Transverse stability with nonlinear space charge

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Transverse stability with nonlinear space charge is studied within the context of coasting beams. For bare tune spreads originating from chromaticity or frequency slip, the space charge tune spread has a fairly small effect and the incoherent space charge force is well modeled by a transverse capacitance. For tune spreads due to octupoles or fringe fields, beams are more stable when the bare tune increases with betatron amplitude.

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I. INTRODUCTION

Early work on transverse coasting beam instability usually treated the direct space charge force as a source of transverse capacitance [1,2]. Somewhat later, attempts to incorporate the (betatron) amplitude dependent nature of the space charge tune shift were made [3,4]. These works generally started with the dispersion relation derived for coasting beams in the presence of a dipolar transverse field [1,2,5–7]. The dispersion relation was then phenomenologically modified to incorporate the nonlinearity in the space charge force.

The present work begins with equations of motion which include nonlinear, interparticle forces [8]. Only one dimension is considered. This simplifies both the notation and the complexities of coupling between the transverse degrees of freedom. Since the intended application is synchrotrons, it is assumed that the magnitude of the tune shift is small compared to the average tune. This allows the equations to be heterodyned. Finally, first-order perturbation theory on the Vlasov equation is used to obtain the dispersion relation. Throughout, the momentum and energy conserving properties of the interparticle forces are retained.

II. THE MODEL

Consider a one-dimensional model with coordinate x. For N interacting particles the equation of motion for particle j is

$$\frac{d^2x_j}{d\theta^2} + Q_j^2 x_j = \frac{1}{N}\sum_{k=1}^{N} f(x_j - x_k) + 2Q_0 W x_k + 2\alpha\frac{dx_k}{d\theta}\,. \tag{1}$$

In Eq. (1), particle j has bare tune Q_j, and the average bare tune is Q_0. The space charge force is characterized by the antisymmetric, nonlinear function $f(x)$. For $Q_j \equiv Q_0$ one can average Eq. (1) over j:

$$\frac{d^2\overline{x}}{d\theta^2} + Q_0^2\overline{x} - 2Q_0 W\overline{x} - 2\alpha\frac{d\overline{x}}{d\theta} = \frac{1}{N^2}\sum_{j,k} f(x_j - x_k) \equiv 0\,. \tag{2}$$

The double sum over the space charge force vanishes due to the antisymmetry of $f(x)$. Assuming $\overline{x} \propto \exp(-iQ\theta)$ and $|W + i\alpha| \ll Q_0$, $Q = \pm(Q_0 - W) + i\alpha$. The wall induced tune shift and growth rate are $-W$ and α, respectively. With no bare tune spread, space charge has no effect on the coherent frequency [3,4,9]. The problem at hand is to determine the behavior of the system when bare tune spread is present. Define the functions $A_j(\theta)$ and $B_j(\theta)$ via

$$x_j(\theta) = A_j(\theta)\cos(Q_0\theta) + B_j(\theta)\sin(Q_0\theta)\,. \tag{3}$$

Since the tune shift is small compared to the tune, the functions A_j and B_j vary slowly compared to the sinusoids. In Eq. (1) make the substitutions

$$\frac{d^2x_j}{d\theta^2} \longrightarrow -Q_0^2 x_j + 2Q_0[B_j'\cos(Q_0\theta) - A_j'\sin(Q_0\theta)]$$

and

$$\frac{dx_k}{d\theta} \longrightarrow Q_0[B_k\cos(Q_0\theta) - A_k\sin(Q_0\theta)]\,,$$

where the $'$ denotes differentiation with respect to θ. Define $\Delta_j = (Q_j^2 - Q_0^2)/2Q_0 \approx Q_j - Q_0$ and $\chi = Q_0\theta$. Equation (1) becomes

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$$B'_j \cos\chi - A'_j \sin\chi = -\Delta_j[A_j \cos\chi + B_j \sin\chi] + W[\overline{A}\cos\chi + \overline{B}\sin\chi] + \alpha[\overline{B}\cos\chi - \overline{A}\sin\chi] + \frac{1}{N}\sum_{k=1}^{N} f([A_j - A_k]\cos\chi + [B_j - B_k]\sin\chi), \tag{4}$$

where

$$\overline{A} = \frac{1}{N}\sum_{k=1}^{N} A_k(\theta),$$

and similarly for $\overline{B}$. To proceed we multiply Eq. (4) by $\cos\chi$ and average over the fast variable, χ. This gives

$$B'_j = -\Delta_j A_j + W\overline{A} + \alpha\overline{B} + \frac{1}{N}\sum_{k=1}^{N}\int_0^{2\pi}\frac{d\chi}{\pi}\cos\chi f([A_j - A_k]\cos\chi + [B_j - B_k]\sin\chi). \tag{5}$$

By Newton's third law $f(-x) = -f(x)$. Defining $(B_j - B_k) + i(A_j - A_k) = R\exp(i\chi_0)$ the integral in Eq. (5) satisfies

$$\begin{aligned}\int_0^{2\pi}\frac{d\chi}{\pi}\cos\chi f([A_j - A_k]\cos\chi + [B_j - B_k]\sin\chi) &= \int_0^{2\pi}\frac{d\chi}{\pi}\cos\chi f\big(R\sin(\chi + \chi_0)\big)\\ &= \int_{-\pi}^{\pi}\frac{d\chi}{\pi}\cos(\chi - \chi_0) f(R\sin\chi)\\ &= R\sin\chi_0\int_{-\pi}^{\pi}\frac{d\chi}{\pi}\frac{\sin\chi}{R} f(R\sin\chi)\\ &= [A_j - A_k]G([A_j - A_k]^2 + [B_j - B_k]^2),\end{aligned}$$

which defines the function $G(x^2)$. After this substitution one obtains

$$B'_j = -\Delta_j A_j + W\overline{A} + \alpha\overline{B} + \frac{1}{N}\sum_{k=1}^{N}[A_j - A_k]G([A_j - A_k]^2 + [B_j - B_k]^2). \tag{6}$$

