Limitations of electron beam conditioning for free-electron lasers

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Several ideas have been proposed to "condition" an electron beam prior to the undulator of a freeelectron laser (FEL) by increasing each particle's energy in proportion to the square of its transverse betatron amplitude. This conditioning enhances FEL gain by reducing the axial velocity spread within the electron bunch. We demonstrate that for symplectic beam lines, and independent of the method, this conditioning is always accompanied by a large head-tail focusing variation which, for short-wavelength FELs, is so severe as to make conditioning completely impractical. We furthermore find that any system added to correct the head-tail focusing variation will also remove the conditioning. As an example, a new method for conditioning is presented and shown to generate exactly the same head-tail focusing problems as in previously published work.

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I. INTRODUCTION

The most demanding requirement for future freeelectron lasers (FELs) in the x-ray regime [1,2] is the reliable generation of a sufficiently small transverse electron emittance. To mitigate this problem, several ideas have been proposed to "condition" an electron beam prior to the undulator by increasing each particle's energy in proportion to the square of its betatron amplitude [3-6]. This conditioning enhances FEL gain by reducing the axial velocity spread within the electron bunch generated over the undulator, due to both energy spread and finite transverse emittance. A common proposal for generating the conditioning correlation is to differentially accelerate off-axis particles with a series of TM₂₁₀-like mode transverse rf structures. But the conditioning can also be accomplished by delaying large amplitude particles in a long, strong focusing channel and differentially accelerating the delayed particles with off crest-phase acceleration [6]. In Ref. [3] an undesirable side effect is briefly mentioned, due to the transverse time-dependent magnetic field, which results in a head-tail focusing variation along the bunch length. A correction is also suggested using rf quadrupoles (RFQ). Unfortunately, as we show in this paper, this side effect is a necessary outcome of conditioning, due to the symplecticity of the map describing the beam line, and it can be prohibitively large. In the conditioner at the end of Ref. [6], although not discussed there, the effect will appear as the chromaticity of a strong focusing channel on a chirped beam. Furthermore, we find that any system added to correct this head-tail focusing variation will necessarily remove the energy conditioning.

In this paper, we briefly review the requirements for FEL beam conditioning and, as an example, present a

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new conditioning scheme using strong solenoid magnets, which at first examination looks promising. But the headtail focusing again appears and we quantify the resulting "projected" (i.e., bunch length integrated) transverse emittance growth, relating it directly to the FEL parameters. A numerical example is used to demonstrate the extreme severity of the effect and the impracticality of conditioning a short-wavelength FEL. We then present a general symplectic beam conditioner using generator functions and show the unavoidable relation between conditioning and projected transverse emittance growth. Finally, we discuss the limitations, provide a more familiar analogy, and summarize our results.

II. FEL BEAM CONDITIONING

Electron beam conditioning, as proposed in [3], increases each particle's energy in proportion to the square of its betatron amplitude. A particle with high energy travels a shorter path in an undulator (increased mean axial velocity), while a large betatron amplitude delays a particle by lengthening its path through the undulator [7]. The conditioning correlation establishes a cancellation of these two effects, resulting in a significant reduction of the axial velocity spread, enhancing the FEL gain, or perhaps allowing the FEL to operate with a larger transverse emittance than required without conditioning. The relative energy conditioning requirement, for natural undulator focusing, can be written as [3]

$$\delta_u = \delta_n + \frac{1}{4\gamma_u} \frac{\epsilon_N}{\beta_u} \frac{\lambda_u}{\lambda_r} r^2, \qquad (1)$$

where $\delta_n \ll \delta_u$ is the nonconditioned component of the particle's relative energy deviation, γ_u is the electron energy in the undulator (in units of rest mass), $\epsilon_N = \gamma_u \epsilon$) is the normalized rms transverse emittance (equal in x and y), $\beta_u = \beta_x = \beta_y$ is the constant beta function in the undulator, λ_u is the undulator period, λ_r is the FEL

