Cumulative beam breakup in linear accelerators with arbitrary beam current profile

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An analytical formalism for the solution of cumulative beam breakup in linear accelerators with arbitrary time dependence of beam current is presented, and a closed-form expression for the time and position dependence of the transverse displacement is obtained. It is applied to the behavior of a single bunch and to the steady-state and transient behavior of dc beams and beams composed of pointlike and finite-length bunches. This formalism is also applied to the problem of cumulative beam breakup in the presence of random displacement of cavities and focusing elements, and a general solution is presented.

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I. INTRODUCTION

The cumulative beam breakup instability (BBU) in linear accelerators results when a beam is injected into an accelerator with a lateral offset or an angular divergence and couples to the dipole modes of the accelerating structures [1]. The dipole modes that are excited in a cavity by the previous bunches can further deflect the following bunches and thereby increase the excitation of the dipole modes in the downstream cavities. In this process the transverse displacement can be amplified and lead to a degradation of beam quality and possibly beam loss. This instability is cumulative since the transverse deflection of a particular bunch or particle results from the additive contributions from all the previous bunches.

Cumulative BBU has been studied in the past mostly in the context of high energy electron accelerators where the beam current profiles were comprised of periodic trains of pointlike bunches [2–8] or for high-current quasi-dc beams [9–13]. Growing interest in high-current superconducting ion accelerators for spallation sources, where the bunches have a finite length, motivated an investigation of cumulative BBU in linear accelerators with periodic beam current profile [14]. The formalism that was developed was found to be applicable to a wide range of problems and could be used to unify and expand previous work on cumulative BBU of continuous beams and beams comprised of pointlike bunches, both in the steady-state and transient regime. With respect to beams with finite-length bunches, which was the original motivation for that work, that formalism was able to investigate their BBU behavior in the steady-state regime. Since all high-current ion accelerators that were under consideration at the time were to operate cw, the steady-state regime was of most interest.

Since then the focus has shifted toward pulsed operation and the first such accelerator, the Spallation Neutron Source, is to be operated at 60 Hz and 7% duty cycle [15]. This has motivated a reexamination of BBU for finite-length bunched beams begun in Ref. [14] and its extension to the transient regime in accelerators with pulsed beams. During the course of this work, a formalism leading to a general expression for cumulative BBU with arbitrary time dependence of the beam current was found. This formalism and the application of the results to various beam and accelerator configurations (single short bunch, steady-state and transient behavior of continuous beams and of beams composed of pointlike and finite-length bunches) are presented here.

This formalism can also lead to a general expression for the transient behavior in the presence of BBU and random displacement of cavities and focusing elements. The general results will be presented, but their development and application to more specific situations will be presented in another paper.

II. EQUATION OF TRANSVERSE MOTION AND GENERAL SOLUTION

In a continuum approximation, the transverse motion of a beam in a misaligned accelerator under the combined influence of focusing and coupling to dipole modes can be modeled by [12,16]

$$\frac{1}{\beta\gamma}\frac{\partial}{\partial\sigma}\left[\beta\gamma\frac{\partial}{\partial\sigma}x(\sigma,\zeta)\right]+\kappa^2[x(\sigma,\zeta)-d_f(\sigma)] = \varepsilon(\sigma)\int_{-\infty}^{\zeta} w(\zeta-\zeta_1)F(\zeta_1)[x(\sigma,\zeta_1)-d_c(\sigma)]d\zeta_1. \quad (1)$$

In this expression β and γ are the usual velocity and energy parameters; $\sigma = s/\mathcal{L}$ is the distance from the entrance of the accelerator normalized to the accelerator length $\mathcal{L}$; κ is the normalized focusing wave number; $\zeta = \omega(t - \int ds/\beta c)$ is the time made dimensionless by an angular frequency ω and measured after the arrival of the head of the beam at location σ; $F(\zeta) = I(\zeta)/\bar{I}$, the current form factor, is the instantaneous current divided by the average current; $w(\zeta)$ is the wake function of the dipole modes; ε is the coupling strength between the

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beam and the dipole modes; $d_f(\sigma)$ and $d_c(\sigma)$ are the lateral displacement of the focusing elements and cavities, respectively, as a function of location along the accelerator.

The continuum model implied in Eq. (1) relies on a certain number of approximations. The cavities that compose the accelerator are assumed to have negligible length and to be electromagnetically decoupled from each other. This implies that the deflecting fields are localized, i.e., they do not propagate along the accelerator. Thus, regenerative and coupled-cavity BBU are outside the scope of this work. The continuum approximation also assumes that the external focusing fields provided by the focusing elements are uniformly smooth along the accelerator. Although only an approximation, this model should be quite accurate when the characteristic length for the transverse dynamics of the beam (BBU-modified betatron period or BBU growth length) is significantly larger than the cavity spacing.

In some high-current, low-energy accelerators [17,18] the BBU growth length is comparable to the cavity spacing. In those cases the continuum model would not be valid and the discreteness of the accelerator needs to be taken into account; thus they fall outside the scope of this work. Nevertheless, even in those cases, the continuum model gives reasonable results when the cavity spacing is not too close to a multiple of a half-betatron wavelength [17].

In what follows we will assume that the dipole deflecting modes, and thereby the wake functions, are constant along the accelerator. It is then natural to choose ω, the normalizing frequency used in the definition of ζ, as the representative frequency in the wake function. In the special case of a single mode, the wake function is then

$$w(\zeta) = u(\zeta)\sin\zeta e^{-\zeta/2Q}, \tag{2}$$

where $u(\zeta)$ is the Heaviside unit step function, and Q is the quality factor of the deflecting mode. The implications of a distribution of deflecting mode frequencies have been investigated in Refs. [14,19,20].

The dimensionless BBU coupling strength ε is a product of quantities representative of the beam, cavities, and linac configuration, respectively, and is given by

$$\varepsilon(\sigma) = \left(\frac{\bar{I}Ze}{2\beta\gamma mc}\right)\left(\frac{\Gamma_\perp}{\omega}\right)\left(\frac{\mathcal{L}}{L}\right), \tag{3}$$

with

$$\Gamma_\perp = \frac{2}{\epsilon_0\omega}\frac{\left|\int_{-L/2}^{L/2} e^{-i\omega z/\beta c}\frac{\partial E_z(0,0,z)}{\partial z}dz\right|^2}{\int_V \mathbf{E}^2(\mathbf{x})d\mathbf{x}}, \tag{4}$$

the transverse shunt impedance of a cavity of length L.

In this paper a number of transforms on functions of distance σ and time ζ will be applied. The following notations for Laplace transform with respect to distance, and for Fourier, Laplace, and z transforms with respect to time will be used:

$$f^\dagger(p,\zeta) = \mathfrak{L}_\sigma[f] = \int_0^\infty f(\sigma,\zeta)e^{-p\sigma}d\sigma, \tag{5a}$$

$$\tilde{f}(\sigma,Z) = \mathfrak{F}_\zeta[f] = \int_{-\infty}^\infty f(\sigma,\zeta)e^{-iZ\zeta}d\zeta, \tag{5b}$$

$$\hat{f}(\sigma,q) = \mathfrak{L}_\zeta[f] = \int_0^\infty f(\sigma,\zeta)e^{-q\zeta}d\zeta, \tag{5c}$$

$$\check{f}(\sigma,z) = \mathfrak{Z}_\zeta[f] = \sum_{k=0}^\infty f(\sigma,k)z^{-k}. \tag{5d}$$

For the main sections of this paper we will assume a coasting beam in a uniform accelerator, i.e., we will assume that $\beta\gamma$, κ, ε, and $w(\zeta)$ are independent of σ. This is not unduly restrictive since, as shown in Appendix A, under some realistic assumptions, variable and coordinate transformations can be found that reduce an accelerated beam to an equivalent coasting beam in a uniform accelerator.

Under these assumptions the equation of motion becomes

$$\frac{\partial^2}{\partial\sigma^2}x(\sigma,\zeta) + \kappa^2[x(\sigma,\zeta) - d_f(\sigma)] = \varepsilon\int_{-\infty}^\zeta w(\zeta-\zeta_1)F(\zeta_1)[x(\sigma,\zeta_1) - d_c(\sigma)]d\zeta_1. \tag{6}$$

As shown in Appendix B, upon introduction of the Laplace transform with respect to the location σ to the transverse displacement $x(\sigma,\zeta)$, the general solution for $x^\dagger(p,\zeta)$ is

$$\begin{aligned} x^\dagger(p,\zeta) = {} & \sum_{n=0}^\infty \frac{\varepsilon^n}{(p^2+\kappa^2)^{n+1}}[x_0 p h_n(\zeta) + x_0' g_n(\zeta)] \\ & - d_c^\dagger(p)\sum_{n=0}^\infty \frac{\varepsilon^{n+1}}{(p^2+\kappa^2)^{n+1}} f_{n+1}(\zeta) \\ & + \kappa^2 d_f^\dagger(p)\sum_{n=0}^\infty \frac{\varepsilon^n}{(p^2+\kappa^2)^{n+1}} f_n(\zeta). \end{aligned} \tag{7}$$

The functions $f_n(\zeta)$, $g_n(\zeta)$, and $h_n(\zeta)$ are defined by the recursion relations

$$f_{n+1}(\zeta) = \int_{-\infty}^\zeta f_n(\zeta_1)F(\zeta_1)w(\zeta-\zeta_1)d\zeta_1, \tag{8a}$$

$$g_{n+1}(\zeta) = \int_{-\infty}^\zeta g_n(\zeta_1)F(\zeta_1)w(\zeta-\zeta_1)d\zeta_1, \tag{8b}$$

$$h_{n+1}(\zeta) = \int_{-\infty}^\zeta h_n(\zeta_1)F(\zeta_1)w(\zeta-\zeta_1)d\zeta_1, \tag{8c}$$

with

$$f_0(\zeta) = 1, \qquad x_0' g_0(\zeta) = x_0'(\zeta) = \frac{\partial}{\partial \sigma} x(\sigma, \zeta) \bigg|_{\sigma=0}, \quad (9)$$
$$x_0 h_0(\zeta) = x_0(\zeta) = x(\sigma = 0, \zeta).$$

$x_0(\zeta)$ and $x_0'(\zeta)$ are the lateral displacement and angular divergence, respectively, of the beam at the entrance of the accelerator. The normalizing constants x_0 and x_0' are introduced to make the functions $h_0(\zeta)$ and $g_0(\zeta)$ dimensionless.

Applying the inverse Laplace transform to Eq. (7) gives the general solution of Eq. (6):

$$\begin{aligned} x(\sigma,\zeta) = &\sum_{n=0}^{\infty} \varepsilon^n [x_0 h_n(\zeta) j_n(\kappa,\sigma) + x_0' g_n(\zeta) i_n(\kappa,\sigma)] \\ &- \sum_{n=0}^{\infty} \varepsilon^{n+1} f_{n+1}(\zeta) \int_0^\sigma i_n(\kappa,u) d_c(\sigma-u) du \\ &+ \kappa^2 \sum_{n=0}^{\infty} \varepsilon^n f_n(\zeta) \int_0^\sigma i_n(\kappa,u) d_f(\sigma-u) du, \quad (10) \end{aligned}$$

where $i_n(\kappa, \sigma)$ and $j_n(\kappa, \sigma)$ are defined in terms of Bessel functions of order integer plus one-half and contain only powers and circular functions,

$$\begin{aligned} i_n(\kappa, \sigma) &= \mathfrak{L}_\sigma^{-1}\left[\frac{1}{(p^2 + \kappa^2)^{n+1}}\right] \\ &= \frac{1}{n!}\left(\frac{\sigma}{2\kappa}\right)^n \frac{1}{\kappa} \sqrt{\frac{\pi\kappa\sigma}{2}} J_{n+(1/2)}(\kappa\sigma), \quad (11a) \end{aligned}$$

$$\begin{aligned} j_n(\kappa, \sigma) &= \mathfrak{L}_\sigma^{-1}\left[\frac{p}{(p^2 + \kappa^2)^{n+1}}\right] = \frac{d}{d\sigma} i_n(\kappa, \sigma) \\ &= \frac{1}{n!}\left(\frac{\sigma}{2\kappa}\right)^n \sqrt{\frac{\pi\kappa\sigma}{2}} J_{n-(1/2)}(\kappa\sigma) = \frac{\sigma}{2n} i_{n-1}(\kappa, \sigma). \quad (11b) \end{aligned}$$

Equation (10) gives the transverse displacement, at location σ and time ζ, of a beam of arbitrary current profile $F(\zeta)$, entering the accelerator with lateral offset $x_0(\zeta)$ and angular divergence $x_0'(\zeta)$, experiencing transverse forces due to a wake field $\varepsilon w(\zeta)$ and focusing κ, and with displacement along the accelerator $d_c(\sigma)$ of the cavities and $d_f(\sigma)$ of the focusing elements.

