Tune shifts of bunch trains due to resistive vacuum chambers without circular symmetry

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We present an explanation for the opposite signs of the horizontal and vertical tune shifts of bunch trains which have been observed recently in several high-energy storage rings. This result can be understood in terms of the long-range *quadrupolar wakes* of noncircular vacuum chambers with finite resistivity. In vacuum chambers with circular cross section, the dominant transverse wake driven by a leading particle and seen by a trailing test particle is *dipolar* and is proportional only to the transverse offset of the driving particle. The contributions of preceding bunches or previous turns tend to cancel, as they add with oscillatory factors. On the other hand, quadrupolar wakes are independent of the offset of the driving particle, and thus the contributions of preceding bunches and turns are strictly additive. Since quadrupole fields are focusing in one plane and defocusing in the plane orthogonal to it, their effects on tune shifts in these planes have opposite signs. Their cumulative effect also explains the large values of the tune shifts measured in PEP-II, which exceeded estimates from other impedance sources by factors of 3 to 4. Our analysis also offers a connection to the familiar Laslett tune shift.

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I. INTRODUCTION

Measurement of the shift of betatron tunes as functions of beam current is often used to estimate the transverse impedances of particle accelerators or storage rings. This is usually expressed as *tune shift per unit beam current* or *tune slope* by dividing the betatron tune shift with the beam current. In general, single bunches are used for these measurements in order to avoid complications of interpretation. It is well known and confirmed by past measurements that for the single-bunch operation with short-range wake fields, both the horizontal and vertical tune slopes are negative.

Positive tune slopes for multibunch trains first came to our attention during commissioning of the High Energy Ring (HER) of the B-factory PEP-II at the Stanford Linear Accelerator Center in early 1998. The results of measurements with 291 equally spaced bunches are shown in Fig. 1. The first explanation proposed for the positive tune slope was the presence of ions. However, a very similar result was observed with the "standard fill" consisting of 1656 bunches in 9 trains separated by large gaps (see Table II). Since ions would not have survived the gaps, this suspicion had to be discarded.

In this paper, we examine the difference in the behavior of tune slopes between single-bunch and multibunch operations. One obvious difference comes from the fact that the single-bunch operation is more sensitive to short-range wake fields, while the multibunch operation is more sensitive to long-range wake fields. In our analysis, however, we point out that in addition to the range of the wake fields, whether the vacuum chamber has a circular symmetry also plays an important role. For a noncircular chamber, there is an additional *quadrupolar* wake field term which enhances the long-range wake effects, and for accelerators with such chambers, the multibunch operation will have a much different tune slope behavior from the single-bunch operation. This effect is then used to explain the PEP-II observation.

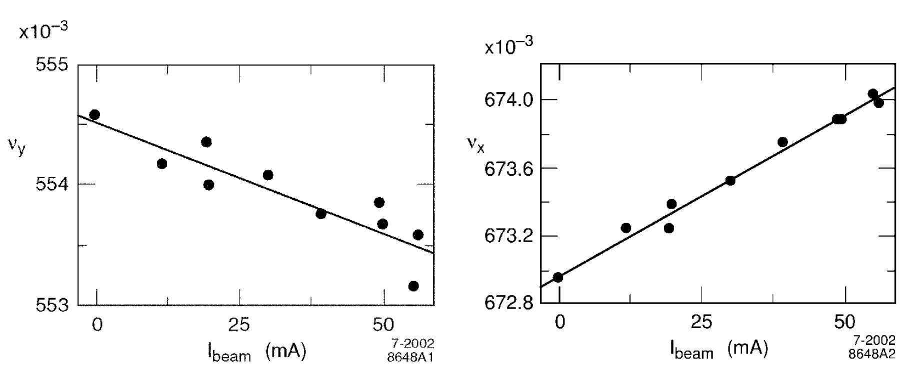

FIG. 1. The vertical (left panel) and horizontal (right panel) tune shifts with beam current in the PEP-II high energy ring. The vertical axis is the fractional part of the betatron tune, and the horizontal axis is the total beam current in mA. The nominal fractional part of the horizontal and vertical tunes are 0.618 and 0.638, respectively.