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radiation wavelength, and r is the invariant normalized 4D betatron amplitude of the particle,

$$r^{2} \equiv \frac{x^{2} + (\beta_{u}x')^{2} + y^{2} + (\beta_{u}y')^{2}}{\beta_{u}\epsilon}.$$
 (2)

The index *u* above indicates the value of the respective quantity in the undulator. The betatron amplitude, *r*, is expressed in terms of a particle's transverse positions, *x* and *y*, and angles, x' and y', with natural focusing where $\alpha_x = \alpha_y = 0$. A conditioner beam line is designed to imprint this $\delta_u \sim r^2$ correlation within the electron bunch, with the coefficient given in Eq. (1).

In a general case, the conditioning might be performed at low energy near the injector where the bunch is still relatively long. For short-wavelength FELs, the bunch is typically compressed and accelerated after the injector. Both effects scale the conditioning, but in the absence of particle mixing, do not alter its correlation character. Acceleration from γ_0 ("energy" at injector) to γ_u ("energy" in undulator) reduces the conditioned relative energy spread, while compression from an initial bunch length, σ_{z_0} , to a shorter final bunch length, σ_{z_0} , amplifies the conditioning. The latter is true because the bunch is shortened after compression and since the longitudinal emittance is preserved, the energy spread must be increased proportionally, independent of the imprinted r^2 correlation. The relative energy deviation, δ , at the location of the conditioner, before acceleration and compression, must then be scaled by the acceleration and compression factors:

$$\delta = \frac{\sigma_{z_f} \gamma_u}{\sigma_{z_0} \gamma_0} \left(\delta_n + \frac{1}{4\gamma_u} \frac{\epsilon_N}{\beta_u} \frac{\lambda_u}{\lambda_r} r^2 \right).$$
(3)

This means that, depending on the level of bunch compression, the relative energy spread immediately after a low-energy conditioner is not necessarily larger than that at the undulator.

III. A ONE-PHASE SOLENOID CONDITIONER

As an example beam conditioner, and to show the ultimate limitations of conditioners, we describe here a simplified system composed of a solenoid magnet and rf accelerating sections. Its specific limitations, however, will be completely applicable to all conditioners which are based on differential acceleration of off-axis particles. The limitations can be described in terms of an effective transverse emittance growth and this growth will be related only to the FEL conditioning requirements, and not to the specific method of conditioning.

The conditioner is shown in Fig. 1 and is composed of a solenoid magnet sandwiched between two rf accelerating sections operated at opposing zero-crossing phases. (A similar idea was proposed at the end of Ref. [6].) The first rf section "chirps" the energy along the bunch length,



FIG. 1. One-phase electron beam conditioner composed of a solenoid magnet sandwiched between rf accelerating sections operated at opposing zero-crossing phases.

and the final section removes the chirp. The conditioning is generated in the solenoid by the delay of particles with large amplitudes in x and y. The solenoid strength is set to produce a +I linear transfer matrix in 6D with the relation $|k|L = n\pi$ (n = 1, 2, 3, ...), where $k \equiv \frac{1}{2}B_z/(B\rho)$, L is the solenoid length, B_z is its axial magnetic field, and ($B\rho$) is the standard magnetic rigidity ($= p_0/e$).

The particle coordinates within the bunch at the entrance to the system are $(x_0, x'_0, y_0, y'_0, z_0, \delta_0)$, where $\delta_0 \equiv \Delta p/p_0$, and we assume these variables are initially uncorrelated and have zero mean. For simplicity, we use a cylindrically symmetric beam with initial Twiss parameters: $\beta_x = \beta_y = \beta$, and $\alpha_x = \alpha_y = 0$. The Twiss parameters are unchanged, to 1st order, across the solenoid and across each "thin" rf section. The electrons are also assumed to be ultrarelativistic. These simplifications are removed in a general treatment in Sec. V.