In this paper we are concerned with the effects of the lateral offset and angular divergence of the beam at the entrance of a perfectly aligned accelerator; therefore, for the remainder of this paper, we will assume $d_f(\sigma) = d_c(\sigma) = 0$, and the equations for $x^\dagger(p, \zeta)$ and $x(\sigma, \zeta)$ are

$$x^\dagger(p, \zeta) = \sum_{n=0}^{\infty} \frac{\varepsilon^n}{(p^2 + \kappa^2)^{n+1}} [p x_0 h_n(\zeta) + x_0' g_n(\zeta)], \quad (12)$$

$$\begin{aligned} x(\sigma, \zeta) = \sum_{n=0}^{\infty} \frac{1}{n!}\left(\frac{\varepsilon\sigma}{2\kappa}\right)^n \sqrt{\frac{\pi\kappa\sigma}{2}} &[x_0 h_n(\zeta) J_{n-(1/2)}(\kappa\sigma) \\ &+ x_0' \frac{g_n(\zeta)}{\kappa} J_{n+(1/2)}(\kappa\sigma)]. \quad (13) \end{aligned}$$

The effects of the misalignment of the focusing elements and cavities will be addressed in another paper.

It can be seen from Eq. (13) that the dependence of the lateral displacement on location σ, focusing strength κ, and coupling strength between the current and the dipole modes ε, is governed by elementary functions. The only functions that need to be computed for a particular accelerator are the functions $f_n(\zeta)$, $g_n(\zeta)$, and $h_n(\zeta)$ that depend only on the beam current profile (not on its magnitude which is included in ε), the initial conditions at the accelerator entrance, and the wake function. Thus they need to be calculated only infrequently and then applied to a wide range of accelerator configurations where the BBU behavior can be determined rapidly at an arbitrary time and location through Eq. (13).

If the lateral displacement $x_0(\zeta)$ and angular divergence $x_0'(\zeta)$ at the entrance of the accelerator have the same time dependence so that they are equal to within a multiplying constant, the same will be true for all $h_n(\zeta)$ and $g_n(\zeta)$. We see from Eqs. (12) and (13) that, in that case, the transverse displacement at location σ and time ζ induced by $x_0(\zeta)$ is the derivative with respect to σ of the transverse displacement induced by $x_0'(\zeta)$. Additionally, if the lateral displacement $x_0(\zeta)$ and angular divergence $x_0'(\zeta)$ at the entrance of the accelerator are constant, then the only functions that need to be determined are the $f_n(\zeta)$.

Equation (13) gives the transverse displacement as a series expansion in powers of ε/κ and thus is more appropriate to the case of weak BBU-strong focusing, which is the most common. Another expression, valid for the strong BBU-weak focusing case, can also be derived from this formalism. From Eq. (12), $x^\dagger(p, \zeta)$ is seen to be of the form $x_1^\dagger(\sqrt{p^2 + \kappa^2}) + p x_2^\dagger(\sqrt{p^2 + \kappa^2})$. The Laplace transform has the property, shown in Appendix C, that, if

$$f(\sigma) = \mathfrak{L}^{-1}[f^\dagger(p)], \quad (14)$$

then

$$\begin{aligned} f(\sigma) - \kappa \int_0^\sigma f_0(u) \frac{J_1[\kappa\sqrt{\sigma^2 - u^2}]}{\sqrt{\sigma^2 - u^2}} u\, du \\ = \mathfrak{L}^{-1}[f^\dagger(\sqrt{p^2 + \kappa^2})], \quad (15a) \end{aligned}$$

$$\begin{aligned} f'(\sigma) - \kappa\sigma \int_0^\sigma f_0'(u) \frac{J_1[\kappa\sqrt{\sigma^2 - u^2}]}{\sqrt{\sigma^2 - u^2}} du + f(0)\delta(\sigma) \\ = \mathfrak{L}^{-1}[p f^\dagger(\sqrt{p^2 + \kappa^2})]. \quad (15b) \end{aligned}$$

Thus, if an expression for the transverse displacement is obtained in the absence of focusing, Eqs. (15a) and (15b) can be used to give the transverse displacement in the presence of focusing. This will be applied in Sec. III. From the expansion of the Bessel function $J_1(\cdot)$ in powers of its argument one can obtain a series expansion of the transverse displacement in powers of the focusing strength κ.

III. SINGLE VERY SHORT BUNCH

The case of a single very short bunch entering an accelerator with a lateral offset was one of the first to be studied [16]; the formalism described above can be applied directly to that case. By short bunch we imply a bunch much shorter than the wavelength of the deflecting mode, so the wake function can be assumed to be linear: $w(\zeta) = \zeta$. For the sake of simplicity we assume that the lateral offset x_0 and angular divergence x_0' are time independent and that the bunch has a constant current distribution $[F(\zeta) = 1]$. With those assumptions the functions $g_n(\zeta)$ and $h_n(\zeta)$ are easily calculated from Eq. (8):

$$g_n(\zeta) = h_n(\zeta) = \frac{\zeta^{2n}}{(2n)!}, \tag{16}$$

and the transverse displacement at an arbitrary location σ in the accelerator and distance ζ from the front of the bunch is given by

$$x(\sigma, \zeta) = \sum_{n=0}^{\infty} \frac{1}{(2n)!n!}\left(\frac{\varepsilon\zeta^2\sigma}{2\kappa}\right)^n \sqrt{\frac{\pi\kappa\sigma}{2}}\left[x_0 J_{n-(1/2)}(\kappa\sigma) + \frac{x_0'}{\kappa} J_{n+(1/2)}(\kappa\sigma)\right]. \tag{17}$$

For large $\kappa\sigma$ the asymptotic expression for the Bessel functions can be used to give

$$x(\sigma, \zeta) = \sum_{n=0}^{\infty} \frac{1}{(2n)!n!}\left(\frac{\varepsilon\zeta^2\sigma}{2\kappa}\right)^n \left[x_0 \cos\left(\kappa\sigma - \frac{n\pi}{2}\right) + \frac{x_0'}{\kappa}\sin\left(\kappa\sigma - \frac{n\pi}{2}\right)\right], \tag{18}$$

a result first obtained in Ref. [16]. It should be noted that the asymptotic expression for the Bessel function is valid for $n^2 \ll 2\kappa\sigma$, so that $\kappa\sigma$ needs to be large for Eq. (18) to be valid.

Similar results can be obtained in the case of a tilted bunch moving parallel to the axis at the accelerator entrance: $x_0(\zeta) = x_0(1 + \theta\zeta)$, $x_0'(\zeta) = 0$. The functions $h_n(\zeta)$ are easily calculated:

$$h_n(\zeta) = \frac{\zeta^{2n}}{(2n)!} + \theta\frac{\zeta^{2n+1}}{(2n+1)!}. \tag{19}$$

In this case the transverse displacement at location σ and time ζ from the front of the bunch is given by

$$x(\sigma, \zeta) = x_0 \sum_{n=0}^{\infty} \frac{1}{(2n)!n!}\left(\frac{\varepsilon\zeta^2\sigma}{2\kappa}\right)^n \left[1 + \theta\frac{\zeta}{2n+1}\right] \times \sqrt{\frac{\pi\kappa\sigma}{2}} J_{n-(1/2)}(\kappa\sigma). \tag{20}$$

The previous expressions for $x(\sigma, \zeta)$ are series expansions in powers of ε/κ and thus are most useful in the case of weak BBU where only a few terms need to be retained. In the case of strong BBU-weak focusing the method described in Sec. II can be applied to the case of the single short bunch. Going back to Eq. (12) for $x^\dagger(p, \zeta)$ with focusing we see that

$$x_\kappa^\dagger(p, \zeta) = (px_0 + x_0') \sum_{n=0}^{\infty} \frac{1}{(2n)!}\frac{(\varepsilon\zeta^2)^n}{(p^2 + \kappa^2)^{n+1}} = \frac{px_0 + x_0'}{p^2 + \kappa^2}\cosh\left[\sqrt{\frac{\varepsilon\zeta^2}{p^2 + \kappa^2}}\right], \tag{21}$$

so that, in the absence of focusing,

$$x_{\kappa=0}^\dagger(p, \zeta) = \left(\frac{x_0}{p} + \frac{x_0'}{p^2}\right)\cosh\left[\frac{\varepsilon^{1/2}\zeta}{p}\right]. \tag{22}$$

Since

$$\mathfrak{L}^{-1}\left[\frac{e^{k/p}}{p^\mu}\right] = \left(\frac{t}{k}\right)^{(\mu-1)/2} I_{\mu-1}(2\sqrt{k\,t}), \tag{23a}$$

$$\mathfrak{L}^{-1}\left[\frac{e^{-k/p}}{p^\mu}\right] = \left(\frac{t}{k}\right)^{(\mu-1)/2} J_{\mu-1}(2\sqrt{k\,t}), \tag{23b}$$

the displacement without focusing is

$$x_{\kappa=0}(\sigma, \zeta) = \frac{x_0}{2}\{J_0[2(\varepsilon^{1/2}\zeta\sigma)^{1/2}] + I_0[2(\varepsilon^{1/2}\zeta\sigma)^{1/2}]\} + \frac{x_0'\sigma}{2}\left(\frac{1}{\varepsilon^{1/2}\zeta\sigma}\right)^{1/2}\{J_1[2(\varepsilon^{1/2}\zeta\sigma)^{1/2}] + I_1[2(\varepsilon^{1/2}\zeta\sigma)^{1/2}]\}, \tag{24}$$

and the displacement with focusing is obtained from Eqs. (15a) and (15b),

$$x_\kappa(\sigma, \zeta) = x_0\chi'(\sigma) + x_0'\chi(\sigma) - \kappa\int_0^\sigma [x_0\sigma\chi'(u) + x_0' u\chi(u)]\frac{J_1[\kappa\sqrt{\sigma^2 - u^2}]}{\sqrt{\sigma^2 - u^2}}du, \tag{25}$$

with

$$\chi'(u) = \frac{1}{2}[J_0(au^{1/2}) + I_0(au^{1/2})],$$
$$\chi(u) = \frac{u^{1/2}}{a}[J_1(au^{1/2}) + I_1(au^{1/2})],$$
$$a = 2\zeta^{1/2}\varepsilon^{1/4}. \tag{26}$$

Equation (25) can also be written as a series expansion in powers of the focusing strength κ

$$x_\kappa(\sigma,\zeta) = x_0\chi'(\sigma) + x_0'\chi(\sigma) - \frac{\kappa^2}{2}\sum_{k=0}^{\infty}\left(\frac{\kappa}{2}\right)^2\frac{(-1)^k}{k!(k+1)!} \times \int_0^\sigma du(\sigma^2-u^2)^k[x_0\sigma\chi'(u) + x_0'u\chi(u)]. \tag{27}$$

IV. STEADY-STATE PERIODIC BEAM

The current form factor $F(\zeta)$ of a purely periodic beam and its Fourier transform $\tilde{F}(Z)$ are given by

$$F(\zeta) = \sum_{k=-\infty}^{\infty} F_k e^{i(2\pi/\omega\tau)k\zeta}, \tag{28}$$

$$\tilde{F}(Z) = 2\pi\sum_{k=-\infty}^{\infty} F_k\delta\left(Z - \frac{2\pi}{\omega\tau}k\right), \tag{29}$$

where τ is the laboratory-frame period of the longitudinal beam modulation.

Since $F(\zeta)$ is the form factor of the current, i.e., the instantaneous current divided by the average current, we have $F_0 = 1$ irrespective of the shape of the bunches.