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We first noted that wake fields of a noncircular chamber contained a quadrupolar term in addition to the dipolar term familiar in the circular case. We also noted that while the dipolar contributions oscillated from turn to turn and tended to cancel among themselves, the quadrupolar term could add up over previous turns. Such an effect was noted by Balbekov [1] in his analysis of bunch-to-bunch tune spread along a bunch train in a noncircular chamber. The vacuum chambers in the magnets of the HER have octagonal cross section, halfway between rectangular and elliptic, but in any case not circularly symmetric. A general analysis of transverse wake fields in such a chamber had recently been completed [2]. There it was found that the transverse wake fields can be represented as superpositions of dipole and quadrupole fields. The quadrupole wakes, which focus in one direction and defocus in the orthogonal direction, could clearly explain the different tune slopes of bunch trains in horizontal and vertical directions. The additive nature of the quadrupole wake effects could explain the large magnitude of these tune slopes.

However, when applied to the long-range resistive wall wake, the infinite sum over the slowly decaying wake fields of previous turns actually diverges unless it is truncated. Our pursuit of this problem resulted with expressions of the Laslett tune shifts [3]. Our analysis therefore also provides a study of the transition from Laslett analysis to the usual analysis using impedances and wake fields.

II. TRANSVERSE IMPEDANCES AND WAKE FIELDS

Tune shift measurements yield only the reactive part (imaginary part) of an impedance, while it is the resistive part (real part) that is more important as it is mainly responsible for the growth rate of instabilities. The reactance shifts the frequency of oscillation and changes tune spread, thus it may still contribute indirectly to an instability by modifying Landau damping. The main types of impedance are discussed below.

The *direct space charge impedance* of a beam is quite negligible at high energies where electric and magnetic forces nearly cancel; it is purely reactive in any case. The *geometric impedances* are caused by variations of the chamber cross section, such as what occurs in cavities or bellows. They can often be modeled as superposition of several resonators, with resonant frequencies determined by the geometric dimensions of the structure. When these frequencies are much higher than the bunch spectral frequencies, their impedance is mainly inductive and the resistive part is negligible. However, when these frequencies are comparable, a large resistive part may occur and one usually has to adopt measures, such as higher-order mode couplers in cavities or shielding of bellows, to reduce it to avoid instabilities. For the *resistive wall impedance*, the resistive and reactive parts are usually equal. In many large high-energy machines, this impedance is the dominant contribution when the geometric impedances have been kept low by careful design. For most storage rings, the betatron frequencies or tunes usually decrease with increasing current due to the inductance of a vacuum chamber wall impedance. The negative tune slopes apply to both the horizontal and the vertical planes.

Since wakes caused by the finite wall resistivity decay quite slowly behind the passage of a bunch, also the contributions of preceding bunches and of previous turns may be significant. However, since dipolar wakes are proportional to the transverse offset of the driving particle, which oscillates around the axis due to betatron oscillations, their sum does not become very large.

On the other hand, in structures with noncircular cross sections, also quadrupolar wakes are excited [2] which are independent of the offset of the driving particle, but proportional to the displacement of the test particle. As an illustration, Fig. 2(a) shows vectorially the integrated transverse wake impulse seen by the test particle in the wake fields of a vertically displaced driving particle (shown by a circle) in bellows with an elliptic cross section. Figure 2(b) shows the quadrupole field excited by a beam on the axis of the same structure. Figure 2(c) is the difference between the two wakes, which is clearly

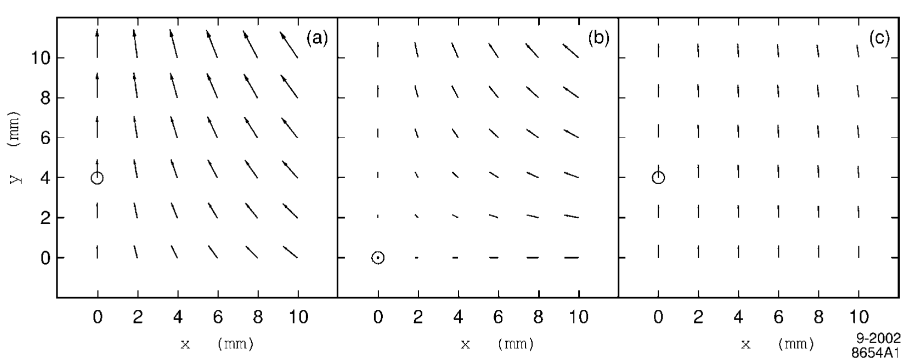

FIG. 2. Transverse wakes of elliptic bellows: (a) total, (b) quadrupolar, and (c) dipolar.

dipolar with very small higher-order distortions for large displacements of the test particle.