Multiplying Eq. (4) by $\sin\chi$ and averaging gives

$$A'_j = +\Delta_j B_j - W\overline{B} + \alpha\overline{A} - \frac{1}{N}\sum_{k=1}^{N}[B_j - B_k]G([A_j - A_k]^2 + [B_j - B_k]^2). \tag{7}$$

If $\overline{A}$ and $\overline{B}$ in Eqs. (6) and (7) are treated as functions of θ alone then these equations can be derived from the Hamiltonian

$$K(\mathbf{A}, \mathbf{B}, \theta) = \sum_{j=1}^{N}\frac{\Delta_j}{2}(A_j^2 + B_j^2) - W\big(A_j\overline{A}(\theta) + B_j\overline{B}(\theta)\big) + \alpha\big(B_j\overline{A}(\theta) - A_j\overline{B}(\theta)\big) + \sum_{j=1}^{N}\sum_{k=1}^{N}\frac{1}{2N}U([A_j - A_k]^2 + [B_j - B_k]^2), \tag{8}$$

where

$$A'_j = \frac{\partial K}{\partial B_j}, \qquad B'_j = -\frac{\partial K}{\partial A_j},$$

and the potential $U(x^2)$ satisfies $dU(x^2)/dx = -xG(x^2)$. If $\alpha = 0$, substituting $W \to W/2$ allows the equations of motion to be derived from this Hamiltonian when the coordinate dependence of $\overline{A}$ and $\overline{B}$ is included. Only the presence of α precludes a full Hamiltonian treatment. This is reasonable given the nonsymplectic motion of a damped oscillator.

Now go to the continuum limit and introduce the distribution function $F(A, B, \Delta, \theta)$ where $F(A, B, \Delta, \theta) \times dA\,dB\,d\Delta$ gives the fraction of particles in phase space volume $dA\,dB\,d\Delta$ so the distribution is normalized to one,

$$\int_{-\infty}^{\infty} dA\int_{-\infty}^{\infty} dB\int_{-\infty}^{\infty} d\Delta\, F(A, B, \Delta, \theta) = 1.$$

Consider a particle with phase space coordinates A, B, Δ. This particle's coordinates evolve according to the continuum versions of Eqs. (6) and (7)

$$B' = -\Delta A + \int dA_1\, dB_1\, d\Delta_1 (+WA_1 + \alpha B_1) F(A_1, B_1, \Delta_1, \theta) + \int dA_1\, dB_1\, d\Delta_1 (A - A_1) G[(A - A_1)^2 + (B - B_1)^2] F(A_1, B_1, \Delta_1, \theta)\,, \tag{9}$$

$$A' = +\Delta B + \int dA_1\, dB_1\, d\Delta_1 (-WB_1 + \alpha A_1) F(A_1, B_1, \Delta_1, \theta) - \int dA_1\, dB_1\, d\Delta_1 (B - B_1) G[(A - A_1)^2 + (B - B_1)^2] F(A_1, B_1, \Delta_1, \theta)\,, \tag{10}$$

$$\Delta' = 0\,, \tag{11}$$

where the equation for Δ has been added for completeness. If A is taken as a position variable with B as its conjugate momentum, and Δ is a position variable with conjugate momentum P_Δ, these equations of motion can be obtained from the Hamiltonian

$$H(A, B, \Delta, \theta) = \frac{\Delta}{2}(A^2 + B^2) - W(A\overline{A} + B\overline{B}) + \alpha(B\overline{A} - A\overline{B}) + \int dA_1\, dB_1\, d\Delta_1\, F(A_1, B_1, \Delta_1, \theta) U\big((A - A_1)^2 + (B - B_1)^2\big)\,, \tag{12}$$

where

$$\overline{A}(\theta) \equiv \int dA_1\, dB_1\, d\Delta_1\, A_1 F(A_1, B_1, \Delta_1, \theta)\,,$$

and similarly for $\overline{B}$. The equations of motion are

$$A' = +\frac{\partial H(A, B, \Delta, \theta)}{\partial B}\,,$$

$$B' = -\frac{\partial H(A, B, \Delta, \theta)}{\partial A}\,,$$

$$\Delta' = \frac{\partial H(A, B, \Delta, \theta)}{\partial P_\Delta} = 0\,.$$

The Hamiltonian given by (12) is not the continuum limit of Eq. (8), but it generates the equations of motion (9)–(11). With Hamiltonian (12) the phase space distribution evolves according to the Vlasov equation,

$$\frac{\partial F}{\partial \theta} + \frac{\partial H}{\partial B}\frac{\partial F}{\partial A} - \frac{\partial H}{\partial A}\frac{\partial F}{\partial B} = 0\,. \tag{13}$$

Since P_Δ is absent from the Hamiltonian, the tune distribution is constant in time (θ),

$$\rho(\Delta, \theta) \equiv \int dA_1\, dB_1\, F(A_1, B_1, \Delta, \theta) = \rho(\Delta)\,.$$

This is clear from the single particle picture and can be verified by integrating Eq. (13) over A and B,

$$\frac{\partial \rho(\Delta, \theta)}{\partial \theta} = -\int dA\, dB\left[\frac{\partial H}{\partial B}\frac{\partial F}{\partial A} - \frac{\partial H}{\partial A}\frac{\partial F}{\partial B}\right] = -\int dA\, dB\left[\frac{\partial H}{\partial B}\frac{\partial F}{\partial A} - \frac{\partial H}{\partial A}\frac{\partial F}{\partial B} - F\frac{\partial H}{dB\partial A} + F\frac{\partial H}{\partial A\partial B}\right]$$
$$= -\int dA\, dB\left[\frac{\partial}{\partial A}\left(\frac{\partial H}{\partial B}F\right) - \frac{\partial}{\partial B}\left(\frac{\partial H}{\partial A}F\right)\right] = 0\,.$$