If we assume time-independent initial conditions at the entrance of the accelerator—$x_0(\zeta) = x_0$ and $x_0'(\zeta) = x_0'$—we see from Eq. (8) that $g_n(\zeta) = h_n(\zeta) = f_n(\zeta)$ so it will be sufficient to determine the functions $f_n(\zeta)$.

From the recursion relation for $f_{n+1}(\zeta)$

$$f_{n+1}(\zeta) = \int_{-\infty}^{\zeta} d\zeta_1 f_n(\zeta_1)F(\zeta_1)w(\zeta-\zeta_1), \tag{30}$$

we obtain the recursion relation for $\tilde{f}(Z) = \mathfrak{F}[f(\zeta)]$, the Fourier transform of $f(\zeta)$:

$$\tilde{f}_{n+1}(Z) = \tilde{w}(Z)\mathfrak{F}[f_n(\zeta)F(\zeta)] = \tilde{w}(Z)\sum_k F_k\tilde{f}_n\left(Z - \frac{2\pi}{\omega\tau}k\right), \tag{31}$$

where $\tilde{w}(Z)$ is the Fourier transform of the wake function $w(\zeta)$, i.e., the impedance.

From Eq. (31), together with $\tilde{f}_0(Z) = 2\pi\delta(Z)$, we obtain the functions $\tilde{f}_n(Z)$ and $f_n(\zeta)$

$$\tilde{f}_n(Z) = 2\pi\tilde{w}(Z)\sum_{k_1}F_{k_1}\tilde{w}\left[Z - \frac{2\pi}{\omega\tau}k_1\right]\cdots\sum_{k_{n-1}}F_{k_{n-1}}\tilde{w}\left[Z - \frac{2\pi}{\omega\tau}(k_1+\cdots+k_{n-1})\right]\sum_{k_n}F_{k_n}\left[Z - \frac{2\pi}{\omega\tau}(k_1+\cdots+k_n)\right], \tag{32}$$

and

$$f_n(\zeta) = \sum_{k_n}\cdots\sum_{k_1}F_{k_n}\cdots F_{k_1}\tilde{w}_{k_n+\cdots+k_1}\cdots\tilde{w}_{k_n+k_{n-1}}\tilde{w}_{k_n}\exp\left[i\frac{2\pi}{\omega\tau}(k_n+\cdots+k_1)\zeta\right] \tag{33}$$

$$= \sum_{k_1}\exp\left[i\frac{2\pi}{\omega\tau}k_1\zeta\right]\tilde{w}_{k_1}\sum_{k_2}\tilde{w}_{k_2}F_{k_1-k_2}\sum_{k_3}\cdots\sum_{k_{n-1}}\tilde{w}_{k_{n-1}}F_{k_{n-2}-k_{n-1}}\sum_{k_n}\tilde{w}_{k_n}F_{k_{n-1}-k_n}F_{k_n}, \tag{34}$$

where $\tilde{w}_k = \tilde{w}[(2\pi/\omega\tau)k]$. Equation (34) is the expression for the Fourier series expansion of $f_n(\zeta)$.

Equation (34) can also be written as

$$f_n(\zeta) = \sum_k \exp\left[i\frac{2\pi}{\omega\tau}k\zeta\right]f_{n,k}, \tag{35}$$

with the following recursion relation resulting from Eq. (31):

$$f_{0,k} = \delta_{0,k}, \tag{36a}$$

$$f_{n+1,k} = \tilde{w}_k\sum_{k_1}F_{k_1}f_{n,k-k_1}. \tag{36b}$$

Thus, the transverse steady-state displacement for an arbitrary periodic beam with time-independent initial conditions is

$$x(\sigma,\zeta) = \sum_{n=0}^{\infty}\frac{1}{n!}\left(\frac{\varepsilon\sigma}{2\kappa}\right)^n\sqrt{\frac{\pi\kappa\sigma}{2}}\left[x_0J_{n-(1/2)}(\kappa\sigma) + \frac{x_0'}{\kappa}J_{n+(1/2)}(\kappa\sigma)\right]f_n(\zeta), \tag{37}$$

where $f_n(\zeta)$ is given by Eqs. (35) and (36).

The same problem was analyzed in Ref. [14] by a different method, but a complete closed-form solution was not found. What was found was a recursion solution for a series expansion in powers of σ valid close to the accelerator entrance. Keeping only the first two terms in Eq. (37) we obtain (assuming $x_0' = 0$)

$$x(\sigma,\zeta) \simeq x_0\left[\cos(\kappa\sigma) + \frac{\varepsilon\sigma}{2\kappa}\sin(\kappa\sigma)\sum_k F_k\tilde{w}_k\exp\left(i\frac{2\pi}{\omega\tau}k\zeta\right)\right]. \tag{38}$$

This expression is valid from the entrance ($\sigma = 0$) to the exit ($\sigma = 1$) of the accelerator; it assumes that ε/κ is sufficiently small and that there are no strong resonances between the beam and the deflecting modes. For $\kappa\sigma \ll 1$ Eq. (38) reduces to Eq. (5.17) of Ref. [14].

As an example, we apply Eq. (38) to a beam composed of bunches of constant current density, separated by $\omega\tau$, of length $\alpha\omega\tau$, and entering the accelerator parallel to but displaced from the axis. The parameter α allows a continuous transition from a dc beam ($\alpha = 1$) to a beam composed of delta-function bunches separated by $\omega\tau$ ($\alpha = 0$). Choosing $\zeta = 0$ as being in the middle of a bunch, the Fourier coefficients of the current form factor are $F_k = [\sin(k\alpha\pi)/k\alpha\pi]$. If we also assume a single deflecting mode where the wake function $w(\zeta)$ is given by Eq. (2) we have

$$\tilde{w}(Z) = \frac{1}{2}\left[\frac{1}{1 - Z + \frac{i}{2Q}} + \frac{1}{1 + Z - \frac{i}{2Q}}\right] = \left[1 - Z^2 + \frac{1}{4Q^2} + i\frac{Z}{Q}\right]^{-1}, \tag{39}$$

$$\tilde{w}_k = \tilde{w}\left[\frac{2\pi}{\omega\tau}k\right] = \frac{1}{[1 - k^2(\frac{2\pi}{\omega\tau})^2 + \frac{1}{4Q^2}] + \frac{i}{Q}\frac{2\pi}{\omega\tau}k}. \tag{40}$$

Rewriting Eq. (38) as

$$x(\sigma, \zeta) \simeq x_0\left[\cos(\kappa\sigma) + \frac{\varepsilon\sigma}{2\kappa}\sin(\kappa\sigma)h_1(\zeta)\right], \tag{41}$$

we have

$$h_1(\zeta) = \frac{\omega\tau}{4\pi}\sum_{k=-\infty}^{\infty}\frac{\sin k\pi\alpha}{k\pi\alpha}\left\{\left[-\frac{\exp(i\frac{2\pi}{\omega\tau}k\zeta)}{k - \frac{\omega\tau}{2\pi}(\frac{i}{2Q} + 1)} + \frac{\exp(i\frac{2\pi}{\omega\tau}k\zeta)}{k - \frac{\omega\tau}{2\pi}(\frac{i}{2Q} - 1)}\right]\right\}. \tag{42}$$

Using the identity

$$\sum_{k=-\infty}^{\infty}\frac{\sin kx}{k(k + a)}e^{iky} = \frac{\pi}{a\sin\pi a}\{\sin\pi a - e^{ia(q\pi - y)}\sin[a\{(p + 1)\pi - x\}]\} + \frac{p\pi}{a}, \tag{43}$$

where

$$2m\pi < x + y < 2(m + 1)\pi, \qquad 2n\pi < x - y < 2(n + 1)\pi, \qquad p = m + n, \qquad q = m - n,$$

we obtain

$$h_1(\zeta) = \frac{1}{2\alpha(1 + \frac{i}{2Q})}\left\{1 - e^{[-(1/2Q)+i][\zeta - q(\omega\tau/2)]}\frac{\sin[\frac{\omega\tau}{2}(1 + \frac{i}{2Q})(p + 1 - \alpha)]}{\sin[\frac{\omega\tau}{2}(1 + \frac{i}{2Q})]} + p\right\} + \text{c.c.}, \tag{44}$$

where p and q take on different values whether ζ is located inside or outside the bunch

$$p = -1, q = -1 \quad \text{for } -\omega\tau/2 \le \zeta \le -\alpha\omega\tau/2,$$

$$p = 0, q = 0 \quad \text{for } -\alpha\omega\tau/2 \le \zeta \le \alpha\omega\tau/2,$$

$$p = -1, q = 1 \quad \text{for } \alpha\omega\tau/2 \le \zeta \le \omega\tau/2.$$

This together with Eq. (41) gives, to first order in ε, the steady-state displacement and shape distortion of the bunches. Equation (44) gives the displacement, not only inside the bunch, but also for particles located outside the bunch, i.e., for a diffuse longitudinal halo that does not contribute to the wake field but experiences it.

As an example Fig. 1 shows the steady-state shape of a train of bunches with $\alpha = 0.1$, $Q = 10^3$, $\kappa = 100.5\pi$, $\varepsilon = 0.2$, and $\sigma = 1$. Figure 1(a) assumes $\omega\tau = 4.005\pi$, and Fig. 1(a) assumes $\omega\tau = 4.001\pi$. Inside the bunch is shown in red while the outside is shown in blue. In this example and others following, the assumed value for κ implies that, at $\sigma = 1$, we have $\cos\kappa\sigma = 0$, and Eq. (41) shows that the bunch displacement and distortion are due entirely to the coupling to the deflecting mode. Figure 2 assumes the same parameters as in Fig. 1 but with $Q = 10^4$. It can be seen from Figs. 1(a) and 2(a), where $\omega\tau = 4.005$, that, if the deflecting mode frequency is sufficiently far from a harmonic of the bunch frequency, then the Q of the deflecting mode has little impact on the transverse steady-state behavior of the beam. On the other hand, Figs. 1(b) and 2(b) show that, in the case of a close resonance, the Q can have a dramatic impact.

Other examples are shown in Figs. 3 and 4. Figure 3 shows the evolution of a beam during the last betatron period of an accelerator whose parameters are $\alpha = 0.1$, $Q = 10^3$, $\kappa = 100\pi$, $\varepsilon = 100$, and $\omega\tau = 5.25\pi$. In Fig. 4 the parameters are $\alpha = 0.1$, $Q = 10^3$, $\kappa = 100\pi$, $\varepsilon = 0.2$, and $\omega\tau = 4.001\pi$.

The steady-state behavior of a dc or δ-function beam that was analyzed in Ref. [14] can also be obtained directly from this formalism.

For a dc beam we have from Eqs. (28), (31), and (34)

$$F_k = \delta_{0k}, \tag{45a}$$

$$\tilde{f}_n(Z) = 2\pi[\tilde{w}(Z)]^n\delta(Z), \tag{45b}$$

$$f_n(\zeta) = \tilde{w}_0^n, \tag{45c}$$

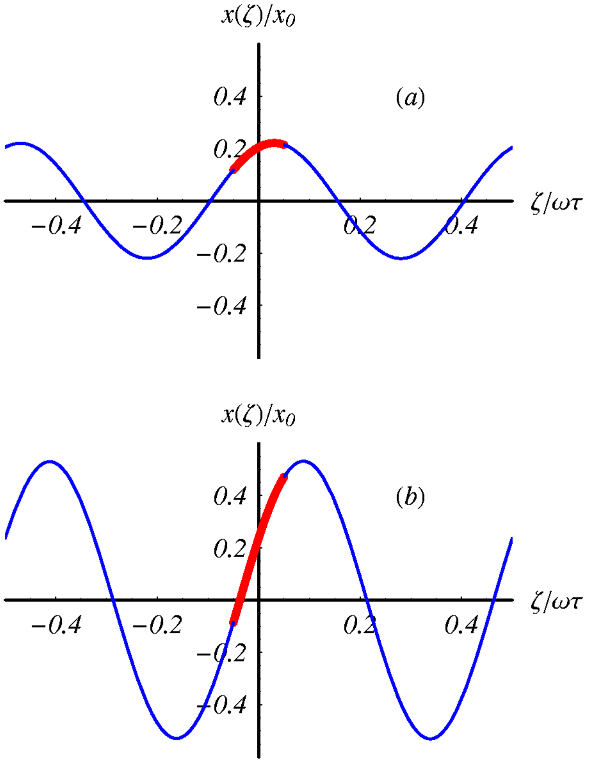

FIG. 1. (Color) Steady-state displacement and shape of an infinite train of finite bunches with $\alpha = 0.1$, $Q = 10^3$, $\kappa = 100.5\pi$, $\varepsilon = 0.2$, and $\sigma = 1$. (a) assumes $\omega\tau = 4.005\pi$, and (b) assumes $\omega\tau = 4.001\pi$. Particles inside the bunch are shown in red while those outside are shown in blue.