The quadrupolar wake fields do not depend on the offset of the driving particle, hence the contributions of all preceding bunches and previous turns are strictly additive, independent of their transverse positions, yielding a large net sum which can change the total tune shift considerably. Pure quadrupolar fields are focusing in one transverse plane and defocusing in the other plane, and therefore the corresponding tune shifts have opposite signs. Depending on the relative strengths of dipolar and quadrupolar wakes, the tune shifts in the horizontal and vertical directions may have the same or opposite signs. The latter behavior has recently been observed in several storage rings using long trains of bunches [4–6].

More quantitatively, for a circular chamber, the impulse received by a test particle of charge e and transverse displacements (x, y) from a driving particle of charge q and transverse displacements (x', y') can be written as [7]

$$\begin{aligned}\Delta p_x &= -\frac{eq}{c} x' W_1(D), \qquad \Delta p_y = -\frac{eq}{c} y' W_1(D),\\ \Delta p_z &= -\frac{eq}{c} W_0'(D) - \frac{eq}{c} W_1'(D)(xx' + yy'),\end{aligned} \tag{1}$$

where D is the longitudinal separation that the test particle is trailing the driving particle, c is the speed of light, $W_{0,1}$ are the transverse and longitudinal wake functions which are functions of D, and $W_{0,1}'$ are their derivatives with respect to D.

For noncircular chambers, and as long as the geometry has an up-down and a left-right symmetry, then the general form of the wake impulse can be written neglecting a small contribution of the higher-order wakes near the beam line as [2]

$$\begin{aligned}\Delta p_x &= -\frac{eq}{c}(a'x' + ax)W_1(D), \qquad \Delta p_y = -\frac{eq}{c}(b'y' + by)W_1(D),\\ \Delta p_z &= -\frac{eq}{c} a_0 W_0'(D) - \frac{eq}{c} W_1'(D)\left(\frac{a}{2}x^2 + a'xx' + dx'^2 + \frac{b}{2}y^2 + b'yy' + ey'^2\right),\end{aligned} \tag{2}$$

where a, a', b, b', a_0, d, e are numerical coefficients. The Panofsky-Wenzel theorem [8] (in the relativistic limit and Cartesian coordinates, it has the form $\partial\Delta p_x/\partial y = \partial\Delta p_y/\partial x$, $\partial\Delta p_x/\partial x = -\partial\Delta p_y/\partial y$, $\partial\Delta p_x/\partial D = \partial\Delta p_z/\partial x$, and $\partial\Delta p_y/\partial D = \partial\Delta p_z/\partial y$) dictates that

$$a = -b, \tag{3}$$

which also follows from what is expected for a pure quadrupolar field.

III. TUNE SHIFTS DUE TO FINITE WALL RESISTIVITY

The *incoherent* tune shift of particles circulating in perfectly conducting chambers can be calculated using the well-known *Laslett coefficients* [3] $\varepsilon_{1,2}$. Since they are solutions of the Laplace equation, they have equal values and opposite signs in the horizontal and vertical directions, $\varepsilon_{1,2H} = -\varepsilon_{1,2V}$. On the other hand, the Laslett coefficients for the *coherent* tune shift $\xi_{1,2H}$ and $\xi_{1,2V}$, which describe the oscillations of the beam as a whole, do not have such a relation and usually have the same sign, although they may have very different values. However, for a thick metallic chamber wall, through which high-frequency fields cannot penetrate, the incoherent coefficients also appear in the calculation of the coherent tune shifts [3,9].

The transverse wake function of a vacuum chamber with resistive walls can be expressed in terms of the distance $D = \beta c\tau$ behind a dipole charge,

$$W_\perp(D) = -\frac{\delta_0}{\pi\varepsilon_0 b^3}\sqrt{\frac{C}{D}}, \tag{4}$$

where $\delta_0 = \sqrt{2/\omega_0\mu_0\sigma_c}$ is the skin depth at the revolution frequency $\omega_0 = \beta c/R$, $C = 2\pi R$ is the machine circumference, and σ_c is the wall conductivity of a circular pipe of radius b. The same equation holds for two parallel plates with vertical separation of $2b$ [10], the wake field coefficients of Eq. (2) are given in Table I. For the wake averaged over one turn around the accelerator, the coefficients a and b are nonzero only in the noncircular part of the beam pipe with the total length L. Therefore, they are weighted below, see Eq. (8), with the factor L/C. The fact that the two cases (and the third case of one single resistive plate a vertical distance of b from the driving particle) have the same value of a_0 has been noted by Piwinski [11].

