Cumulative beam breakup in linear accelerators with random displacement of cavities and focusing elements

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A formalism presented in a previous paper for the analysis of cumulative beam breakup with arbitrary time dependence of the beam current [J. R. Delayen, Phys. Rev. ST Accel. Beams **6**, 084402 (2003)] is applied to the problem of beam breakup in the presence of random displacements of cavities and focusing elements. A closed-form solution is obtained and is applied to the behavior of a single bunch and to the steady-state and transient behavior of dc beams and beams composed of pointlike bunches.

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I. INTRODUCTION

Cumulative beam breakup (BBU) in linear accelerators results when a beam traverses the accelerating structures off axis and thus couples to the dipole modes of the structure. This can occur when the beam enters the accelerator with a lateral offset or angular divergence. This is the case studied most often and a general analysis was presented in Ref. [1].

The coupling between beam and dipole modes can also occur when the structures themselves or the focusing elements are displaced from the nominal accelerator beam line. Such displacements occur in a random fashion and the displacement of the beam along the accelerator will exhibit a random behavior. The formalism presented in [1] can also be applied to the situation of random displacements of the cavities and focusing elements and a general solution was presented in that paper. This general solution is investigated here, its statistical properties are determined and applied here to specific examples.

II. EQUATION OF MOTION AND GENERAL SOLUTION

In a continuum approximation, the transverse motion of a beam in a misaligned accelerator under the combined influence of focusing and coupling to dipole modes can be modeled by [2–5]

$$\frac{\partial^2}{\partial\sigma^2}x(\sigma,\zeta)+\kappa^2[x(\sigma,\zeta)-d_f(\sigma)] = \varepsilon\int_{-\infty}^{\zeta} w(\zeta-\zeta_1)F(\zeta_1)[x(\sigma,\zeta_1)-d_c(\sigma)]d\zeta_1. \quad (1)$$

In this expression $\sigma = s/\mathcal{L}$ is the distance from the entrance of the accelerator normalized to the accelerator length $\mathcal{L}$; κ is the normalized focusing wave number; $\zeta = \omega(t-\int ds/\beta c)$ is the time made dimensionless by an angular frequency ω and measured after the arrival of the head of the beam at location σ; $F(\zeta) = I(\zeta)/\bar{I}$, the current form factor, is the instantaneous current divided by the average current; $w(\zeta)$ is the wake function of the dipole modes; ε is the coupling strength between the beam and the dipole modes; $d_f(\sigma)$ and $d_c(\sigma)$ are the lateral displacement of the focusing elements and cavities, respectively, as a function of location along the accelerator.

The dimensionless BBU coupling strength ε is given by

$$\varepsilon = \frac{w_0\bar{I}e\mathcal{L}^2}{\gamma mc^2\omega}, \quad (2)$$

where w_0 is the wake amplitude. With these definitions the wake function $w(\zeta)$ is a dimensionless function of a dimensionless variable and includes only the functional dependence on ζ.

The continuum model assumed in Eq. (1) relies on a number of approximations that are addressed in [1]. Equation (1) also assumes a coasting beam in a uniform accelerator but, as shown in Appendix A of [1], an accelerated beam can, under general assumptions, be reduced to a coasting beam with the introduction of appropriate variable and coordinate transformations, and the extension of the results presented in the body of this paper to an accelerated beam is presented in the Appendix. Equation (1) can then be solved through the use of the Laplace transform with respect to σ, and the solutions for $x^\dagger(p,\zeta) = \mathfrak{L}_\sigma[x(\sigma,\zeta)]$ and $x(\sigma,\zeta)$ are [1]

$$x^\dagger(p,\zeta) = \sum_{n=0}^{\infty}\frac{\varepsilon^n}{(p^2+\kappa^2)^{n+1}}[x_0 p h_n(\zeta)+x_0' g_n(\zeta)] - d_c^\dagger(p)\sum_{n=0}^{\infty}\frac{\varepsilon^{n+1}}{(p^2+\kappa^2)^{n+1}}f_{n+1}(\zeta) + \kappa^2 d_f^\dagger(p)\sum_{n=0}^{\infty}\frac{\varepsilon^n}{(p^2+\kappa^2)^{n+1}}f_n(\zeta), \quad (3)$$

and

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$$x(\sigma, \zeta) = \sum_{n=0}^{\infty} \varepsilon^n [x_0 h_n(\zeta) j_n(\kappa, \sigma) + x_0' g_n(\zeta) i_n(\kappa, \sigma)] - \sum_{n=0}^{\infty} \varepsilon^{n+1} f_{n+1}(\zeta) i_n(\kappa, \sigma) * d_c(\sigma) + \kappa^2 \sum_{n=0}^{\infty} \varepsilon^n f_n(\zeta) i_n(\kappa, \sigma) * d_f(\sigma). \tag{4}$$

The functions $f_n(\zeta)$, $g_n(\zeta)$, and $h_n(\zeta)$ are defined by the recursion relations

$$\begin{Bmatrix} f_{n+1}(\zeta) \\ g_{n+1}(\zeta) \\ h_{n+1}(\zeta) \end{Bmatrix} = \int_{-\infty}^{\zeta} \begin{Bmatrix} f_n(\zeta_1) \\ g_n(\zeta_1) \\ h_n(\zeta_1) \end{Bmatrix} w(\zeta - \zeta_1) F(\zeta_1) d\zeta_1, \tag{5}$$

with

$$f_0(\zeta) = 1, \tag{6a}$$

$$x_0' g_0(\zeta) = x_0'(\zeta) = \frac{\partial}{\partial \sigma} x(\sigma, \zeta) \Big|_{\sigma=0}, \tag{6b}$$

$$x_0 h_0(\zeta) = x_0(\zeta) = x(\sigma = 0, \zeta), \tag{6c}$$

where $x_0(\zeta)$ and $x_0'(\zeta)$ are the lateral displacement and angular divergence, respectively, of the beam at the entrance of the accelerator. The normalizing constants x_0 and x_0' are introduced to make the functions $h_0(\zeta)$ and $g_0(\zeta)$ dimensionless.

The functions $i_n(\kappa, \sigma)$ and $j_n(\kappa, \sigma)$ are defined in terms of Bessel functions of order integer plus one-half,

$$i_n(\kappa, \sigma) = \mathfrak{L}_\sigma^{-1}\left[\frac{1}{(p^2 + \kappa^2)^{n+1}}\right] = \frac{1}{n!}\left(\frac{\sigma}{2\kappa}\right)^n \frac{1}{\kappa} \sqrt{\frac{\pi \kappa \sigma}{2}} J_{n+(1/2)}(\kappa \sigma), \tag{7a}$$

$$j_n(\kappa, \sigma) = \mathfrak{L}_\sigma^{-1}\left[\frac{p}{(p^2 + \kappa^2)^{n+1}}\right] = \frac{d}{d\sigma} i_n(\kappa, \sigma) = \frac{1}{n!}\left(\frac{\sigma}{2\kappa}\right)^n \sqrt{\frac{\pi \kappa \sigma}{2}} J_{n-(1/2)}(\kappa \sigma) = \frac{\sigma}{2n} i_{n-1}(\kappa, \sigma), \tag{7b}$$

and

$$i_n(\kappa, \sigma) * d(\sigma) = \int_0^\sigma i_n(\kappa, u)\, d(\sigma - u)\, du$$

is the convolution of $i_n(\kappa, \sigma)$ and $d(\sigma)$.

Equation (4) gives the transverse displacement, at location σ and time ζ, of a beam of arbitrary current profile $F(\zeta)$, entering the accelerator with lateral offset $x_0(\zeta)$ and angular divergence $x_0'(\zeta)$, experiencing transverse forces due to a wakefield $\varepsilon w(\zeta)$ and focusing κ, and with displacement along the accelerator $d_c(\sigma)$ of the cavities and $d_f(\sigma)$ of the focusing elements.