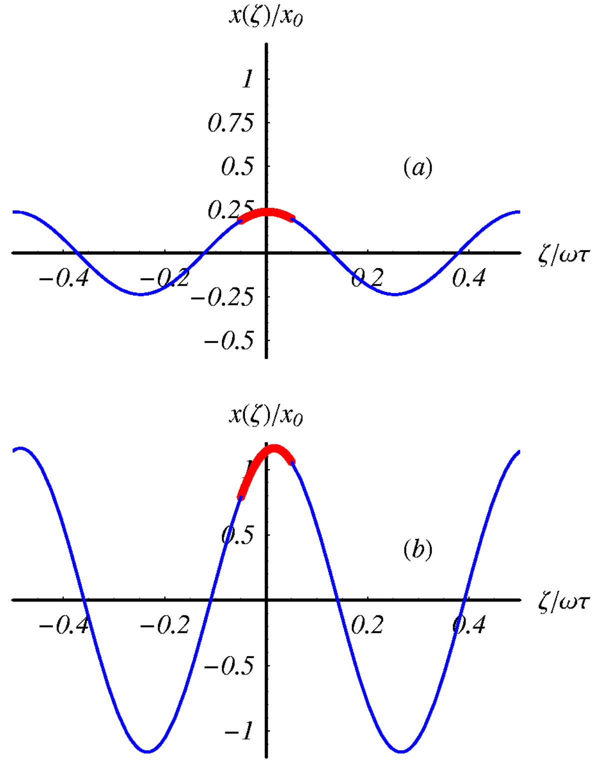

FIG. 2. (Color) Same parameters as in Fig. 1 but with $Q = 10^4$.

and the steady-state solution is

$$x(\sigma, \zeta) = \sqrt{\frac{\pi\kappa\sigma}{2}} \sum_{n=0}^{\infty} \frac{1}{n!}\left(\frac{\varepsilon\sigma\tilde{w}_0}{2\kappa}\right)^n \times \left[x_0 J_{n-(1/2)}(\kappa\sigma) + \frac{x_0'}{\kappa} J_{n+(1/2)}(\kappa\sigma)\right]. \tag{46}$$

Using the identity

$$\sum_{k=0}^{\infty} \frac{t^k}{k!} J_{k+\nu}(z) = z^{\nu/2}(z - 2t)^{-\nu/2} J_\nu(\sqrt{z^2 - 2tz}), \tag{47}$$

we obtain

$$x(\sigma, \zeta) = x_0 \cos[\sigma\sqrt{\kappa^2 - \varepsilon\tilde{w}_0}] + x_0' \frac{\sin[\sigma\sqrt{\kappa^2 - \varepsilon\tilde{w}_0}]}{\sqrt{\kappa^2 - \varepsilon\tilde{w}_0}}, \tag{48}$$

which is Eq. (3.10) of [14].

In the case of a beam comprised of δ-function bunches we have from Eqs. (28), (31), and (34)

$$F_k = 1, \tag{49}$$

$$\tilde{f}_n(Z) = \begin{cases} 2\pi\delta(Z) & n = 0, \\ 2\pi\tilde{w}(Z)[\tilde{W}(Z)]^{n-1}\sum_{k=-\infty}^{\infty}\delta(Z - \frac{2\pi}{\omega\tau}k) & n \geq 1, \end{cases} \tag{50}$$

$$f_n(\zeta) = \begin{cases} 1 & n = 0, \\ \tilde{W}_0^{n-1}\sum_{k=-\infty}^{\infty}\tilde{w}_k \exp(i\frac{2\pi}{\omega\tau}k\zeta) & n \geq 1, \end{cases} \tag{51}$$

where we have defined

$$\tilde{W}(Z) = \sum_{k=-\infty}^{\infty}\tilde{w}\left(Z - \frac{2\pi}{\omega\tau}k\right) = \omega\tau\sum_{k=0}^{\infty} w(k\omega\tau)e^{-ik\omega\tau Z}, \tag{52}$$

and

$$\tilde{W}_0 = \tilde{W}(0) = \sum_{k=-\infty}^{\infty}\tilde{w}_k = \omega\tau\sum_{k=0}^{\infty} w(k\omega\tau). \tag{53}$$

For a single mode we have

$$\tilde{W}(Z) = \frac{\omega\tau}{2}\frac{\sin\omega\tau}{\cosh[\omega\tau(\frac{1}{2Q} + iZ)] - \cos\omega\tau}, \tag{54}$$

and $\tilde{W}_0$ is the well-known resonance function.

The steady-state solution for an infinite train of point-like bunches is then

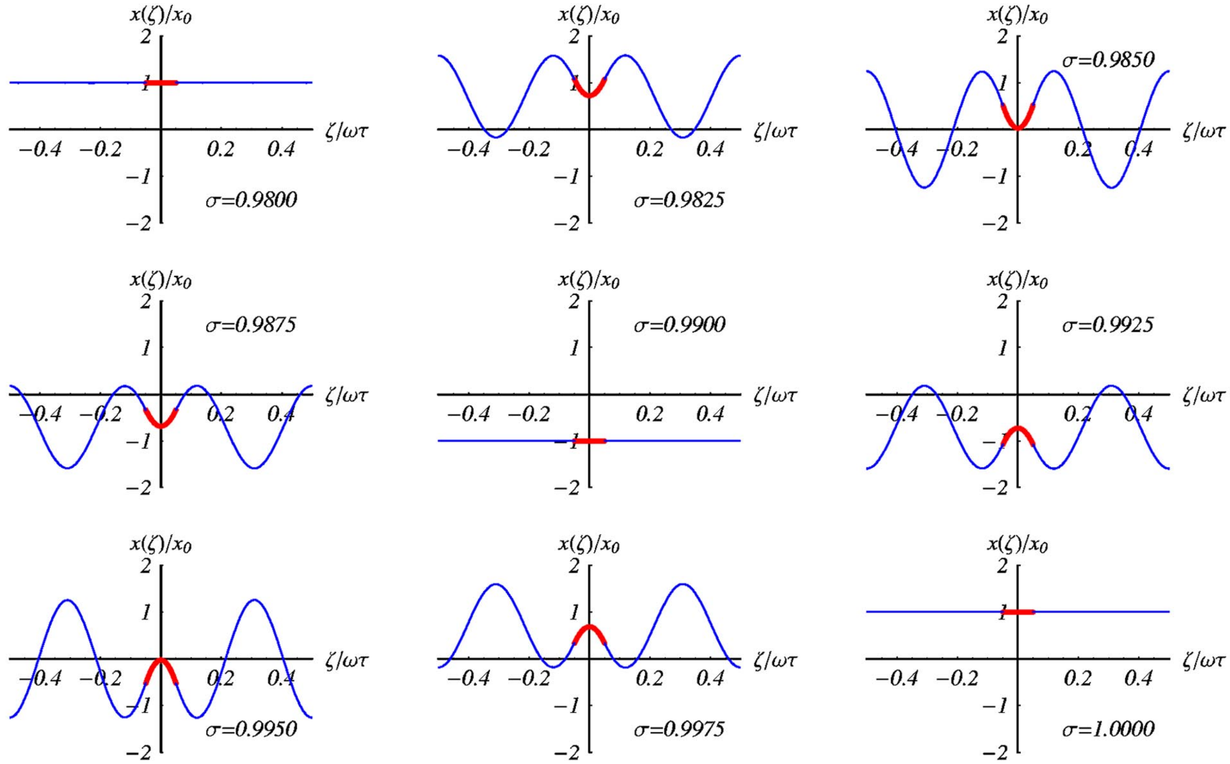

FIG. 3. (Color) Steady-state evolution of an infinite bunch train with $\alpha = 0.1$, $Q = 10^3$, $\kappa = 100\pi$, $\varepsilon = 100$, and $\omega\tau = 5.25\pi$ during the last betatron period of the accelerator.

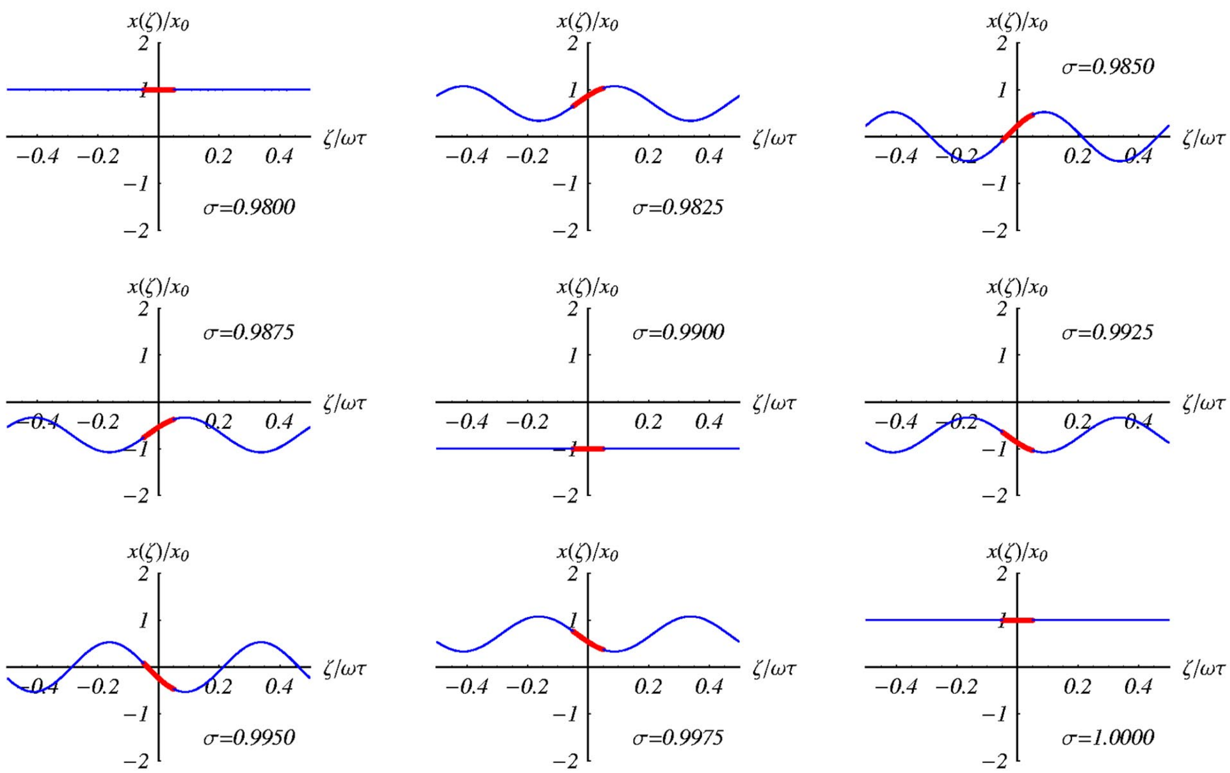

FIG. 4. (Color) Steady-state evolution of an infinite bunch train with $\alpha = 0.1$, $Q = 10^3$, $\kappa = 100\pi$, $\varepsilon = 0.2$, and $\omega\tau = 4.001\pi$ during the last betatron period of the accelerator.

$$x(\sigma,\zeta) = x_0\cos\kappa\sigma + \frac{x_0'}{\kappa}\sin\kappa\sigma + \frac{1}{\tilde{W}_0}\left[\sum_{k=-\infty}^{\infty}\tilde{w}_k\exp\left(i\frac{2\pi}{\omega\tau}k\zeta\right)\right] \times\left\{\sqrt{\frac{\pi\kappa\sigma}{2}}\sum_{n=0}^{\infty}\frac{1}{n!}\left(\frac{\varepsilon\sigma\tilde{W}_0^n}{2\kappa}\right)\left[x_0J_{n-(1/2)}(\kappa\sigma) + \frac{x_0'}{\kappa}J_{n+(1/2)}(\kappa\sigma)\right] - x_0\cos\kappa\sigma + \frac{x_0'}{\kappa}\sin\kappa\sigma\right\}, \tag{55}$$

and, using the same identity as above, it can be written as

$$x(\sigma,\zeta) = x_0\cos\kappa\sigma + \frac{x_0'}{\kappa}\sin\kappa\sigma + \frac{1}{\tilde{W}_0}\left[\sum_{k=-\infty}^{\infty}\tilde{w}_k\exp\left(i\frac{2\pi}{\omega\tau}k\zeta\right)\right] \times\left[x_0\{\cos[\sigma\sqrt{\kappa^2-\varepsilon\tilde{W}_0}] - \cos\kappa\sigma\} + x_0'\left\{\frac{\sin[\sigma\sqrt{\kappa^2-\varepsilon\tilde{W}_0}]}{\sqrt{\kappa^2-\varepsilon\tilde{W}_0}} - \frac{\sin\kappa\sigma}{\kappa}\right\}\right], \tag{56}$$

which is Eq. (4.11) of Ref. [14].