Evaluating expressions from Refs. [12,13], we have numerically calculated the wake fields for an upright elliptical geometry for PEP-II and have found the values of the coefficients very close to those of the two-parallel-plate geometry. (Instead of $-\pi^2/24$, for

TABLE I. Expansion coefficients for Eq. (2).

Geometry	a'	a	b'	b	a_0	d	e
Circular pipe of radius b	1	0	1	0	1	0	0
Two parallel plates with vertical separation of $2b$	$\pi^2/24$	$-\pi^2/24$	$\pi^2/12$	$\pi^2/24$	1	$-\pi^2/48$	$\pi^2/48$

example, we found −0.40; instead of $\pi^2/24$, we found 0.38; and instead of $\pi^2/12$, we found 0.85, etc.) In the following, we will use the coefficients of two parallel plates to represent the results for PEP-II.

For a circular chamber, the wake field contains only the dipolar terms ($a = b = d = e = 0$). The complex coherent mode frequency shifts for the most affected modes in the horizontal (Ω_x) and the vertical (Ω_y) planes are given by the expressions [7]

$$\begin{aligned}\Delta\Omega_x &= -\frac{6\sqrt{2}}{\pi^2\omega_x}\Gamma[g(\nu_x) + if(\nu_x)],\\ \Delta\Omega_y &= -\frac{6\sqrt{2}}{\pi^2\omega_y}\Gamma[g(\nu_y) + if(\nu_y)],\end{aligned} \tag{5}$$

where mode index $[\nu]$ is the integer nearest to ν,

$$\Gamma = \frac{\pi}{6}\frac{Nr_0c^2}{\gamma b^3}\left(\frac{\delta_0}{2R}\right), \tag{6}$$

with N being the total number of particles in the beam, $C = 2\pi R$ the machine circumference, r_0 the classical particle radius, and γ the Lorentz energy factor. The functions g and f are, respectively, the real and the imaginary parts of

$$\frac{1}{n_b}\sum_{m=1}^{n_b}\sum_{k=1}^{\infty}\sqrt{\frac{2}{k+\frac{mS_b}{C}}}e^{i2\pi\nu[k+(mS_b/c)]} = g(\nu) + if(\nu), \tag{7}$$

where n_b is the number of equally populated and evenly spaced bunches in the beam, and S_b is the bunch spacing. The real part of Eq. (5) gives the mode frequency shift, and the imaginary part of Eq. (5) gives the instability growth rate.

In the case of two parallel plates, the wake field contains an additional quadrupolar term. The complex coherent mode frequency shift is given by

$$\begin{aligned}\Delta\Omega_x &= -\frac{\Gamma}{2\sqrt{2}\omega_x}\left[g(\nu_x) + if(\nu_x) - \frac{L}{C}\sqrt{2}h\right],\\ \Delta\Omega_y &= -\frac{\Gamma}{2\sqrt{2}\omega_y}\left[2g(\nu_y) + 2if(\nu_y) + \frac{L}{C}\sqrt{2}h\right],\end{aligned} \tag{8}$$

where

$$h = \frac{1}{n_b}\sum_{m=1}^{n_b}\sum_{k=0}^{\infty}\sqrt{\frac{1}{k+\frac{mS_b}{C}}} \tag{9}$$

is a divergent sum over quadrupolar wakes of all previous turns. In Eq. (8), the terms involving f and g are due to the dipolar wake terms and are similar to those in Eq. (5), but the terms involving h are new terms due to quadrupolar wake fields. The divergence of h is to be removed by considering a resistive pipe with finite thickness, as will be done in the next section. The reason of the divergence is that the quadrupolar wake fields strictly add turn after turn. This is not the case for the dipolar wake fields because the dipolar wake fields are proportional to the transverse offset of the driving particle which oscillates at the betatron frequency.

It should be mentioned here that this new quadrupolar term involving the quantity h is purely real, indicating that it affects only the coherent mode frequency shifts and not the instability growth rates. The noncircular shape of the vacuum chamber, therefore, mainly affects the mode frequency shifts and not the instability, unless, as noted earlier, through a Landau damping mechanism.