Equivalent expressions for $x'(\sigma, \zeta) = \frac{\partial}{\partial \sigma} x(\sigma, \zeta)$ and $x'^{\dagger}(p, \zeta)$—the angular divergence and its Laplace transform—are

$$x'^{\dagger}(p, \zeta) = x_0\left\{-\frac{\kappa^2}{p^2 + \kappa^2} h_0(\zeta) + \sum_{n=1}^{\infty} \varepsilon^n \left[\frac{1}{(p^2 + \kappa^2)^n} - \frac{\kappa^2}{(p^2 + \kappa^2)^{n+1}}\right] h_n(\zeta)\right\} + x_0' \sum_{n=0}^{\infty} \varepsilon^n \frac{p}{(p^2 + \kappa^2)^{n+1}} g_n(\zeta) - d_c^{\dagger}(p) \sum_{n=0}^{\infty} \varepsilon^{n+1} \frac{p}{(p^2 + \kappa^2)^{n+1}} f_{n+1}(\zeta) + \kappa^2 d_f^{\dagger}(p) \sum_{n=0}^{\infty} \varepsilon^n \frac{p}{(p^2 + \kappa^2)^{n+1}} f_n(\zeta), \tag{8}$$

and

$$x'(\sigma, \zeta) = x_0\left\{-\kappa^2 i_0(\kappa, \sigma) h_0(\zeta) + \sum_{n=1}^{\infty} \varepsilon^n h_n(\zeta)[i_{n-1}(\kappa, \sigma) - \kappa^2 i_n(\kappa, \sigma)]\right\} + x_0' \sum_{n=0}^{\infty} \varepsilon^n g_n(\zeta) j_n(\kappa, \sigma) - \sum_{n=0}^{\infty} \varepsilon^{n+1} f_{n+1}(\zeta) j_n(\kappa, \sigma) * d_c(\sigma) + \kappa^2 \sum_{n=0}^{\infty} \varepsilon^n f_n(\zeta) j_n(\kappa, \sigma) * d_f(\sigma). \tag{9}$$

In this paper we are mainly concerned with the effects of misalignment of the cavities and focusing elements, and not of the lateral offset and angular divergence of the beam at the entrance of the accelerator which were investigated in [1] . Therefore, in the remainder of this paper (with the exception of Sec. IV where we will examine the behavior of a beam under the combined influence of injection offsets and misalignments), we will assume that $x_0(\zeta) = x_0'(\zeta) = 0$, and the equations for $x^{\dagger}(p, \zeta)$, $x'^{\dagger}(p, \zeta)$, $x(\sigma, \zeta)$, and $x'(\sigma, \zeta)$ are

$$x^{\dagger}(p, \zeta) = -d_c^{\dagger}(p) \sum_{n=0}^{\infty} \frac{\varepsilon^{n+1}}{(p^2 + \kappa^2)^{n+1}} f_{n+1}(\zeta) + \kappa^2 d_f^{\dagger}(p) \sum_{n=0}^{\infty} \frac{\varepsilon^n}{(p^2 + \kappa^2)^{n+1}} f_n(\zeta), \tag{10}$$

$$x'^{\dagger}(p, \zeta) = -d_c^{\dagger}(p) \sum_{n=0}^{\infty} \frac{p\varepsilon^{n+1}}{(p^2 + \kappa^2)^{n+1}} f_{n+1}(\zeta) + \kappa^2 d_f^{\dagger}(p) \sum_{n=0}^{\infty} \frac{p\varepsilon^n}{(p^2 + \kappa^2)^{n+1}} f_n(\zeta), \tag{11}$$

$$x(\sigma, \zeta) = -\sum_{n=0}^{\infty} \varepsilon^{n+1} f_{n+1}(\zeta) i_n(\kappa, \sigma) * d_c(\sigma) + \kappa^2 \sum_{n=0}^{\infty} \varepsilon^n f_n(\zeta) i_n(\kappa, \sigma) * d_f(\sigma), \quad (12)$$

$$x'(\sigma, \zeta) = -\sum_{n=0}^{\infty} \varepsilon^{n+1} f_{n+1}(\zeta) j_n(\kappa, \sigma) * d_c(\sigma) + \kappa^2 \sum_{n=0}^{\infty} \varepsilon^n f_n(\zeta) j_n(\kappa, \sigma) * d_f(\sigma). \quad (13)$$

Complete knowledge of the transverse displacement $d_c(\sigma)$ of the cavities and $d_f(\sigma)$ of the focusing elements would allow, in principle, determination of the transverse displacement $x(\sigma, \zeta)$ and angular divergence $x'(\sigma, \zeta)$ of the beam at location σ and time ζ. Such knowledge is often neither available nor necessary; what is usually available or needed are the statistical properties of the transverse displacements.

It can be seen from Eqs. (10)–(13) that, although not identical, the effects of the displacements of the cavities and of the focusing elements have similar features. For this reason we will treat in detail only the effects resulting from the displacements of the cavities. The effects resulting from the displacement of the focusing elements can be treated in a similar fashion and only the final results will be presented. There is also another configuration of interest where the displacements of the cavities and focusing elements are identical at all locations; this could occur, for example, in a room-temperature linear-collider-like accelerator where both are located on the same girders. In this case also we will present only the final results.

The effects of the misalignments on the lateral displacement and on the angular divergence are also similar and, for the latter, only the final results will be presented.

When only the cavities are displaced by $d_c(\sigma)$ the displacement of the beam is

$$x(\sigma, \zeta) = -\varepsilon \sum_{n=0}^{\infty} \varepsilon^n f_{n+1}(\zeta) i_n(\kappa, \sigma) * d_c(\sigma). \quad (14)$$

Defining $R_x(\sigma_1, \sigma_2; \zeta)$ as the autocorrelation function of $x(\sigma, \zeta)$ we have

$$R_x(\sigma_1, \sigma_2; \zeta) = \langle x(\sigma_1, \zeta) x(\sigma_2, \zeta) \rangle = \varepsilon^2 \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \varepsilon^{m+n} f_{m+1}(\zeta) f_{n+1}(\zeta) \int_0^{\sigma_1} \int_0^{\sigma_2} i_m(\kappa, \sigma_1 - u) i_n(\kappa, \sigma_2 - v) \langle d_c(u) d_c(v) \rangle du\, dv. \quad (15)$$

The mean-square displacement $\overline{x^2}(\sigma, \zeta) = R_x(\sigma, \sigma; \zeta) = \langle x(\sigma, \zeta) x(\sigma, \zeta) \rangle$ is then

$$\overline{x^2}(\sigma, \zeta) = \varepsilon^2 \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \varepsilon^{m+n} f_{m+1}(\zeta) f_{n+1}(\zeta) \int_0^{\sigma} \int_0^{\sigma} du\, dv\, i_m(\kappa, \sigma - u) i_n(\kappa, \sigma - v) R_{d_c}(u, v), \quad (16)$$

where $R_{d_c}(u, v)$ is the autocorrelation of $d_c(\sigma)$.

Assuming that the cavities have a normalized length $\sigma_0 \ll 1$, are displaced parallel to the accelerator axis, that their displacements are uncorrelated from one cavity to the others and follow a normal distribution with 0 mean and standard deviation d_{c0}, then the autocorrelation function of $d_c(\sigma)$ is

$$R_{d_c}(u - v) = \begin{cases} d_{c0}^2 (1 - |\frac{u-v}{\sigma_0}|) & \text{for } |u - v| < \sigma_0 \\ 0 & \text{for } |u - v| > \sigma_0. \end{cases} \quad (17)$$

In this expression d_{c0} is the standard deviation of the cavity displacement and has the dimension of a length, while σ_0 is the ratio of a cavity length and accelerator length and, therefore, is dimensionless. σ_0 is also the packing factor of the accelerator divided by the number of cavities.

If one assumes that the cavities are much shorter than any other characteristic length (betatron period, growth length of BBU) then the autocorrelation function can be simplified to

$$R_{d_c}(u - v) = d_{c0}^2 \sigma_0 \delta(u - v), \quad (18)$$

and the mean-square displacement is

$$\overline{x^2}(\sigma, \zeta) = d_{c0}^2 \sigma_0 \varepsilon^2 \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \varepsilon^{m+n} f_{m+1}(\zeta) f_{n+1}(\zeta) \times \int_0^{\sigma} i_m(\kappa, u) i_n(\kappa, u) du\,. \quad (19)$$

Defining

$$y_{m,n}(t) = \frac{1}{m! n!} \int_0^t t^{m+n} \left(\frac{\pi t}{2}\right) J_{m+(1/2)}(t) J_{n+(1/2)}(t)\, dt, \quad (20)$$

the mean-square displacement for the beam then becomes

$$\frac{\overline{x^2}(\sigma, \zeta)}{d_{c0}^2 \sigma_0} = \frac{\varepsilon^2}{\kappa^3} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \left(\frac{\varepsilon}{2\kappa^2}\right)^{m+n} y_{m,n}(\kappa\sigma) f_{m+1}(\zeta) f_{n+1}(\zeta) = \frac{\varepsilon^2}{\kappa^3} \sum_{N=0}^{\infty} \left(\frac{\varepsilon}{2\kappa^2}\right)^N \sum_{k=0}^{N} y_{k,N-k}(\kappa\sigma) f_{k+1}(\zeta) f_{N+1-k}(\zeta). \quad (21)$$

The functions $y_{m,n}(\cdot)$ defined in Eq. (20) can, in principle, be calculated for arbitrary m and n and involve only products of powers and circular functions of $\kappa\sigma$. In particular we have

$$y_{0,0}(t)=\frac{t}{2}-\frac{1}{4}\sin 2t, \tag{22a}$$

$$y_{1,0}(t)=y_{0,1}(t)=\frac{t}{2}+\frac{t}{4}\cos 2t-\frac{3}{8}\sin 2t, \tag{22b}$$

$$y_{11}(t)=\frac{t^3}{6}+\frac{t}{2}+\frac{3}{4}t\cos 2t+\frac{2t^2-5}{8}\sin 2t, \tag{22c}$$

$$y_{20}(t)=y_{02}(t)=-\frac{t^3}{12}+\frac{3}{4}t+\frac{t}{2}\cos 2t+\frac{t^2-5}{8}\sin 2t. \tag{22d}$$