As a final check of the physical equivalence of the Vlasov equation and Eqs. (6) and (7), consider the evolution of the Klimontovich distribution [10]

$$F(A, B, \Delta, \theta) = \frac{1}{N}\sum_{k=1}^{N} \delta\big((A - A_k(\theta)\big)\delta\big(B - B_k(\theta)\big)\delta\big(\Delta - \Delta_k(\theta)\big)\,,$$

where the parameters $A_k(\theta)$, $B_k(\theta)$, and $\Delta_k(\theta)$ can depend on θ, but not on A, B, or Δ. With its sum of delta functions, this is the distribution for N identical particles. Substituting this distribution into the Vlasov equation (13) with Hamiltonian (12) gives Eqs. (6) and (7) and $d\Delta_k(\theta)/d\theta = 0$.

III. SOLUTIONS

To solve to the Vlasov equation, start by introducing the action angle variables J and ψ defined implicitly by

$$A = \sqrt{2J}\sin\psi, \qquad B = \sqrt{2J}\cos\psi\,.$$

The coordinate change is canonical so $dAdB = dJd\psi$.

The threshold for instability is determined using first order perturbation theory. Let $F = F_0 + F_1$ where

$$F_0(J, \Delta) = \frac{1}{2\pi}\hat{F}_0(J)\rho(\Delta)\,,$$

with ρ and $\hat{F}_0$ each normalized to unity. The perturbation is parametrized as

$$F_1(J,\psi,\Delta,\theta) = \frac{1}{2\pi}\hat{F}_1(J,\Delta)e^{i\psi - i\nu\theta}.$$

The ψ dependence of F_1 was chosen to generate nonzero values of $\overline{A}$ and $\overline{B}$. If one had chosen a dependence $\exp(ik\psi)$ with $|k| \neq 1$ then $\overline{A} = \overline{B} \equiv 0$.

Substituting the expression for F in the Hamiltonian one gets $H = H_0 + H_1$ where

$$H_0 = \Delta J + \int_0^\infty dJ_1\,\hat{F}_0(J_1)\int_0^{2\pi}\frac{d\psi_1}{2\pi}\,U(2J + 2J_1 - 4\sqrt{JJ_1}\cos\psi_1) = \Delta J + U_0(J) \tag{14}$$

and

$$\begin{aligned} H_1 &= \nu_0\sqrt{J}e^{i\psi - i\nu\theta}\int dJ_1\,d\Delta_1\sqrt{J_1}\,\hat{F}_1(J_1,\Delta_1) \\ &\quad + e^{i\psi - i\nu\theta}\int d\Delta_1\,dJ_1\hat{F}_1(J_1,\Delta_1)\int_0^{2\pi}\frac{e^{-i\psi_1}d\psi_1}{2\pi}\,U(2J + 2J_1 - 4\sqrt{JJ_1}\cos\psi_1) \\ &= [\nu_0\sqrt{J}\,g_1 + U_1(J)]e^{i\psi - i\nu\theta}, \end{aligned} \tag{15}$$

where $\nu_0 = -W + i\alpha$ is the tune shift for $\Delta \equiv 0$ and $g_1 = g_1[F_1]$ does not vary with J or Δ:

$$g_1 = \int dJ_1\,d\Delta_1\sqrt{J_1}\,\hat{F}_1(J_1,\Delta_1)\,. \tag{16}$$

Applying first order perturbation theory gives

$$-i\nu F_1 + \frac{\partial H_0}{\partial J}\frac{\partial F_1}{\partial\psi} - \frac{\partial H_1}{\partial\psi}\frac{\partial F_0}{\partial J} = 0\,. \tag{17}$$

With the simple ψ dependence the solution is straightforward giving

$$\hat{F}_1(J,\Delta) = \frac{\nu_0\sqrt{J}\,g_1 + U_1(J)}{\Delta + \frac{dU_0}{dJ} - \nu}\frac{d\hat{F}_0}{dJ}\,\rho(\Delta)\,. \tag{18}$$

Both g_1 and $U_1(J)$ depend on F_1 which in turn depends on ν.

First consider the case where $\rho(\Delta)$ is a Lorentzian,

$$\rho(\Delta) = \frac{\epsilon}{\pi(\epsilon^2 + \Delta^2)}\,.$$

Integrating Eq. (18) over Δ yields

$$\int_{-\infty}^{\infty} d\Delta\,\hat{F}_1(J,\Delta) \equiv \tilde{F}_1(J) = \frac{\nu_0\sqrt{J}\,g_1 + U_1(J)}{-i\epsilon + \frac{dU_0}{dJ} - \nu}\frac{d\hat{F}_0}{dJ}\,. \tag{19}$$

The right-hand side of Eq. (19) depends on F_1 only through $\tilde{F}_1$. Multiplying both sides by the denominator on the right and $\sqrt{J}$ and integrating with respect to J yields

$$\begin{aligned} &(\nu_0 - i\epsilon - \nu)g_1 \\ &\quad = \int_0^\infty dJ\sqrt{J}\left[U_1(J)\frac{d\hat{F}_0}{dJ} - \tilde{F}_1(J)\frac{dU_0}{dJ}\right] \equiv X\,. \end{aligned} \tag{20}$$

In fact, $X \equiv 0$, and is a statement of Newton's third law. To show this, consider the first term in brackets in the integrand. Integrate this by parts, which yields

$$X = -\int_0^\infty dJ\left[\hat{F}_0(J)\frac{d\sqrt{J}\,U_1(J)}{dJ} + \sqrt{J}\,\tilde{F}_1(J)\frac{dU_0}{dJ}\right].$$