IV. RESISTIVE WALL OF FINITE THICKNESS

We have yet to deal with the divergent sum (9). The expression for the wake function of a wall with finite resistivity, Eq. (4), is valid only when the wall thickness is large compared to the skin depth, i.e., when rf fields do not penetrate through the pipe wall. Since this wake function decays only as the inverse square root of distance behind the source, the infinite sum (9) over preceding turns diverges.

However, since the skin depth $\delta = (2/\mu\sigma_c\omega)^{1/2}$ is proportional to the inverse square root of frequency, for a finite wall thickness this condition is always violated when the frequency is low enough. In the time domain, this corresponds to a sufficiently long time, after which the rf fields will penetrate the chamber wall.

Analytic expressions for the transverse impedance and hence for the tune shift caused by a resistive wall with finite thickness are known for the case of circular chambers [14–16]. However, the penetration of fields will not be very different in chambers without circular symmetry, as has been verified by numerical computation of field penetration through resistive walls in a parallel plate geometry [17].

The transverse impedance for a thin wall of thickness t and radius b with conductivity σ_c can be written [14]

$$Z_\perp(\omega) = \frac{CZ_0}{\pi b^2}\left(\frac{L}{C}\right)\frac{1}{\omega/\omega_c + i/r}, \qquad \frac{\omega_c}{c} = \frac{\delta_0^2}{2btR}, \tag{10}$$

where $r = 1 + b^2/d^2$ for a perfect magnet and $r = 1 - b^2/d^2$ for a perfect conductor at radius $d > b + t$. For $d \to \infty$, i.e., no outer wall, $r = 1$. From the impedance we can obtain the wake function by a Fourier transformation. Combined with Eq. (4), we obtain

$$W_1(D) = -\frac{\delta_0}{\pi\varepsilon_0 b^3}\frac{L}{C}\begin{cases}|\frac{C}{D}|^{1/2}, & \text{for small } |D|,\\ \pi\frac{\delta_0}{t}\exp(-\frac{\pi\delta_0^2 D}{rbtC}), & \text{for large } |D|,\end{cases} \tag{11}$$

where δ_0 is the skin depth at the revolution frequency $\omega = \omega_0$. The value of D that makes the transition from one regime to the other in Eq. (11) is determined by the point when the two expressions have equal magnitude. This is given by

$$xe^{-2x} = \frac{t}{\pi rb}, \quad \text{where } x = \pi \frac{\delta_0^2}{rbt} \left| \frac{D}{C} \right|. \tag{12}$$

When $t \ll b$, we have $x \ll 1$ and therefore, $x \approx t/(\pi rb)$, which corresponds to $|D/C| \approx t^2/(\pi^2 \delta_0^2)$. When $t \gg \delta_0$, we have $|D/C| \gg 1$ at the point of transition, i.e., the transition occurs after many turns.

With the modification (11) of the long-range wake, the function h of Eq. (9) is to be replaced by

$$h \approx \frac{1}{n_b} \sum_{m=1}^{n_b} \sum_{k=1}^{\hat{k}-1} \sqrt{\frac{1}{k + \frac{mS_b}{C}}} + \pi \frac{\delta_0}{t} \int_{\hat{k}}^{\infty} dk \exp\left(-\frac{\pi \delta_0^2 k}{rbt}\right), \tag{13}$$

where $\hat{k} = t^2/(\pi^2 \delta_0^2)$, and we have replaced a sum by an integral in the second term above. Physically this value of $\hat{k}$ corresponds to the diffusion time needed for the wake field to penetrate through the pipe thickness t with a diffusion coefficient of $c/Z_0\sigma_c = \omega_0 \delta_0^2/2$.

In Eq. (13), the second term dominates when $b \gg t$ for $t \ll \pi rb$. This means the long-range resistive wall wake dominates the tune shifts in a noncircular beam pipe, although the instability growth rate is still dominated by the wake fields left behind by the latest few revolutions. Dropping the small first term, and performing the remaining integration, we obtain

$$h \approx \frac{rb}{\delta_0}. \tag{14}$$

Substituting into Eq. (8), and using Eq. (6), gives the quadrupolar contribution of mode frequency shifts:

$$\Delta \nu_{x,y} = \pm \frac{\Gamma}{2\omega_0 \omega_{x,y}} h \approx \pm \frac{\Gamma}{2\omega_0 \omega_{x,y}} \frac{rb}{\delta_0} = \pm \frac{1}{48} \frac{rNr_0 L}{\gamma b^2 \nu_{x,y}}, \tag{15}$$

where L is the length of the octagonal chamber in dipoles. The tune slope then is

$$\frac{d\nu_{x,y}}{dI} = \pm \left(\frac{\pi r}{48 \nu_{x,y}}\right)\left(\frac{Z_0}{E/e}\right)\left(\frac{R}{b}\right)^2\left(\frac{L}{C}\right), \qquad r = 1 + \frac{b^2}{d^2}. \tag{16}$$