If, instead, the focusing elements are displaced with standard deviation d_{f0} the mean-square displacement of the beam can be obtained in a similar manner and is given by

$$\begin{aligned}\frac{\overline{x^2}(\sigma,\zeta)}{d_{f0}^2\sigma_0} &= \kappa\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\left(\frac{\varepsilon}{2\kappa^2}\right)^{m+n} y_{m,n}(\kappa\sigma)f_m(\zeta)f_n(\zeta)\\ &= \kappa\sum_{N=0}^{\infty}\left(\frac{\varepsilon}{2\kappa^2}\right)^{N}\sum_{k=0}^{N} y_{k,N-k}(\kappa\sigma)f_k(\zeta)f_{N-k}(\zeta).\end{aligned} \tag{23}$$

If the cavities and the focusing elements are identically displaced (and therefore perfectly correlated) with standard deviation d_0^2 the mean-square displacement of the beam is given by

$$\begin{aligned}\frac{\overline{x^2}(\sigma,\zeta)}{d_0^2\sigma_0} &= \frac{1}{\kappa^3}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\left(\frac{\varepsilon}{2\kappa^2}\right)^{m+n} y_{m,n}(\kappa\sigma)[\kappa^2 f_m(\zeta)-\varepsilon f_{m+1}(\zeta)][\kappa^2 f_n(\zeta)-\varepsilon f_{n+1}(\zeta)]\\ &= \frac{1}{\kappa^3}\sum_{N=0}^{\infty}\left(\frac{\varepsilon}{2\kappa^2}\right)^{N}\sum_{k=0}^{N} y_{k,N-k}(\kappa\sigma)[\kappa^2 f_k(\zeta)-\varepsilon f_{k+1}(\zeta)][\kappa^2 f_{N-k}(\zeta)-\varepsilon f_{N+1-k}(\zeta)].\end{aligned} \tag{24}$$

If both the cavities and focusing elements are displaced from the beam line but their displacement is totally uncorrelated, then the beam displacement is given by the sum of Eqs. (21) and (23).

Similar results can be obtained for the mean-square value of the angular divergence. For the case of displaced cavities, displaced focusing elements, and identical displacement for both, they are, respectively,

$$\frac{\overline{x'^2}(\sigma,\zeta)}{\kappa^2 d_{c0}^2\sigma_0} = \frac{\varepsilon^2}{\kappa^3}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\left(\frac{\varepsilon}{2\kappa^2}\right)^{m+n} z_{m,n}(\kappa\sigma)f_{m+1}(\zeta)f_{n+1}(\zeta) = \frac{\varepsilon^2}{\kappa^3}\sum_{N=0}^{\infty}\left(\frac{\varepsilon}{2\kappa^2}\right)^{N}\sum_{k=0}^{N} z_{k,N-k}(\kappa\sigma)f_{k+1}(\zeta)f_{N+1-k}(\zeta), \tag{25}$$

$$\frac{\overline{x'^2}(\sigma,\zeta)}{\kappa^2 d_{f0}^2\sigma_0} = \kappa\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\left(\frac{\varepsilon}{2\kappa^2}\right)^{m+n} z_{m,n}(\kappa\sigma)f_m(\zeta)f_n(\zeta) = \kappa\sum_{N=0}^{\infty}\left(\frac{\varepsilon}{2\kappa^2}\right)^{N}\sum_{k=0}^{N} z_{k,N-k}(\kappa\sigma)f_k(\zeta)f_{N-k}(\zeta), \tag{26}$$

$$\begin{aligned}\frac{\overline{x'^2}(\sigma,\zeta)}{\kappa^2 d_0^2\sigma_0} &= \frac{1}{\kappa^3}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\left(\frac{\varepsilon}{2\kappa^2}\right)^{m+n} z_{m,n}(\kappa\sigma)[\kappa^2 f_m(\zeta)-\varepsilon f_{m+1}(\zeta)][\kappa^2 f_n(\zeta)-\varepsilon f_{n+1}(\zeta)]\\ &= \frac{1}{\kappa^3}\sum_{N=0}^{\infty}\left(\frac{\varepsilon}{2\kappa^2}\right)^{N}\sum_{k=0}^{N} z_{k,N-k}(\kappa\sigma)[\kappa^2 f_k(\zeta)-\varepsilon f_{k+1}(\zeta)][\kappa^2 f_{N-k}(\zeta)-\varepsilon f_{N+1-k}(\zeta)],\end{aligned} \tag{27}$$

where

$$z_{m,n}(t) = \frac{1}{m!n!}\int_0^t t^{m+n}\left(\frac{\pi t}{2}\right)J_{m-(1/2)}(t)J_{n-(1/2)}(t)dt, \tag{28}$$

$$z_{0,0}(t)=\frac{t}{2}+\frac{1}{4}\sin 2t, \tag{29a}$$

$$z_{1,0}(t)=z_{0,1}(t)=-\frac{t}{4}\cos 2t+\frac{1}{8}\sin 2t, \tag{29b}$$

$$z_{11}(t)=\frac{t^3}{6}-\frac{t}{4}\cos 2t+\frac{-2t^2+1}{8}\sin 2t, \tag{29c}$$

$$z_{20}(t)=z_{02}(t)=-\frac{t^3}{12}-\frac{t}{4}\cos 2t+\frac{-t^2+1}{8}\sin 2t. \tag{29d}$$

The expressions for the mean-square displacements given by Eqs. (21) and (23)–(27) are quite general since no assumptions have been made so far on the ζ-dependencies of the beam current $F(\zeta)$ or the wake function $w(\zeta)$. These dependencies are incorporated in the functions of $f_n(\zeta)$ defined by Eq. (5). The functions $f_n(\zeta)$ were calculated for a number of beam and accelerator configurations in [1] and, in the remainder of this paper, we will apply Eqs. (21) and (23)–(27) to the same configurations.

III. SINGLE VERY SHORT BUNCH

As in [1], by very short bunch we imply that the bunch is much shorter than the wavelength of the deflecting mode—so the wake function can be assumed to be linear in ζ—and that the current density is constant during the bunch. Under these assumptions the functions $f_n(\zeta)$ are

$$f_n(\zeta) = \frac{\zeta^{2n}}{(2n!)}, \tag{30}$$

and the mean-square displacement is given, from Eq. (21), by

$$\overline{x^2}(\sigma,\zeta) = \frac{d_{c0}^2\sigma_0\varepsilon^2}{\kappa^3}\sum_{N=0}^{\infty}\left(\frac{\varepsilon}{2\kappa^2}\right)^N\sum_{k=0}^{N} y_{k,N-k}(\kappa\sigma)f_{k+1}(\zeta)f_{N+1-k}(\zeta)$$
$$= \frac{d_{c0}^2\sigma_0\varepsilon^2\zeta^4}{\kappa^3}\sum_{N=0}^{\infty}\left(\frac{\varepsilon\zeta^2}{2\kappa^2}\right)^N\sum_{k=0}^{N}\frac{\int_0^{\kappa\sigma} du u^N(\frac{\pi u}{2})J_{k+(1/2)}(u)J_{N-k+(1/2)}(u)}{k!(N-k)!(2k+2)!(2N+2-2k)!}. \tag{31}$$

The sum over k in the above equation is, in principle, calculable to arbitrary order N and involve only combinations of powers and circular functions of $\kappa\sigma$. The dominant terms for the evolution of $\overline{x^2}(\sigma,\zeta)$, however, are the polynomials of $\kappa\sigma$. Keeping only those terms and ignoring the circular functions we obtain for $\overline{x^2}(\sigma,\zeta)$

$$\overline{x^2}(\sigma,\zeta) = \frac{d_{c0}^2\sigma_0\varepsilon^2\zeta^4\sigma}{8\kappa^2}\left[1+\frac{1}{12}\frac{\varepsilon\zeta^2}{\kappa^2} + \frac{1}{2880}\left(\frac{\varepsilon\zeta^2}{\kappa^2}\right)^2(11+\kappa^2\sigma^2)+\cdots\right]. \tag{32}$$

This result is similar to that first obtained in Ref. [2] but with small differences. In [2] the second term in the bracket is missing and the coefficient 2880 is replaced by 1728.

