\vec{z}, \theta)$ is an action variable. However, there is no compelling need for working in a Hamiltonian framework. We will rather work with the equations of spin-orbit motion and try to analyze them by the most appropriate tools available, and these tools do not have to be based on a Hamiltonian.

II. THE ADIABATIC SPIN INVARIANT ON THE CLOSED ORBIT

A. The equation of spin motion on the closed orbit

As a first step for finding an adiabatic invariant of spin motion on the closed orbit we bring the equation of spin motion into an appropriate form.

On the closed orbit $\vec{z}(\theta + 2\pi) = \vec{z}(\theta)$ and thus $\mathbf{A}_0(\theta) := \mathbf{A}(\theta, \vec{z}(\theta))$ is 2π periodic, so Eq. (1) becomes

$$\dot{\vec{S}} = \mathbf{A}_0(\theta)\vec{S} = \vec{\Omega}_0(\theta) \times \vec{S}, \quad (10)$$

$$\mathbf{A}_0(\theta + 2\pi) = \mathbf{A}_0(\theta), \qquad \mathbf{A}_0^T = -\mathbf{A}_0, \quad (11)$$

and by the Floquet theorem there exists a 2π-periodic solution $\vec{n}_0(\theta)$. Physically, a particle which travels along the closed orbit with its spin initially parallel to $\vec{n}_0$ at θ_0 satisfies $\vec{S}(\theta) = \vec{n}_0(\theta)$. Thus for this particle not only the orbit but also the spin motion is 2π periodic. The rotation axis, $\vec{n}_0(\theta)$, of the one-turn spin transport matrix is therefore sometimes called the spin closed orbit [30]. We note that rings are often designed so that $\vec{n}_0$ is not everywhere vertical, for example, in rings where spin rotators are used to provide longitudinal polarization for experiments.

It was conjectured in Sec. I that spins which are nearly parallel to $\vec{n}_0$ will follow slow changes of $\vec{n}_0$ at θ_0 whenever a parameter is slowly varied. We will prove this by showing that $\vec{S}(\theta) \cdot \vec{n}_0(\theta)$, the component of an arbitrary spin vector $\vec{S}(\theta)$ along $\vec{n}_0(\theta)$, is an adiabatic invariant. To do this, it is convenient to introduce a coordinate system with $\vec{n}_0(\theta)$ as one of its coordinate vectors and in which the spin motion on the closed orbit is as simple as possible. Two unit vectors $\vec{m}_0(\theta)$ and $\vec{l}_0(\theta)$ are now chosen which at θ_0 make up the right-hand coordinate system $(\vec{m}_0(\theta_0), \vec{l}_0(\theta_0), \vec{n}_0(\theta_0))$ and propagate around the ring according to the T-BMT equation on the closed orbit,

$$\dot{\vec{m}}_0 = \vec{\Omega}_0(\theta) \times \vec{m}_0, \qquad \dot{\vec{l}}_0 = \vec{\Omega}_0(\theta) \times \vec{l}_0. \quad (12)$$

The three unit vectors will always constitute a right-hand coordinate system, since all three are rotated by the same precession equation. Whereas $\vec{n}_0$ is periodic around the ring, the vectors $\vec{m}_0$ and $\vec{l}_0$ rotate around $\vec{n}_0$ by the angle $2\pi\nu_0$ after one turn, and the unit vectors are therefore in general not 2π periodic in θ. Here, ν_0 is the nontrivial positive Floquet exponent associated with Eq. (10). Now a 2π-periodic coordinate system is defined by rotating $\vec{m}_0$ and $\vec{l}_0$ uniformly by $2\pi\nu_0$ during one turn [24,25,31]; thus

$$\vec{m} + i\vec{l} := e^{i\nu_0\theta}(\vec{m}_0 + i\vec{l}_0), \quad (13)$$

and Eq. (12) becomes

$$\dot{\vec{m}} = (\vec{\Omega}_0 - \nu_0\vec{n}_0) \times \vec{m}, \qquad \dot{\vec{l}} = (\vec{\Omega}_0 - \nu_0\vec{n}_0) \times \vec{l}. \quad (14)$$

In this coordinate system, the spin is written as

$$\vec{S}(\theta) = s_1(\theta)\vec{m}(\theta) + s_2(\theta)\vec{l}(\theta) + s_3(\theta)\vec{n}_0(\theta), \tag{15}$$

with $s_1^2 + s_2^2 + s_3^2 = 1$. From the equation of spin motion we have $\vec{\Omega}_0 \times \vec{S} = \dot{\vec{S}} = \dot{s}_1\vec{m} + \dot{s}_2\vec{l} + \dot{s}_3\vec{n}_0 + s_1(\vec{\Omega}_0 - \nu_0\vec{n}_0) \times \vec{m} + s_2(\vec{\Omega}_0 - \nu_0\vec{n}_0) \times \vec{l} + s_3\vec{\Omega}_0 \times \vec{n}_0$ which gives $\dot{s}_1\vec{m} + \dot{s}_2\vec{l} + \dot{s}_3\vec{n}_0 = \nu_0(\vec{n}_0 \times \vec{S})$. Thus we obtain $\dot{s}_1 = -\nu_0 s_2$, $\dot{s}_2 = -\nu_0 s_1$, $\dot{s}_3 = 0$, which can be written compactly as

$$\dot{\vec{s}} = \nu_0\mathbf{J}\vec{s}, \quad \text{and so} \quad \vec{s}(\theta) = \exp(\mathbf{J}\nu_0\theta)\vec{s}(0). \tag{16}$$

Here,

$$\mathbf{J} = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \exp(\mathbf{J}t) = \begin{pmatrix} \cos t & -\sin t & 0 \\ \sin t & \cos t & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{17}$$

Since s_3 is constant, Eq. (15) describes a uniform rotation around $\vec{n}_0$ and we expect it to be an adiabatic invariant when parameters of the system are slowly varied. Equation (15) can be written $\vec{S} = \mathbf{U}(\theta)\vec{s}$ and thus the solution of Eq. (10) is

$$\vec{S}(\theta) = \mathbf{R}_0(\theta)\vec{S}(0), \tag{18}$$

$$\mathbf{R}_0(\theta) = \mathbf{U}(\theta)\exp(\mathbf{J}\nu_0\theta)\mathbf{U}(0)^T, \qquad \mathbf{R}_0(0) = \mathbf{I}. \tag{19}$$

Although this equation was derived by a physical argument, it is also a consequence of the Floquet theorem which states that the principal solution matrix for Eq. (10) can be written as in Eq. (19), where $\mathbf{U}$ is a 2π-periodic SO(3) matrix. Since $\dot{\mathbf{R}}_0 = \dot{\mathbf{U}}\mathbf{U}^T\mathbf{R}_0 + \mathbf{U}\mathbf{J}\nu_0\mathbf{U}^T\mathbf{R}_0 = \mathbf{A}_0\mathbf{R}_0$ implies

$$\dot{\mathbf{U}} + \nu_0\mathbf{U}\mathbf{J} = \mathbf{A}_0\mathbf{U}, \tag{20}$$

the Floquet transformation $\vec{S} \mapsto \vec{s}$ defined by $\vec{S} = \mathbf{U}(\theta)\vec{s}$ yields $\dot{\vec{S}} = \dot{\mathbf{U}}\,\vec{s} + \mathbf{U}\dot{\vec{s}} = (\mathbf{A}_0\mathbf{U} - \nu_0\mathbf{U}\mathbf{J})\vec{s} + \mathbf{U}\dot{\vec{s}} = \mathbf{A}_0\mathbf{U}\vec{s}$, and so

$$\dot{\vec{s}} = \nu_0\mathbf{J}\vec{s}, \tag{21}$$

which agrees with Eq. (16).