These tune shifts are independent of the vacuum chamber pipe thickness t or the pipe conductivity σ_c. As mentioned earlier, they are the incoherent Laslett tune shifts [3,7]. Note that the reason it corresponds to the *incoherent* Laslett tune shift is that the wake fields do not depend on the offset due to the beam's betatron motion. Note also that the horizontal tune shift is positive while the vertical tune shift is negative. These signs agree with the PEP-II observation in the multibunch operation.

It may be useful to make two side remarks here.

The first term in Eq. (13) contributes to h approximately by $2t/\pi\delta_0$ which is much smaller than the second term if $b \gg t$.

The modification on long-range wake also affects Eq. (7), and this modification also affects the instability growth rate. However, due to the oscillatory nature of the terms, the modification is usually small.

V. TUNE SLOPES MEASURED IN PEP-II

Tune slopes which were measured during commissioning of the HER of the SLAC B-factory PEP-II in January 1998 by one of the authors (B. Z.) are shown in Table II. As can be seen in the table, for the fills with equally spaced 291 bunches, with and without transverse feedback, the vertical tune slope was reduced by a factor of nearly 40, while the horizontal tune slope became positive. (Note that the values of the currents in Table II were the total beam currents, not currents per bunch.) The first suspicion for the sign change was the effect of ions, which had been estimated to contribute a positive tune slope of up to 0.8 A^{-1}. A somewhat lower vacuum pressure could then have explained the reduction of the vertical tune slope of a single bunch to the values measured with multibunch fillings by 1.2 A^{-1}. However, the tune shifts in the standard fill with 1656 bunches, in 9 trains and with a 180-bunch gap, remained essentially unchanged. Such a large gap would certainly have eliminated any ions if they had been present. Hence the hypothesis of ions causing positive tune slopes had to be abandoned.

The copper chambers in the magnets of the High Energy Ring of PEP-II are octagonal with 9 cm $\times$ 5 cm overall dimensions. As mentioned earlier, approximating them with an elliptic cross section yields the form factor coefficients very close to those of a two-parallel-plate geometry.

For the HER we use the following parameters: $\omega_0 = 2\pi \times 137$ KHz, $b = 25$ mm, $t = 6$ mm, $\sigma_c = 6 \times 10^7$ s/m, $d = 35.4$ mm, $L/C = 0.59$, and tunes $\nu_x = 24.62$ and $nu_y = 24.64$. This yields a skin depth at a revolution frequency of $\delta_0 = 0.17$ mm and an exponential decay length of $btr/(\pi\delta_0^2) \approx 2500$ turns [see Eq. (11)]. However, the slow and fast decays already become equal after $\hat{k} = [t/(\pi\delta_0)]^2 \approx 126$ turns. According to Eq. (16), the calculated tune shifts become $d\nu_x/dI = -d\nu_y/dI = 0.0195$ A^{-1}, in quite good agreement with the observed results shown in Table II.

TABLE II. Measured tune slopes in HER of PEP-II.

Filling	Current (mA)	Horizontal A^{-1}	Vertical A^{-1}
Single bunch	1	−0.45	−1.22
Low resolution	1	−0.32	−1.0
291 bunches	50	+0.0204	−0.0256
Feedback on	100	+0.0200	−0.0262
1656 bunches	100	+0.018	−0.018

VI. CONCLUSIONS

Positive slopes of the horizontal tune shift with beam current, observed recently in several storage rings operated with bunch trains, can be explained by the quadrupolar wake fields excited in vacuum chambers without circular symmetry but with walls of finite resistivity. In chambers with circular cross section, only the more familiar dipolar wakes, which are proportional to the offsets of the driving particle, are excited. In the presence of transverse betatron oscillations, a strong cancellation occurs when dipolar wakes excited in subsequent turns are added, including the phase factor.

