Bunch lengthening by a betatron motion in quasi-isochronous storage rings

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(Received 21 January 2004; published 28 September 2004)

A simple analytical formula is described for bunch lengthening by a linear horizontal betatron motion in an electron storage ring. An example of calculation shows that this lengthening is larger than the intrinsic bunch shortening limit for most dispersive sections, which strongly limits the wavelength region of coherent radiation.

DOI: 10.1103/PhysRevSTAB.7.090703 PACS numbers: 29.20.Dh, 29.27.–a, 41.75.–i

I. INTRODUCTION

A quasi-isochronous electron synchrotron, which means a very small momentum compaction factor, can store a very short electron bunch. When its bunch length is a few mm or less, an extremely strong coherent synchrotron radiation in the THz frequency region is emitted even with a low stored beam current. Expanding on the observations of stable coherent radiation by Abo-Bakr *et al.* at BESSY-II [1], we examined new uses for this strong THz radiation. It is important to study limitations on the bunch shortening at a sufficiently low current. To date, two types of fundamental limitations in a linear system are theoretically predicted. One is the intrinsic bunch shortening limit from longitudinal radiation excitation [2]. The other is the effect of linear betatron motion. An electron in a storage ring passing through bending magnets at the outer side or inner side of a central orbit according to its betatron oscillation amplitude and phase has a deviation in the path length and produces a bunch lengthening. Deacon estimated this effect in 1981 [3], but he used a very rough approximation and did not consider that the lengthening depends on the location in the storage ring. I show that equations used to calculate synchrobeta resonance [4] give simple and accurate formulas for the bunch lengthening. I also show an example calculation using an isochronous lattice of an existing ring, NewSUBARU [5].

II. THEORETICAL FORMULAS

In this report my discussions are focused on the linear effect of the horizontal betatron motion. I assume that a vertical betatron motion and an energy displacement are negligible and higher order effects of betatron motion [6] are ignored. Coordinates x and s are the displacement from the reference electron in radial direction and the azimuthal coordinate, respectively. A displacement from the reference electron in the s direction is referred to by z, the Twiss parameters by $\alpha(s)$, $\beta(s)$, and $\gamma(s)$, and the betatron phase by $\psi(s)$.

The change in path length from the reference in one revolution δL when higher order terms are ignored is given by

$$\delta L = \int_{s_S}^{s_S+L_0} [x(s)/\rho(s)]ds. \tag{1}$$

Here L_0 is a circumference, s_S is s at the light source point, and $\rho(s)$ is the curvature of the radius of the reference orbit. When the radial displacement is produced by the betatron oscillation, $x(s)$ is written as

$$x(s) = \sqrt{\varepsilon_{\mathrm{CSI}}\beta(s)}\sin\psi(s). \tag{2}$$

Here ϵ_{CSI} is the Courant-Snyder invariant of a particle. If $s_S \le s < s_S + L_0$ the betatron phase $\psi(s)$ after the m revolutions around the ring is replaced by $\psi(s) + 2m\pi\nu$, where ν is the betatron tune. The total change of the path length after n revolutions, δL_n, is given by

$$\delta L_n(s_S) = \sqrt{\varepsilon_{\mathrm{CSI}}}\sum_{m=0}^{n-1}\int_{s_S}^{s_S+L_0}[\sqrt{\beta(s)}/\rho(s)]\sin[\psi(s) + 2m\pi\nu]ds. \tag{3}$$

Here I will use the following equations,

$$\sum_{m=0}^{n-1}\sin[(2m+1)\pi\nu + \psi(s_S)] = (\sin n\pi\nu/\sin\pi\nu)\sin[n\pi\nu + \psi(s_S)], \tag{4a}$$

$$\sum_{m=0}^{n-1}\cos[(2m+1)\pi\nu + \psi(s_S)] = (\sin n\pi\nu/\sin\pi\nu)\cos[n\pi\nu + \psi(s_S)], \tag{4b}$$

dispersion function $\eta(s)$, and dispersion angle $\eta'(s)$ given by

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$$\eta(s_S) = \frac{\sqrt{\beta(s_S)}}{2\sin\pi\nu}\int_{s_S}^{s_S+L_0}\frac{\sqrt{\beta(s)}}{\rho(s)}\cos[\psi(s)-\psi(s_S)-\pi\nu]ds, \tag{5a}$$

$$\alpha(s_S)\eta(s_S)+\beta(s_S)\eta'(s_S) = \frac{\sqrt{\beta(s_S)}}{2\sin\pi\nu}\int_{s_S}^{s_S+L_0}\frac{\sqrt{\beta(s)}}{\rho(s)}\sin[\psi(s)-\psi(s_S)-\pi\nu]ds. \tag{5b}$$

With these equations Eq. (3) is simplified to [4]

$$\delta L_n(s_S) = 2\sin(n\pi\nu)\sqrt{\varepsilon_{\rm CSI}}\{[\eta(s_S)/\sqrt{\beta(s_S)}]\sin[n\pi\nu + \psi(s_S)] + [\eta(s_S)\alpha(s_S)/\sqrt{\beta(s_S)} + \eta'(s_S) \times\sqrt{\beta(s_S)}]\cos[n\pi\nu+\psi(s_S)]\}. \tag{6}$$