Now suppose that $\mathbf{A}_0$ depends on a slowly varying quantity, τ,

$$\mathbf{A}_0 = \mathbf{A}_0(\theta, \tau), \qquad \tau = \varepsilon\theta, \tag{22}$$

where ε is small and positive. For fixed τ, we can write Eq. (19) as

$$\mathbf{R}_0(\theta, \tau) = \mathbf{U}(\theta, \tau)\exp\{\mathbf{J}\nu_0(\tau)\theta\}\mathbf{U}(0, \tau)^T. \tag{23}$$

Transforming $\vec{S} \mapsto \vec{s}$ via

$$\vec{S} = \mathbf{U}(\theta, \varepsilon\theta)\vec{s} \tag{24}$$

as before, we obtain

$$\begin{aligned}\dot{\vec{S}} &= (D_1\mathbf{U} + D_2\mathbf{U}\varepsilon)\vec{s} + \mathbf{U}\dot{\vec{s}} \\ &= (\mathbf{A}_0\mathbf{U} - \nu_0(\varepsilon\theta)\mathbf{U}\mathbf{J} + D_2\mathbf{U}\varepsilon)\vec{s} + \mathbf{U}\dot{\vec{s}} = \mathbf{A}_0\mathbf{U}\vec{s}.\end{aligned} \tag{25}$$

Therefore $\mathbf{U}\dot{\vec{s}} = (\nu_0\mathbf{U}\mathbf{J} - D_2\mathbf{U}\varepsilon)\vec{s}$ and we obtain

$$\dot{\vec{s}} = \nu_0(\tau)\mathbf{J}\vec{s} - \varepsilon\mathbf{U}^T(\theta, \tau)D_2\mathbf{U}(\theta, \tau)\vec{s}, \qquad \dot{\tau} = \varepsilon. \tag{26}$$

To see that $\mathbf{U}^T(\theta, \tau)D_2\mathbf{U}(\theta, \tau)$ is skew symmetric, we write $\mathbf{U}^T(\theta, \tau)\mathbf{U}(\theta, \tau) = 1$, which implies $(D_2\mathbf{U})^T\mathbf{U} + \mathbf{U}^T D_2\mathbf{U} = 0$, so $(\mathbf{U}^T D_2\mathbf{U})^T = (D_2\mathbf{U})^T\mathbf{U} = -\mathbf{U}^T D_2\mathbf{U}$. Thus we define $\vec{\eta}(\theta, \tau) = (\eta_1, \eta_2, \eta_3)^T$ via

$$\mathbf{U}^T(\theta, \tau)D_2\mathbf{U}(\theta, \tau) =: \begin{pmatrix} 0 & -\eta_3 & \eta_2 \\ \eta_3 & 0 & -\eta_1 \\ -\eta_2 & \eta_1 & 0 \end{pmatrix}. \tag{27}$$

Clearly $\vec{\eta}(\theta, \tau)$ is 2π periodic in θ. The skew symmetry of the matrix means that the basis $(\vec{m}, \vec{l}, \vec{n}_0)$ remains orthonormal for all values of τ and Eq. (27) can be written

$$\begin{aligned}&\partial_\tau\vec{n}_0 = \vec{\eta}(\theta, \tau) \times \vec{n}_0, \\ &\partial_\tau\vec{m} = \vec{\eta}(\theta, \tau) \times \vec{m}, \; \partial_\tau\vec{l} = \vec{\eta}(\theta, \tau) \times \vec{l},\end{aligned} \tag{28}$$

so that the variation of the coordinate vectors with τ is a rotation about $\vec{\eta}$. Our basic equations of motion now become

$$\begin{aligned}\dot{s}_1 &= -\nu_0(\tau)s_2 + \varepsilon[\eta_3(\theta, \tau)s_2 - \eta_2(\theta, \tau)s_3], \\ \dot{s}_2 &= \nu_0(\tau)s_1 + \varepsilon[-\eta_3(\theta, \tau)s_1 + \eta_1(\theta, \tau)s_3], \\ \dot{s}_3 &= \varepsilon[\eta_2(\theta, \tau)s_1 - \eta_1(\theta, \tau)s_2], \qquad \dot{\tau} = \varepsilon.\end{aligned} \tag{29}$$

We now put these in a standard form for averaging of autonomous systems with three slow variables and two fast variables.

Let $\vec{y} = (s_1, s_2)^T$ and $\vec{\zeta}(\theta, \tau) = (\eta_1, \eta_2)^T$; then

$$\begin{aligned}\dot{\vec{y}} &= [\nu_0(\tau) - \varepsilon\eta_3(\theta, \tau)]\mathbf{J}_2\vec{y} + \varepsilon\mathbf{J}_2\vec{\zeta}(\theta, \tau)s_3, \\ \dot{s}_3 &= \varepsilon\vec{\zeta}^T(\theta, \tau)\mathbf{J}_2\vec{y}, \qquad \dot{\tau} = \varepsilon, \\ \dot{\psi} &= \nu_0(\tau) - \varepsilon\eta_3(\theta, \tau),\end{aligned} \tag{30}$$

where

$$\mathbf{J}_2 := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \tag{31}$$

and where the equation for ψ was introduced for convenience. When $s_3 = 0$, the first of the four equations (30) describes a rotation of $\vec{y}$ by the angle ψ. Our final transformation eliminates this rotation by $\vec{y} \mapsto \vec{x}$ via

$$\vec{y} = \exp\left\{\mathbf{J}_2\int_0^\theta[\nu_0(\varepsilon\theta') - \varepsilon\eta_3(\theta', \varepsilon\theta')]d\theta'\right\}\vec{x}, \tag{32}$$

and our final set of equations becomes

$$\begin{aligned}\dot{\vec{x}} &= \varepsilon \exp(-\mathbf{J}_2\psi)\mathbf{J}_2\vec{\zeta}(\tilde{\theta},\tau)s_3,\\ \dot{s_3} &= \varepsilon\vec{\zeta}^T(\tilde{\theta},\tau)\exp(\mathbf{J}_2\psi)\mathbf{J}_2\vec{x}, \qquad \dot{\tau}=\varepsilon,\\ \dot{\psi} &= \nu_0(\tau)-\varepsilon\eta_3(\tilde{\theta},\tau), \qquad \dot{\tilde{\theta}}=1,\end{aligned} \tag{33}$$

where $\tilde{\theta}$ has been introduced to make the system autonomous.

B. Averaging for two-frequency systems

Equations (33) are in a standard form for the method of averaging and we now state the appropriate averaging theorem. We state a theorem for systems with two fast variables which allows for crossing of resonances (and we apply it in the next subsection to spin motion on the closed orbit). Various multiphase averaging theorems could be used [29], Chaps. 4–6, [32]; here theorem 3 of [29], Section 4.1 is used, which is attributed to [33]. The application of two-phase averaging to the simple problem of spin motion on the closed orbit might seem more complicated than necessary, but by going into considerable detail here while dealing with the closed orbit, the stage is set for adiabatic invariants in the case of spin motion on a general trajectory, treated later below.

Theorem.—Consider a system of the form

$$\frac{d}{d\theta}\vec{I} = \varepsilon\vec{f}(\tilde{\theta},\psi,\vec{I},\varepsilon), \tag{34}$$

$$\frac{d}{d\theta}\psi = \nu(\vec{I}) + \varepsilon g(\tilde{\theta},\psi,\vec{I},\varepsilon), \tag{35}$$

$$\frac{d}{d\theta}\tilde{\theta} = 1, \tag{36}$$

where $\vec{I}$ belongs to a regular compact subset of Euclidean $\mathbb{R}^m$. Each function on the right-hand side is real valued, C^1 (first-order differentials exist and are continuous) in $\vec{I}$ and ε, periodic with period 2π in ψ and $\tilde{\theta}$, and each possesses an analytic extension for $\psi \in \mathbb{C}$, with imaginary part $\Im\{\psi\} < \sigma$ and $\tilde{\theta} \in \mathbb{C}$, $\Im\{\tilde{\theta}\} < \sigma$ for some $\sigma > 0$. The associated (slow) averaged system is

$$\begin{aligned}\frac{d}{d\theta}\bar{\vec{I}} &= \varepsilon\bar{\vec{f}}(\bar{\vec{I}}),\\ \bar{\vec{f}}(\bar{\vec{I}}) &= \frac{1}{(2\pi)^2}\int_0^{2\pi}\int_0^{2\pi}\vec{f}(\tilde{\theta},\psi,\bar{\vec{I}},0)d\psi d\tilde{\theta},\end{aligned} \tag{37}$$

with initial condition $\bar{\vec{I}}(0) = \vec{I}(0)$. Let every trajectory of the exact system for which $\vec{I}$ stays in the range of definition for $\theta \in [0, 1/\varepsilon]$ have a strictly monotonic variation of $\nu(\vec{I})$ with θ: $|\frac{d}{d\theta}\nu| > c_1\varepsilon$ for some $c_1 \in \mathbb{R}^+$. Then there exists a positive constant c such that for sufficiently small ε, all trajectories satisfy

$$\sup_{\theta\in[0,1/\varepsilon]}|\vec{I}(\theta) - \bar{\vec{I}}(\theta)| < c\sqrt{\varepsilon}. \tag{38}$$

In the absence of resonance, the error estimate in Eq. (38) may often be improved [in some cases, up to $O(\varepsilon)$]. The $O(\sqrt{\varepsilon})$ estimate above is however typical of *passage through* resonance. At the other extreme, *remaining at* resonance typically leads to a complete decorrelation between behavior of the exact and averaged systems [i.e., the error estimate deteriorates to $O(1)$]. We give two simple but explicit examples to illustrate the latter cases.

A basic model of passage through (a single) resonance is the system $\dot{s}_3 = \varepsilon\cos(\psi - m\theta)$ and $\dot{\psi} = m - 1/2 + \varepsilon\theta$ (m an integer), since it begins well away from the resonance m and reaches it halfway through the interval $[0, 1/\varepsilon]$. The exact solution is $s_3(\theta) = s_3(0) + \varepsilon\int_0^\theta \cos((\varepsilon t^2 - t)/2 + \psi(0))dt$. Using a stationary phase argument, it is easy to show that this deviates from the solution of the averaged system $\bar{s}_3(\theta) = s_3(0)$ by $O(\sqrt{\varepsilon})$ at $O(1/\varepsilon)$ times.