The expression for $\overline{x^2}(\sigma,\zeta)$ given in Eq. (31) is valid in the case of strong focusing since it is a series expansion in powers of ε/κ^2 . In the absence of focusing we have

$$i_n(0,\sigma) = \frac{\sigma^{2n+1}}{(2n+1)!}, \tag{33}$$

and

$$\int_0^{\sigma} i_m(0,u)i_n(0,u)du = \frac{\sigma^{2m+2n+3}}{(2m+2n+3)(2m+1)!(2n+1)!}. \tag{34}$$

The mean-square displacement is then

$$\overline{x^2}(\sigma,\zeta) = d_{c0}^2\sigma_0\varepsilon^2\sigma^3\zeta^4\sum_{N=0}^{\infty}\frac{(\varepsilon\sigma^2\zeta^2)^N}{2N+3}\sum_{k=0}^{N}\frac{1}{(2k+1)!(2k+2)!(2N-2k+1)!(2N-2k+2)!}$$
$$= \frac{d_{c0}^2\sigma_0\varepsilon^2\sigma^3\zeta^4}{12}\left[1+\frac{\varepsilon\sigma^2\zeta^2}{60}+\frac{11}{302\,400}(\varepsilon\sigma^2\zeta^2)^2+\cdots\right]. \tag{35}$$

In the case of displaced focusing elements the mean-square displacement of the bunch can be obtained in a fashion similar to the case of displaced cavities but is now given by

$$\overline{x^2}(\sigma,\zeta) = d_{f0}^2\sigma_0\kappa\sum_{N=0}^{\infty}\varepsilon^N\sum_{k=0}^{N} y_{k,N-k}(\kappa\sigma)f_k(\zeta)f_{N-k}(\zeta)$$
$$= d_{f0}^2\sigma_0\kappa\sum_{N=0}^{\infty}\left(\frac{\varepsilon\zeta^2}{2\kappa^2}\right)^N\sum_{k=0}^{N}\frac{\int_0^{\kappa\sigma} du\, u^N(\frac{\pi u}{2})J_{k+(1/2)}(u)J_{N-k+(1/2)}(u)}{k!(N-k)!(2k)!(2N-2k)!}, \tag{36}$$

which, to lowest order, gives

$$\overline{x^2}(\sigma,\zeta) = \frac{d_{f0}^2\sigma_0\kappa^2\sigma}{2}\left[1+\frac{1}{2}\frac{\varepsilon\zeta^2}{\kappa^2}+\frac{1}{288}\left(\frac{\varepsilon\zeta^2}{\kappa^2}\right)^2(27+5\kappa^2\sigma^2)\ +\ \cdots\right]. \tag{37}$$

In the case of displaced cavities we see from Eq. (32) that there is no rms displacement of the bunch unless there is coupling between the bunch and the deflecting mode ($\varepsilon \neq 0$), that the front of the bunch ($\zeta = 0$) is not deflected, and that the bunch develops a tail that varies as ζ^2 and grows as $\kappa^{-1}\varepsilon\sigma^{1/2}$.

In the case of displaced focusing elements we see from Eq. (37) that there is an rms displacement of the front of the bunch that grows as $\kappa\sigma^{1/2}$ and that the bunch develops a tail that varies as ζ^4 and grows as $\kappa^{-1}\varepsilon^2\sigma^{5/2}$.

This clearly shows the influence of the focusing strength κ. Strong focusing reduces the effect of the coupling between the beam and the dipole modes and inhibits the formation of the tail; on the other hand, it increases the sensitivity of the displacement of the bunch as a whole to the displacement of the focusing elements.

IV. STEADY-STATE DC AND DELTA-FUNCTION BEAMS

As was shown in [1], in the case of a dc or delta-function beam (beam composed of pointlike bunches separated by τ in the laboratory frame), the functions $f_n(\zeta)$ in the steady state regime are easily calculated. For the dc beam they are

$$\lim_{\zeta\to+\infty} f_n(\zeta) = [\tilde{w}(0)]^n = \left[\int_0^\infty w(\zeta)d\zeta\right]^n, \tag{38}$$

where $\tilde{w}(Z)$ is the Fourier transform of the wake function $w(\zeta)$ and, in the case of a delta-function beam, they are

$$\lim_{M\to+\infty} f_n(M\omega\tau) = [\tilde{W}(0)]^n = \left[\omega\tau\sum_{k=0}^{\infty} w(k\omega\tau)\right]^n, \tag{39}$$

where $\tilde{W}(Z) = \sum_{k=-\infty}^{\infty}\tilde{w}[Z - \frac{2\pi}{\omega\tau}k]$. So, in both cases, $f_n(\infty)$ is of the form

$$f_n(\infty) = [f_1(\infty)]^n \equiv \varpi^n. \tag{40}$$

In the case of a single deflecting mode, where $w(\zeta) = u(\zeta)\sin\zeta e^{-\zeta/(2Q)}$, we have for the dc beam

$$\varpi = \int_0^\infty e^{-\zeta/(2Q)}\sin\zeta\, d\zeta = \frac{4Q^2}{4Q^2+1}, \tag{41}$$

and for the δ-function beam

$$\varpi = \omega\tau\sum_{k=0}^{\infty} e^{-k\omega\tau/(2Q)}\sin k\omega\tau = \frac{\omega\tau}{2}\frac{\sin\omega\tau}{\cosh\frac{\omega\tau}{2Q} - \cos\omega\tau}. \tag{42}$$

Assuming that only the cavities are displaced, Eq. (10) yields

$$\begin{aligned} x^\dagger(p,\infty) &= -d_c^\dagger(p)\sum_{n=0}^{\infty}\left(\frac{\varepsilon\varpi}{p^2+\kappa^2}\right)^{n+1} \\ &= -\varepsilon\varpi\frac{d_c^\dagger(p)}{p^2+\kappa^2-\varepsilon\varpi}, \end{aligned} \tag{43}$$

and

$$x(\sigma,\infty) = -\frac{\varepsilon\varpi}{\lambda}\int_0^\sigma d_c(\sigma-u)\sin\lambda u\, du, \tag{44}$$

where

$$\lambda^2 = \kappa^2 - \varepsilon\varpi. \tag{45}$$

The mean-square displacement is then

$$\overline{x^2}(\sigma,\infty) = \frac{\varepsilon^2\varpi^2}{\lambda^2}\int_0^\sigma dv\int_0^\sigma du \sin\lambda(\sigma-v)\sin\lambda(\sigma-u) \times R_{d_c}(u-v) \tag{46}$$

$$= d_{c0}^2\sigma_0\frac{\varepsilon^2\varpi^2}{\lambda^2}\int_0^\sigma \sin^2\lambda u\, du, \tag{47}$$

$$\overline{x^2}(\sigma,\infty) = \begin{cases} \frac{d_{c0}^2\sigma_0}{2}\frac{\varepsilon^2\varpi^2}{\kappa^2-\varepsilon\varpi}\{\sigma - \frac{\sin[2\sigma(\kappa^2-\varepsilon\varpi)^{1/2}]}{2(\kappa^2-\varepsilon\varpi)^{1/2}}\}, & \kappa^2-\varepsilon\varpi > 0, \\ \frac{d_{c0}^2\sigma_0\varepsilon^2\varpi^2\sigma^3}{3}, & \kappa^2-\varepsilon\varpi = 0, \\ \frac{d_{c0}^2\sigma_0}{2}\frac{\varepsilon^2\varpi^2}{\varepsilon\varpi-\kappa^2}\{\frac{\sinh[2\sigma(\varepsilon\varpi-\kappa^2)^{1/2}]}{2(\varepsilon\varpi-\kappa^2)^{1/2}} - \sigma\}, & \kappa^2-\varepsilon\varpi < 0. \end{cases} \tag{48}$$

In a configuration where the beam would be inherently stable ($\kappa^2 - \varepsilon\varpi > 0$) the rms displacement of the beam increases as $\sigma^{1/2}$, while, for a beam that would be unstable ($\kappa^2 - \varepsilon\varpi < 0$), it increases exponentially with σ.

When only the focusing elements are displaced, the position dependence of the mean-square displacement of the beam is given by

$$\overline{x^2}(\sigma,\infty) = \begin{cases} \frac{d_{f0}^2\sigma_0}{2}\frac{\kappa^4}{\kappa^2-\varepsilon\varpi}\{\sigma - \frac{\sin[2\sigma(\kappa^2-\varepsilon\varpi)^{1/2}]}{2(\kappa^2-\varepsilon\varpi)^{1/2}}\}, & \kappa^2-\varepsilon\varpi > 0, \\ \frac{d_{f0}^2\sigma_0\kappa^4\sigma^3}{3}, & \kappa^2-\varepsilon\varpi = 0, \\ \frac{d_{f0}^2\sigma_0}{2}\frac{\kappa^4}{\varepsilon\varpi-\kappa^2}\{\frac{\sinh[2\sigma(\varepsilon\varpi-\kappa^2)^{1/2}]}{2(\varepsilon\varpi-\kappa^2)^{1/2}} - \sigma\}, & \kappa^2-\varepsilon\varpi < 0. \end{cases} \tag{49}$$

When both the cavities and the focusing elements are simultaneously and identically displaced the mean-square displacement of the beam is

$$\overline{x^2}(\sigma,\infty) = \begin{cases} \frac{d_0^2\sigma_0}{2}(\kappa^2-\varepsilon\varpi)\{\sigma - \frac{\sin[2\sigma(\kappa^2-\varepsilon\varpi)^{1/2}]}{2(\kappa^2-\varepsilon\varpi)^{1/2}}\}, & \kappa^2-\varepsilon\varpi > 0, \\ 0, & \kappa^2-\varepsilon\varpi = 0, \\ \frac{d_0^2\sigma_0}{2}(\varepsilon\varpi-\kappa^2)\{\frac{\sinh[2\sigma(\varepsilon\varpi-\kappa^2)^{1/2}]}{2(\varepsilon\varpi-\kappa^2)^{1/2}} - \sigma\}, & \kappa^2-\varepsilon\varpi < 0. \end{cases} \tag{50}$$

In this case a circumstance exists where there is no steady-state displacement of the beam anywhere along the accelerator although the transient displacement is finite.