Insert the integral expressions for U_0 and U_1. In the double integral interchange J and J_1 in the second term in brackets. The result is

$$\begin{aligned} X &= -\int_0^\infty dJ\,dJ_1\,\hat{F}_0(J)\tilde{F}_1(J_1) \\ &\quad\times\int_0^{2\pi}\frac{d\psi}{2\pi}\left\{\frac{\cos\psi}{2\sqrt{J}}\,U + 2\sqrt{J_1}\sin^2\psi\,U'\right\}, \end{aligned}$$

where the argument of U is $(2J + 2J_1 - 4\cos\psi\sqrt{JJ_1})$ and U' denotes differentiation with respect to argument. Now,

$$\left\{\frac{\cos\psi}{2\sqrt{J}}\,U + 2\sqrt{J_1}\sin^2\psi\,U'\right\} = \frac{\partial}{\partial\psi}\frac{U\sin\psi}{2\sqrt{J}}\,,$$

and the integral is zero since $U\sin\psi$ has period 2π. Therefore $X \equiv 0$ and one concludes that the coherent frequency with a Lorentzian distribution in Δ is given by

$$\nu = \nu_0 - i\epsilon\,. \tag{21}$$

Notice that the nonlinearity in the space charge force as well as the shape of the unperturbed distribution are irrelevant in Eq. (21). Only $\rho(\Delta)$ was constrained, and with this constraint the space charge force has no effect on the coherent frequency.

Next consider the case where $\hat{F}_0(J)$ is a step function which is constant for $J < J_0$ and vanishes for $J > J_0$. The right-hand side of Eq. (18) is proportional to $\delta(J - J_0)$. This gives $\hat{F}_1(J,\Delta) = \delta(J - J_0)\rho_1(\Delta)$, where $\rho_1(\Delta)$ is unknown. Again, one is left with a straightforward dispersion relation. Define the constant

$$\begin{aligned} Q_{sc}^e &= -\left(\frac{dU_0}{dJ}\right)_{J=J_0} \\ &= \frac{1}{J_0}\int_0^{2\pi} d\psi\,\frac{\cos\psi}{2\pi}\,U\big(4J_0(1 - \cos\psi)\big)\,, \end{aligned} \tag{22}$$

where the equality of the two expressions is a special case of the argument used to obtain $X \equiv 0$. With this definition Eq. (18) becomes

$$\rho_1(\Delta) = \frac{\nu_0 + Q_{sc}^e}{\nu + Q_{sc}^e - \Delta}\,\rho(\Delta)\int_{-\infty}^{\infty} d\Delta_1\,\rho_1(\Delta_1)\,. \quad (23)$$

Integrating Eq. (23) over Δ yields a slightly modified dispersion integral from the result with no space charge. Suppose the space charge free dispersion relation is $\nu_0 = D(\nu)$. When space charge is included the dispersion relation is $\nu_0 + Q_{sc}^e = D(\nu + Q_{sc}^e)$. Generally, Q_{sc}^e is positive. The peak of the threshold curve given by $\mathrm{Im}(\nu) = 0$ shifts toward negative values of $\mathrm{Re}(\nu_0)$. Both this result and the previous one using the Lorentzian distribution support the usual prescription of treating the space charge force as a source of transverse capacitance. The final problem is to understand the effects of space charge tune spread when the distributions $\rho(\Delta)$ and $F_0(J)$ are more realistic.

Solving the problem with more realistic distributions of $\rho(\Delta)$ and $F_0(J)$ appears to require a model of the pairwise space charge force. A simple model, in the raw position variable, which includes nonlinearity is given by

$$f(x) = \kappa\left(2x - \frac{8x^3}{3\sigma^2}\right). \quad (24)$$

For real space charge both κ and σ are positive and the coefficients are chosen to simplify later expressions. After performing the averaging procedures the space charge potentials are given by

$$U_0(J) = -\kappa\left(J - \frac{J^2}{\sigma^2} - \frac{4J}{\sigma^2}\int_0^{\infty}\hat{F}_0(J_1)J_1\,dJ_1\right), \quad (25)$$

$$U_1(J) = \kappa\int_0^{\infty} dJ_1\sqrt{JJ_1}\left[1 - \frac{2(J + J_1)}{\sigma^2}\right] \times\int_{-\infty}^{\infty} d\Delta_1\,\hat{F}_1(J_1,\Delta_1) = \kappa\left[g_1\sqrt{J} - \frac{2}{\sigma^2}(g_1J^{3/2} + g_2\sqrt{J})\right]. \quad (26)$$

The parameter g_1 is defined in Eq. (16) and

$$g_2 = \int dJ_1\,d\Delta_1\,J_1^{3/2}\hat{F}_1(J_1,\Delta_1)\,. \quad (27)$$

Multiplying Eq. (18) by $\sqrt{J}$ and integrating over J and Δ gives g_1, while multiplying by $J^{3/2}$ and integrating yields g_2. The dependence on F_1 is thus reduced to two numbers. Define the general dispersion integral

$$I_k(\nu) = \int_0^{\infty} dJ\,J^k\,\frac{d\hat{F}_0(J)}{dJ}\int_{-\infty}^{\infty} d\Delta\,\frac{\rho(\Delta)}{\nu - \Delta - \frac{dU_0}{dJ}}\,. \quad (28)$$

The dispersion relation is equivalent to the two simultaneous equations,

$$g_1\left[1 + (\nu_0 + \kappa)I_1(\nu) - \frac{2\kappa}{\sigma^2}I_2(\nu)\right] = g_2\,\frac{2\kappa}{\sigma^2}\,I_1(\nu)\,, \quad (29)$$

and

$$g_1\left[(\nu_0 + \kappa)I_2(\nu) - \frac{2\kappa}{\sigma^2}I_3(\nu)\right] = g_2\left[\frac{2\kappa}{\sigma^2}I_2(\nu) - 1\right]. \quad (30)$$