An example of a closely related system which *remains* at resonance is $\dot{s}_3 = \varepsilon\cos(\psi - m\theta)$ and $\dot{\psi} = \nu_0$. The averaged system has the constant solution $\bar{s}_3 = s_3(0)$, whereas the solution of the exact system for $\nu_0 = m$ (an integer) is $s_3 = s_3(0) + \varepsilon\theta\cos(\psi(0))$. The change in s_3 for $\theta \in [0, 1/\varepsilon]$ is clearly $O(1)$, and s_3 is therefore not an adiabatic invariant.

The last example shows how large changes in the slow variables can build up at resonances. The simplest way of avoiding this behavior is to consider only systems in which resonances are traversed quickly so that capture into resonance is avoided. This is the reason why the averaging theorem for two-frequency systems requires $|\frac{d}{d\theta}\nu| > c_1\varepsilon$, which is often called condition A (the terminology derives from the Russian literature). This condition excludes systems where trajectories pass arbitrarily slowly through a resonance or cross the same resonance several times.

For a detailed proof of the two-phase averaging theorem used above, see [29], Sec. 4.1, or [33]. We end this subsection with a brief discussion of the ingredients in that proof; we hope this gives an idea of the methods used and the obstacles to be overcome along the way.

For simplicity we consider the third component of Eq. (34) [the equation governing s_3 in Eq. (33)]. The right-hand side $f_3(\theta, \psi, \vec{x}, \tau)$ is a 2π-periodic function of θ and ψ, with zero mean:

$$\bar{f}_3 = \frac{1}{(2\pi)^2}\int_0^{2\pi}\int_0^{2\pi} f_3(\theta,\psi,\vec{x},\tau)d\psi d\theta = 0. \tag{39}$$

In the usual approach to averaging, one tries to simplify the equation for s_3 by transforming to a coordinate $\bar{s}_3 = s_3 + \varepsilon u(\theta, \psi, \vec{x}, \tau)$ where the perturbation εu is periodic in ψ and θ (as f_3 is). Now $\dot{\bar{s}}_3 = \dot{s}_3 + \varepsilon\frac{d}{d\theta}u(\theta,\psi,\vec{x},\tau) = \varepsilon(f_3(\theta,\psi,\vec{x},\tau) + D_1u(\theta,\psi,\vec{x},\tau) + \nu_0(\tau)D_2u(\theta,\psi,\vec{x},\tau)) + O(\varepsilon^2)$. One thus chooses u so that

$$\hat{f} := D_1 u + \nu_0 D_2 u + f_3 \tag{40}$$

is as simple as possible. If $\hat{f}$ can be made to vanish identically, then $\dot{\bar{s}}_3 = O(\varepsilon^2)$, from which adiabatic invariance of s_3 easily follows.

Making $\hat{f}$ vanish (or as simple as possible) is a form of the so-called fundamental (or ''homological'') equation of perturbation theory, and it is relatively easy to solve when there is only a single phase. But when there are two or more phases (in the present case we have the phases ψ and θ), the homological equation is much more difficult to solve in general because of the presence of ''small'' or ''zero divisors.'' We illustrate this difficulty as follows.

We first expand f_3 and u in terms of their Fourier series as

$$f_3 = \sum_{k=-\infty}^{\infty} f_k^+(\vec{x}, \tau) e^{i(k\theta + \psi)} + f_k^-(\vec{x}, \tau) e^{i(k\theta - \psi)}, \tag{41}$$

$$u = \sum_{k=-\infty}^{\infty} u_k^+(\vec{x}, \tau) e^{i(k\theta + \psi)} + u_k^-(\vec{x}, \tau) e^{i(k\theta - \psi)}, \tag{42}$$

where we have taken into account the fact that f_3 contains only the terms $\exp(\pm i\psi)$ and we have written u in the same form since we only want to find a particular solution and not the most general. When these are inserted into the homological equation $\hat{f} = 0$, we get

$$[k \pm \nu_0(\tau)] u_k^\pm(\vec{x}, \tau) = i f_k^\pm(\vec{x}, \tau), \tag{43}$$

and we have a *zero* divisor problem when $\nu_0(\tau)$ takes integer values. If $\nu_0(\tau)$ is not an integer, then one can choose $u_k^\pm(s_3, \tau) = i f_k^\pm/[k \pm \nu_0(\tau)]$ for all k and we obtain $\dot{\bar{s}}_3 = O(\varepsilon^2)$. It follows that $\bar{s}_3 = O(\varepsilon^2 \theta)$, and if arbitrarily *small* divisors can be avoided—i.e., if $\nu_0(\tau)$ remains bounded away from integers (a primitive form of the so-called Diophantine conditions used in more general situations below), then $u = O(1)$, and therefore $s_3 = O(\varepsilon)$ for $0 \le \theta \le O(1/\varepsilon)$.

On the other hand, if $\nu_0(\tau) = m$ (a positive integer) for some τ, the best we can do is take $u_{-m}^+ = u_m^- = 0$, in which case $\hat{f} = 2\mathrm{Re}\{f_m^- e^{i(m\theta - \psi)}\} = 2 f_m^- \cos(m\theta - \psi)$ if f_m^- is real. We thus get $\dot{\bar{s}}_3 = \varepsilon 2 f_m^- \cos(m\theta - \psi) + O(\varepsilon^2)$ and $\dot{\psi} = \nu_0(\tau) + O(\varepsilon)$, a system very similar to the basic model for passage through resonance discussed above (after the statement of the two-phase averaging theorem).

Despite these difficulties, it turns out that resonant values of $\nu_0(\tau)$ can be tolerated provided they occur only over brief time intervals (condition A is of course one way to ensure this). Once the time intervals spent at (or near) resonant values are known to be limited, with some effort it is possible to estimate the cumulative effect of all ''passages through resonance,'' and to conclude that the averaging approximation remains valid over the interval $[0, 1/\varepsilon]$. Finally, we note that in the present application, since the closed-orbit spin tune changes with energy ($\nu_0 = G\gamma$ in a flat ring), the assumption of condition A is likely to be physically realistic in situations where the beam energy is increasing at a minimum rate.

C. Adiabatic invariance of spin motion via two-phase averaging

We now apply the above theorem to obtain our main result of this section. For spin motion on the closed orbit, we take the exact system to be Eq. (33), so that $\vec{x}$, s_3, and τ form the (four) components of $\vec{I}$ in Eq. (34). Since $I_3(\theta) = s_3(\theta)$, the averaged system includes the equation $\frac{d}{d\theta}\bar{I}_3 = \bar{f}_3 = 0$ and leads to $\bar{I}_3(\theta) = \bar{I}_3(0) = s_3(0)$. If $|\frac{d}{d\theta}\nu_0(\varepsilon\theta)| > c_1\varepsilon$, the theorem then guarantees that

$$\lim_{\varepsilon \to 0} \sup_{\theta \in [0, 1/\varepsilon]} |s_3(\theta) - s_3(0)| = 0 \tag{44}$$

which we interpret as adiabatic invariance of s_3 according to our definition at the end of Sec. I.

Several remarks can be made here. We note that the theorem gives more than adiabatic invariance of s_3; it additionally gives an estimate of the *rate* of convergence toward 0 in Eq. (44), so we have some knowledge of how slowly τ must be varied in order to achieve a desired degree of constancy in s_3. The theorem also says that $\vec{x}$ is adiabatically invariant, which, in view of Eq. (32), gives an approximate evolution equation for $\vec{y} = (s_1, s_2)^T$, namely

$$\vec{y} = \exp\left\{\mathbf{J}_2 \int_0^\theta [\nu_0(\varepsilon\theta') - \varepsilon\eta_3(\theta', \varepsilon\theta')] d\theta'\right\} \vec{x}(0). \tag{45}$$

We further note that, although condition A ($|\partial_\tau \nu_0(\tau)| > c_1$) requires the spin tune to change quickly enough to keep ν_0 from remaining very long at integer values, this condition could be dropped altogether if $\nu_0(\tau)$ remains bounded away from integers (one then applies an averaging theorem for nonresonant domains; cf. [32], Sec. 1.6).