Equivalent expressions for $\overline{x'^2}(\sigma,\infty)$ can be obtained in a similar fashion.

V. FINITE DELTA-FUNCTION BEAM

In Sec. IV the steady-state displacement of an infinite train of pointlike bunches was obtained in closed form. In this section we will analyze the transient displacement of a finite train of bunches.

In this case, as in all others, the transient rms displacement of a delta-function beam under the combined influence of coupling to a deflecting mode and displacement of the cavities is given by Eq. (21) where the functions $y_{m,n}(\kappa\sigma)$ are defined in Eq. (20). For a delta-function beam this can be rewritten as

$$\begin{aligned}\overline{x_M^2}(\sigma) &= \frac{d_{c0}^2\sigma_0\varepsilon^2}{\kappa^3}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\left(\frac{\varepsilon}{2\kappa^2}\right)^{m+n} y_{m,n}(\kappa\sigma)f_{m+1}(M\omega\tau)f_{n+1}(M\omega\tau)\\ &= \frac{d_{c0}^2\sigma_0\varepsilon^2}{\kappa^3}\sum_{N=0}^{\infty}\left(\frac{\varepsilon}{2\kappa^2}\right)^{N}\sum_{k=0}^{N} y_{k,N-k}(\kappa\sigma)f_{k+1}(M\omega\tau)f_{N+1-k}(M\omega\tau). \qquad (51)\end{aligned}$$

As shown in [1], the functions $f_n(M\omega\tau)$ can be defined with the help of the z transform. Defining $\check{w}(z) = \sum_{k=0}^{\infty} w(k\omega\tau)z^{-k}$ as the z transform of the wake function, the z transform of $f_n(M\omega\tau)$ is

$$\check{f}_n(z) = \frac{z}{z-1}[\omega\tau\check{w}(z)]^n. \qquad (52)$$

The functions $f_n(M\omega\tau)$ are then obtained through the use of the inverse z transform

$$f_n(M\omega\tau) = \frac{1}{2\pi i}\oint dz\, z^{M-1}\check{f}_n(z), \qquad (53)$$

where the path of integration is a circle centered at the origin which encloses all the singularities of the integrand.

In the case of a single deflecting mode, the functions $f_n(M\omega\tau)$ can be obtained to arbitrary order n through the use of the inverse z transform and symbolic manipulation software. In the case of more complicated wake functions, the functions $f_n(M\omega\tau)$ can be tabulated through the use of the recursion relations

$$f_0(M\omega\tau) = 1, \qquad (54a)$$

$$f_{n+1}(M\omega\tau) = \omega\tau\sum_{k=0}^{M} f_n(k\omega\tau)w[(M-k)\omega\tau]. \qquad (54b)$$

Since the wake functions under consideration satisfy $w(0) = 0$, the recursion relations (54) imply that $f_n(M\omega\tau) = 0$ for $n > M$. As a consequence, the infinite sums in (51) reduce to finite sums $\sum_{m=0}^{M}\sum_{n=0}^{M}$.

As an example we will apply these analytical results to a beam representative of a room temperature linear collider. For comparison we will use the same parameters as those used in [1,6,7], and which are listed in Table I. Since this is an accelerated beam, the transformations described in the Appendix, and in particular Eqs. (A20)–(A24), will be used. Converting the parameters in Table I to those used in this paper we have $\varepsilon(0) = [w_0 qe\mathcal{L}^2/\gamma(0)mc^2\omega\tau] = 38.02$, $\kappa(0) = 1100\pi$, $\varsigma(\sigma = 1) = 2/11$, and $\omega\tau = 263.014$.

Figure 1 shows the normalized rms lateral displacements (upper graphs, blue points) and angular divergence (lower graphs, red points) due to misalignment of the cavities (left column) and focusing elements (right column). These plots were obtained from Eq. (51), and equivalent ones for $\overline{x_M'^2}$ and for displaced focusing elements, with the sums for m and n extending from 0 to 6.

Of note is the fact that, for this example and for the same average rms value, the displacement of the focusing elements induces rms displacements and angular divergences of the beam that are about 2 orders of magnitude larger than those induced by the displacement of the cavities.

There is an obvious correlation between the rms lateral displacement and angular divergence for all bunches. This is shown clearly in Fig. 2 where the motion in phase space of the bunch train follows closely a straight line going through the origin. It should be remembered that Figs. 1 and 2 are plots of the rms displacements for an ensemble of accelerators. For an actual accelerator the motion of the successive bunches in phase space may not follow a straight line; as shown in Sec. IV the straight line motion in phase space is true to first order in ε but may not be to higher order.

Figure 3 shows, for the same example, the rms lateral displacement and angular divergence of the last bunch ($M = 90$) as it travels along the accelerator for random displacement of the cavities and focusing elements. In this figure, as in previous ones, the effects of acceleration including adiabatic damping described in the Appendix [Eqs. (A20)–(A24)] have been included.

The results of Eqs. (21) and (23)–(27) exemplified in Figs. 1 and 2 can be used to calculate the emittance growth caused by the random displacement of the accelerator components. Similar estimates have been done

TABLE I. Nominal top-level linear-collider design parameters [6,7]

Parameter	Value
Total initial energy $\gamma(0)mc^2$	10 GeV
Total final energy $\gamma(1)mc^2$	1 TeV
Linac length $\mathcal{L}$	10 km
Number of betatron periods	100
Bunch charge	1 nC
Number of bunches in train M	90
Bunch spacing τ	2.8 ns
Deflecting-wake frequency $\omega/2\pi$	14.95 GHz
Deflecting-wake quality factor Q	∞
Deflecting-wake amplitude w_0	10^{15} V C^{-1} m^{-2}

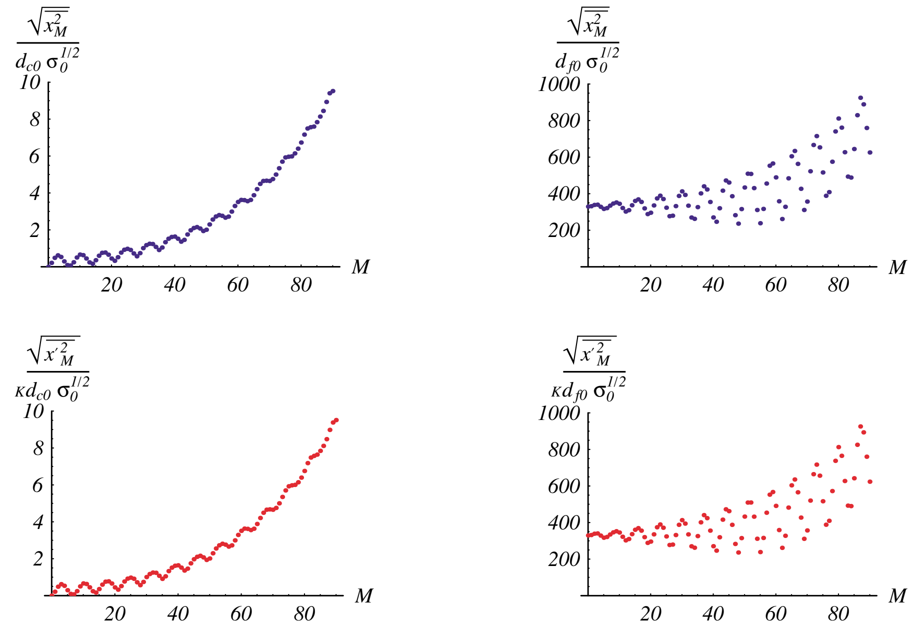

FIG. 1. (Color) Normalized rms lateral displacement (upper row, blue points) and angular divergence (lower row, red points) of a finite train of pointlike bunches at the exit of a representative linear collider due to misalignment of the cavities (left column) or focusing elements (right column). See Table I for the choice of parameters.

previously [8,9]. However, in those previous analyses, only the displacement of the cavities was considered and it was assumed that, in the language of [8], the "betatron term" was small compared to the "displacement term"; this corresponds to neglecting $x(\sigma, \zeta_1)$ with respect to $d_c(\sigma)$ in the right-hand side of Eq. (1). This is also equivalent to keeping only the term $m = n = 0$ (or $N = 0$) in Eq. (21) and similar ones. Although possibly pessimistic, the example of Table I required keeping terms up to $m = n = 6$ in order to obtain convergence.