Note that Eq. (56) is valid for all values of ζ and is not limited to $\zeta = M\omega\tau$. In particular, it gives the transverse displacement of particles that may be located outside the bunches. Such particles do not contribute to the transverse fields generated by the bunches but are deflected by them.

For $\zeta = M\omega\tau$, i.e., for bunch M, we have $f_n(M\omega\tau) = \tilde{W}_0{}^n$, $\exp[i(2\pi/\omega\tau)k\zeta] = 1$, and the displacement becomes

$$x(\sigma, M\omega\tau) = x_0\cos[\sigma\sqrt{\kappa^2-\varepsilon\tilde{W}_0}] + x_0'\frac{\sin[\sigma\sqrt{\kappa^2-\varepsilon\tilde{W}_0}]}{\sqrt{\kappa^2-\varepsilon\tilde{W}_0}}, \tag{57}$$

which is identical to that of a dc beam but with $\tilde{w}_0$ replaced by $\tilde{W}_0$.

V. TRANSIENT PERIODIC BEAM

In this section we will analyze the BBU behavior of a periodic beam that was turned on at $\zeta = 0$. For the sake of simplicity we will assume that $x_0(\zeta) = x_0$ and $x_0'(\zeta) = 0$.

The general solution is still given by

$$x(\sigma,\zeta) = x_0\sum_{n=0}^{\infty}\frac{1}{n!}\left(\frac{\epsilon\sigma}{2\kappa}\right)^n\sqrt{\frac{\pi\kappa\sigma}{2}}J_{n-(1/2)}(\kappa\sigma)h_n(\zeta), \tag{58}$$

where now

$$h_0(\zeta) = u(\zeta), \tag{59a}$$

$$h_{n+1}(\zeta) = \int_0^{\zeta} h_n(\zeta_1)F(\zeta_1)w(\zeta-\zeta_1)d\zeta_1. \tag{59b}$$

As in the steady-state case there are three main beams of interest that will be analyzed in this section: the dc beam, the beam comprised of pointlike bunches (delta-function beam), and the beam comprised of bunches of finite length.

A. dc beam

A dc beam turned on at $\zeta = 0$ is represented by the current form factor $F(\zeta) = u(\zeta)$, and the functions $h_n(\zeta)$ are now

$$h_0(\zeta) = u(\zeta), \tag{60a}$$

$$h_{n+1}(\zeta) = \int_0^{\zeta} w(\zeta-\zeta_1)h_n(\zeta_1)d\zeta_1. \tag{60b}$$

Since $w(\zeta < 0) = 0$ the integrals can be extended to $+\infty$ and the functions $h_n(\zeta)$ can be defined more simply through their Laplace transform with respect to ζ,

$$\mathfrak{L}_{\zeta}[h_n] = \hat{h}_n(q) = \int_0^{\infty} h_n(\zeta)e^{-q\zeta}d\zeta, \tag{61a}$$

$$\mathfrak{L}_{\zeta}[w] = \hat{w}(q) = \int_0^{\infty} w(\zeta)e^{-q\zeta}d\zeta. \tag{61b}$$

From Eq. (60) we then obtain

$$\hat{h}_n(q) = \frac{1}{q}[\hat{w}(q)]^n. \tag{62}$$

In the case of a single mode we have

$$w(\zeta) = e^{-(\zeta/2Q)}\sin\zeta, \tag{63a}$$

$$\hat{w}(q) = \frac{1}{(q+\frac{1}{2Q})^2+1}, \tag{63b}$$

which yields

$$h_n(\zeta) = \begin{cases} u(\zeta) & n=0, \\ \frac{1}{(n-1)!}\int_0^{\zeta}d\zeta_1 e^{-(\zeta_1/2Q)}(\frac{\zeta_1}{2})^{n-1}\sqrt{\frac{\pi\zeta_1}{2}}J_{n-(1/2)}(\zeta_1) & n>0. \end{cases} \tag{64}$$

This together with Eq. (58) defines completely the transverse displacement at an arbitrary location σ and time ζ. The functions $h_n(\zeta)$ can, in principle, be calculated to arbitrary n, and $h_1(\zeta)$ and $h_2(\zeta)$ are

$$h_1(\zeta) = \frac{1}{1+\frac{1}{4Q^2}}\left[1 - e^{-(\zeta/2Q)}\left(\cos\zeta + \frac{1}{2Q}\sin\zeta\right)\right], \tag{65}$$

$$h_2(\zeta) = \left(\frac{1}{1+\frac{1}{4Q^2}}\right)^2\left[1 - e^{-(\zeta/2Q)}\left(\cos\zeta\left\{1 - \frac{\zeta}{4Q}\left(1+\frac{1}{4Q^2}\right)\right\} + \sin\zeta\left\{\frac{1}{4Q}\left(3+\frac{1}{4Q^2}\right) + \frac{\zeta}{2}\left(1+\frac{1}{4Q^2}\right)\right\}\right)\right]. \tag{66}$$

It can be noted that, from Eq. (64), $h_n(\zeta \ll 1) \sim [\zeta^{2n}/(2n)!]$, which is what was obtained in Eq. (16) for the single very short bunch and that, from Eq. (62), $\lim_{\zeta\to\infty} h_n(\zeta) = \lim_{q\to 0} q\hat{h}_n(q) = [\hat{w}(0)]^n = \tilde{w}_0^n$, which is what was obtained in Eq. (45) for the steady state.

B. Delta-function beam

In the case of a beam comprised of pointlike bunches turned on at $\zeta = 0$ we will consider only the displacement of the bunches themselves, and bunch M will correspond to $\zeta = M\omega\tau$.

In this case the current form factor is

$$F(\zeta) = \omega\tau \sum_{k=0}^{[\zeta/\omega\tau]} \delta[\zeta - k\omega\tau], \tag{67}$$

where $[\zeta/\omega\tau]$ is the largest integer less than or equal to $\zeta/\omega\tau$, and the recursion relations (59) then become

$$h_0(M\omega\tau) = 1, \tag{68a}$$

$$h_{n+1}(M\omega\tau) = \omega\tau \sum_{k=0}^{M} h_n(k\omega\tau) w[(M-k)\omega\tau]. \tag{68b}$$

The displacement of bunch M at location σ is given by

$$x_M(\sigma) = x_0 \sum_{n=0}^{\infty} \frac{1}{n!}\left(\frac{\varepsilon\sigma}{2\kappa}\right)^n \sqrt{\frac{\pi\kappa\sigma}{2}} h_n(M\omega\tau) J_{n-(1/2)}(\kappa\sigma). \tag{69}$$

Since the wake functions under consideration satisfy $w(0) = 0$, the recursion relations (68) imply that $f_n(M\omega\tau) = 0$ for $n > M$. As a consequence, the infinite sum in (69) reduces to a finite sum $\sum_{n=0}^{M}$.

The functions $h_n(M\omega\tau)$ can also be defined through the use of the z transform:

$$\mathfrak{Z}[f] = \check{f}(z) = \sum_{k=0}^{\infty} f(k) z^{-k}. \tag{70}$$

The z transform of the wake function $w(\zeta)$ is then

$$\mathfrak{Z}[w] = \check{w}(z) = \sum_{k=0}^{\infty} z^{-k} w(k\omega\tau), \tag{71}$$

and the recursion relation (68) for $h_n(M\omega\tau)$ leads to the following recursion relations for $\check{h}_n(z)$:

$$\check{h}_{n+1}(z) = \omega\tau \check{h}_n(z)\check{w}(z), \tag{72}$$

with

$$\check{h}_0(z) = \sum_{k=0}^{\infty} z^{-1} = \frac{z}{z-1}, \tag{73}$$

so that

$$\check{h}_n(z) = \frac{z}{z-1}[\omega\tau\check{w}(z)]^n. \tag{74}$$

The functions $h_n(M\omega\tau)$ can, in principle, be obtained by applying the inverse z transform.

$$h_n(M\omega\tau) = \mathfrak{Z}^{-1}[\check{h}_n(z)] = \frac{1}{2\pi i}\oint dz\, z^{M-1}\check{h}_n(z). \tag{75}$$

In the case of a single deflecting mode where the wake function is $w(\zeta) = u(\zeta)e^{-(\zeta/2Q)}\sin\zeta$, its z transform is

$$\check{w}(z) = \frac{\sin\omega\tau}{ze^{(\omega\tau/2Q)} - 2\cos\omega\tau + z^{-1}e^{(-\omega\tau/2Q)}}. \tag{76}$$

There is no simple unique closed-form representation for the functions $h_n(M\omega\tau)$, but they can be obtained to arbitrary order n using symbolic manipulation software to calculate the inverse z transform. For example, $h_1(M\omega\tau)$ and $h_2(M\omega\tau)$ are

$$h_1(M\omega\tau) = \frac{\omega\tau\sin\omega\tau}{2[\cosh\frac{\omega\tau}{2Q} - \cos\omega\tau]}\left[1 - e^{-M(\omega\tau/2Q)}\frac{\sin(M+1)\omega\tau}{\sin\omega\tau} + e^{-(M+1)(\omega\tau/2Q)}\frac{\sin M\omega\tau}{\sin\omega\tau}\right], \tag{77}$$

$$\begin{aligned} h_2(M\omega\tau) = \frac{(\omega\tau)^2}{4[\cosh\frac{\omega\tau}{2Q} - \cos\omega\tau]}\Bigg\{&\frac{(\sin\omega\tau)^2}{\cosh\frac{\omega\tau}{2Q} - \cos\omega\tau} - e^{-M(\omega\tau/2Q)}\left[\frac{\sin\omega\tau\sin(M+1)\omega\tau}{\cosh\frac{\omega\tau}{2Q} - \cos\omega\tau} - M\cos(M+1)\omega\tau + \frac{\sin M\omega\tau}{\sin\omega\tau}\right] \\ &- e^{-(M+1)(\omega\tau/2Q)}\left[\frac{\sin\omega\tau\sin M\omega\tau}{\cosh\frac{\omega\tau}{2Q} - \cos\omega\tau} - (M+1)\cos M\omega\tau + \frac{\sin(M+1)\omega\tau}{\sin\omega\tau}\right]\Bigg\}. \end{aligned} \tag{78}$$

Taking the limits $\omega\tau \to 0$ and $M\omega\tau \to \zeta$ we recover $h_1(\zeta)$ and $h_2(\zeta)$ for the dc beam given by Eqs. (65) and (66). In the case of more complicated wake functions, the functions $h_n(M\omega\tau)$ can be tabulated using Eq. (68).

TABLE I. Nominal top-level linear-collider design parameters [21,22].