III. THE AMPLITUDE-DEPENDENT SPIN TUNE AND THE UNIQUENESS OF $\vec{n}(\vec{z})$

The closed-orbit spin tune ν_0 was introduced as the spin-rotation angle divided by 2π for particles which travel one turn around the closed orbit. For particles which oscillate around the closed orbit, this rotation angle can depend on the amplitude of their oscillation. For the case where the orbit motion can be described in terms of action and angle variables $\vec{J}$ and $\vec{\Phi}$ (as is always the case for stable linear motion), and where the tunes Q_j are not in resonance on the invariant torus described by $\vec{J}$, we will now show how to define a spin-rotation angle which is independent of $\vec{\Phi}$ on that torus. Assuming that an $\vec{n}$ axis exists at an azimuth θ_0, one can introduce two unit vectors $\vec{\tilde{u}}_1(\vec{z})$ and $\vec{\tilde{u}}_2(\vec{z})$ to create a right-hand coordinate system $(\vec{\tilde{u}}_1, \vec{\tilde{u}}_2, \vec{n})$. The vectors $\vec{\tilde{u}}_1$ and $\vec{\tilde{u}}_2$ are therefore defined up to a rotation around the $\vec{n}$ axis by an arbitrary phase-space-dependent angle $\phi(\vec{z})$. The spin direction $\vec{S}$ is expressed in terms of this coordinate system by the relation $\vec{S} = s_1\vec{\tilde{u}}_1 + s_2\vec{\tilde{u}}_2 + J_S\vec{n}$. The coefficient J_S is called the spin action and does not change during the particle motion around the ring, since the particle transport matrix $\mathbf{R}(\vec{z})$ is orthogonal and ensures that $J_S = \vec{S}\cdot\vec{n}$ is invariant. This property can be used to define the invariant function of spin-orbit motion $J_S(\vec{z},\vec{S}) = \vec{S}\cdot\vec{n}(\vec{z})$. Using Eq. (7) the invariance is given by

$$\begin{aligned} J_S(\vec{z}_f,\vec{S}_f) &= J_S(\vec{M}(\vec{z}_i),\mathbf{R}(\vec{z}_i)\vec{S}_i) = (\mathbf{R}(\vec{z}_i)\vec{S}_i)\cdot\vec{n}(\vec{M}(\vec{z}_i)) \\ &= (\mathbf{R}(\vec{z}_i)\vec{S}_i)\cdot(\mathbf{R}(\vec{z}_i)\vec{n}(\vec{z}_i)) = \vec{S}_i\cdot\vec{n}(\vec{z}_i) \\ &= J_S(\vec{z}_i,\vec{S}_i). \end{aligned} \tag{46}$$

For one turn, the spin motion in the coordinate system with $(\vec{\tilde{u}}_1,\vec{\tilde{u}}_2,\vec{n})$ is a rotation around the $\vec{n}$ axis by a phase-space-dependent angle $2\pi\tilde{\nu}(\vec{z})$,

$$s_{f1} + is_{f2} = e^{i2\pi\tilde{\nu}(\vec{z})}(s_{i1} + is_{i2}). \tag{47}$$

By reexpressing this in terms of the complex quantity $\hat{s} = e^{i\phi(\vec{z})}(s_1 + is_2)$, where $\phi(\vec{z})$ is the arbitrary angle of $\vec{\tilde{u}}_1$ and $\vec{\tilde{u}}_2$, one obtains

$$e^{-i\phi(\vec{M}(\vec{z}))}\hat{s}_f = e^{i(2\pi\tilde{\nu}(\vec{z})-\phi(\vec{z}))}\hat{s}_i. \tag{48}$$

The one-turn transport of phase-space motion is described by $\vec{J}_f = \vec{J}_i$ and $\vec{\Phi}_f = \vec{\Phi}_i + 2\pi\vec{Q}$. Using the symbols $2\pi\tilde{\nu}_{\vec{J}}(\vec{\Phi})$ and $\phi_{\vec{J}}(\vec{\Phi})$ to indicate the spin-rotation angle and the free phase of the coordinate system for motion on the invariant torus characterized by $\vec{J}$, we have

$$\hat{s}_f = e^{i(2\pi\tilde{\nu}_{\vec{J}}(\vec{\Phi})-\phi_{\vec{J}}(\vec{\Phi})+\phi_{\vec{J}}(\vec{\Phi}+2\pi\vec{Q}))}\hat{s}_i. \tag{49}$$

The goal of the subsequent manipulation is to choose $\phi_{\vec{J}}(\vec{\Phi})$ so that the spin motion characterized by the exponent is simplified and the rotation angle becomes independent of $\vec{\Phi}$. As with any function of phase space, the rotation $e^{i2\pi\tilde{\nu}_{\vec{J}}(\vec{\Phi})}$ is 2π periodic in all components Φ_j. Therefore the rotation angle can have a 2π-periodic contribution and a contribution linear in the phases

$$2\pi\tilde{\nu}_{\circ\vec{J}}(\vec{\Phi}) + \vec{j}\cdot\vec{\Phi}, \tag{50}$$

for some vector $\vec{j}$ with integer components. We choose the rotation angle

$$2\pi\tilde{\nu}_{\vec{J}}(\vec{\Phi}) = 2\pi\tilde{\nu}_{\circ\vec{J}}(\vec{\Phi}) + \vec{j}\cdot\vec{\Phi}\,\mathrm{mod}\,2\pi \tag{51}$$

which is 2π periodic and can thus be Fourier expanded. The rotation $e^{i\phi_{\vec{J}}(\vec{\Phi})}$ is also 2π periodic in all components Φ_j. Therefore the rotation angle $\phi_{\vec{J}}(\vec{\Phi})$ can also have a 2π-periodic contribution $\phi_{\circ\vec{J}}(\vec{\Phi})$ and a contribution linear in the phases

$$\phi_{\vec{J}}(\vec{\Phi}) = \phi_{\circ\vec{J}}(\vec{\Phi}) + \vec{j}\cdot\vec{\Phi}. \tag{52}$$

If the orbit tunes $\vec{Q}$ are not in resonance, then $\phi_{\vec{J}}(\vec{\Phi})$ can be chosen to eliminate the phase dependence of the exponent in Eq. (49) completely. This can be seen by Fourier transforming the periodic functions $\tilde{\nu}_{\vec{J}}(\vec{\Phi})$ and $\phi_{\circ\vec{J}}(\vec{\Phi})$ to obtain the following exponent in Eq. (49):

$$2\pi\vec{j}\cdot\vec{Q} + \sum_{\vec{k}}(2\pi\check{\nu}_{\vec{J}}(\vec{k}) - \check{\phi}_{\circ\vec{J}}(\vec{k})(1 - e^{i2\pi\vec{k}\cdot\vec{Q}}))e^{i\vec{k}\cdot\vec{\Phi}}. \tag{53}$$

By choosing the Fourier coefficients $\check{\phi}_{\circ\vec{J}}(\vec{k})$ so that $\check{\phi}_{\circ\vec{J}}(\vec{k}) = 2\pi\check{\nu}_{\vec{J}}(\vec{k})/(1 - e^{i2\pi\vec{k}\cdot\vec{Q}})$, one can eliminate all Fourier coefficients $\check{\nu}_{\vec{J}}(\vec{k})$ except those with $\vec{k} = 0$. For this special choice of $\phi(\vec{z})$ the one-turn spin-rotation angle becomes $2\pi\nu(\vec{J}) = 2\pi(\check{\nu}_{\vec{J}}(0) + \vec{j}\cdot\vec{Q})$ and does not depend on $\vec{\Phi}$ but only on the action variables $\vec{J}$. Therefore this rotation angle is the same for all particles on one invariant torus and thus does not change during particle motion. This spin precession rate $\nu(\vec{J})$ is a characteristic of the torus and allows the degree of coherence between spin and orbital motion to be quantified. In particular we expect coherent excitations of spin motion when the amplitude-dependent spin tune $\nu(\vec{J})$ is in resonance with the orbital tunes as in Eq. (8). Other angles which might be alternatively proposed [34,35] do not correlate with resonance effects [36–39].

To guarantee the convergence of the Fourier series of $\phi(\vec{z})$, we require the orbit tunes to be strongly incommensurable with 1 ([32], Sec. 1.5), which implies that they are strongly non–orbit-resonant, defined as follows using the distance to the nearest integer $[\ldots]_d$ and the 1-norm $|\vec{k}|_1 = \sum_{n=1}^{3}|k_n|$:

Strongly non–orbit-resonant.—The particle motion is said to be strongly non–orbit-resonant if $C, r \in \mathbb{R}^+$ exist with $(\vec{k}\cdot\vec{Q}) \geq C|\vec{k}|_1^{-r}d$ for all nonzero vectors $\vec{k}$ with integer components.

This strong incommensurability is a common requirement in perturbation theories, and it is known that for $r > \dim(\vec{k}) - 1$ [here $\dim(\vec{k}) = 3$], the set of $\vec{Q}$ for which there is no such C has Lebesgue measure 0 ([29], Appendix 4, [40,41]). In mathematical settings, the infinitely many inequalities above are called "Diophantine conditions" and may often be reduced to finitely many conditions by so-called ultraviolet cutoff techniques (see [29], Secs. 7.2, 7.4).