In the previous example the Q of the deflecting mode was infinite and thus no steady state was achieved. Figure 4 shows the mean-square displacement at the exit of the accelerator due to cavity misalignment in the case of modest BBU coupling ($\varepsilon = 0.1$), and close to a resonance with a low-Q deflecting mode ($\omega\tau = 4.01\pi$, $Q = 1500$).

Figure 5 shows, for the same parameters, the mean-square displacement along the accelerator of bunch $M = 500$ (blue curve), $M = 600$ (green curve), and $M = \infty$ (red curve). The curves for $M = 500$ and $M = 600$ were obtained from Eq. (51) , while the curve for $M = \infty$ was obtained from Eq. (48). All three curves clearly show a $\sigma^{1/2}$ dependence of the rms displacement.

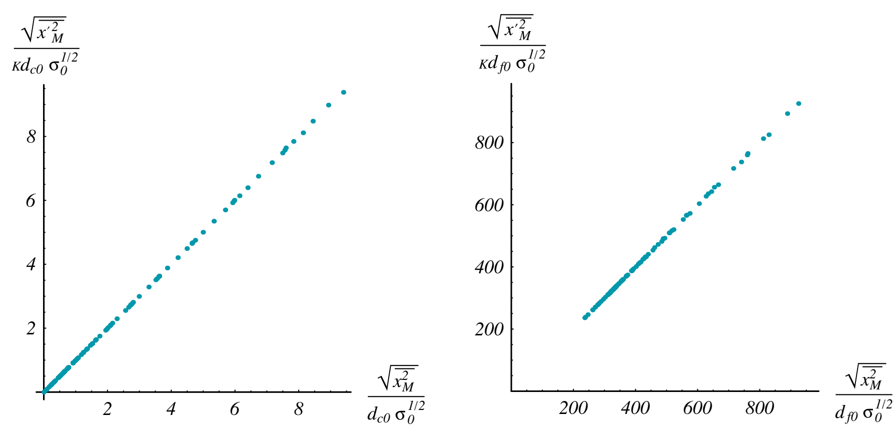

FIG. 2. (Color) Phase space motion of the normalized rms lateral displacement and angular divergence of a finite train of pointlike bunches at the exit of a representative linear collider due to misalignment of the cavities (left) or focusing elements (right). See Table I for the choice of parameters.

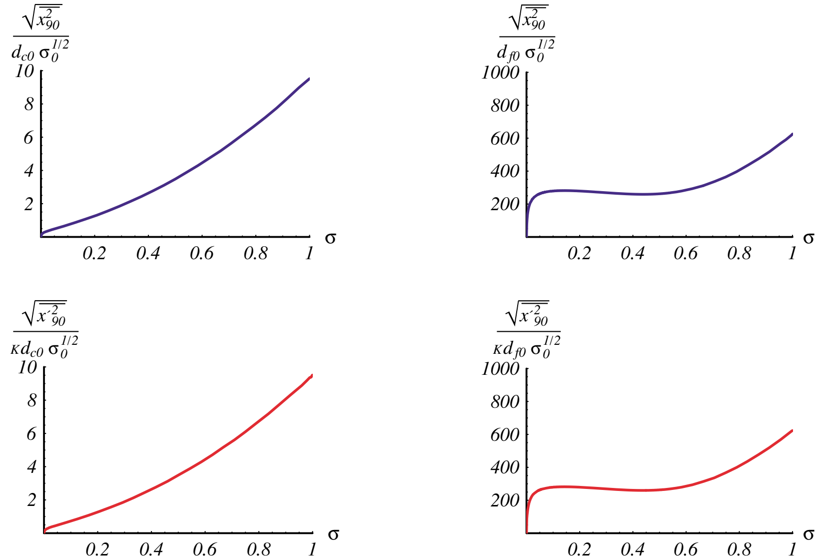

FIG. 3. (Color) Normalized rms lateral displacement (upper row, blue curves) and angular divergence (lower row, red curves) for the last bunch as a function of position in the accelerator due to misalignment of the cavities (left column) and focusing elements (right column). See Table I for the choice of parameters.

IV. CUMULATIVE BBU UNDER THE COMBINED INFLUENCE OF INJECTION OFFSETS AND MISALIGNMENTS

In Ref. [1] cumulative BBU in the presence of injection offsets was analyzed in detail and, in the previous sections of this paper, the analysis was restricted to the case of a beam injected in a misaligned accelerator without lateral or angular offsets. In this section the analysis is extended to the behavior of a beam experiencing BBU in the presence of injection offsets and misalignments. As has been observed before [2,10], the addition of injection offsets can reduce the BBU-induced emittance growth that would be produced from misalignments alone.

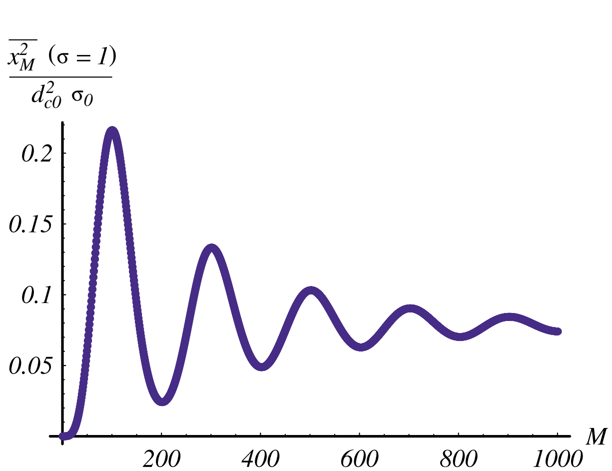

FIG. 4. (Color) Mean-square displacement of a train of pointlike bunches at the exit of the accelerator due to cavity misalignment for $\varepsilon = 0.1$, $\kappa = 100$, $\omega\tau = 4.01\pi$, $Q = 1500$.

If the injection offsets are assumed to be time independent—$x_0(\zeta) = x_0$, $x_0'(\zeta) = x_0'$—Eqs. (5) and (6) imply $f_n(\zeta) = g_n(\zeta) = h_n(\zeta)$ and, in particular, $f_0(\zeta) = g_0(\zeta) = h_0(\zeta) = 1$, and expressions for $x(\sigma, \zeta)$ and $x'(\sigma, \zeta)$ are

$$\begin{aligned} x(\sigma, \zeta) = & \sum_{n=0}^{\infty} \varepsilon^n f_n(\zeta)[x_0 j_n(\kappa, \sigma) + x_0' i_n(\kappa, \sigma)] \\ & - \sum_{n=0}^{\infty} \varepsilon^{n+1} f_{n+1}(\zeta) i_n(\kappa, \sigma) * d_c(\sigma) \\ & + \kappa^2 \sum_{n=0}^{\infty} \varepsilon^n f_n(\zeta) i_n(\kappa, \sigma) * d_f(\sigma), \end{aligned} \tag{55}$$

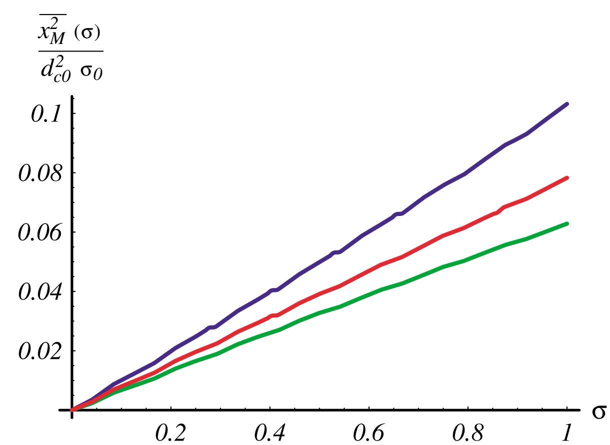

FIG. 5. (Color) Mean-square displacement along the accelerator for bunch $M = 500$ (blue curve), $M = 600$ (green curve), and $M = \infty$ (red curve). The parameters are the same as in Fig. 4.

$$x'(\sigma, \zeta) = -x_0\left\{\kappa^2 i_0(\kappa, \sigma) + \sum_{n=1}^{\infty} \varepsilon^n f_n(\zeta)[i_{n-1}(\kappa, \sigma) - \kappa^2 i_n(\kappa, \sigma)]\right\} + x_0' \sum_{n=0}^{\infty} \varepsilon^n f_n(\zeta) j_n(\kappa, \sigma)$$
$$- \sum_{n=0}^{\infty} \varepsilon^{n+1} f_{n+1}(\zeta) j_n(\kappa, \sigma) * d_c(\sigma) + \kappa^2 \sum_{n=0}^{\infty} \varepsilon^n f_n(\zeta) j_n(\kappa, \sigma) * d_f(\sigma). \tag{56}$$

To zeroth order in ε this reduces to

$$x(\sigma, \zeta) = x_0 j_0(\kappa, \sigma) + x_0' i_0(\kappa, \sigma) + \kappa^2 i_0(\kappa, \sigma) * d_f(\sigma) + O(\varepsilon), \tag{57}$$

$$x'(\sigma, \zeta) = -x_0 \kappa^2 i_0(\kappa, \sigma) + x_0' j_0(\kappa, \sigma) + \kappa^2 j_0(\kappa, \sigma) * d_f(\sigma) + O(\varepsilon). \tag{58}$$

This indicates that, to zeroth order in ε, the lateral and angular displacements at any location in the accelerator are independent of time ζ. This implies, in particular, that a finite-length bunch will not be distorted and that, in a train of pointlike bunches, all will be displaced by the same amount. In other words, the zeroth order terms will produce a constant displacement in phase space and will not contribute to emittance growth, and can be ignored.