Parameter	Value
Total initial energy $\gamma(0)mc^2$	10 GeV
Total final energy $\gamma(1)mc^2$	1 TeV
Linac Length $\mathcal{L}$	10 km
Number of betatron periods	100
Bunch charge	1 nC
Number of bunches in train M	90
Bunch spacing τ	2.8 ns
Deflecting-wake frequency $\omega/2\pi$	14.95 GHz
Deflecting-wake quality factor Q	∞
Deflecting-wake amplitude w_0	10^{15} VC^{-1} m^{-2}

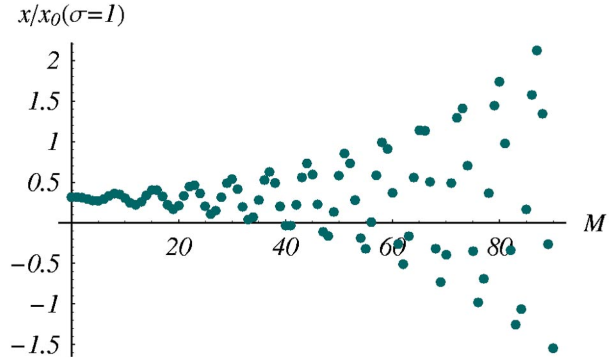

FIG. 5. (Color) Normalized lateral displacement of a finite train of pointlike bunches at the exit of a representative linear collider. See Table I for the choice of parameters.

Equation (69) allows direct calculation of the lateral displacement of an arbitrary bunch M at an arbitrary location σ. As an example we will apply these analytical results to a beam representative of a linear collider. For comparison we will use the same parameters as those used in [21,22], and which are listed in Table I. Since this is an accelerated beam, the transformations described in Appendix A, and, in particular, Eqs. (A10) and (A11), will be used. Converting the parameters in Table I to those used in this paper we have $\varepsilon(0) = [w_0 q e \mathcal{L}^2/\gamma(0)\mathrm{mc}^2\omega\tau] = 38.02$, $\kappa(0) = 1100\pi$, and $\omega\tau = 263.014$. The validity of the assumption (A7) is easily verified since, in this example, $\kappa_r^2 \simeq 1.2 \times 10^7$ while $-(\psi''/4) + (\psi'^2/16\psi)$ varies from 613 at the entrance of the accelerator to 6.13 at the exit.

The transverse displacements of the bunches at the exit of the accelerator are shown in Fig. 5. Note that the normalized displacement of the first, and all subsequent, bunches is reduced due to adiabatic damping. These results, which we obtained analytically, are identical to those shown in Fig. 1 of [22] which were obtained by tracking successive bunches along the accelerator. The only difference is that, in [22], the steady-state displacement was subtracted.

C. Finite train of finite bunches

In the case of a beam composed of finite but identical bunches turned on at $\zeta = 0$, the beam current form factor is

$$F(\zeta) = u(\zeta) \sum_{k=-\infty}^{\infty} F_k \exp\left(i\frac{2\pi}{\omega\tau}k\zeta\right), \qquad (79)$$

so that

$$\begin{aligned} h_1(\zeta) &= \int_{-\infty}^{\zeta} u(\zeta_1)F(\zeta_1)w(\zeta-\zeta_1)d\zeta_1 = \int_0^{\zeta} F(\zeta-\zeta_1)w(\zeta_1)d\zeta_1 \\ &= \sum_k F_k \tilde{w}_k \exp\left(i\frac{2\pi}{\omega\tau}k\zeta\right) - \sum_k F_k \exp\left(i\frac{2\pi}{\omega\tau}k\zeta\right)\int_{\zeta}^{\infty}\exp\left(-i\frac{2\pi}{\omega\tau}k\zeta_1\right)w(\zeta_1)d\zeta_1. \end{aligned} \qquad (80)$$

The first term in Eq. (80) is simply the steady state that was obtained in Eq. (38) while the second is the transient that decays when $\zeta \to +\infty$.

In the case of a single deflecting mode we obtain

$$\begin{aligned} h_1(\zeta) &= \frac{\omega\tau}{4\pi}\sum_k F_k\left\{\left[-\frac{\exp(i\frac{2\pi}{\omega\tau}k\zeta)}{k-\frac{\omega\tau}{2\pi}(\frac{i}{2Q}+1)} + \frac{\exp(i\frac{2\pi}{\omega\tau}k\zeta)}{k-\frac{\omega\tau}{2\pi}(\frac{i}{2Q}-1)}\right] + e^{-(\zeta/2Q)}\left[\frac{e^{i\zeta}}{k-\frac{\omega\tau}{2\pi}(\frac{i}{2Q}+1)} - \frac{e^{-i\zeta}}{k-\frac{\omega\tau}{2\pi}(\frac{i}{2Q}-1)}\right]\right\} \\ &= \sum_k \frac{F_k}{[1-k^2(\frac{2\pi}{\omega\tau})^2 + \frac{1}{4Q^2}] + \frac{i}{Q}\frac{2\pi}{\omega\tau}k}\left\{\exp\left(i\frac{2\pi}{\omega\tau}k\zeta\right) - e^{-(\zeta/2Q)}\left[\left(\frac{1}{2Q} + i\frac{2\pi}{\omega\tau}k\right)\sin\zeta + \cos\zeta\right]\right\}. \end{aligned} \qquad (81)$$

As an example we will use the same beam profile first used in Sec. IV for the steady state, i.e., a beam composed of bunches of constant density, separated by $\omega\tau$, and of length $\alpha\omega\tau$. In this case, however, since the beam was turned on at $\zeta = 0$, we will assume that $\zeta = 0$ is the front of the first bunch, and the Fourier coefficients of the current form factor are

$$F_k = e^{-ik\pi\alpha}\frac{\sin k\pi\alpha}{k\pi\alpha}. \qquad (82)$$

We will look for the transverse displacement of a particle located within bunch M and a (normalized) time $\alpha\omega\tau\phi$ behind the head of that bunch so that

$\zeta = \omega\tau[M + \alpha\phi]$. The variable ϕ will represent the location of the particle within the bunch, with $\phi = 0$ being the head of the bunch while $\phi = 1$ will be the tail. With these assumptions we have

$$h_1(\zeta) = \frac{\omega\tau}{4\pi}\sum_{k=-\infty}^{\infty} e^{-ik\pi\alpha}\frac{\sin k\pi\alpha}{k\pi\alpha}\left\{\left[-\frac{e^{i2\pi k\alpha\phi}}{k - \frac{\omega\tau}{2\pi}(\frac{i}{2Q}+1)} + \frac{e^{i2\pi k\alpha\phi}}{k - \frac{\omega\tau}{2\pi}(\frac{i}{2Q}-1)}\right] + e^{-(\zeta/2Q)}\left[\frac{e^{i\zeta}}{k - \frac{\omega\tau}{2\pi}(\frac{i}{2Q}+1)} - \frac{e^{-i\zeta}}{k - \frac{\omega\tau}{2\pi}(\frac{i}{2Q}-1)}\right]\right\}. \tag{83}$$

Using the identity (43) we obtain

$$h_1(\zeta) = \frac{1}{2\alpha(1+\frac{i}{2Q})}\left\{1 - e^{\omega\tau\alpha[\phi-(1/2)][-(1/2Q)+i]}\frac{\sin[\frac{\omega\tau}{2}(1+\frac{i}{2Q})(1-\alpha)]}{\sin[\frac{\omega\tau}{2}(1+\frac{i}{2Q})]} - e^{\zeta[-(1/2Q)+i]}\left[1 - e^{-(\omega\tau\alpha/2)[-(1/2Q)+i]}\frac{\sin[\frac{\omega\tau}{2}(1+\frac{i}{2Q})(1-\alpha)]}{\sin[\frac{\omega\tau}{2}(1+\frac{i}{2Q})]}\right]\right\} + \text{c.c.} \tag{84}$$

Again, the first term in the bracket is the steady state, while the second is the transient that decays when $\zeta \to +\infty$.

Setting $\alpha = 1$ in Eq. (84) we recover $h_1(\zeta)$ given in Eq. (65) for the dc beam, and, taking the limit $\alpha \to 0$ with $\zeta = M\omega\tau$, we recover $h_1(M\omega\tau)$ given in Eq. (77) for the delta-function beam. As before the displacement and shape of the bunches, to first order in ε, are given by

$$x(\sigma, \zeta) \simeq x_0\left[\cos(\kappa\sigma) + \frac{\varepsilon\sigma}{2\kappa}\sin(\kappa\sigma)h_1(\zeta)\right], \tag{85}$$

where now $h_1(\zeta)$ is given by Eq. (84).

Examples are shown in Figs. 6–9. Figures 6 and 7 are examples of a beam with weak coupling to the dipole mode ($\varepsilon = 0.2$) but with the beam frequency in close resonance to a high-Q dipole mode ($\omega\tau = 4.005\pi$, $Q = 10^4$). As in previous examples, we chose $\kappa = 100.5\pi$ so that, at the accelerator exit ($\sigma = 1$), the bunch displacement and shape distortion are due exclusively to the coupling to the dipole mode. Figure 6 show the displacement of the front, middle, and tail of the bunch train. Figure 7 shows the evolution of the shape of the bunches as the bunch number increases. The steady-state behavior ($M \to \infty$) is shown in Fig. 2(a).

Figures 8 and 9 are similar to Figs. 6 and 7, but are examples of bunches that are more strongly coupled to a lower-Q dipole mode in almost antiresonance ($\varepsilon = 100$, $\omega\tau = 3.01\pi$, $Q = 2 \times 10^3$). The effect of the antiresonance is clearly seen in the shape of successive bunches. For example, the tail of bunch $M = 1$ is not deflected since the wake field generated by the preceding particles in the same bunch $M = 1$ cancel almost exactly the wake field generated by the preceding bunch $M = 0$.

VI. DISCUSSION

This paper presents a formalism to address analytically cumulative beam breakup in linear accelerators under fairly general conditions. It allows, in principle, direct calculation, at any time and location, of the transverse displacement of beams of arbitrary current distribution in the transient and steady-state regime. When applied to finite-length bunched beams it reveals the transverse

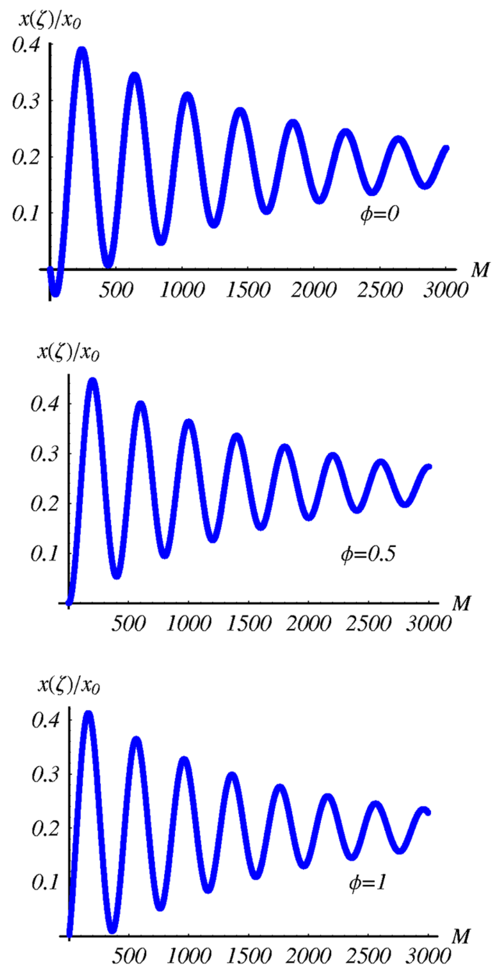

FIG. 6. (Color) Normalized displacement of a train of finite-length bunches with $\alpha = 0.1$, $Q = 10^4$, $\kappa = 100.5\pi$, $\varepsilon = 0.2$, $\omega\tau = 4.005\pi$, and $\sigma = 1$. The upper, middle, and lower figures are for the front, middle, and tail of the bunches, respectively.