The first rf section changes the relative energy deviation of a particle to $\delta_1 = \delta_0 + h_1 z_0$, where h_1 is the linear rf-induced slope ($h \equiv d\delta/dz$). For simplicity, the rf sections are treated as thin elements which do not alter the transverse coordinates. After the solenoid, the coordinates are unchanged to 1st order, but a chromatic 2ndorder aberration is added to the angles with $\Delta x' =$ $2T_{216}x_0(\delta_0 + h_1z_0)$ and $\Delta y' = 2T_{436}y_0(\delta_0 + h_1z_0)$. All other 2nd-order transverse aberrations are small in comparison for the case $|k|\beta \gg 1$, $|k|L = n\pi$ [8].

The energy is not changed in the solenoid, but the longitudinal coordinate is delayed by the helical trajectory according to $z_1 = z_0 + T_{511}x_0^2 + T_{533}y_0^2$ (bunch head at z > 0). Similarly, all other 2nd-order longitudinal aberrations are small for the case $|k|\beta \gg 1$. The 2nd-order coefficients of a solenoid with $|k|L = n\pi$ are related to each other by $T_{511} = T_{533} = -T_{216} = -T_{436} = -k^2L/2$ [8], which, as shown in Sec. V, is an unavoidable connection for symplectic systems. The final rf section, h_2 , changes the energy according to $\delta = \delta_1 + h_2 z_1 \approx (h_1 + h_2)z_0 - \frac{1}{2}k^2Lh_2(x_0^2 + y_0^2)$. The second chirp is chosen equal and opposite to the first, $h_1 = -h_2 \equiv h$, and the final coordinate map across the conditioner, to second order and for $|\delta_0| \ll |hz_0|$, becomes

$$\begin{aligned} x &= x_0, \qquad x' = x_0' + k^2 L h z_0 x_0, \\ z &= z_0 - \frac{1}{2} k^2 L (x_0^2 + y_0^2), \qquad \delta = \delta_0 + \frac{1}{2} k^2 L h (x_0^2 + y_0^2), \end{aligned}$$

with similar relations in y and y'. The final energy deviation, δ , is clearly conditioned (for h > 0) in both planes but in only one betatron phase (i.e., x_0 , but not x'_0). This system provides spatial (but not angular) conditioning described by

$$\delta = \delta_0 + \frac{1}{2}k^2 Lh\beta\epsilon_0 r^2, \qquad r^2 \equiv \frac{x_0^2 + y_0^2}{\beta\epsilon_0}.$$
 (5)

Two solenoids can also be used, separated by a $\pi/2$ transformer to condition both betatron phases, but here we simplify the description by considering only a one-phase conditioner.

The bunch-length coordinate, z, in Eqs. (4) also includes a nonlinear distortion due to the solenoid delay of large amplitude particles. This can easily be removed, without changing the energy conditioning, by adding a four-dipole chicane, or similar section, with $R_{56} = 1/h > 0$, after the final rf section, resulting in $z = z_0 + R_{56}\delta_0 \approx z_0$. The bunch is then restored to its initial length with no significant change in distribution. This point is not relevant here, so we ignore this correctable distortion and instead examine the energy conditioning and the associated transverse aberrations.

IV. ENERGY CONDITIONING AND TRANSVERSE EMITTANCE GROWTH

The conditioning coefficient in Eq. (3) can be equated to that in Eqs. (5) producing the conditioning requirement for the solenoid system

$$k^{2}Lh\beta\sigma_{z_{0}} = \frac{1}{2}\frac{\lambda_{u}}{\lambda_{r}}\frac{\sigma_{z_{f}}}{\beta_{u}} \equiv a,$$
(6)

where the solenoid-conditioner parameters are on the left side and the FEL parameters are on the right, and here we define the dimensionless conditioning coefficient, a. In the typical case of a short-wavelength FEL, the conditioning parameter a is large, $a \gg 1$ (see numerical example below). The chirp parameter, h, is more easily interpreted by relating it to the rms relative energy spread in the solenoids: $\sigma_{\delta_1} \approx |h| \sigma_{z_0}$. This also makes it clear that the transverse aberrations in Eqs. (4) are chromatic $(\delta_1 \approx hz_0)$, and it is useful at this point to quantify the aberrations in terms of an effective transverse emittance growth. The rms emittance after the solenoid is calculated using the first two of Eqs. (4) and the first and second moments of the particle ensemble:

$$\boldsymbol{\epsilon}_x^2 = \langle (x - \overline{x})^2 \rangle \langle (x' - \overline{x}')^2 \rangle - \langle (x - \overline{x})(x' - \overline{x}') \rangle^2.$$
(7)