To first order in ε we then have

$$x(\kappa, \sigma) = \varepsilon f_1(\zeta)[x_0 j_1(\kappa, \sigma) + x_0' i_1(\kappa, \sigma) - i_0(\kappa, \sigma) * d_c(\sigma) + \kappa^2 i_1(\kappa, \sigma) * d_f(\sigma)] + O(\varepsilon^2), \tag{59}$$

$$x'(\kappa, \sigma) = \varepsilon f_1(\zeta)\{x_0[i_0(\kappa, \sigma) - \kappa^2 i_1(\kappa, \sigma)] + x_0' j_1(\kappa, \sigma) - j_0(\kappa, \sigma) * d_c(\sigma) + \kappa^2 j_1(\kappa, \sigma) * d_f(\sigma)\} + O(\varepsilon^2). \tag{60}$$

To first order in ε, $x(\kappa, \sigma)$ and $x'(\kappa, \sigma)$ are proportional to $f_1(\zeta)$. This implies that, as a function of time, the particles will move in phase space on a straight line through the origin [8]. This will not be true to higher order in ε.

Equations (59) and (60) also imply that, by a judicious choice of injection offsets x_0 and x_0', the motion in phase space can be eliminated to first order in ε. This will occur when

$$x_0 = \frac{j_1[i_0 * d_c - \kappa^2 i_1 * d_f] - i_1[j_0 * d_c - \kappa^2 j_1 * d_f]}{j_1^2 - i_1[i_0 - \kappa^2 i_1]}, \qquad x_0' = \frac{j_1[j_0 * d_c - \kappa^2 j_1 * d_f] - (i_0 - \kappa^2 i_1)[i_0 * d_c - \kappa^2 i_1 * d_f]}{j_1^2 - i_1[i_0 - \kappa^2 i_1]}. \tag{61}$$

With these values of x_0 and x_0', the motion of the particles in phase space, and therefore the emittance growth, will be of second order in ε. It can be noted that the values of x_0 and x_0' that eliminate the first order displacement in phase space are functions of σ. This cancellation occurs only at a predeterminate location in the accelerator but not everywhere.

Since $d_c(\sigma)$ and $d_f(\sigma)$ are not usually known, the values of x_0 and x_0' that cancel the first order displacement cannot be calculated *a priori* but must be determined experimentally. It is also often the case that $d_c(\sigma)$ and $d_f(\sigma)$ are time-dependent drifts due to the environment, and x_0 and x_0' will need to be adjusted periodically or continuously to maintain the first order cancellation.

In situations where BBU is sufficiently strong, the emittance growth is not dominated by the first order effect in ε, and the injection offsets given in Eq. (61) will not lead to a substantial reduction in emittance growth. Nevertheless, values of x_0 and x_0' can be found to cancel the effect of a particular order or, more generally, minimize the emittance growth.

ACKNOWLEDGMENTS

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APPENDIX: ACCELERATED BEAM

In the body of this paper we investigate BBU for a coasting beam in a uniform accelerator. As shown in [1], under reasonable assumptions, the results for a coasting beam can be extended to the case of an accelerated beam by the introduction of suitable variable and coordinate transformations.

In the general case of an accelerated relativistic beam, where now γ, κ, and ε can vary with σ, the equation for the transverse displacement is

$$\frac{1}{\gamma}\frac{\partial}{\partial\sigma}\left[\gamma\frac{\partial}{\partial\sigma}x(\sigma, \zeta)\right] + \kappa^2[x(\sigma, \zeta) - d_f(\sigma)] = \varepsilon\int_{-\infty}^{\zeta} w(\zeta - \zeta_1)F(\zeta_1)[x(\sigma, \zeta_1) - d_c(\sigma)]d\zeta_1. \tag{A1}$$

Assuming that $\varepsilon \propto \gamma^{-1}$ and $\kappa \propto \gamma^{-1/2}$, and defining γ_r, ε_r, and κ_r as the values of $\gamma(\sigma)$, $\varepsilon(\sigma)$, and $\kappa(\sigma)$ at an arbitrary reference location σ_r in the accelerator, and $\psi(\sigma)$ as

$$\psi(\sigma) = \frac{\gamma(\sigma)}{\gamma_r}, \tag{A2}$$

we have $\varepsilon(\sigma) = \varepsilon_r/\psi(\sigma)$ and $\kappa(\sigma) = \kappa_r/\psi^{1/2}(\sigma)$. We now introduce new variables ξ, δ_f, and δ_c for the transverse displacements

$$x(\sigma,\zeta) = \xi(\sigma,\zeta)[\psi(\sigma)]^{-1/4}, \tag{A3a}$$

$$d_f(\sigma,\zeta) = \delta_f(\sigma,\zeta)[\psi(\sigma)]^{-1/4}, \tag{A3b}$$

$$d_c(\sigma,\zeta) = \delta_c(\sigma,\zeta)[\psi(\sigma)]^{-1/4}, \tag{A3c}$$

and ς for the longitudinal location along the linac,

$$\varsigma = \int_0^{\sigma}[\psi(\sigma')]^{-1/2}d\sigma'. \tag{A4}$$

With these new variables the equation of motion becomes

$$\frac{\partial^2}{\partial\varsigma^2}\xi(\varsigma,\zeta) + \left[-\frac{\psi''}{4} + \frac{\psi'^2}{16\psi} + \kappa_r^2\right]\xi(\varsigma,\zeta) - \kappa_r^2\delta_f(\varsigma,\zeta) = \varepsilon_r\int_{-\infty}^{\zeta} w(\zeta-\zeta_1)F(\zeta_1)[\xi(\varsigma,\zeta_1) - \delta_c(\varsigma,\zeta_1)]d\zeta_1. \tag{A5}$$

Since in most applications we have $-\frac{\psi''}{4} + \frac{\psi'^2}{16\psi} \ll \kappa_r^2$, the equation of motion for $\xi(\varsigma,\zeta)$ in the accelerated case is identical to that for $x(\sigma,\zeta)$ in the coasting case.

Thus, with the assumptions made above, the results obtained for a coasting beam are directly applicable to an accelerated beam after the appropriate change of variables and coordinates, and the general solution for $\xi(\varsigma,\zeta)$ is given by [1]

$$\xi(\varsigma,\zeta) = \sum_{n=0}^{\infty}\varepsilon_r^n[\xi_0\mathfrak{h}_n(\zeta)j_n(\kappa_r,\varsigma) + \xi_0'\mathfrak{g}_n(\zeta)i_n(\kappa_r,\varsigma)(\zeta)] - \sum_{n=0}^{\infty}\varepsilon_r^{n+1}f_{n+1}(\zeta)i_n(\kappa_r,\varsigma)*\delta_c(\varsigma) + \kappa_r^2\sum_{n=0}^{\infty}\varepsilon_r^n f_n(\zeta)i_n(\kappa_r,\varsigma)*\delta_f(\varsigma). \tag{A6}$$

The functions $\mathfrak{h}_n(\zeta)$ and $\mathfrak{g}_n(\zeta)$ obey the same recursion relations as $h_n(\zeta)$ and $g_n(\zeta)$ but we now have

$$\xi_0\mathfrak{h}_0(\zeta) = \xi_0(\zeta) = x_0(\zeta)[\psi(0)]^{1/4} = x_0h_0(\zeta)[\psi(0)]^{1/4}, \tag{A7a}$$

$$\xi_0'\mathfrak{g}_0(\zeta) = \frac{d}{d\varsigma}\xi(\varsigma,\zeta)|_{\varsigma=0} = \left[x_0'(\zeta) + \frac{1}{4}\frac{\psi'(0)}{\psi(0)}x_0(\zeta)\right][\psi(0)]^{3/4} = \left[x_0'g_0(\zeta) + \frac{1}{4}\frac{\psi'(0)}{\psi(0)}x_0h_0(\zeta)\right][\psi(0)]^{3/4}. \tag{A7b}$$

As we did in the body of the paper, we will assume that the beam enters the accelerator without lateral displacement and angular divergence. Since the case of displaced focusing elements can be treated in a fashion similar to the case of displaced cavities, we will assume that only the cavities are displaced and $\xi(\varsigma,\zeta)$ is given by

$$\xi(\varsigma,\zeta) = -\sum_{n=0}^{\infty}\varepsilon_r^{n+1}f_{n+1}(\zeta)i_n(\kappa_r,\varsigma)*\delta_c(\varsigma). \tag{A8}$$

By analogy with the results obtained for the coasting beam in Sec. II, the mean-square displacement for an accelerated beam is then

$$\overline{\xi^2}(\varsigma,\zeta) = \varepsilon_r^2\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\varepsilon_r^{m+n}f_{m+1}(\zeta)f_{n+1}(\zeta)\int_0^{\varsigma}\int_0^{\varsigma}du\,dv\,i_m(\kappa_r,\varsigma-u)i_n(\kappa_r,\varsigma-v)R_{\delta_c}(u,v), \tag{A9}$$

where $R_{\delta_c}(\varsigma_1,\varsigma_2)$ is the autocorrelation function of $\delta_c(\varsigma)$.