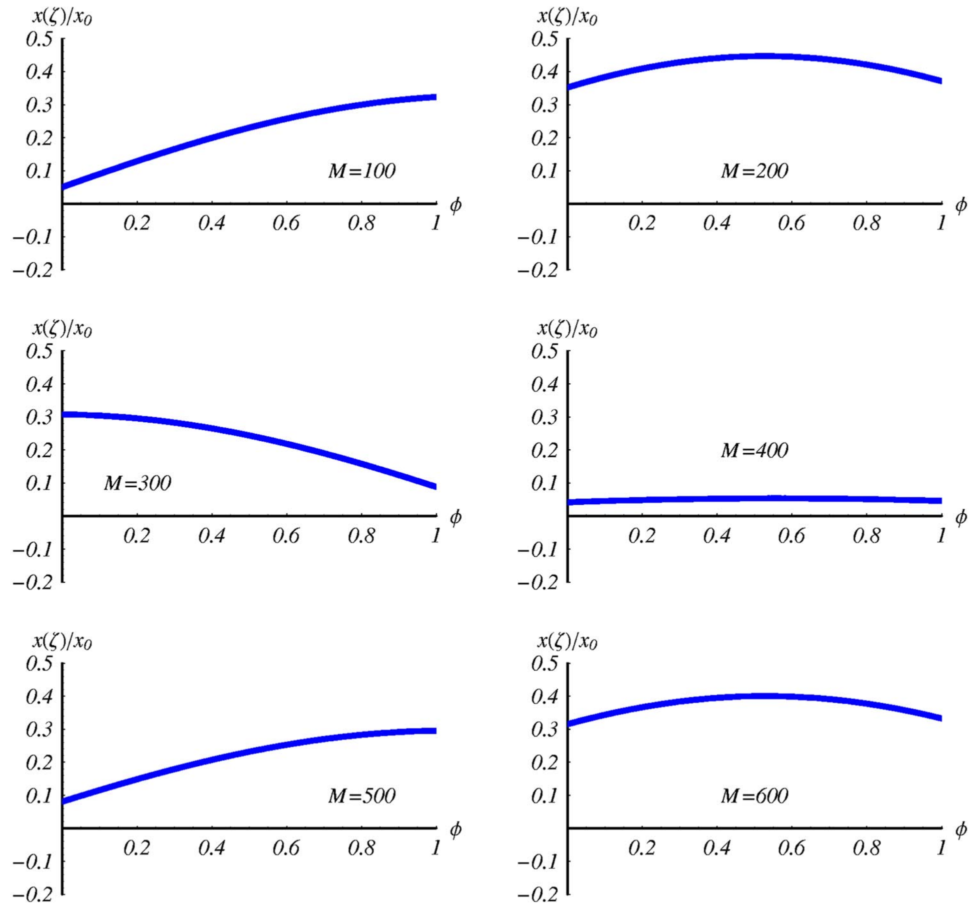

FIG. 7. (Color) Bunch shapes in a train of finite-length bunches. The parameters are the same as in Fig. 6.

distortion of the bunches. When applied to a colliderlike accelerator, the analytical results reproduce exactly numerical simulations that were done previously.

While this paper has concentrated on cumulative beam breakup, the same formalism and analytical results can be applied directly to other types of instabilities, such as resistive wall instabilities, by an appropriate choice of the wake functions.

The influence of displacement of the cavities and focusing elements are incorporated in these results but were addressed only briefly. This will be the subject of a follow-on communication.

While the analytical results presented in this paper assume time-independent focusing and coupling to the dipole modes, the same formalism can be extended to include such time dependence and the results will be presented in another paper.

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APPENDIX A: ACCELERATED BEAM

In the main sections of this paper we assume a coasting beam where $\beta\gamma$, κ, and ε are independent of σ. However, under realistic assumptions, the results can be straightforwardly applied to the case of an accelerated beam.

The full equation of motion for an accelerated beam is

$$\frac{1}{\beta\gamma}\frac{\partial}{\partial\sigma}\left[\beta\gamma\frac{\partial}{\partial\sigma}x(\sigma,\zeta)\right]+\kappa^2[x(\sigma,\zeta)-d_f(\sigma)] = \varepsilon\int_{-\infty}^{\zeta} w(\zeta-\zeta_1)F(\zeta_1)[x(\sigma,\zeta_1)-d_c(\sigma)]d\zeta_1, \quad \text{(A1)}$$

where now $\beta\gamma$, κ, and ε can vary with σ.

From the definition of ε given in Eq. (3) we see that ε has an explicit dependence on $\beta\gamma$. It also has an implicit dependence on β through the shunt impedance $\Gamma_\perp$ and the cavity length L, but this would occur only for nonrelativistic beams, and, even in that case, one may assume that the shunt impedance per unit length is constant. Thus, it is reasonable to assume $\varepsilon \propto (\beta\gamma)^{-1}$. Similarly, we can assume that $\kappa \propto (\beta\gamma)^{-1/2}$.

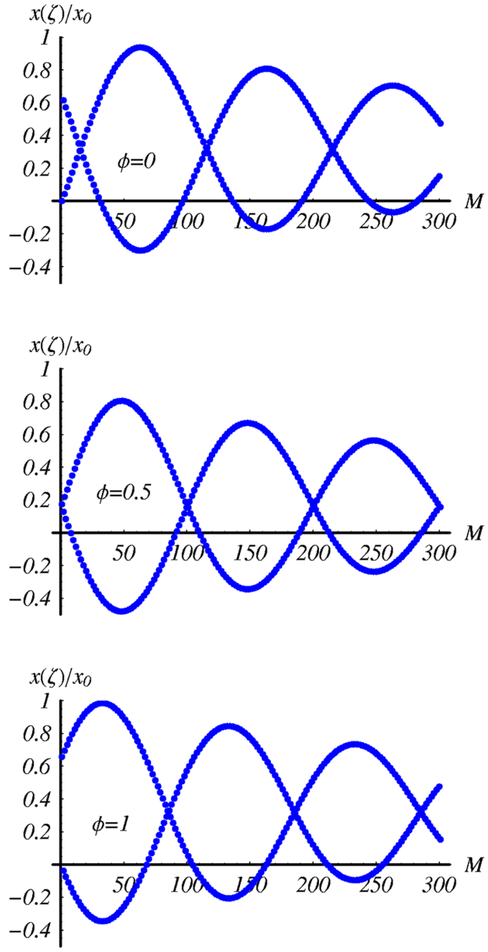

FIG. 8. (Color) Normalized displacement of a train of finite-length bunches with $\alpha = 0.1$, $Q = 2 \times 10^3$, $\kappa = 100.5\pi$, $\varepsilon = 100$, $\omega\tau = 3.01\pi$, and $\sigma = 1$. The upper, middle, and lower figures are for the front, middle, and tail of the bunches, respectively.

Defining β_r, γ_r, ε_r, and κ_r as the values of $\beta(\sigma)$, $\gamma(\sigma)$, $\varepsilon(\sigma)$, and $\kappa(\sigma)$ at an arbitrary reference location σ_r in the accelerator, and $\psi(\sigma)$ as

$$\psi(\sigma) = \frac{\beta(\sigma)\gamma(\sigma)}{\beta_r\gamma_r}, \tag{A2}$$

we have $\varepsilon(\sigma) = \varepsilon_r/\psi(\sigma)$ and $\kappa(\sigma) = \kappa_r/\psi^{1/2}(\sigma)$. We now introduce new variables ξ, δ_f, and δ_c for the transverse displacements

$$x(\sigma, \zeta) = \xi(\sigma, \zeta)[\psi(\sigma)]^{\mu}, \tag{A3a}$$

$$d_f(\sigma, \zeta) = \delta_f(\sigma, \zeta)[\psi(\sigma)]^{\mu}, \tag{A3b}$$

$$d_c(\sigma, \zeta) = \delta_c(\sigma, \zeta)[\psi(\sigma)]^{\mu}, \tag{A3c}$$

and ς for the longitudinal location along the linac

$$\varsigma = \int_0^{\sigma} [\psi(\sigma)]^{\nu} d\sigma. \tag{A4}$$

With these new variables the equation of motion becomes

$$\psi^{2\nu+1}\frac{\partial^2}{\partial\varsigma^2}\xi(\varsigma, \zeta) + (2\mu + \nu + 1)\psi'\psi^2\frac{\partial}{\partial\varsigma}\xi(\varsigma, \zeta) + [\mu\psi'' + \mu^2\psi'^2\psi^{-1} + \kappa_r^2]\xi(\varsigma, \zeta) - \kappa_r^2\delta_f(\varsigma, \zeta) = \varepsilon_r\int_{-\infty}^{\zeta} w(\zeta - \zeta_1)F(\zeta_1)[\xi(\varsigma, \zeta_1) - \delta_c(\varsigma, \zeta_1)]d\zeta_1. \tag{A5}$$

If we choose $\nu = -1/2$ and $\mu = -1/4$ then

$$\frac{\partial^2}{\partial\varsigma^2}\xi(\varsigma, \zeta) + \left[-\frac{\psi''}{4} + \frac{\psi'^2}{16\psi} + \kappa_r^2\right]\xi(\varsigma, \zeta) - \kappa_r^2\delta_f(\varsigma, \zeta) = \varepsilon_r\int_{-\infty}^{\zeta} w(\zeta - \zeta_1)F(\zeta_1)[\xi(\varsigma, \zeta_1) - \delta_c(\varsigma, \zeta_1)]d\zeta_1. \tag{A6}$$

Since in most applications we have

$$-\frac{\psi''}{4} + \frac{\psi'^2}{16\psi} \ll \kappa_r^2, \tag{A7}$$

the equation of motion for $\xi(\varsigma, \zeta)$ in the accelerated case is identical to that for $x(\sigma, \zeta)$ in the coasting case.

Thus, with the assumptions made above, the results obtained for a coasting beam are directly applicable to an accelerated beam after the appropriate change of variables and coordinates, and the general solution for $\xi(\varsigma, \zeta)$ is given by

$$\xi(\varsigma, \zeta) = \sum_{n=0}^{\infty}\varepsilon_r^n[\xi_0\mathfrak{h}_n(\zeta)j_n(\kappa_r, \varsigma) + \xi_0'\mathfrak{g}_n i_n(\kappa_r, \varsigma)(\zeta)] - \sum_{n=0}^{\infty}\varepsilon_r^{n+1}f_{n+1}(\zeta)\int_0^{\varsigma} i_n(\kappa_r, u)\delta_c(\varsigma - u)du + \kappa_r^2\sum_{n=0}^{\infty}\varepsilon_r^n f_n(\zeta)\int_0^{\varsigma} i_n(\kappa_r, u)\delta_f(\varsigma - u)du. \tag{A8}$$

The functions $\mathfrak{h}_n(\zeta)$ and $\mathfrak{g}_n(\zeta)$ obey the same recursion relations as $h_n(\zeta)$ and $g_n(\zeta)$ but we now have

$$\xi_0\,\mathfrak{h}_0(\zeta) = \xi_0(\zeta) = x_0(\zeta)[\psi(0)]^{1/4} = x_0\,h_0(\zeta)[\psi(0)]^{1/4}, \tag{A9a}$$

$$\xi_0'\mathfrak{g}_0(\zeta) = \frac{d}{d\varsigma}\xi(\varsigma, \zeta)|_{\varsigma=0} = \left[x_0'(\zeta) + \frac{1}{4}\frac{\psi'(0)}{\psi(0)}x_0(\zeta)\right][\psi(0)]^{3/4} = \left[x_0'g_0(\zeta) + \frac{1}{4}\frac{\psi'(0)}{\psi(0)}x_0h_0(\zeta)\right][\psi(0)]^{3/4} \tag{A9b}$$