The denominator $1 - e^{i2\pi\vec{k}\cdot\vec{Q}}$ then decreases according to a power law: $|1 - e^{i2\pi\vec{k}\cdot\vec{Q}}| = 2|\sin(\pi\vec{k}\cdot\vec{Q})| \geq 4(\vec{k}\cdot\vec{Q}) \geq 4C|\vec{k}|_1^{-r}d$. We further require that the spin-rotation angle $2\pi\tilde{\nu}_{\vec{J}}(\vec{\Phi})$ have an analytic extension and therefore that its Fourier coefficients fall off exponentially with $|\vec{k}|_1$ [29], Appendix 1.1, so as to counterbalance the denominator and lead to a convergent Fourier series. Alternatively one could require sufficient differentiability of $\tilde{\nu}_{\vec{j}}(\vec{\Phi})$, which would lead to a sufficiently strong power law falloff for the $\check{\phi}_{\vec{j}}(\vec{k})$ [29], Appendix 1.2.

The coordinate vectors $\vec{\tilde{u}}_1$ and $\vec{\tilde{u}}_2$ for this special choice of $\phi(\vec{z})$ are referred to as $\vec{u}_1$ and $\vec{u}_2$. The exponent reduces to $\nu(\vec{J}) = \check{\nu}_{\vec{J}}(0) + \vec{j}\cdot\vec{Q}$ and the spin rotation of Eq. (49) simplifies to

$$\hat{s}_f = e^{i2\pi\nu(\vec{J})}\hat{s}_i. \tag{54}$$

The goal of constructing a spin rotation depending only on orbital actions but not on the angle variables $\vec{\Phi}$ has now been achieved. The function $\nu(\vec{J})$ is called the amplitude-dependent spin tune. It is not unique, since one can add an integer j_0 and a linear combination $\vec{j}\cdot\vec{Q}$ of the orbit tunes by choosing different integers for $\vec{j}$ in Eq. (53). It is interesting to note that for action variables $\vec{J}$ where the integer components of $\vec{j}$ can be chosen so that

$$\nu(\vec{J}) + \vec{j}\cdot\vec{Q} = 0 \bmod 1, \tag{55}$$

one can eliminate the spin rotation completely. These are the resonances described in Eq. (8). Several of the subsequent statements will be restricted to cases where this resonance condition is not satisfied. Furthermore, when $-\vec{n}(\vec{z})$ is chosen as the $\vec{n}$ axis for defining the spin tune on a torus, the spin tune ν also changes sign and $-\nu + j_0 + \vec{j}\cdot\vec{Q}$ could alternatively be chosen as the spin tune. This leads to the following conclusion:

Existence of $\nu(\vec{J})$: Given that an $\vec{n}$ axis exists, that the system is strongly non–orbit-resonant, and that the above-mentioned analytic extension of the spin-rotation angle $2\pi\tilde{\nu}$ exists, then a coordinate system $(\vec{u}_1, \vec{u}_2, \vec{n})$ can be specified which defines an amplitude-dependent spin tune $\nu(\vec{J})$. This choice is not unique, since $\pm\nu + j_0 + \vec{j}\cdot\vec{Q}$ can also be chosen as the spin tune.

Usually the integers are chosen so that the limit for small amplitudes is equal to the closed-orbit spin tune ν_0, which is $G\gamma$ for an unperturbed flat ring without solenoids, spin rotators, or snakes.

It is worth noting that for the single resonance model (SRM) of spin motion the $\vec{n}$ axis can be computed and $\nu(\vec{J})$ exists even on orbit resonances.

To analyze the uniqueness of the $\vec{n}$ axis, the periodicity condition (7) is written in the coordinate system $(\vec{u}_1, \vec{u}_2, \vec{n})$,

$$\vec{n}(\vec{M}(\vec{z})) = \begin{pmatrix} \cos(2\pi\nu) & -\sin(2\pi\nu) & 0 \\ \sin(2\pi\nu) & \cos(2\pi\nu) & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \vec{n}(\vec{z}), \tag{56}$$

with the obvious solution $\vec{n}(\vec{z}) = (0, 0, 1)^T$ for all $\vec{z}$. If another $\vec{n}$ axis $\vec{n}_2(\vec{z})$ exists, then $\vec{n}_2 - \vec{n}(\vec{n}\cdot\vec{n}_2)$ is nonzero at least at one phase-space point and on all iterations of this point which can be reached during particle motion. This difference vector at these phase-space points is normalized and written as $\cos(\alpha(\vec{z}))\vec{u}_1 + \sin(\alpha(\vec{z}))\vec{u}_2$, or as $e^{i\alpha(\vec{z})}$. In orbital action-angle variables, the function $\alpha_{\vec{J}}(\vec{\Phi}) = \alpha_{\circ\vec{J}}(\vec{\Phi}) + \vec{j}\cdot\vec{\Phi}$ has a 2π-periodic contribution and a linear contribution, and in complex notation, the periodicity condition (56) reads as

$$e^{i\alpha_{\vec{J}}(\vec{\Phi}+2\pi\vec{Q})} = e^{i(2\pi\nu(\vec{J})+\alpha_{\vec{J}}(\vec{\Phi}))}. \tag{57}$$

This requires that all Fourier coefficients of $\alpha_{\circ\vec{J}}(\vec{\Phi})$ vanish except $\check{\alpha}_{\circ\vec{J}}(0)$. The resulting equation $\nu(\vec{J}) = \vec{j}\cdot\vec{Q} \bmod 1$ shows that the periodicity condition (57) for $\vec{n}_2(\vec{z})$ can only be satisfied when a spin-orbit resonance occurs. Otherwise the ISF is unique. This is summarized as follows:

Uniqueness of $\vec{n}(\vec{z})$*—no spin-orbit resonance*—If an $\vec{n}$ axis and basis vectors $\vec{u}_1$, $\vec{u}_2$ exist and the spin-rotation angle in one turn is not a linear combination of orbit phase advances modulo 2π, then the $\vec{n}$ axis is unique up to a sign.

If the orbital tunes are rational, one can also formulate some statements about the uniqueness of the $\vec{n}$ axis. Given that the phase-space motion can be described by action-angle variables and that the orbital tunes on an invariant torus in the Poincaré section at azimuth θ_0 are rational numbers $Q_j = \frac{n_j}{m_j}$, where the smallest possible integer denominators are used, let N be the smallest common multiple of these denominators. Then $\vec{M}^N(\vec{z})$ is the identity map, whereas $\vec{M}^n(\vec{z})$ is not the identity map for any $n < N$, and the following conclusions can be drawn:

Uniqueness of $\vec{n}(\vec{z})$*—rational tunes.*—If for some $N \in \mathbb{N}$, the N-turn spin transport matrix on an invariant torus is not the identity matrix but the N-turn orbital transport map is the identity, then an $\vec{n}$ axis exists on this invariant torus and is unique up to a sign.

To show this, the spin transport matrix $\mathbf{R}_N(\vec{z})$ for N turns around the ring starting at θ_0 is used. Since it is not the identity matrix, it describes a rotation around a vector $\vec{e}_N(\vec{z})$ which is unique up to a sign. After N turns, the phase-space transport map is the identity map and the periodicity condition (7) for an $\vec{n}$ axis $\vec{n}_N(\vec{z})$ of the N-turn spin-orbit system becomes $\vec{n}_N(\vec{z}) = \mathbf{R}_N(\vec{z})\vec{n}_N(\vec{z})$. The rotation vector $\vec{e}_N(\vec{z})$ is therefore the $\vec{n}$ axis $\vec{n}_N(\vec{z})$, unique up to a sign. The rotation vector $\vec{e}_N(\vec{M}(\vec{z}))$ of $\mathbf{R}_N(\vec{M}(\vec{z}))$ is given by $\pm\mathbf{R}(\vec{z})\vec{e}_N(\vec{z})$ since

$$\begin{aligned}\mathbf{R}_N(\vec{M}(\vec{z}))\mathbf{R}(\vec{z})\vec{e}_N(\vec{z}) &= \mathbf{R}(\vec{M}^N(\vec{z}))\mathbf{R}_N(\vec{z})\vec{e}_N(\vec{z}) \\ &= \mathbf{R}(\vec{z})\vec{e}_N(\vec{z}),\end{aligned} \tag{58}$$

where the fact that $\vec{M}^N(\vec{z})$ is the identity map was used. The rotation vectors are unique up to a sign and $\vec{e}_N(\vec{z})$ satisfies the periodicity condition (7) of an $\vec{n}$ axis up to a possible sign change,

$$\vec{e}_N(\vec{M}(\vec{z})) = \pm\mathbf{R}(\vec{z})\vec{e}_N(\vec{z}). \tag{59}$$

Since $\vec{M}^N(\vec{z})$ is the identity map, a particle with initial phase-space point $\vec{z}_i$ can only reach the N phase-space points $Z(\vec{z}_i) = \{\vec{z}|\vec{M}^n(\vec{z}_i), n \in \{1, \ldots, N\}\}$ which will be called the trajectory through $\vec{z}_i$. Given that the sign for $\vec{e}_N(\vec{z}_i)$ has been chosen, the sign of the rotation vectors on the trajectory through $\vec{z}_i$ is chosen so that the $+$ sign in Eq. (59) is obtained. This equation is then the periodicity condition (7) for $\vec{e}_N(\vec{z})$ and shows that $\vec{n}(\vec{z}) = \vec{e}_N(\vec{z})$ is an ISF which is unique up to a sign for each trajectory. Assuming sufficient smoothness of $\mathbf{R}_N(\vec{z})$, the rotation vector $\vec{e}_N(\vec{z})$ also varies smoothly over phase space and the signs on each trajectory are chosen so that $\vec{n}(\vec{z}) = \vec{e}_N(\vec{z})$ is a smooth function on the invariant torus.

