Comment on "Nonlinear Compton scattering and electron acceleration in interfering laser beams"

Yousef I. Salamin*
Theoretische Quantendynamik, Fakultät für Physik, Universität Freiburg, Hermann-Herder-Strasse 3, D-79104 Freiburg, Germany
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We point out that, the way it is reported, the solution to the equation of motion of a relativistic electron in the field of two electromagnetic waves advanced recently by Amatuni and Pogorelsky [Phys. Rev. ST Accel. Beams **1**, 034001 (1998)] does not handle the case of two copropagating waves differing in frequency. An equivalent form for that solution is rigorously developed.

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In an interesting and potentially important article, Amatuni and Pogorelsky [1] have recently discussed in detail various situations involving the acceleration of, and the (nonlinear Compton) scattering of radiation by, a single electron in two interfering plane-wave laser beams. Their discussion is based on an exact solution to the Lorentz equation of the electron in the given fields which they develop following an approach advanced for the single plane-wave case many years ago [2]. In [1] the Lorentz equation is broken into two coupled nonlinear equations holding under conditions that apply to situations involving two waves polarized linearly along the same direction or to two collinear (or anticollinear) waves of arbitrary linear polarization.

The purpose of this paper is to demonstrate that, in the way it is reported in [1], the solution in question does not seem to handle the case of two copropagated (collinear) waves of different frequencies. We develop a form for the solution to the problem at hand that is capable of handling the cases considered in [1] as well as the one just described. Our form of the solution reproduces all the working equations derived and discussed with considerable detail in [1] for the cases considered there and, in this sense, the present paper does not raise any questions about the validity of the conclusions of Amatuni and Pogorelsky.

We employ the system of units in which $\hbar = c = 1$ throughout this paper, and we start by slightly changing the notation from that of Ref. [1]. The scalar product of two four-vectors $a = (a_0, \boldsymbol{a})$ and $b = (b_0, \boldsymbol{b})$ will be denoted by $a \cdot b = (a_0 b_0 - \boldsymbol{a} \cdot \boldsymbol{b})$. A dot on a four-vector will denote differentiation with respect to τ, the proper time of the particle, such as $\dot{a}_1 = da_1/d\tau$, and so on. On the other hand, a prime will mean differentiation with respect to the phase η, as in $a_1' = da_1/d\eta_1$, ..., etc. The particle's four-vector momentum at any (proper) time τ will be given by $p(\tau) = (\mathcal{E}, \boldsymbol{p})$ and its initial value by $p(\tau_i) = p_i = (\mathcal{E}_i, \boldsymbol{p}_i)$. Let the two plane waves be modeled by the vector potentials $a_1(\eta_1)$ and $a_2(\eta_2)$, where the phases are $\eta_1 = k_1 \cdot x$ and $\eta_2 = k_2 \cdot x$; $x = (t, \boldsymbol{x})$ being the particle's coordinate four-vector and $k_j = (\omega_j, \boldsymbol{k}_j) = \omega_j(1, \hat{\boldsymbol{e}}_j)$, $j = 1, 2$, the waves' propagation four-vectors. Furthermore, $\hat{\boldsymbol{e}}_j$ is a unit vector in the direction of propagation of the jth wave.

With $A = a_1 + a_2$, the Lorentz equation of motion of the particle, whose mass is m and whose (negative) charge is e, in the two waves reads

$$\frac{dp_\mu}{d\tau} = \frac{e}{m} F_{\mu\nu} p^\nu, \tag{1}$$

where $\mu = 0, 1, 2, 3$, and the electromagnetic field tensor is given by

$$\begin{aligned} F_{\mu\nu} &= \partial_\mu A_\nu - \partial_\nu A_\mu \\ &= (k_{1\mu} a_{1\nu}' + k_{2\mu} a_{2\nu}') - (k_{1\nu} a_{1\mu}' + k_{2\nu} a_{2\mu}'). \end{aligned} \tag{2}$$

In principle, a solution to Eq. (1) will yield the particle's momentum four-vector p. From p one then works out a parametric representation, using τ as a parameter, for the associated coordinate four-vector x via

$$x = x_i + \int_{\tau_i}^{\tau} \frac{p(\tau')\, d\tau'}{m}, \tag{3}$$

where x_i is the value of x initially, at τ_i or before the particle has been injected into the region of interaction with the waves. Following Amatuni and Pogorelsky [1] we will seek a solution to Eq. (1) of the form

$$p(\tau) = p_i - e(a_1 + a_2) + k_1 f_1(\tau) + k_2 f_2(\tau), \tag{4}$$

where f_1 and f_2 are unknown functions of the proper time which will be determined by substituting Eq. (4) into Eq. (1). Note that the initial conditions at τ_i, before the particle is subjected to the waves, demand that these two functions vanish identically as $\tau \to \tau_i$, owing to the fact that k_1 and k_2 are independent. Substituting Eq. (4) into Eq. (1), and after some lengthy algebra, one obtains

$$[(k_1 \cdot p_i) + (k_1 \cdot k_2) f_2]\dot{f}_1 = e[p_i - e(a_1 + a_2)] \cdot \dot{a}_1, \tag{5}$$