On the other hand, quadrupolar wakes depend on the transverse offset of the driven particle itself and hence the contributions of previous bunches and preceding turns are strictly additive. In structures without circular symmetry, they focus in one plane and defocus in the other, hence horizontal and vertical tune shifts due to these wakes will have opposite signs. The total tune shifts are the sum of both dipolar and quadrupolar contributions, and thus their signs depend on their relative strengths. In PEP-II operated with bunch trains, the quadrupolar tune shifts dominate and that explains the sign difference of the tune shifts in the two transverse planes. The quadrupolar wakes give the dominant contribution for a long train of bunches as it is clear from Eq. (8) where the first two turns given by the dipolar wakes roll off with the number of bunches n_b while the contribution of the quadrupolar wakes is independent of n_b.

Our analysis of the multiturn tune shifts also provides a connection between two treatments on the topic, namely, the Laslett analysis and the analysis using impedances and wake fields.

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APPENDIX: TRANSVERSE IMPEDANCE AND WAKE FUNCTION OF THIN METAL WALLS

The transverse impedance of a thin, circular conducting "shell" inside a surrounding cylinder can be expressed in closed form, starting from the simplified model analyzed in Ref. [16]. The approximation used consists of neglecting the variation of the longitudinal electric field across the thin shell—which assumption becomes better, the thinner the shell or the lower the frequency with a larger skin depth. Here we are mainly concerned with very long-ranged wake fields; hence extremely low frequencies are of interest. The change of the magnetic field across the shell equals the current induced in it, which in turn can be expressed by the conductivity and the electric field. This set of assumptions leads to the expressions Eqs. (6.53/55) of Ref. [16], which for a full machine circumference $C = 2\pi R$ can be written[1]

$$\begin{aligned} Z_\perp(\omega) &= -j\frac{CZ_0}{\beta\gamma^2 a^2} I_1^2(s)\left[\frac{K_1(s)}{I_1(s)} - \frac{K_1(x)}{I_1(x)} + M\frac{G_1(x)}{I_1(x)}\right], \\ M^{-1} &= 1 + i\zeta G_1(x) I_1(x)\left(1 + \frac{\beta\gamma^2}{x^2\Delta}\right), \\ \Delta &= 1 + i\zeta I_1'(x) H_1'(x), \end{aligned} \tag{A1}$$

where $s = ka/\gamma$ for a dipole modulated ring charge of radius a, and $x = kb/\gamma$ for the shell of radius b. The first two terms in the bracket for the impedance express the direct space charge contributions which are proportional to γ^{-2} and thus are of little concern for high-energy machines. Only the last term is due to the finite wall resistance and will be evaluated below. The parameter ζ is given by Eq. (6.8) of Ref. [16] and can be rewritten

$$\zeta = \frac{2tb}{\beta^2\gamma^2\delta_b^2} = Z_0\sigma_c t\frac{x^2}{\beta\gamma}, \tag{A2}$$

where σ_c is the conductivity of the metal shell of thickness t. One still needs to determine the factors G and H, which were defined in Eqs. (6.36/37) of Ref. [16] as

$$\begin{aligned} G_1(u) &= K_1(u) - \alpha_G I_1(u), \\ H_1(u) &= K_1(u) - \alpha_H I_1(u), \end{aligned} \tag{A3}$$

where $u = kr/\gamma$ and the coefficients α_G and α_H are determined by the boundary conditions at the outer cylinder at radius d or $y = kd/\gamma$. For a perfectly conducting wall, the tangential electric field components E_z and E_θ must vanish, and Eqs. (6.38) and (6.39) then yield $G_1(y) = H_1'(y) = 0$. Therefore $\alpha_G = K_1(y)/I_1(y)$, and $\alpha_H = K_1'(y)/I_1'(y)$

Similarly, at the surface of a perfect magnet ($r = d$), the tangential magnetic field components H_z and H_θ are assumed to vanish. Equations (6.40) and (6.41) then yield $G_1'(y) = H_1(y) = 0$. Therefore $\alpha_G = K_1'(y)/I_1'(y)$, and $\alpha_H = K_1(y)/I_1(y)$. Finally, if the outer region extends to infinity, both coefficients α_G and α_H must be zero to avoid divergence of the expressions for G and H at infinity. For a large energy factor $\gamma \gg 1$ and low frequency $\omega = k\beta c$, the arguments of the modified Bessel functions are very small and one may use the first order approximations

$$I_1(u) \approx \frac{u}{2}, \qquad (u) \approx \frac{1}{u}, \tag{A4}$$

and the functions G and H' can be written

[1] For consistency with the main body of the report, we replaced j by $-i$, corrected a missing β in the last term of Eq. (6.53), and replaced F_1 by I_1 in the second term of the bracket in Eq. (6.55).