Taking the ratio of the right-hand and left-hand sides of Eqs. (29) and (30) results in the final dispersion integral,

$$\nu_0 = -\kappa + \left(\frac{2\kappa}{\sigma^2}\right)^2 I_3(\nu) - \frac{[1 - \frac{2\kappa}{\sigma^2}I_2(\nu)]^2}{I_1(\nu)}\,. \quad (31)$$

Given $F_0(J,\Delta)$, κ, σ, and ν, Eq. (31) gives ν_0. When combined with Eq. (29) or (30) one obtains g_2/g_1 which via Eq. (26) gives $U_1(J)/g_1$ and finally via Eq. (18) the eigenfunction $\hat{F}_1(J,\Delta)/g_1$. The constant g_1 cancels and the problem is, in principle, solved. Of course, one is generally interested in the problem of finding ν given ν_0 or, less ambitiously, whether a given value of ν_0, which is proportional to the transverse impedance, corresponds to an unstable system. This will be addressed later using threshold curves.

The next problem is to find a suitable expression for Eq. (28). In principle one could do the double dispersion integral numerically for arbitrary ρ and F_0 but, given the cubic approximation to the space charge force, any reasonable functions would be equally realistic. Toward this end, expressions which yield a final result in terms of simple functions will be obtained.

The bare tune distribution is approximated as

$$\rho(\Delta) = \sum_{m=1}^{M} C_m\,\frac{\epsilon_m}{\pi(\epsilon_m^2 + \Delta^2)}\,. \quad (32)$$

Given a set $\epsilon_1 < \epsilon_2 < \cdots \epsilon_M$ the C_k's are chosen to keep the integral over Δ equal to unity and to cancel the tails of the distribution. The latter is done by demanding

$$\lim_{\Delta\to\infty}\Delta^{2N}\rho(\Delta) = 0\,,$$

for $N = 1, 2, \ldots, M - 1$. The full set of constraint equations is given by

$$1 = \sum_{m=1}^{M} C_m, \qquad 0 = \sum_{m=1}^{M} C_m\epsilon_m^{2N-1},$$

where $N = 1, 2, \ldots, M - 1$. For large M the set of linear equations is solved numerically. The resulting distribution is

$$\rho(\Delta) = \rho(0)\prod_m \frac{1}{(1 + \Delta^2/\epsilon_m^2)}\,. \quad (33)$$

With $\rho(\Delta)$ given by Eq. (32) the integral with respect to Δ in Eq. (28) is easily done and results in

$$I_k(\nu) = \sum_{m=1}^{M} C_m\int_0^{\infty} dJ\,J^k\,\frac{d\hat{F}_0(J)}{dJ}\,\frac{1}{\nu + i\epsilon_m - \frac{dU_0}{dJ}} = \sum_{m=1}^{M} C_m I_k^m(\nu)\,, \quad (34)$$

where $I_k^m(\nu)$ is the integral over J for each value of m.

There are many possible choices for $\hat{F}_0(J)$. For definiteness take

$$\hat{F}_0(J) = \frac{3}{J_0}\begin{cases}(1 - J/J_0)^2 & \text{if } J < J_0\,,\\ 0 & \text{otherwise}\,.\end{cases} \tag{35}$$

This representation of $\hat{F}_0(J)$ is smooth enough to give bounded values of $I_k^m(\nu)$ even if $\mathrm{Im}(\nu + \epsilon_m) = 0$ [11]. The space charge tune shift as a function of action is given by

$$-\frac{dU_0}{dJ} = \kappa\left(1 - \frac{2J + J_0}{\sigma^2}\right). \tag{36}$$

Define $q = 2\kappa J_0/\sigma^2$ and let $y = J/J_0$. The required dispersion integrals are

$$I_k^m = \frac{6J_0^{k-1}}{q}\int_0^1 \frac{y^k dy(1 - y)}{y - \frac{1}{q}(\nu_R + \kappa - q/2 + i[\nu_I + \epsilon_m])}\,, \tag{37}$$

where the real and imaginary parts of ν are explicit. The integral in Eq. (37) is along the real axis and only results with $\nu_I + \epsilon_m \geq 0$ are meaningful. The integrals are elementary but tedious. An efficient representation involves the function

$$G_k(z) = 6\int_0^1 \frac{y^k dy(1 - y)}{y - z}\,,$$

where

$$z = u + iv = \frac{1}{q}(\nu_R + \kappa - q/2 + i[\nu_I + \epsilon_m])$$

and $I_k^m = J_0^{k-1}G_k(z)/q$. With these definitions [12],

$$G_1(u + iv) = 3 - 6z + 6z(1 - z)\left\{i\left[\tan^{-1}\left(\frac{1 - u}{v}\right) + \tan^{-1}\left(\frac{u}{v}\right)\right] + \frac{1}{2}\ln\left(\frac{v^2 + (1 - u)^2}{v^2 + u^2}\right)\right\}. \tag{38}$$

The remaining integrals are given by $G_2(z) = 1 + zG_1(z)$ and $G_3(z) = 1/2 + zG_2(z)$. The real and imaginary parts of these functions for $v = 0^+$ are shown in Figs. 1 and 2, respectively. The functions are bounded and continuous. Notice that $\mathrm{Im}(G_1) = 6\pi u(1 - u)$ for $0 < u < 1$, and that the discontinuity in its derivative at $u = 0$ would be present for any $\hat{F}_0(J)$ that had a nonzero derivative as $J \to 0^+$. For instance, a thermal distribution would yield a G_1 with a discontinuous derivative at $u = 0$ [11]. Therefore, the present choice of $\hat{F}_0(J) = 3(1 - J/J_0)^2/J_0$ has no unrealistic singularities and should fairly represent real beams.