*On leave from: Physics Department, Birzeit University, P.O. Box 14, Birzeit, West Bank, Palestinian Authority. Email address: ysalamin@science.birzeit.edu

$$[(k_2 \cdot p_i) + (k_1 \cdot k_2) f_1] \dot{f}_2 = e[p_i - e(a_1 + a_2)] \cdot \dot{a}_2 . \tag{6}$$

Apart from the slight change in notation, Eqs. (5) and (6) are identical, respectively, to the two lines of Eq. (9) in Ref. [1]. Adding them as suggested in [1], carrying out the single integration, and solving the result for f_2 in terms of f_1 yields

$$f_2 = \alpha_1 \left[\frac{b - f_1}{\alpha_2 + f_1} \right], \tag{7}$$

where

$$\alpha_1 = \frac{k_1 \cdot p_i}{k_1 \cdot k_2}, \tag{8}$$

$$\alpha_1 (k_1 \cdot k_2) b = e p_i \cdot (a_1 + a_2) - \frac{e^2}{2} (a_1 + a_2)^2, \tag{9}$$

$$\alpha_2 = \frac{k_2 \cdot p_i}{k_1 \cdot k_2}. \tag{10}$$

Alternatively, the same result could have been arrived at earlier by squaring both sides of Eq. (4), taking note of the fact that $p^2 = p_i^2 = m^2$.

Because of the exchange symmetry exhibited by Eqs. (5) and (6), namely, that one of them may be obtained from the other by merely interchanging the indices 1 and 2, one needs only to find a solution to one of them; the solution of the other then follows by letting $1 \longrightarrow 2$ and $2 \longrightarrow 1$. To find an exact expression for f_1, for example, one first uses Eq. (7) in order to eliminate f_2 from Eq. (5). With the subscripts temporarily suspended, the resulting differential equation is then written in the form

$$\frac{df}{d\tau} + P f = R . \tag{11}$$

This equation has the following solution, obtained via the standard integrating-factor technique,

$$f(\tau) = e^{-I} \left[\int^{\tau} R(\tau') e^{I(\tau')} \, d\tau' + C \right], \tag{12}$$

$$I(\tau) = \int^{\tau} P(\tau') \, d\tau', \tag{13}$$

where C is a constant to be determined from the initial condition, namely, the vanishing of f at τ_i. Employing the initial condition formally amounts to giving (for f_1)

$$f_1(\tau) = e^{-I_1} \int_{\tau_i}^{\tau} R_1(\tau') e^{I_1(\tau')} \, d\tau', \tag{14}$$

where

$$P_1(\tau) = - \frac{e \dot{a}_1 \cdot [p_i - e(a_1 + a_2)]}{\alpha_3 + [e p_i \cdot (a_1 + a_2) - \frac{e^2}{2} (a_1 + a_2)^2]}, \tag{15}$$

$$R_1(\tau) = \frac{e \dot{a}_1 \cdot [p_i - e(a_1 + a_2)] \alpha_2}{\alpha_3 + [e p_i \cdot (a_1 + a_2) - \frac{e^2}{2} (a_1 + a_2)^2]}. \tag{16}$$

In these equations,

$$\alpha_3 = \frac{(k_1 \cdot p_i)(k_2 \cdot p_i)}{k_1 \cdot k_2}. \tag{17}$$

As was noted above, similar expressions for f_2 and the related quantities may easily be written down from their f_1 counterparts by interchanging the indices $1 \longrightarrow 2$ and $2 \longrightarrow 1$.

With the change in notation introduced above, the corresponding expressions for f_1 and f_2 reported in Ref. [1] as Eqs. (10) and (11), respectively, may alternatively be written as

$$f_1 = \alpha_2 \left[\exp\left(\int_{\tau_i}^{\tau} F_1 \, d\tau \right) - 1 \right], \tag{18}$$

$$f_2 = \alpha_1 \left[\exp\left(\int_{\tau_i}^{\tau} F_2 \, d\tau \right) - 1 \right], \tag{19}$$

where

$$F_1 = \frac{e \dot{a}_1 \cdot [p_i - e(a_1 + a_2)]}{\alpha_3 + [e p_i \cdot (a_1 + a_2) - \frac{e^2}{2} (a_1 + a_2)^2]}, \tag{20}$$

$$F_2 = \frac{e \dot{a}_2 \cdot [p_i - e(a_1 + a_2)]}{\alpha_3 + [e p_i \cdot (a_1 + a_2) - \frac{e^2}{2} (a_1 + a_2)^2]}. \tag{21}$$