If functions $H(s)$ and $\psi_H(s)$ are defined by

$$\sqrt{H}\sin\psi_H = \eta(\alpha/\sqrt{\beta}) + \eta'\sqrt{\beta}, \tag{7a}$$

$$\sqrt{H}\cos\psi_H = \eta/\sqrt{\beta}, \tag{7b}$$

Eq. (6) can be rewritten as

$$\delta L_n(s_S) = \sqrt{\varepsilon_{\rm CSI}H(s_S)}\{-\cos[2n\pi\nu+\psi(s_S)+\psi_H(s_S)] + \cos[\psi(s_S)+\psi_H(s_S)]\}. \tag{8}$$

The function H is given by

$$H = \gamma\eta^2 + 2\alpha\eta\eta' + \beta\eta'^2. \tag{9}$$

The longitudinal displacement from the reference z after n revolutions is given by

$$z = -\delta L_n - z_S. \tag{10}$$

Here z_S is an initial displacement, which depends on the initial phase $\psi(s_S)$. If the reference is defined as an average over $\psi(s_S)$, z_S is determined and the longitudinal displacement $z(s_S)$ is given by

$$z(s_S) = \sqrt{\varepsilon_{\rm CSI}H(s_S)}\cos[2n\pi\nu+\psi(s_S)+\psi_H(s_S)], \tag{11}$$

where the radial displacement $x(s_S)$ is given by

$$x(s_S) = \sqrt{\varepsilon_{\rm CSI}\beta(s_S)}\sin[2n\pi\nu+\psi(s_S)]. \tag{12}$$

Notice that H and ψ_H are unique functions of s_S but ψ is not. Equation (9) shows that the amplitude of the oscillation of z is $\sqrt{\varepsilon_{\rm CSI}H}$. The bunch lengthening is zero at a dispersion free location. The $\psi_H - \pi/2$ is the phase difference between the radial and the longitudinal oscillations. The particle moves on an ellipse in the z-x plane. When ψ_H is $\pm\pi/2$ the ellipse shrinks to a line.

The density distribution function in the z axis of a bunch of particles with the same $\epsilon_{\rm CSI}$ and uniform distribution in ψ_S is given by

$$D(z,\varepsilon_{CSI})dz = \begin{cases}\frac{1}{\pi}\frac{dz}{(\varepsilon_{\rm CSI}H-z^2)} & \text{for } z^2 < \varepsilon_{\rm CSI}H\\ 0 & \text{for } z^2 \geq \varepsilon_{\rm CSI}H.\end{cases} \tag{13}$$

In a realistic model, the particles have $\epsilon_{\rm CSI}$ variations and the density distribution function in the $\epsilon_{\rm CSI}$ axis is

$$F(\varepsilon_{\rm CSI})d\varepsilon_{\rm CSI} = \frac{1}{2\varepsilon}e^{-\varepsilon_{\rm CSI}/2\varepsilon}d\varepsilon_{\rm CSI}. \tag{14}$$

Here a Gaussian distribution is assumed in a betatron oscillation phase space with rms emittance of ε. The realistic distribution is given by a convolution of Eq. (13) with Eq. (14),

$$G(z)dz = \left[\int_0^{\sqrt{\varepsilon H}}F(\varepsilon_{\rm CSI})D(z,\varepsilon_{\rm CSI})d\varepsilon_{\rm CSI}\right]dz \tag{15a}$$

$$= \frac{1}{\sqrt{2\pi\varepsilon H}}e^{-z^2/2\varepsilon H}dz. \tag{15b}$$

It is a Gaussian distribution with standard deviation of $\sqrt{\varepsilon H}$.

III. CALCULATION AT NEWSUBARU

NewSUBARU is a 1.5 GeV racetrack-type synchrotron radiation ring at the SPring-8 site. Its bending cell is a modified double bend achromat with one $-8°$ invert bend between two $34°$ normal bends. This cell facilitates control of the linear momentum compaction factor (α_p) while maintaining the achromatic cell with only a small change in the natural emittance. Table I lists the parameters of the ring used in the calculation when the ring was operated in quasi-isochronous operation mode [7].

Figure 1 shows the horizontal beta function β and the dispersion function η in 1/4 of the ring. Figure 2 shows the H and the ψ_H in 1/4 of the ring. The maximum rms bunch lengthening by the natural emittance (ε_{XN}) is 0.10 mm at the inverse bends. The bunch lengthening was 0.06 mm at the light source point of the beam line used to measure the bunch length.

TABLE I. Main parameters of NewSUBARU.