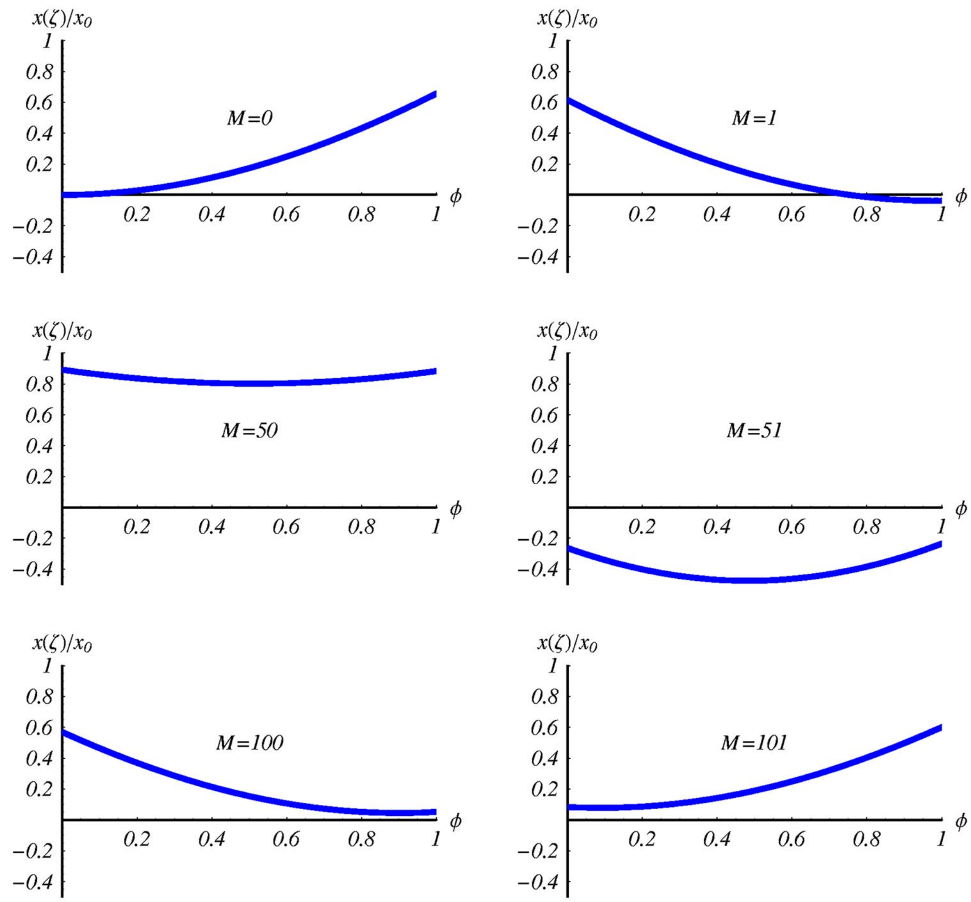

FIG. 9. (Color) Bunch shapes in a train of finite-length bunches. The parameters are the same as in Fig. 8.

As an example we consider a relativistic uniformly accelerated beam and we ignore the transverse displacements of cavities and focusing elements. Although the expression for $\xi(\varsigma, \zeta)$ in Eq. (A8) includes κ_r and ε_r—values of $\kappa(\sigma)$ and $\varepsilon(\sigma)$ at $\sigma = \sigma_r$—the actual value of $\xi(\varsigma, \zeta)$ ought to be, and is, as confirmed by algebra, independent of the choice of σ_r. For this reason, in the definition of $\psi(\sigma)$, we will choose $\sigma_r = 0$. With these assumptions we have

$$x(\sigma, \zeta) = (1 + \bar{\gamma}\sigma)^{-1/4} \sum_{n=0}^{\infty} \varepsilon_0^n \left\{ x_0 h_n(\zeta) j_n(\kappa_0, \varsigma) + \left[x_0' g_n(\zeta) + x_0 \frac{\bar{\gamma}}{4} h_n(\zeta) \right] i_n(\kappa_0, \varsigma) \right\}, \tag{A10}$$

with

$$\gamma(\sigma) = \gamma_0(1 + \bar{\gamma}\sigma), \tag{A11a}$$

$$\varsigma = \int_0^\sigma du(1 + \bar{\gamma}u)^{-(1/2)} = \frac{2\sigma}{(1 + \bar{\gamma}\sigma)^{1/2} + 1}, \tag{A11b}$$

$$x_0' g_0(\zeta) = x_0'(\zeta), \qquad g_{n+1}(\zeta) = \int_{-\infty}^{\zeta} g_n(\zeta_1) F(\zeta_1) w(\zeta - \zeta_1) d\zeta_1, \tag{A11c}$$

$$x_0 h_0(\zeta) = x_0(\zeta), \quad h_{n+1}(\zeta) = \int_{-\infty}^{\zeta} h_n(\zeta_1) F(\zeta_1) w(\zeta - \zeta_1) d\zeta_1. \tag{A11d}$$

APPENDIX B: LAPLACE-TRANSFORMED EQUATION OF MOTION

The transverse equation of motion for a coasting beam is

$$\frac{\partial^2}{\partial \sigma^2} x(\sigma, \zeta) + \kappa^2[x(\sigma, \zeta) - d_f(\sigma)] = \varepsilon \int_{-\infty}^{\zeta} w(\zeta - \zeta_1) F(\zeta_1)[x(\sigma, \zeta_1) - d_c(\sigma)] d\zeta_1. \tag{B1}$$

Applying the Laplace transform with respect to σ to the transverse displacements

$$x^{\dagger}(p,\zeta) = \mathfrak{L}_{\sigma}[x(\sigma,\zeta)], \qquad d_f^{\dagger}(p,\zeta) = \mathfrak{L}_{\sigma}[d_f(\sigma,\zeta)], \qquad d_c^{\dagger}(p,\zeta) = \mathfrak{L}_{\sigma}[d_c(\sigma,\zeta)], \tag{B2}$$

and defining $x_0(\zeta) = x_0 h_0(\zeta) = x(\sigma = 0, \zeta)$ and $x_0'(\zeta) = x_0' g_0(\zeta) = (\partial/\partial\sigma)x(\sigma,\zeta)|_{\sigma=0}$ as the lateral displacement and angular divergence of the beam at the entrance of the accelerator, the equation for $x^{\dagger}(p,\zeta)$ is, from Eq. (B1),

$$\begin{aligned} x^{\dagger}(p,\zeta) = {} & \frac{px_0}{p^2+\kappa^2}h_0(\zeta) + \frac{x_0'}{p^2+\kappa^2}g_0(\zeta) + \kappa^2\frac{d_f^{\dagger}(p)}{p^2+\kappa^2} - \varepsilon\frac{d_c^{\dagger}(p)}{p^2+\kappa^2}\int_{-\infty}^{\zeta} w(\zeta-\zeta_1)F(\zeta_1)d\zeta_1 \\ & + \varepsilon\frac{1}{p^2+\kappa^2}\int_{-\infty}^{\zeta} w(\zeta-\zeta_1)F(\zeta_1)x^{\dagger}(p,\zeta_1)d\zeta_1. \end{aligned} \tag{B3}$$

The variables x_0 and x_0' are normalizing constants that establish the scale of the lateral displacement and angular divergence so that the time dependencies $h_0(\zeta)$ and $g_0(\zeta)$ are dimensionless. If the initial conditions are time independent we then have $h_0(\zeta) = g_0(\zeta) = 1$.

Using the right-hand side of Eq. (B3) to replace $x^{\dagger}(p,\zeta_1)$ in the last integral in Eq. (B3) and assuming that κ and ε do not depend on ζ we obtain

$$\begin{aligned} x^{\dagger}(p,\zeta) = {} & \frac{px_0}{p^2+\kappa^2}\left[h_0(\zeta) + \frac{\varepsilon}{p^2+\kappa^2}\int_{-\infty}^{\zeta} w(\zeta-\zeta_1)F(\zeta_1)h_0(\zeta_1)d\zeta_1\right] \\ & + \frac{x_0'}{p^2+\kappa^2}\left[g_0(\zeta) + \frac{\varepsilon}{p^2+\kappa^2}\int_{-\infty}^{\zeta} w(\zeta-\zeta_1)F(\zeta_1)g_0(\zeta_1)d\zeta_1\right] \\ & + \kappa^2\frac{d_f^{\dagger}(p)}{p^2+\kappa^2}\left[1 + \frac{\varepsilon}{p^2+\kappa^2}\int_{-\infty}^{\zeta} w(\zeta-\zeta_1)F(\zeta_1)d\zeta_1\right] \\ & - \varepsilon\frac{d_c^{\dagger}(p)}{p^2+\kappa^2}\left[\int_{-\infty}^{\zeta} w(\zeta-\zeta_1)F(\zeta_1)d\zeta_1 + \frac{\varepsilon}{p^2+\kappa^2}\int_{-\infty}^{\zeta} w(\zeta-\zeta_1)F(\zeta_1)d\zeta_1\int_{-\infty}^{\zeta_1} w(\zeta_1-\zeta_2)F(\zeta_2)d\zeta_2\right] \\ & + \frac{\varepsilon^2}{p^2+\kappa^2}\int_{-\infty}^{\zeta} d\zeta_1 w(\zeta-\zeta_1)F(\zeta_1)\int_{-\infty}^{\zeta_1} w(\zeta_1-\zeta_2)F(\zeta_2)x^{\dagger}(p,\zeta_2)d\zeta_2. \end{aligned} \tag{B4}$$

Again, the right-hand side of Eq. (B3) can be used to replace $x^{\dagger}(p,\zeta_2)$ in the last integral in Eq. (B4) and the process can be repeated indefinitely; Eq. (7) then follows directly.

Note that Eq. (B3) makes no assumption on the ζ dependence of κ and ε, while the next steps to Eq. (B4) and ultimately Eq. (7) assume that κ and ε are independent of ζ. The extension of this formalism to time-dependent focusing $[\kappa(\zeta)]$ or coupling to the wake field $[\varepsilon(\zeta)]$, such that may result from the presence of rf focusing or an energy chirp imposed on the beam, will be addressed in another paper.

APPENDIX C: A PROPERTY OF LAPLACE TRANSFORMS

In this appendix we find a relationship between the function $f_k(\sigma)$ of Laplace transform $f^{\dagger}(\sqrt{p^2+\kappa^2})$ and the function $f_{k=0}(\sigma)$ of Laplace transform $f^{\dagger}(p)$. This can be used to obtain the transverse displacement in the presence of focusing from the expression for the transverse displacement in the absence of focusing.

Assuming that the function $f_{\kappa=0}(\sigma)$ can be expanded in a power series

$$f_{\kappa=0}(\sigma) = \sum_{k=0}^{\infty} a_k \sigma^k, \tag{C1}$$

we have

$$f^{\dagger}(p) = \mathfrak{L}[f_{\kappa=0}(\sigma)] = \sum_{k=0}^{\infty} a_k \frac{\Gamma(k+1)}{p^{k+1}}. \tag{C2}$$

The function $f^{\dagger}(\sqrt{p^2+\kappa^2})$ is then

$$f^{\dagger}(\sqrt{p^2+\kappa^2}) = \sum_{k=0}^{\infty} a_k \frac{\Gamma(k+1)}{(p^2+\kappa^2)^{(k+1)/2}}, \tag{C3}$$

and its inverse Laplace transform is

$$\begin{aligned} f_{\kappa}(\sigma) &= \mathfrak{L}^{-1}[f^{\dagger}(\sqrt{p^2+\kappa^2})] \\ &= \sum_{k=0}^{\infty} a_k \Gamma\left(\frac{k}{2}+1\right)\left(\frac{2\sigma}{\kappa}\right)^{k/2} J_{(k/2)}(\kappa\sigma). \end{aligned} \tag{C4}$$

In order to relate $f_{\kappa}(\sigma)$ to $f_{\kappa=0}(\sigma)$ we need to find a linear relationship between σ^k and $\Gamma(k/2+1) \times (2\sigma/\kappa)^{k/2} J_{(k/2)}(\kappa\sigma)$ that is independent of k. This is provided by Eq. (2.12.4.5) of Ref. [23]

$$\int_0^a x^{1-n}(a^2 - x^2)^{\beta-1} J_n(cx)dx = (-1)^n 2^{\beta-1}\Gamma(\beta)\frac{a^{\beta-n}}{c^\beta}\left[J_{\beta-n}(ac) - \sum_{k=0}^{n-1}\frac{(-1)^k (a\,c/2)^{\beta-n+2k}}{k!\Gamma(\beta-n+k+1)}\right]. \tag{C5}$$

With the appropriate assignment to the parameters we obtain

$$\Gamma\left(\frac{k}{2}+1\right)\left(\frac{2\sigma}{\kappa}\right)^{k/2} J_{(k/2)}(\kappa\sigma) = \sigma^k - \kappa\int_0^\sigma (\sigma^2 - x^2)^{k/2} J_1(\kappa x)dx. \tag{C6}$$

After a change of variable in the integral in Eq. (C6), Eq. (15a) follows directly, and Eq. (15b) is obtained from the formula for the Laplace transform of a derivative.

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