In Fig. 3, distributions from the series

$$\epsilon_j^2 = 2\sigma_G^2(M + \sqrt{M}(j - 1))\ln(1 + \sqrt{M})/\sqrt{M}$$

for $j = 1, 2, \ldots, M$ with $\sigma_G = 1$ and $M = 5, 10$ are shown. A Gaussian distribution with unit standard deviation is plotted for comparison. Using Eq. (33) one may show that the density distributions $\rho_M(\Delta)$ converge uniformly to Gaussians with standard deviation σ_G as $M \to \infty$. Figure 4 shows the dispersion diagrams at threshold for the distributions in Fig. 3. These are plots of $\nu_0 = -1/I_1(\nu)$ with ν varying over real values and I_1

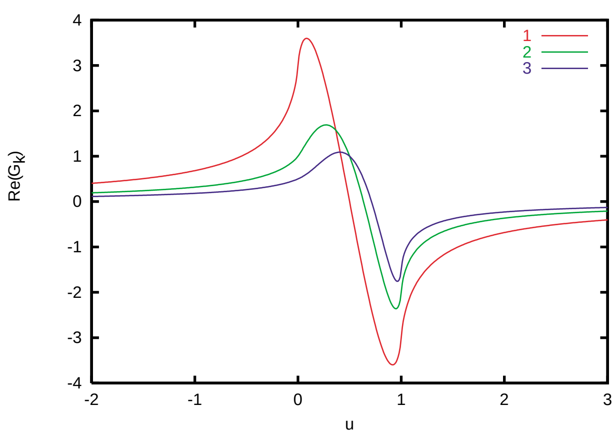

FIG. 1. (Color) $\mathrm{Re}(G_k(u + i0))$ for $k = 1, 2, 3$.

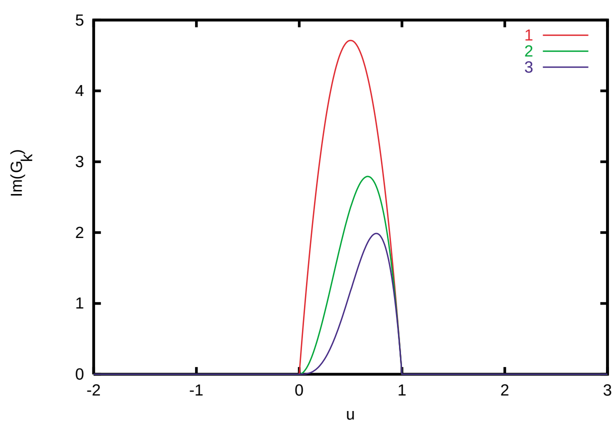

FIG. 2. (Color) $\mathrm{Im}(G_k(u + i0))$ for $k = 1, 2, 3$.

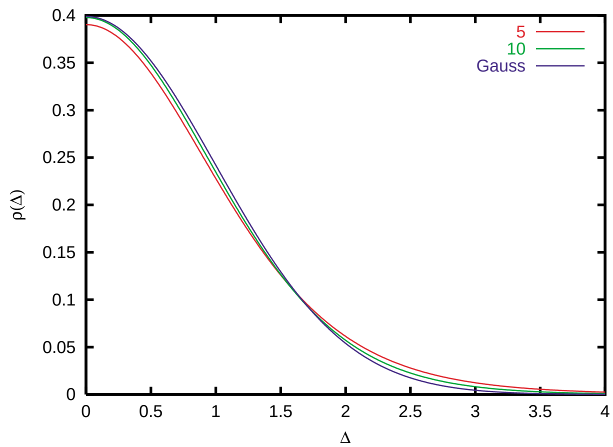

FIG. 3. (Color) The bare tune distributions for five- and ten-term Gaussian series with $\sigma_G = 1$. A Gaussian distribution with unit standard deviation is shown as well.

given by Eq. (28) with no space charge. For a given curve, giving ν a positive imaginary part always results in a value of ν_0 which lies above the curve so those values of ν_0 which lie on and below the curve result in no exponential growth. In fact, solutions corresponding to values of ν_0 below the curve are damped [13].

Adding space charge to the picture requires choosing both κ and q. To get the maximum effect in one dimension choose q/κ so that particles at opposite edges of the beam exert no force on each other. The difference in their positions is $x = 2\sqrt{2J_0}$, and setting Eq. (24) to zero gives $q/\kappa = 3/16$. Real beams have a two-dimensional cross section. Take an unperturbed distribution $F_0(J_x, J_y) \propto (J_0 - J_x - J_y)$. This yields a round beam with a one-dimensional projection $F_0(J_x) \propto (J_0 - J_x)^2$ and similarly for y. The x component of the space charge force is

$$F_x(x, y) = Kx\left[1 - \frac{x^2 + y^2}{R^2} + \frac{(x^2 + y^2)^2}{3R^4}\right], \quad (39)$$

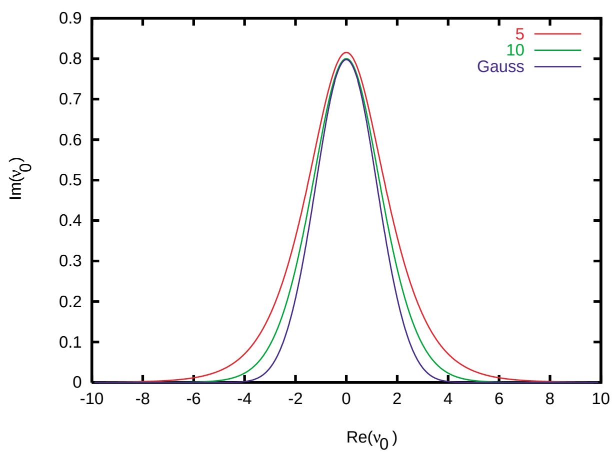

FIG. 4. (Color) Threshold diagrams for the distributions shown in Fig 3.