Even though it is straightforward to find an $\vec{n}$ axis for such a system with rational orbital tunes, it is important to note that an amplitude-dependent spin tune cannot in general be computed for this $\vec{n}$ axis, since the dependence of the spin-rotation angle $2\pi\tilde{\nu}_{\vec{J}}(\vec{\Phi})$ on $\vec{\Phi}$ can only be eliminated in Eq. (53) when there is no resonance for the orbital tunes. The phase-dependent rotation angle has been computed analytically for some systems with rational tunes in [42,43].

Nonuniqueness of $\vec{n}(\vec{z})$.—If for some $N \in \mathbb{N}$, the N-turn spin transport matrix on the invariant torus and also the N-turn orbital transport map are identity maps, then an $\vec{n}$ axis exists but is not unique.

In such a system all orbital tunes are rational and for each tune Q_j, the smallest possible denominator is denoted here by m_j. Let the angle variables $\vec{\Phi}_0 = 0$ correspond to the point $\vec{z}_0$ in phase space at some azimuth θ_0. There is a number k of turns after which a particle, starting at $\vec{z}_0$, reaches the phase-space point $\vec{z}_k$ with angle variables $\Phi_j = \frac{2\pi}{m_j}$. We introduce the set of phase-space points $P_1 = \{\vec{z}|\Phi_j \in [0, \frac{2\pi}{m_j})$ for all $j\}$ and note that the entire torus is covered by the sets $P_n = \{\vec{z}|\Phi_j - nQ_j \in [0, \frac{2\pi}{m_j}]$ for all $j\}$ which are obtained by transporting the phase-space points of P_1 n times through the one-turn transport map $\vec{M}$. At the fixed azimuth θ_0, the trajectory $Z(\vec{z}_i)$ through a point $\vec{z}_i$ contains N points, each of which is located in one of the P_n.

Consider a spin field $\vec{f}(\vec{z})$ which is arbitrarily chosen for all $\vec{z} \in P_1$. For all points $\vec{M}^n(\vec{z}) \in P_{n+1}$, it is chosen to satisfy $\vec{f}(\vec{M}^n(\vec{z})) = \mathbf{R}^n(\vec{z})\vec{f}(\vec{z})$ for $n \in \{1, \ldots, N-1\}$. This defines the spin field on the entire torus.

According to this definition, the periodicity condition $\vec{f}(\vec{M}(\vec{z})) = \mathbf{R}(\vec{z})\vec{f}(\vec{z})$ is satisfied for all $\vec{z} \in P_n$ with $n \in \{1, \ldots, N-1\}$. But due to the assumption that $\vec{M}^N$ and $\mathbf{R}^N$ are identity maps, it is also satisfied for P_N and thus for all of the torus. Therefore $\vec{f}(\vec{z})$ is an ISF, and since it was chosen arbitrarily for a set of points on the torus, it is not unique.

IV. THE ADIABATIC SPIN INVARIANT ON PHASE-SPACE TRAJECTORIES

A. The equation of spin motion on general orbits

As a first step for finding an adiabatic invariant of spin motion on phase-space trajectories we bring the equation of spin motion into an appropriate form.

In general the ISF on the Poincaré section at θ_0 changes when parameters such as the beam energy or quadrupole settings are changed. In other words, if the variable τ describes one of these parameters, then $\vec{n}(\vec{z}, \tau)$ changes when τ varies from the initial setting $\tau_i = 0$. Of course it is assumed that an $\vec{n}$ axis exists for every value this parameter might take. A beam which is polarized along the ISF with initial polarization $|\langle\vec{n}(\vec{z}, 0)\rangle|$ will remain polarized closely parallel to $\vec{n}(\vec{z}, \tau)$ while τ is changed, provided the change is slow enough and no very strong resonance effects diminish the polarization. For example, when the beam is accelerated slowly, the beam polarization may be low while $P_{\lim} = |\langle\vec{n}\rangle|$ is small, but when the spins follow the slow change of $\vec{n}(\vec{z}, \tau)$ with energy parameter τ, the beam may have high polarization later, once the energy has reached a value where $P_{\lim}$ is large.

We will prove in this section that spins follow slow changes of the invariant spin field by showing that the product $J_S = \vec{S} \cdot \vec{n}(\vec{z})$ is an adiabatic invariant. On the closed orbit, the ISF $\vec{n}(\vec{z})$ is parallel to the one-turn rotation vector $\vec{n}_0$. The angle between $\vec{n}_0(\theta)$ and the spin $\vec{S}(\theta)$ of a particle traveling on the closed orbit changes little when the system changes slowly, and therefore also $s_3(\theta) = \vec{S}(\theta) \cdot \vec{n}_0(\theta)$ was shown to be an adiabatic invariant. This proof will now be generalized to show that $J_S = \vec{S} \cdot \vec{n}(\vec{z}, \tau)$ changes little along a particle trajectory while the spin motion $\vec{S}$ and the phase-space motion $\vec{z}$ are subject to equations of motion which change slowly with the parameter $\tau = \varepsilon\theta$,

$$\frac{d}{d\theta}\vec{z} = \vec{v}(\vec{z}, \theta, \tau), \qquad \frac{d}{d\theta}\vec{S} = \vec{\Omega}(\vec{z}, \theta, \tau) \times \vec{S}. \tag{60}$$

For Hamiltonian motion generated by $H_{\text{tot}} = \vec{Q}^T(\vec{J}, \tau)\vec{J} + \varepsilon H(\theta, \vec{\Phi}, \vec{J}, \tau)$, these equations of motion have the form

$$\begin{aligned} \dot{\vec{S}} &= \mathbf{A}(\theta, \vec{\Phi}, \vec{J}, \tau)\vec{S}, \qquad \dot{\vec{J}} = -\varepsilon\vec{H}_\Phi(\theta, \vec{\Phi}, \vec{J}, \tau), \\ \dot{\vec{\Phi}} &= \vec{Q}(\vec{J}, \tau) + \varepsilon\vec{H}_J(\theta, \vec{\Phi}, \vec{J}, \tau), \qquad \dot{\tau} = \varepsilon, \end{aligned} \tag{61}$$

with $H_{\Phi,i} = \frac{\partial}{\partial\Phi_i}H$, $H_{J,i} = \frac{\partial}{\partial J_i}H$ and $\mathbf{A} = -\mathbf{A}^T$.

We assume that action-angle variables $\vec{J}$, $\vec{\Phi}$ and an ISF $\vec{n}(\vec{z}, \tau)$ exist for each $\tau \in [0, 1]$ at some azimuth θ_0. Since the $\vec{n}$ axis is a spin field, it is propagated around the ring like $\vec{S}$ in Eq. (60). Note that therefore $\vec{S} \cdot \vec{n}$ is invariant as long as τ does not change. If τ changes with θ, then $\vec{S}$ is still a solution of Eq. (60), but $\vec{n}(\vec{z}(\theta), \tau(\theta))$ is not. Even though $\vec{S} \cdot \vec{n}$ is not an invariant, we will show that it is an adiabatic invariant.

We also assume that a coordinate system $(\vec{u}_1, \vec{u}_2, \vec{n})$ exists at θ_0 for each $\tau \in [0, 1]$. To obtain a 2π-periodic coordinate system, we propagate $\vec{u}_1$ and $\vec{u}_2$ by

$$\frac{d}{d\theta}\vec{u}_i = (\vec{\Omega} - \nu\vec{n}) \times \vec{u}_i. \tag{62}$$

This lets spins rotate uniformly around $\vec{n}$ during one turn, after which their rotation is described by Eq. (56). This leads to an amplitude-dependent spin tune $\nu(\vec{J}, \tau)$ which in general depends on τ. These are nontrivial assumptions, since during the analysis of the existence of $\nu(\vec{J})$ in Sec. III, the existence of $\vec{u}_1$ and $\vec{u}_2$ was only guaranteed when the system is strongly non–orbit-resonant, even though there are simple models like the SRM where $\nu(\vec{J})$ exists for orbit resonances. We also take into account that the tunes $\vec{Q}(\vec{J}, \tau)$ of phase-space motion depend on the action variables and on the parameter τ which slowly changes the system.