The mean values, $\overline{x} = \langle x \rangle$ and $\overline{x}' = \langle x' \rangle$, are zero since the initial coordinates are uncorrelated and have zero mean. The correlation $\langle xx' \rangle$ is zero for the same reasons, so the *x* emittance after the solenoid is

$$\epsilon_x^2 = \langle x^2 \rangle \langle x'^2 \rangle \approx \langle x_0^2 \rangle \langle (x_0' + k^2 L h z_0 x_0)^2 \rangle$$

= $\epsilon_{x0}^2 [1 + (k^2 L h \beta \sigma_{z_0})^2],$ (8)

where $\langle x_0^2 \rangle = \beta \epsilon_{x0}, \langle x_0^2 \rangle = \epsilon_{x0}/\beta$, and $\langle z_0^2 \rangle = \sigma_{z_0}^2$, with a similar form in y. The relative emittance growth after the solenoid is

$$\frac{\epsilon_x}{\epsilon_{x0}} \approx k^2 L h \beta \sigma_{z_0} = a \gg 1, \tag{9}$$

which is identical to the conditioning relation in Eq. (6), providing a direct connection between transverse emittance growth and FEL conditioning requirements.

For parameters of the LCLS [1] shown in Table I [using a beta function for natural focusing, to be consistent with Eq. (1)], the relative emittance growth is extremely large at $\epsilon_x/\epsilon_{x0} \approx 33$. The parameters for the VISA FEL [9] are also included showing that conditioning may still be possible at longer wavelengths.

This growth is actually an increase of the projected transverse emittance integrated over the bunch length. The second of Eqs. (4) shows that the bunch head $(z_0 > 0)$ is defocused (equating $k^2 Lh z_0 = 1/f$), while the bunch tail $(z_0 < 0)$ is focused. The significance of the focusing variation is evaluated by comparing the focal length, f, at $z_0 = \pm \sigma_{z_0}$, to β . The ratio is $\beta/f(\pm \sigma_{z_0}) = \pm k^2 L\beta h \sigma_{z_0} = \pm a$, which is a very strong effect $(\beta/|f| \gg 1)$ for short-wavelength FELs, and is precisely the conditioning coefficient given in Eq. (6).

TABLE I. FEL and conditioner parameters for the LCLS [1] and for VISA [9].

Parameter	Symbol	LCLS	VISA	Units
Electron energy/ mc^2 in undulator	γ_u	28 000	140	
Undulator period	λ_{u}	3	1.8	cm
Radiation wavelength	λ_r	1.5	8500	Å
Undulator beta function (natural focusing)	β_u	72	0.6	m
Final rms bunch length	$\sigma_{z_{\ell}}$	24	100	μ m
Conditioning coefficient	a	33	1.8	

With a chirped energy spread, the chromatic effects of the solenoid are equivalent to the effects of an RFQ. It is interesting to compare this result with that of Ref. [3], where a completely different conditioner beam line, employing transverse rf cavities, produced an undesirable RFQ effect. In fact, as shown in the next section, FEL beam conditioning in a symplectic beam line always produces an undesirable RFQ effect, which is extremely large for short-wavelength FELs, as given in Eq. (9).

V. A GENERAL CONDITIONER

In this section we will show that the transverse emittance growth associated with conditioning is not related to the specific design outlined in the previous section, but is a general feature of any conditioner, and is due to the symplecticity of the map between the entrance to and exit from the conditioner.