where R is the beam radius. Neglect the third term in brackets yielding a force which vanishes at the edge of the beam, and a soft upper limit for realistic distributions. Using first-order perturbation theory the x tune shift is given by

$$Q_{sc}(J_x, J_y) = Q_{sc}^0\left(1 - \frac{3}{4}\frac{J_x}{J_0} - \frac{1}{2}\frac{J_y}{J_0}\right), \quad (40)$$

where Q_{sc}^0 is the x tune depression in the center of the beam. The probability distribution of Q_{sc} is

$$P(Q) = \int_0^{J_0} dJ_x \times \int_0^{J_0 - J_x} dJ_y\, F_0(J_x, J_y)\delta(Q - Q_{sc}(J_x, J_y)),$$

where

$$\int_{Q_1}^{Q_2} P(Q)\, dQ = \text{fraction of particles with } \{Q_1 < Q_{sc} < Q_2\}.$$

With Eq. (40) the distribution of Q_{sc} may be obtained in closed form. Plots of this expression as well as the distribution for a 1D case with the same average and standard deviation are shown in Fig. 5. The effective nonlinearity is $q/\kappa = 3/5$, just more than 3 times the one-dimensional case.

Figure 6 shows the threshold curves for the ten-term Gaussian distribution with $\kappa = 0, 5, 10$ and $q/\kappa = 3/16$. Adding space charge shifts the curves to the left, which is expected since space charge is capacitive. These curves are shifted by $\kappa - q$, which is slightly less than the space charge tune shift averaged over F_0, which is $\kappa - 3q/4$. For fixed q, the width of the dispersion curve is independent of κ. Figure 7 shows similar curves for $q/\kappa = 3/5$.

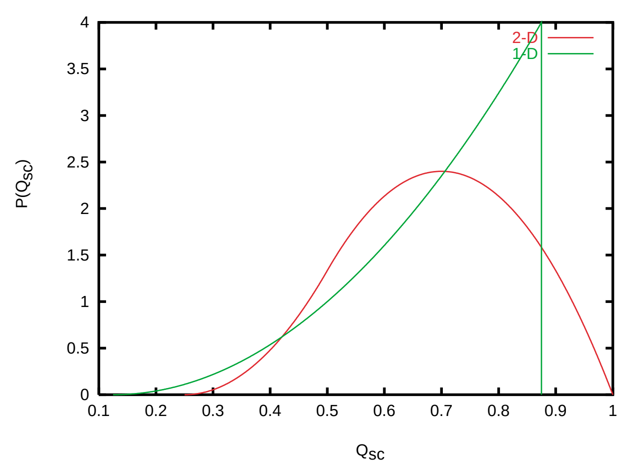

FIG. 5. (Color) Probability distributions for 1D and 2D beams with the same average and standard deviation. The peak value of the tune shift for the 2D beam is 1 while the 1D beam has $\kappa = 5/4$ and $q = 3/4$.

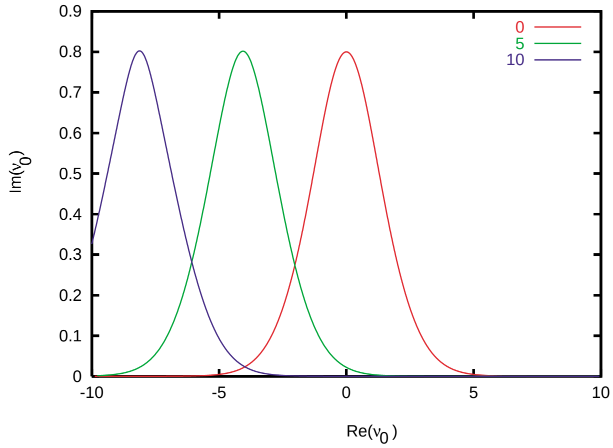

FIG. 6. (Color) Threshold diagrams for ten-term Gaussian series for $\kappa = 0, 5, 10$ with $q/\kappa = 3/16$.

The center of the curve is shifted by half the peak value of the space charge tune shift in the 2D beam. The shape of the distribution changes noticeably as well, but for $|\mathrm{Re}(\nu_0)| \lesssim q$ the threshold value of the transverse resistance is reduced.

For space charge dominated machines, reducing the effective space charge tune shift by half roughly doubles the intensity threshold estimates for transverse instabilities. An independent estimate of this effect can be obtained as follows. Consider a beam composed of rigid cylinders instead of points. The force per unit length of cylinder 1 on cylinder 2 is given by

$$\mathbf{F}_{1,2} = \int n_2(\mathbf{x}')\, d^2x' \int n_1(\mathbf{x}) \frac{1}{2\pi\epsilon_0\gamma^2} \frac{\mathbf{x}' - \mathbf{x}}{|\mathbf{x}' - \mathbf{x}|^2}\, d^2x\,,$$

where $n_1(\mathbf{x})$ and $n_2(\mathbf{x}')$ are the two-dimensional charge distributions of the generator and recipient of the Coulomb force, respectively. Assume $n_1(\mathbf{x}) = (3/\pi) \times (1 - |\mathbf{x}|^2/R^2)^2$ with a total charge of 1. The horizontal electric field generated by n_1 is proportional to the space charge force given by Eq. (39). Make the same approximation as before, neglecting the term proportional to $\mathbf{x}|\mathbf{x}|^4$. Next, one calculates $F_{1,2}$ for $n_2(\mathbf{x}) = \delta(\mathbf{x} - \mathbf{x}_2)/x_2$ and $n_2(\mathbf{x}) = n_1(\mathbf{x} - \mathbf{x}_2)/x_2$ where $\mathbf{x}_2$ is the offset . For $|\mathbf{x}_2| \ll R$ the total force on the extended distribution is exactly half the total force on the offset delta function. This coincides with the shift observed using the dispersion integrals and is easily extendable to other distributions. For a Kapchinskij-Vladimirskij distribution which has constant charge density within the beam, the force on the appropriate extended distribution is equal to the force on the delta distribution. This follows from Eq. (23) and is found in [1,2,5,6]. For a Gaussian cross section the force on the extended distribution is exactly half the force on a delta distribution with the same dipole moment. This last result is valid even if the beam is elliptical.