Using the phase-space-dependent 2π-periodic coordinate system, the spin of a particle with phase-space coordinate $\vec{z}$ at azimuth θ is described by $\vec{S} = s_1\vec{u}_1 + s_2\vec{u}_2 + J_S\vec{n}$.

To make use of this new coordinate system, we assume for $\varepsilon = 0$ (which includes τ fixed) the existence of a uniform invariant frame field

$$\mathbf{U}(\theta, \vec{\Phi}; \vec{J}, \tau) = (\vec{u}_1, \vec{u}_2, \vec{u}_3) \tag{63}$$

with the properties:

P1. $\vec{u}_3$ is an invariant spin field $\vec{n}$ (ISF)

P2. $\mathbf{U}$ has an SO(3) matrix representation 2π periodic in θ and all the orbital phases Φ_i.

P3. For $\varepsilon = 0$, $\dot{\vec{S}} = \mathbf{A}(\theta, \vec{Q}(\vec{J}_0, \tau)\theta + \vec{\Phi}_0, \vec{J}_0, \tau_0)\vec{S}$ which has a principal solution matrix

$$\mathbf{R}(\theta; \vec{\Phi}_0, \vec{J}_0, \tau_0) = \mathbf{U}(\theta, \vec{Q}(\vec{J}_0, \tau_0)\theta + \vec{\Phi}_0, \vec{J}_0, \tau_0) \times \exp(\nu(\vec{J}_0, \tau_0)\mathbf{J}\theta)\mathbf{U}^T(0, \vec{\Phi}_0, \vec{J}_0, \tau_0) \tag{64}$$

which is analogous to Eq. (19).

P4. The basic relation analogous to Eq. (20) is

$$D_1\mathbf{U} + \vec{Q}^T(\vec{J}, \tau)\vec{D}_2\mathbf{U} = \mathbf{A}(\theta, \vec{\Phi}, \vec{J}, \tau)\mathbf{U} - \nu(\vec{J}, \tau)\mathbf{U}\mathbf{J}. \tag{65}$$

Here $\vec{D}_2$ is the gradient operator with respect to the dependence of $\mathbf{U}(\theta, \vec{\Phi}; \vec{J}, \tau)$ on $\vec{\Phi}$.

This is discussed in detail in [6]. Note that the ISF is 2π periodic and satisfies $D_1\vec{u}_3(\theta, \vec{\Phi}) + \vec{Q}^T\vec{D}_2\vec{u}_3(\theta, \vec{\Phi}) = \mathbf{A}(\theta, \vec{\Phi})\vec{u}_3$.

Again we use the transformation $\vec{S} \mapsto \vec{s}$ via

$$\vec{S} = \mathbf{U}(w)\vec{s}, \qquad w = (\theta, \vec{\Phi}, \vec{J}, \tau), \tag{66}$$

$$\dot{\vec{S}} = \dot{\mathbf{U}}(w)\vec{s} + \mathbf{U}(w)\dot{\vec{s}}, \tag{67}$$

$$\begin{aligned}\dot{\mathbf{U}} &= D_1\mathbf{U} + \dot{\vec{\Phi}}^T\vec{D}_2\mathbf{U} + \dot{\vec{J}}^T\vec{D}_3\mathbf{U} + D_4\mathbf{U}\varepsilon \\ &= D_1\mathbf{U} + \vec{Q}^T\vec{D}_2\mathbf{U} + \varepsilon(\vec{H}_J^T\vec{D}_2\mathbf{U} - \vec{H}_\Phi^T\vec{D}_3\mathbf{U} + D_4\mathbf{U}) \\ &= \mathbf{A}\mathbf{U} - \nu\mathbf{U}\mathbf{J} + \varepsilon(\vec{H}_J^T\vec{D}_2\mathbf{U} - \vec{H}_\Phi^T\vec{D}_3\mathbf{U} + D_4\mathbf{U}).\end{aligned} \tag{68}$$

Now it follows from Eqs. (61) and (68) that

$$(\mathbf{A}\mathbf{U} - \nu\mathbf{U}\mathbf{J})\vec{s} + \varepsilon(\vec{H}_J^T\vec{D}_2\mathbf{U} - \vec{H}_\Phi^T\vec{D}_3\mathbf{U} + D_4\mathbf{U})\vec{s} + \mathbf{U}\dot{\vec{s}} = \mathbf{A}\mathbf{U}\vec{s}. \tag{69}$$

But now $\mathbf{U}^T(w)\mathbf{U}(w) = \mathbf{I}$ and so as before, between Eqs. (26) and (27), we have

$$\mathbf{U}^T(\vec{H}_J^T\vec{D}_2\mathbf{U} - \vec{H}_\Phi^T\vec{D}_3\mathbf{U} + D_4\mathbf{U}) =: \begin{pmatrix} 0 & -\eta_3 & \eta_2 \\ \eta_3 & 0 & -\eta_1 \\ -\eta_2 & \eta_1 & 0 \end{pmatrix}. \tag{70}$$

This again means that the periodic unit vectors depend on τ, and, when τ is changed, their variation with τ can only be a rotation around some vector $\vec{\eta}(\vec{z}, \theta, \tau)$,

$$\partial_\tau\vec{n} = \vec{\eta} \times \vec{n}, \qquad \partial_\tau\vec{u}_1 = \vec{\eta} \times \vec{u}_1, \qquad \partial_\tau\vec{u}_2 = \vec{\eta} \times \vec{u}_2. \tag{71}$$

With the following notation,

$$\vec{\eta}(w) = (\eta_1, \eta_2, \eta_3)^T, \quad \vec{\zeta}(w) = (\eta_1, \eta_2)^T, \quad \vec{y} = (s_1, s_2)^T \tag{72}$$

as before in Eq. (30) we have

$$\begin{aligned}\dot{\vec{y}} &= (\nu(\vec{J}, \tau) - \varepsilon\eta_3(w))\mathbf{J}_2\vec{y} + \varepsilon\mathbf{J}_2\vec{\zeta}(w)J_S, \\ \dot{J}_S &= \varepsilon\vec{\zeta}^T(w)\mathbf{J}_2\vec{y}, \qquad \dot{\vec{J}} = -\varepsilon\vec{H}_\Phi(w), \\ \dot{\vec{\Phi}} &= \vec{Q}(\vec{J}, \tau) + \varepsilon\vec{H}_J(w), \qquad \dot{\tau} = \varepsilon.\end{aligned} \tag{73}$$

The transformation $\vec{y} \mapsto \vec{x}$ via

$$\vec{y} = \exp\left\{\mathbf{J}_2\int_0^\theta (\nu(\vec{J}(\theta'), \varepsilon\theta') - \varepsilon\eta_3(w(\theta')))d\theta'\right\}\vec{x} \tag{74}$$

gives as in Eq. (33)

$$\begin{aligned}\dot{\vec{x}} &= \varepsilon\exp(-\mathbf{J}_2\psi)\mathbf{J}_2\vec{\zeta}(w)J_S, \\ \dot{J}_S &= \varepsilon\vec{\zeta}^T(w)\mathbf{J}_2\exp(\mathbf{J}_2\psi)\vec{x}, \qquad \dot{\vec{J}} = -\varepsilon\vec{H}_\Phi(w), \\ \dot{\tau} &= \varepsilon, \qquad \dot{\psi} = \nu(\vec{J}, \tau) - \varepsilon\eta_3(\theta, \tau), \\ \dot{\vec{\Phi}} &= \vec{Q}(\vec{J}, \tau) + \varepsilon\vec{H}_J(w), \qquad \dot{\theta} = 1.\end{aligned} \tag{75}$$

The perturbations to the motion of the action and angle variables are due to the variation of the equation of phase-space motion (60) with the parameter τ.

B. Averaging for *n*-frequency systems

Equations (75) are in a standard form for the method of averaging. The average over the angle variable $\vec{\Phi}$ of the perturbation to $\vec{J}$ vanishes for Hamiltonian systems, since there $\langle \partial_{\vec{\Phi}} H(\tilde{\theta}, \vec{\Phi}, \vec{J}, \tau)\rangle_{\vec{\Phi}} = 0$, where the fact is used that the derivative of a periodic function has zero mean. The average over the phase ψ of the perturbation to $\vec{J}_S$ also vanishes. For accelerators, with 6-dimensional phase space, this system of ordinary differential equations therefore has 7 slowly changing variables (J_S, τ, and the 5 components of $\vec{x}$ and $\vec{J}$) and 5 rapidly changing variables (ψ, θ, and the 3 components of $\vec{\Phi}$), for small ε. The system is in the standard form of n frequency (or "multiphase") averaging theorems ([29], Chap. 6, [32], Sec. 1.9), which permit the crossing of resonances between all fast variables. Here we use theorem 2 of [29], Sec. 6.1 which is attributed to [44].