To simplify consideration, we assume that the conditioner does not introduce coupling between the vertical and horizontal planes, and we consider only the horizontal plane with the initial values of coordinates (x_0, x'_0) at the entrance, and the final values (x, x') at the exit. Consideration of the vertical coordinates y, y' can be carried out analogously to x, x'. We will also assume that the initial and final values of the longitudinal coordinate are the same: $z = z_0$. Instead of using the variables x_0 , x'_0 and x, x', it is convenient and more general to introduce new variables u_0 , v_0 and u, v, such that

$$\begin{pmatrix} u_0 \\ v_0 \end{pmatrix} = Q_0 \begin{pmatrix} x_0 \\ x'_0 \end{pmatrix}, \qquad \begin{pmatrix} u \\ v \end{pmatrix} = Q \begin{pmatrix} x \\ x' \end{pmatrix}, \qquad (10)$$

where the matrices Q_0 and Q are

$$Q_0 = \frac{1}{\sqrt{\beta_0}} \begin{pmatrix} 1 & 0\\ \alpha_0 & \beta_0 \end{pmatrix}, \qquad Q = \frac{1}{\sqrt{\beta}} \begin{pmatrix} 1 & 0\\ \alpha & \beta \end{pmatrix}, \quad (11)$$

with β_0 , α_0 and β , α the Twiss parameters at the entrance and exit of the conditioner, respectively. Being symplectic linear transformations, Q and Q_0 conserve the symplecticity of the map from (u_0, v_0) to (u, v). Note that in linear approximation this map has a form

$$\binom{u}{v} = A\binom{u_0}{v_0},\tag{12}$$

where

$$A = \begin{pmatrix} \cos\psi & \sin\psi \\ -\sin\psi & \cos\psi \end{pmatrix},$$
 (13)

with ψ the betatron phase advance across the conditioner. Also note that the contribution of the *x* coordinate $x_0^2/(\beta\epsilon_0)$ to the parameter *r* in Eqs. (5) is equal to u_0^2/ϵ_0 , and the conditioning requirement Eqs. (5) can be written as

$$\delta = \delta_0 + \frac{1}{2}bu_{0}^2,\tag{14}$$

where $b = a/\sigma_{z_0}$, and the conditioning constant *a* is given by Eq. (6).

To derive a general symplectic map which in linear approximation reduces to the linear map Eq. (12) and also includes the conditioning given by Eq. (14), we will use a method of generating functions [10]. We choose a generating function which depends on initial coordinates u_0 and z_0 and final momenta v and δ , $F(u_0, z_0, v, \delta)$. The map is defined by the relations

$$v_0 = \frac{\partial F}{\partial u_0}, \qquad \delta_0 = \frac{\partial F}{\partial z_0}, \qquad u = \frac{\partial F}{\partial v}, \qquad z = \frac{\partial F}{\partial \delta}.$$
(15)

In paraxial approximation all coordinates and momenta are considered small and we can expand F in a Taylor series. The linear terms in this expansion vanish because zero initial coordinates and momenta map to zero final ones. The expansion begins from the second order terms

$$F \approx F_2 + F_3 + \cdots, \tag{16}$$

where F_2 is a quadratic, and F_3 is a cubic function of the coordinates and momenta. The function F_2 should generate the linear map Eq. (12) for u and v with a unit transformation for z and δ —a direct calculation shows that

$$F_2 = \frac{1}{2}(u_0^2 + v^2)\tan\psi + u_0v\sec\psi + \delta z_0.$$
 (17)

The function F_3 generates 2nd-order abberations in the system, out of which we choose only a term responsible for the conditioning:

$$F_3 = -\frac{1}{2}bz_0 u_0^2. \tag{18}$$

Indeed, using the second of Eqs. (15) with Eqs. (17) and (18) we find

$$\delta_0 = \delta - \frac{1}{2} b u_0^2, \tag{19}$$

in agreement with Eq. (14). At the same time the first and the third equations of (15) yield

$$\boldsymbol{v}_0 = \boldsymbol{u}_0 \tan \boldsymbol{\psi} + \boldsymbol{v} \sec \boldsymbol{\psi} - \boldsymbol{b} \boldsymbol{z}_0 \boldsymbol{u}_0, \quad (20)$$

$$u = v \tan \psi + u_0 \sec \psi. \tag{21}$$

These equations can be easily solved for u and v:

$$u = u_0 \cos\psi + v_0 \sin\psi + bz_0 u_0 \sin\psi,$$

$$v = -u_0 \sin\psi + v_0 \cos\psi + bz_0 u_0 \cos\psi.$$
(22)