It may easily be shown that Eqs. (18)–(21) satisfy Eqs. (5) and (6). They also satisfy the initial condition, namely, $f_1(\tau_i) = f_2(\tau_i) = 0$. As a check, two special cases are considered in Ref. [1], namely, the case of a single plane wave and that in which the two waves are components of the same wave. According to [1] the first case, obtained a long time ago [2], may be realized by letting $a_2 = 0$, thus eliminating the second wave. Note that this elimination may not be complete without simultaneously setting $k_2 = 0$, which in turn leads to indeterminate values for f_1 and f_2. Fortunately, one need not do that; rather, k_2 may be left everywhere in place in this special case, as the integration in Eq. (18) (with $a_2 = 0$) may be carried out, and when the result is substituted into Eq. (20) all dependence upon k_2 drops out naturally. The same thing also applies verbatim in the second case. However, other cases not considered in [1] and of which an example will be considered here shortly cannot be treated in this way. Rewriting Eqs. (8), (10), and (17) explicitly, the terms that stand to cause trouble are

$$\alpha_1 = \frac{(\mathcal{E}_i - \hat{\boldsymbol{e}}_1 \cdot \boldsymbol{p}_i)}{\omega_2 (1 - \hat{\boldsymbol{e}}_1 \cdot \hat{\boldsymbol{e}}_2)}, \tag{22}$$

$$\alpha_2 = \frac{(\mathcal{E}_i - \hat{\boldsymbol{e}}_2 \cdot \boldsymbol{p}_i)}{\omega_1 (1 - \hat{\boldsymbol{e}}_1 \cdot \hat{\boldsymbol{e}}_2)}, \tag{23}$$

$$\alpha_3 = \frac{(\mathcal{E}_i - \hat{\boldsymbol{e}}_1 \cdot \boldsymbol{p}_i)(\mathcal{E}_i - \hat{\boldsymbol{e}}_2 \cdot \boldsymbol{p}_i)}{1 - \hat{\boldsymbol{e}}_1 \cdot \hat{\boldsymbol{e}}_2}. \tag{24}$$

Recall that $\hat{\boldsymbol{e}}_j$ is a unit vector in the direction of propagation of the jth wave. Obviously, α_1, α_2, and α_3 all diverge in the case of two collinear beams $\hat{\boldsymbol{e}}_1 \cdot \hat{\boldsymbol{e}}_2 = 1$. An exact solution to this case along the lines outlined above, for the two special cases, does not seem to be possible. Worse yet, if one lets, say, $\alpha_3 \to \infty$ in Eq. (20) one gets $F_1 = 0$, $\alpha_2 \to \infty$, and, hence, f_1 will be indeterminate. So a different trick seems to be needed in this particular case. The otherwise useful trick, based on Eq. (32) of [1], which merely reproduces our Eq. (7), does not help either.

This case is important in, for example, the vacuum beat wave laser accelerator [3]. Our equations offer to handle the problem as follows. Letting $\alpha_3 \to \infty$ in Eq. (15) gives $P_1 = 0$ and, hence, $I_1 = 0$. Furthermore, when one divides the numerator and denominator of Eq. (16) by α_2 and subsequently takes the limit $\hat{\boldsymbol{e}}_1 \cdot \hat{\boldsymbol{e}}_2 \to 1$ in the result, one gets

$$R_1(\tau) = \frac{e\dot{a}_1 \cdot [p_i - e(a_1 + a_2)]}{(k_1 \cdot p_i)}. \qquad (25)$$

Hence, assuming $a_1(\tau_i) = a_2(\tau_i) = 0$, one finally obtains

$$\begin{aligned} f_1(\tau) &= \frac{e}{k_1 \cdot p_i} \int_{\tau_i}^{\tau} \dot{a}_1 \cdot [p_i - e(a_1 + a_2)]\, d\tau \\ &= \frac{[e(p_i \cdot a_1) - \frac{e^2}{2} a_1^2]}{k_1 \cdot p_i} \\ &\quad - \frac{e^2}{k_1 \cdot p_i} \int_{\tau_i}^{\tau} (\dot{a}_1 \cdot a_2)\, d\tau . \end{aligned} \qquad (26)$$

This result could have been obtained from Eq. (5) by setting $k_1 \cdot k_2 = 0$ and subsequently carrying out the remaining integration. One may now suspect that the two forms of the solution, Eqs. (14) and (18), are equivalent. They are. Close inspection of Eqs. (14)–(16) reveals that $P_1 = -F_1$ and that $R_1 = \alpha_2 F_1$. Hence, the integrand in Eq. (14) is actually a total differential

$$R_1(\tau') e^{I_1(\tau')} = -\alpha_2 \frac{d}{d\tau'} [e^{-\int^{\tau} F_1\, d\tau'}]. \qquad (27)$$

Recognizing this, one may now rewrite Eq. (14) in terms of F_1 alone and establish its equivalence with Eq. (20).

For the sake of completeness, note that the special case of only one wave present and the (related) situation in which the two waves are components of the same wave follow immediately from Eq. (26). In general, however, the remaining integral in Eq. (26) needs to be evaluated, perhaps utilizing further restrictions on the two waves. For example, it vanishes if the two waves are modeled by vector potentials polarized, say, linearly, perpendicular to each other.

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[1] A. Ts. Amatuni and I. V. Pogorelsky, Phys. Rev. ST Accel. Beams **1**, 034001 (1998).
[2] A. I. Nikishov and V. I. Ritus, Zh. Eksp. Teor. Fiz. **46**, 776 (1964); **46**, 1768 (1964).
[3] E. Esarey, P. Sprangle, and J. Krall, Phys. Rev. E **52**, 5443 (1995).