Finally, the effect of amplitude dependent tune spread due to octupoles or other nonlinear elements is addressed. These nonlinearities are with respect to the center of the beam pipe and introduce tune spread which can damp dipole modes. For one transverse dimension, nonlinearities add a term to the Hamiltonian $U_{\mathrm{oct}}(J) = rJ^2/2J_0$, where r is proportional to the strength of the octupoles. Factorizing $F_0 = \hat{F}_0(J)\rho(\Delta)/2\pi$ the calculation proceeds as before and Eq. (14) becomes $H_0 = \Delta J + rJ^2/2J_0 + U_0(J)$. The net effect is to make the substitution $dU_0/dJ \rightarrow dU_0/dJ + rJ/J_0$ in the dispersion integrals for $I_k(\nu)$ in Eq. (28). Equations (31) and (34) stand with the substitutions $I_k^m = J_0^{k-1}G_k(z)/(q + r)$ and

$$z = u + iv = \frac{1}{q + r}(\nu_R + \kappa - q/2 + i[\nu_I + \epsilon_m])\,.$$

Figure 8 shows threshold curves with $\kappa = q$ and $\Delta \equiv 0$ for various values of r and q. In real beams the curves would be shifted to the left by $\gtrsim 5q/3$ so that, with space charge, the point $\nu_0 = 0$ would be only marginally stable

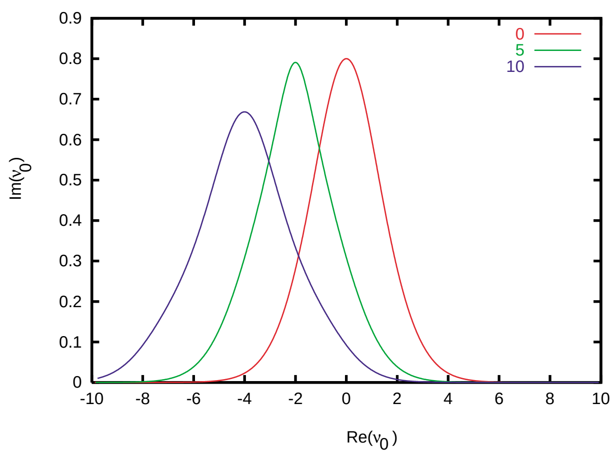

FIG. 7. (Color) Threshold diagrams for ten-term Gaussian series for $\kappa = 0, 5, 10$ with $q/\kappa = 3/5$.

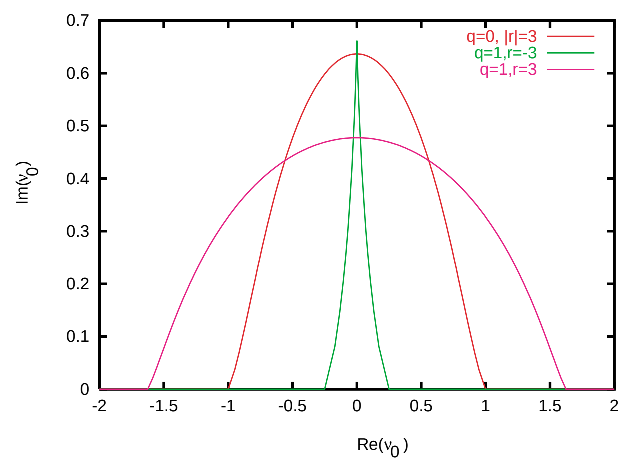

FIG. 8. (Color) Threshold diagrams for tune spread due to octupoles (r) with space charge tune spread (q). In a real beam the curves would be shifted to the left by $\sim 2q$.

for either sign of r. However, for $r < 0$, space charge nonlinearity dramatically reduces the area of the stable region and $r > 0$ is the best choice. The octupoles should cause the tune to increase with betatron amplitude, as found previously [4].

In two dimensions, nonlinear elements add a term $\delta H = r_{xx} J_x^2 + r_{yy} J_y^2 + r_{xy} J_x J_y$ to the Hamiltonian. Making all three coefficients positive requires three families of octupoles, and the benefits to stability need to be weighed against any reductions in dynamic aperture. This is especially important when space charge tune shifts $\gtrsim 0.1$ are expected. Another possibility is to exploit the effect of quadrupole end fields. Equation (17) in [14] shows that fringe field effects in quadrupoles almost always lead to a betatron tune increase with amplitude. While that reference suggested lengthening magnets to reduce nonlinear effects, introducing tune spread by using short, strong quadrupoles may be an attractive alternative to strong octupoles.

IV. CONCLUSIONS

A momentum and energy conserving model of the direct space charge force has been used to predict the effects of this force on transverse stability. It was found that the nonlinearity in the space charge force can modify dispersion diagrams even though it causes no Landau damping on its own. Even using a soft upper limit for the amount of space charge tune spread, it was found that space charge reduces stability for a reasonable machine impedance. When using traditional dispersion diagrams with smooth beams and tune spreads created by momentum spread, using half of the peak space charge tune shift in the dispersion relations gives fairly good estimates. Earlier work, showing that nonlinear elements should cause the betatron tune to increase in amplitude, were confirmed.

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