With the assumed form for the autocorrelation function $R_{d_c}(\sigma_1,\sigma_2)$ of $d_c(\sigma)$ shown in Eq. (18), $R_{\delta_c}(\varsigma_1,\varsigma_2)$ can be simply related to $R_{d_c}(\sigma_1,\sigma_2)$:

$$R_{\delta_c}(\sigma_1,\sigma_2) = \langle\delta_c(\sigma_1)\delta_c(\sigma_2)\rangle = [\psi(\sigma_1)\psi(\sigma_2)]^{1/4}\langle(d_c(\sigma_1)d_c(\sigma_2)\rangle = \psi^{1/2}(\sigma_1)d_{c0}^2\sigma_0\delta(\sigma_1-\sigma_2). \tag{A10}$$

Since

$$\frac{d\varsigma}{d\sigma} = \psi^{-1/2}(\sigma), \tag{A11}$$

we have

$$R_{\delta_c}(\varsigma_1,\varsigma_2) = \psi^{1/2}(\sigma_1)d_{c0}^2\sigma_0\delta(\varsigma_1-\varsigma_2)\psi^{-1/2}(\sigma_1) = d_{c0}^2\sigma_0\delta(\varsigma_1-\varsigma_2). \tag{A12}$$

A similar expression was used in [11].

Equation (21) is then directly applicable to $\overline{\xi^2}(\varsigma,\zeta)$,

$$\overline{\xi^2}(\varsigma,\zeta) = d_{c0}^2\sigma_0\frac{\varepsilon_r^2}{\kappa_r^3}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\left(\frac{\varepsilon_r}{2\kappa_r^2}\right)^{m+n}y_{m,n}(\kappa_r\varsigma)f_{m+1}(\zeta)f_{n+1}(\zeta), \tag{A13}$$

and the mean-square displacement $\overline{x^2}(\sigma,\zeta)$ becomes

$$\overline{x^2}(\sigma, \zeta) = \psi^{-1/2}(\sigma)\overline{\xi^2}(\varsigma, \zeta), \tag{A14}$$

$$\frac{\overline{x^2}(\sigma, \zeta)}{d_{c0}^2\sigma_0} = \psi^{-1/2}(\sigma)\frac{\varepsilon_r^2}{\kappa_r^3}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\left(\frac{\varepsilon_r}{2\kappa_r^2}\right)^{m+n} y_{m,n}(\kappa_r\varsigma)f_{m+1}(\zeta)f_{n+1}(\zeta). \tag{A15}$$

An expression for the mean-square angular divergence $\overline{x'^2}(\sigma, \zeta)$ can be obtained in a similar fashion. In the case of displaced cavities, $\frac{d\xi}{d\varsigma}$ is given by

$$\frac{d\xi}{d\varsigma}(\varsigma, \zeta) = -\sum_{n=0}^{\infty}\varepsilon_r^{n+1}f_{n+1}(\zeta)j_n(\kappa_r, \varsigma) * \delta_c(\varsigma), \tag{A16}$$

which leads to

$$\overline{\left(\frac{d\xi}{d\varsigma}\right)^2}(\varsigma, \zeta) = d_{c0}^2\sigma_0\frac{\varepsilon_r^2}{\kappa_r}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\left(\frac{\varepsilon_r}{2\kappa_r^2}\right)^{m+n} z_{m,n}(\kappa_r\varsigma)f_{m+1}(\zeta)f_{n+1}(\zeta), \tag{A17}$$

$$\overline{\left(\frac{dx}{d\sigma}\right)^2}(\sigma, \zeta) = \psi^{-3/2}(\sigma)d_{c0}^2\sigma_0\frac{\varepsilon_r^2}{\kappa_r}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\left(\frac{\varepsilon_r}{2\kappa_r^2}\right)^{m+n} z_{m,n}(\kappa_r\varsigma)f_{m+1}(\zeta)f_{n+1}(\zeta). \tag{A18}$$

Since $\kappa_r^2\psi^{-1}(\sigma) = \kappa^2(\sigma)$, we have

$$\frac{1}{\kappa^2(\sigma)d_{c0}^2\sigma_0}\overline{\left(\frac{dx}{d\sigma}\right)^2}(\sigma, \zeta) = \psi^{-1/2}(\sigma)\frac{\varepsilon_r^2}{\kappa_r^3}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\left(\frac{\varepsilon_r}{2\kappa_r^2}\right)^{m+n} z_{m,n}(\kappa_r\varsigma)f_{m+1}(\zeta)f_{n+1}(\zeta). \tag{A19}$$

In the case of a uniformly accelerated relativistic beam we have

$$\frac{\overline{x^2}(\sigma, \zeta)}{d_{c0}^2\sigma_0} = (1 + \bar{\gamma}\sigma)^{-1/2}\frac{\varepsilon_0^2}{\kappa_0^3}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\left(\frac{\varepsilon_0}{2\kappa_0^2}\right)^{m+n} y_{m,n}(\kappa_0\varsigma)f_{m+1}(\zeta)f_{n+1}(\zeta), \tag{A20}$$

$$\frac{\overline{x'^2}(\sigma, \zeta)}{\kappa^2(\sigma)d_{c0}^2\sigma_0} = (1 + \bar{\gamma}\sigma)^{-1/2}\frac{\varepsilon_0^2}{\kappa_0^3}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\left(\frac{\varepsilon_0}{2\kappa_0^2}\right)^{m+n} z_{m,n}(\kappa_0\varsigma)f_{m+1}(\zeta)f_{n+1}(\zeta), \tag{A21}$$

where

$$\gamma(\sigma) = \gamma_0(1 + \bar{\gamma}\sigma), \qquad \varsigma = \int_0^{\sigma} du(1 + \bar{\gamma}u)^{-1/2} = \frac{2\sigma}{(1 + \bar{\gamma}\sigma)^{1/2} + 1}, \tag{A22}$$

and where ε_0 and κ_0 are the BBU and focusing strengths, respectively, at the entrance of the accelerator. Equation (A20) is identical to Eq. (21) but with the addition of the adiabatic damping prefactor and the replacement of ε, κ, and σ with ε_0, κ_0, and ς, respectively. Equation (A21) is identical to Eq. (25) with the same changes; notice, however, that in the left-hand side, κ has been replaced by $\kappa(\sigma)$, the focusing wave number at the location in the accelerator under consideration.

In the case of misaligned focusing elements we have

$$\frac{\overline{x^2}(\sigma, \zeta)}{d_{f0}^2\sigma_0} = (1 + \bar{\gamma}\sigma)^{-1/2}\kappa_0\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\left(\frac{\varepsilon_0}{2\kappa_0^2}\right)^{m+n} y_{m,n}(\kappa_0\varsigma)f_m(\zeta)f_n(\zeta), \tag{A23}$$

$$\frac{\overline{x'^2}(\sigma, \zeta)}{\kappa^2(\sigma)d_{f0}^2\sigma_0} = (1 + \bar{\gamma}\sigma)^{-1/2}\kappa_0\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\left(\frac{\varepsilon_0}{2\kappa_0^2}\right)^{m+n} z_{m,n}(\kappa_0\varsigma)f_m(\zeta)f_n(\zeta). \tag{A24}$$

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[11] K. Y. Ng and C. L. Bohn, in *Proceedings of the Second Asian Particle Accelerator Conference, Beijing, 2001* (IHEP, Beijing, 2001), p. 372. Equation (5) of that paper includes a typographical error and should read $\langle x_{Q,A}(\sigma_1) x_{Q,A}(\sigma_2)\rangle = \frac{d_g^2}{N_g}\Sigma(1)(\frac{d\Sigma(\sigma_1)}{d\sigma_1})^{-1}\delta(\sigma_1 - \sigma_2)$. This is identical to our Eqs. (A9) and (A10) with the exception of the additional factor $\Sigma(1) = \int_0^1 d\sigma' \sqrt{\gamma(0)/\gamma(\sigma')}$. This extra factor is due to the fact that Ng and Bohn assume that the length of the displaced elements scales along the accelerator as the betatron wavelength while we assume that it remains constant.