We emphasize here that the same term in the symplectic map Eq. (18) that is responsible for the conditioning of the beam also introduces in Eq. (22) the transverse deflection that varies along the bunch. This also means that adding a system that "fixes" this deflection downstream of the conditioner would inevitably remove the conditioning itself.

Finally, we will calculate the emittance increase of the beam due to the conditioning. Using Eq. (7) for the emittance, with $\bar{u} = \bar{v} = 0$, we find

$$\epsilon_x^2 = \langle u^2 \rangle \langle v^2 \rangle - \langle uv \rangle^2. \tag{23}$$

Substituting the map, Eqs. (22), into this yields

$$\epsilon_x^2 = \epsilon_{x0}^2 (1 + b^2 \sigma_{z_0}^2) = \epsilon_{x0}^2 (1 + a^2), \qquad (24)$$

in agreement with Eq. (8), but now in a general case with arbitrary phase advance, ψ , and nonzero initial alpha function, α_0 . For the specific conditioner described in Sec. III, we have $\psi = 2n\pi$, $\beta_0 = \beta$, $\alpha_0 = \alpha = 0$, and Eqs. (22) reproduce the first two of Eqs. (4).

VI. DISCUSSION

We have demonstrated for a general one-phase conditioner that a strong head-tail focusing variation will always accompany the energy conditioning correlation, and that this focusing variation is set solely by the FEL parameters, and not the conditioner. An analogous situation can be described by considering the addition of a skew quadrupole to an uncoupled transport line. If the desired effect (the "conditioning") is an $\langle x'y \rangle$ correlation, the skew quadrupole will suffice, but an accompanying $\langle xy' \rangle$ correlation will also result and may be an extremely important side effect, depending on the x-to-y emittance ratio. The two effects are, of course, scaled by the same skew-quadrupole strength and are not independent. This example is a linear one, while energy conditioning is 2nd order. However, both cases are based on symplectic transformations including coupling between two planes.

A two-phase conditioner is more complicated, but does not qualitatively change the arguments presented here. If a second solenoid is added to Fig. 1, with a $\pi/2$ transformer separating it from the first solenoid, a two-phase conditioner is formed. The chromatic aberrations, however, generated in the first solenoid will distort the beam in the second solenoid and the energy conditioning will then include large terms higher than 2nd order, which will likely dominate and spoil the conditioning. A twophase system magnifies the projected emittance growth and also degrades the character of the conditioning itself.

We have also studied various forms of two-phase conditioners and, while not presenting a rigorous proof of it here, find the emittance growth much larger than a onephase system. An analytical calculation requires care in retaining the symplectic map to very high order. Tracking the LCLS case through the two-phase solenoid system shows a projected transverse emittance growth of $\epsilon_x/\epsilon_{x0} > 10^4$. We believe this result will be the same for any symplectic energy conditioner designed for the LCLS parameters.

Finally, conditioning is even less practical for an undulator with strong FODO-cell focusing [7]. In this case, the path length increase with increased betatron amplitude [see Eq. (1)] is nearly twice that of natural focusing, and the energy spread necessary for conditioning is then twice larger. This effect, in conjunction with the reduced mean beta function, β_u , in a FODO-cell undulator, makes the conditioning coefficient, *a*, and therefore the projected transverse emittance growth, even larger than for natural focusing.

For these reasons, we conclude that conditioning by adding an energy correlation with the square of the betatron amplitude is impractical if not impossible in shortwavelength FELs.

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