Theorem.—Consider a system of the form

$$\frac{d}{d\theta}\vec{I} = \varepsilon \vec{f}(\vec{\phi}, \vec{I}, \varepsilon), \tag{76}$$

$$\frac{d}{d\theta}\vec{\phi} = \vec{\nu}(\vec{I}) + \varepsilon g(\vec{\phi}, \vec{I}, \varepsilon), \tag{77}$$

where $\vec{I}$ belongs to a regular compact subset of Euclidean $\mathbb{R}^m$ and $\vec{\phi} \in \mathbb{R}^n$. Each function on the right-hand side is real valued, C^1 in $\vec{I}$ and ε, periodic with period 2π in all ϕ_j, and each possesses an analytic extension for $\phi_j \in \mathbb{C}$, $\Im\{\phi_j\} < \sigma$ with $\sigma > 0$. The associated averaged system is

$$\frac{d}{d\theta}\vec{\bar{I}} = \varepsilon \vec{\bar{f}}(\vec{\bar{I}}), \qquad \vec{\bar{f}}(\vec{\bar{I}}) = \frac{1}{(2\pi)^n}\int_0^{2\pi} \vec{f}(\vec{\phi}, \vec{\bar{I}}, 0) d\vec{\phi}, \tag{78}$$

with $\vec{\bar{I}}(0) = \vec{I}(0)$. Let the following nondegeneracy condition (called Arnold's condition) be satisfied: Assuming the frequency $\nu_n(\vec{I}) \neq 0$ (with no loss of generality, since in every region at least one frequency will be nonzero), then the map $\vec{I} \mapsto (\nu_1(\vec{I}), \ldots, \nu_{n-1}(\vec{I}))/\nu_n(\vec{I})$ has maximal rank, equal to $n-1$. Then for every continuous function $\rho(\varepsilon)$ with $C_1\sqrt{\varepsilon} \leq \rho(\varepsilon) \leq C_2$, $C_1, C_2 \in \mathbb{R}^+$, the set of allowed initial conditions V is partitioned as $V = V'[\varepsilon, \rho(\varepsilon)] \times \bigcup V''[\varepsilon, \rho(\varepsilon)]$ for sufficiently small ε such that

$$\mathrm{Sup}_{\theta \in [0,1/\varepsilon]} |\vec{I}(\theta) - \vec{\bar{I}}(\theta)| < \rho(\varepsilon) \tag{79}$$

for $(\vec{I}(0), \vec{\phi}(0)) \in V'$; i.e., for initial conditions in V', the separation between the exact solution and the solution of the averaged system is less than $\rho(\varepsilon)$. Moreover, the measure of $V''[\varepsilon, \rho(\varepsilon)]$ is smaller than $C\sqrt{\varepsilon}/\rho(\varepsilon)$ for some $C \in \mathbb{R}^+$.

We point out that there is a slight error in the proof of this theorem in [44] [because the equality in Eq. (17) of that paper does not hold in general]; however, this is easily repaired, as is done between Eqs. (4.9) and (4.13) of [45].

When the frequencies are in resonance, the slow variables $\vec{I}$ can accumulate large changes and the solution of the averaged system may not approximate the original system very well. In the above theorem, Arnold's condition ensures that no slow variable I_j can change at a resonance without moving the system out of this resonance. Arnold's condition also has important geometric consequences: requiring the reduced frequency map $\vec{I} \mapsto (\nu_1(\vec{I}), \ldots, \nu_{n-1}(\vec{I}))/\nu_n(\vec{I})$ to be of maximal rank means that preimages (in I-space) of sets in reduced frequency space are nicely structured; in particular, preimages of resonances among the frequencies are not "too large." This is made more precise by the so-called preimage theorem (see for example [46], Sec. 4).

C. Adiabatic invariance of spin motion via multiphase averaging

We now apply the averaging theorem for n-frequency systems to Eq. (75) for spin-orbit motion. The frequency of the variable $\tilde{\theta}$ is 1 and can therefore be used as ν_n of Arnold's condition. The four frequencies $(\vec{Q}(\vec{J}, \tau), \nu(\vec{J}, \tau))$ depend on four of the five slow variables and we assume that the rank is 4 so that the Jacobian matrix of the four frequencies has nonvanishing determinant, $\det(\partial_{(\vec{J},\tau)}(\vec{Q}, \nu)) \neq 0$.

Choosing $\rho(\varepsilon) = \varepsilon^{1/4}$ one finds, for Hamiltonian systems where $\vec{\bar{J}} = \vec{J}(0)$ since $\langle \partial_{\vec{\Phi}} H(\tilde{\theta}, \vec{\Phi}, \vec{J}, \tau)\rangle_{\vec{\Phi}} = 0$, that the set of initial conditions for which $\mathrm{Sup}_{\theta \in [0,1/\varepsilon]}|\vec{J}(\theta) - \vec{J}(0)| \geq \varepsilon^{1/4}$ has measure smaller than $C\varepsilon^{1/4}$. The variation of the action variables $\vec{J}$ for $\theta \in [0, 1/\varepsilon]$ therefore tends to 0 with ε, except for initial conditions from a set with measure that also tends to 0 with ε. The action variables are therefore adiabatic invariants as defined in Sec. I, which is a well known fact.

In addition we find with $\langle \exp(\mathbf{J}_2\psi)\rangle_\psi = 0$ that the set of initial conditions for which $\mathrm{Sup}_{\theta \in [0,1/\varepsilon]}|J_S(\theta) - J_S(0)| \geq \varepsilon^{1/4}$ has measure smaller than $C\varepsilon^{1/4}$. Therefore $J_S = \vec{S} \cdot \vec{n}(\vec{z})$ is an adiabatic invariant as defined in Sec. I.

It should be noted that, as announced in Sec. I, there is no need to show that J_S is an action variable of a Hamiltonian to establish it as an adiabatic invariant. We establish it as such by an analysis of the equations of spin-orbit motion, which is independent of the existence of a Hamiltonian for spin-orbit motion.

V. SUMMARY AND CONCLUSION

In our investigations of spin-orbit motion, we have used mathematical techniques to rigorously demonstrate the existence of adiabatic invariants, first for the simple case of motion on the closed orbit (Sec. II), then for the more complex case of motion on general orbits (Sec. IV). The last case entails a discussion of the amplitude-dependent spin tune ν and the $\vec{n}$ axis (Sec. III).

For convenience, we provide here a summary of our main results on invariance, highlighting the relevant mathematical restrictions and their physical meanings.

First, consider spin-orbit motion on the closed orbit of a particle accelerator. In order to study the case where the spin motion (including the spin tune) depends on a slowly varying parameter τ, we transform the spin-orbit equations of motion into a standard form for a two-frequency averaging principle. In order to apply the averaging principle, we need to assume that the τ-dependent spin tune spends only brief time intervals near resonance. Mathematically, this is "condition A" of Sec. II B, and, as we explain at the end of that section, this weak assumption corresponds to reasonable physical conditions in an accelerator where particles are slowly brought to operating energies. We therefore apply the two-frequency averaging theorem (beginning of Sec. II B), and conclude that $s_3(\theta)$, the projection of a particle's spin on the spin-rotation axis $\vec{n}_0$, is an adiabatic invariant. This is expressed more precisely in Eq. (44) and the remarks that follow.

Consider now the more general situation in a particle accelerator where the spin-orbit dynamics depends on a slowly varying parameter τ, with the property that for fixed τ, the orbital dynamics is integrable and an invariant spin field exists. We assume the existence of a uniform invariant frame field $\mathbf{U}(\theta, \vec{\Phi}; \vec{J}, \tau) = (\vec{u}_1, \vec{u}_2, \vec{u}_3)$ satisfying the properties P1–P4 following Eq. (63). Mathematical proof of the existence of this frame field is a difficult and as-yet-unresolved issue, however numerical experiments indicate that real machines have such fields to good approximation. Assuming its existence, we use this frame field to transform the equations of motion into a standard form for multiphase averaging. Under relatively mild yet explicit assumptions, the multiphase averaging theorem of Sec. IV B applies to the transformed equations. Most important is the assumption that the motion is strongly non-–orbit-resonant [following Eq. (53)], which means physically that the orbit tunes remain away from low-order resonances. It then follows from the theorem that $J_S = \vec{S} \cdot \vec{n}(\vec{z})$ is an adiabatic invariant of spin-orbit motion. To reiterate, this means that for all but a small set of "exceptional" initial conditions, J_S remains close to its initial value on a time scale of $O(1/\varepsilon)$. As $\varepsilon \to 0$, both the measure of the exceptional set and the nearness to the initial value also vanish, while the length of the time scale of nearness grows unboundedly.

We believe that an understanding of the issues discussed in this paper will be important in the design and operation of particle accelerators with polarized beams, both present and future, and, as mentioned at the end of the introduction, we are continuing this work by investigating a more direct approach to the existence of adiabatic invariants for spin-orbit motion.

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