$$G_1(x) \approx \frac{r}{x}, \qquad H_1'(x) \approx -\frac{r}{x^2}. \tag{A5}$$

Here $r = 1 - \kappa^2$ for a perfectly conducting cylinder, $r = 1 + \kappa^2$ for a perfectly magnetic cylinder, and $r = 1$ if the outer region extends to infinity, where $\kappa = b/d$.

With these approximations one finds $\Delta = 1 + iA/u$ and $M^{-1} = 1 + iAu + iA/\beta(u - iA)$, where $A = Z_0\sigma_c tr/2$ and $u = x/\beta\gamma$. For a metal, $A \gg 1$, while for high energy, $u \ll 1$. Thus we may write approximately $M^{-1} = -B + iAu$ with $B = 1/2\beta\gamma^2$, and the expression for the impedance finally becomes

$$Z_\perp(\omega) = \frac{CZ_0}{\pi b^2}\frac{1}{\omega/\omega_c + i/r}, \tag{A6}$$

where $\omega_c = 2\beta c/Z_0\sigma_c tb = \omega_0\delta_0^2/bt$, when δ_0 is the skin depth at the revolution frequency $\omega_0 = \beta c/R$.

While these expressions are valid for a circular-cylindrical thin wall, we assume that they can be applied with little error to thin walls of different cross sections such as the octagonal chambers inside the magnets of PEP-II. We thus use this expression, multiplied by a circumference factor L/C for the length L of the octagonal chamber, to calculate the very long-ranged wake function which will modify the short-range $1/\sqrt{t}$ dependence and thus avoid the divergence of the infinite sum over the wakes of previous turns.

The transverse wake function at distance τ behind a point charge is the Fourier transform of the impedance, i.e.,

$$W_\perp(\tau) = i\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}Z_\perp(\omega)\exp(-i\omega\tau). \tag{A7}$$

For a "thick" metal wall (compared to the skin depth), the impedance due to the finite resistivity in the "asymptotic region," which extends typically from a fraction of a millimeter to many 100's of meters, is given by the expression [7]

$$Z_\perp(\omega) = [\mathrm{sgn}(\omega) - i]\frac{\mu_r R Z_0 \delta}{b^3}, \tag{A8}$$

where $\delta = \sqrt{2/\mu_r\mu_0\omega\sigma_c} = \delta_0\sqrt{\omega_0/\omega}$ is the skin depth at frequency ω. The relative permeability μ_r of the vacuum chamber is usually equal to unity. One then gets for the transverse wake function

$$W_\perp(\tau) = -\frac{2\beta\delta_0\omega_0}{\varepsilon_0 b^3\sqrt{\omega_0\tau}}\int_{-\infty}^{\infty}\frac{dx}{\sqrt{|x|}}[\sin x + \cos x]. \tag{A9}$$

The integral can be evaluated [18] and the wake function becomes

$$W_\perp(\tau) = -\sqrt{\frac{2}{\pi}}\frac{\beta\delta_0}{\varepsilon_0 b^3}\frac{1}{\sqrt{\omega_0\tau}}. \tag{A10}$$

Converting the distance in time to that in space, $D = \beta c\tau$ yields Eq. (4).

On the other hand, for a thin wall, the impedance given above leads to a wake function proportional to a negative exponential, which avoids the divergence problems encountered with the "thick wall" wake function which decreases only very slowly. The integral can be evaluated from its residue at $\omega = -i\omega_c/r$ to yield, at the distance $D = \beta c\tau$ behind the source,

$$W_\perp(D) = \frac{\delta_0^2}{\varepsilon_0 b^3 t}\exp\left(-\frac{\pi\delta_0^2 D}{rbtC}\right), \tag{A11}$$

in agreement with the second of Eq. (11) in the text. A slightly better approximation to the true wake function is obtained if the integral is evaluated over the "thin-wall" impedance up to the approximate transition frequency ω_c, and from there on over the "thick-wall" impedance to infinity. The result can be expressed in terms of Fresnel functions but gives only a small correction to the expressions used above.

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