Stored electron energy E (GeV)	1.0
Circumference L_0 (m)	118.73
rf frequency $f_{\rm rf}$ (MHz)	499.956
Natural emittance ε_{XN} (nm)	30.1
Horizontal betatron tune ν	6.30
Linear momentum compaction factor α_p	0
Natural energy spread $\sigma_{\rm EN}$ (%)	0.047
Intrinsic bunch shortening limit $\sigma_{\rm TI}$ (ps)	0.08

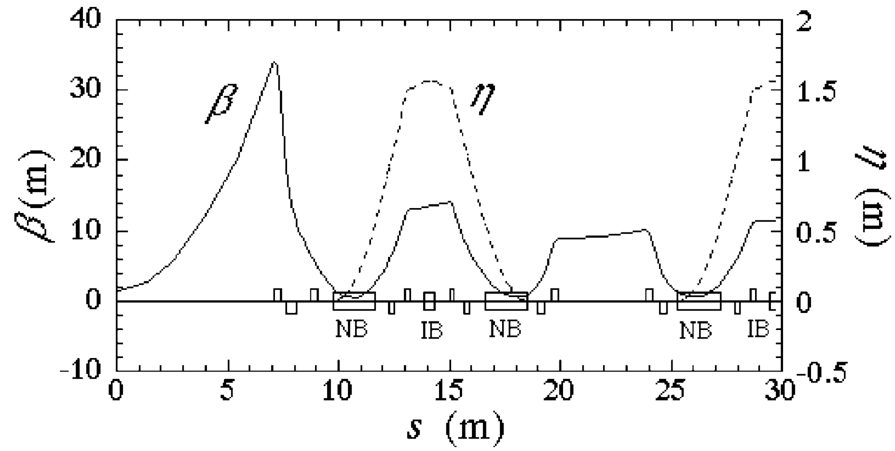

FIG. 1. Beta function (β, solid line) and dispersion function (η, broken line) in 1/4 of NewSUBARU. The boxes indicate locations and lengths of the bending magnets (indicated by NB and IB) and quadrupole magnets.

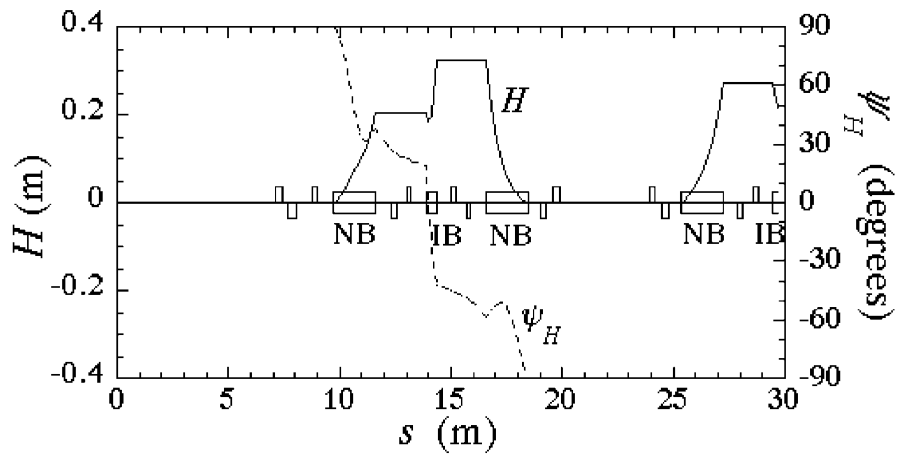

FIG. 2. The amplitude function (H, solid line) and the phase difference between the longitudinal and transversal oscillation (ψ_H, broken line) along 1/4 of NewSUBARU.

IV. DISCUSSION

This bunch lengthening effect should be considered in extreme cases beyond those realized at BESSY-II ($\sigma_L = 0.45$ mm) [8] and at NewSUBARU ($\sigma_L = 0.48$ mm). For coherent radiation Eq. (15b) gives a strong limitation to the wavelength of radiation. The time structure of a bunch, produced by the rf potential well distortion or any method, is smeared in bending magnets.

I will compare the calculated $\sqrt{\varepsilon_{XN} H}$ with the intrinsic bunch shortening limit coming from the longitudinal radiation excitation. The idea is that a stochastic fluctuation where the photoemission takes place produces a fluctuation of rf phase and enlarges the equilibrium bunch length and energy spread as the energy spread excitation does. Because of this radiation excitation the bunch length cannot be larger than

$$\sigma_{\rm TI} = T_0 \sigma_{\rm EN} \sqrt{I_\alpha} \tag{16}$$

at any locations of the ring. This is the intrinsic limit of a storage ring determined only by T_0 (revolution period), $\sigma_{\rm EN}$ (natural energy spread), and I_α [a variance of partial momentum compaction factor: $\breve{\alpha}(s)$]. The I_α and $\breve{\alpha}(s)$ are given by the following equations:

$$\breve{\alpha}(s_S) = \frac{1}{L_0} \int_{s_S}^{L_0} \frac{\eta(s)}{\rho(s)} ds, \tag{17}$$

$$I_\alpha = \langle [\breve{\alpha}(s_S) - \langle \breve{\alpha} \rangle]^2 \rangle. \tag{18}$$

In NewSUBARU σ_{TI} is calculated to be 0.02 mm, which is smaller than $\sqrt{\varepsilon_{XN}H}$ in most locations of the bending magnets.

ACKNOWLEDGMENTS

The author thanks Dr. S. Daté, Mr. T. Nakamura, and Dr. H. Tanaka of SPring-8 for their helpful discussions on this subject and proofreading this article.

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