Spread-frequency model of the fast beam-ion instability in an electron storage ring

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When a gap in the electron bunch train prevents the trapping of ions, a transverse electron-ion instability may result from the ions created and lost during a single passage of the bunch train. A spread-frequency model is used to study this instability when the ions have a broad distribution of natural oscillation frequencies about the center of the electron beam. A growing disturbance saturates from Balakin-Novokhatsky-Smirnov damping at approximately the same time, and with the same total growth, as in the case without an ion frequency spread. At the tail of the bunch train, an unstable disturbance is amplified by a factor $\sim \exp(\omega_i \tau_b)$ before saturation occurs, where ω_i is a typical ion oscillation frequency and τ_b is the duration of the bunch train. Initially, the instability displays exponential growth in time, unlike the case where the ion-frequency spread is neglected. For a broad distribution of ion frequencies, instability may be prevented by a betatron damping rate that exceeds the incoherent betatron frequency shift induced by ions at the tail of the bunch train.

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I. INTRODUCTION

The space charge of electrons circulating in an electron storage ring may attract ions created by electron-neutral collisions, resulting in a channel of trapped ions [1]. The trapped ions produce a tune shift of the circulating electrons and may cause a transverse ion-electron instability, predominantly in the vertical direction when the beam width exceeds its height. By introducing a gap in the bunch train, the trapped ion density may be greatly reduced. However, transient ions (which are created and lost within a single bunch train) may still give rise to an instability, termed the "fast beam-ion instability" [2–9]. In this paper, we model the fast beam-ion instability when there is a broad distribution of ion "bounce" frequencies, where the bounce frequencies are the natural frequencies of transverse ion oscillations about the electron orbit [10]. The results are compared with modeling that neglects the spread in ion bounce frequencies.

In both cases, the amplitude of an unstable disturbance saturates after growing by a factor $\sim \exp(\omega_i \tau_b)$ at the tail of the bunch train, where ω_i is a typical ion oscillation frequency and τ_b is the duration of the bunch train. The saturation is a result of Balakin-Novokhatsky-Smirnov (BNS) damping [8,11] because the ion-induced incoherent betatron frequency shift (i.e., the tune shift) increases toward the tail of the bunch train.

With a large spread in ion bounce frequencies, the instability initially displays exponential growth in time, with a growth rate comparable to the ion-induced incoherent betatron frequency shift. Consequently, the fast beam-ion instability may be prevented by betatron damping (or feedback) that exceeds the incoherent betatron frequency shift induced by ions at the tail of the bunch train.

II. SINGLE BOUNCE FREQUENCY

Let us first consider the case where there is no spread in ion bounce frequencies, for an ion density that grows linearly with the passage of the bunch train, using a smooth approximation for the betatron focusing. We model the bunch train as an electron beam of duration τ_b, using the propagation time $Z \equiv z/v$ to describe the propagation distance z divided by beam velocity v, while coordinate $T \equiv t - z/v$ denotes the time after passage of the head of the electron bunch train. For small rigid displacements of the electrons and ions, we have the approximate equations of motion for the electron beam vertical position $b(Z,T)$ and average ion vertical position $c(Z,T)$,

$$\begin{aligned} \left[\frac{\partial^2}{\partial Z^2} + \omega_\beta^2 + \omega_e^2(T)\right] b(Z,T) &= \omega_e^2(T) c(Z,T), \\ \left[\frac{\partial^2}{\partial T^2} + \omega_i^2\right] c(Z,T) &= \omega_i^2 b(Z,T), \end{aligned} \tag{1}$$

where ω_β is the electron betatron frequency in the absence of ions and ω_i is the vertical ion bounce frequency given by

$$\omega_i = \left(\frac{\langle n_e\rangle_t Z e^2}{\varepsilon_0 m_i} \frac{\sigma_x}{\sigma_x + \sigma_y}\right)^{1/2}, \tag{2}$$

in which $\langle n_e\rangle_t$ is the time-averaged electron density on axis during the passage of the bunch train, m_i is the ion mass, Z is the ion charge, and σ_x and σ_y are the horizontal and vertical beam dimensions. Because of the dependence upon the ion mass and the position-dependent quantities $\langle n_e\rangle_t$, σ_x, and σ_y, a large range of ion bounce frequencies may be expected in a typical electron storage ring. The electron bounce frequency in the ion channel (for $\omega_\beta = 0$), denoted $\omega_e(T)$, is

$$\omega_e(T) = \left(\frac{Ze^2}{\varepsilon_0 \gamma m_e} \frac{n_i(T)\sigma_x}{\sigma_x + \sigma_y} \right)^{1/2}, \tag{3}$$

where $n_i(T)$ is the ion density. For ions created by collision of the electrons with neutral molecules and lost on a time scale large compared to the bunch train duration τ_b, $n_i(T)$ is proportional to T, so we have

$$\omega_e^2(T) = KT\,, \tag{4}$$

where $K = \omega_e^2(\tau_b)/\tau_b$.

When $\omega_\beta = 0$, Eq. (1) describes the "ion hose" instability of an electron beam focused by an ion channel [12–15]. In an electron storage ring, where ion effects are a perturbation to the betatron motion in applied magnetic fields, we instead have $\omega_\beta \gg \omega_e(T)$. In the left-hand side of the lower equation in Eq. (1), we have neglected the term $(1/T)\partial_T c(Z,T)$ which describes the damping of collective ion oscillations that results from the constant creation rate of stationary ions [13]. This term is relatively small for $\omega_i T \gg 1$; neglecting it restricts the validity of our results to $T \gg 1/\omega_i$, i.e., more than one ion oscillation period behind the head of the bunch train.

Because Eq. (1) is not invariant in T and Z, a solution of the form $\exp(-i\Omega Z - i\omega T)$, which applies in a region where $\omega_e^2(T)$ is approximately constant, will not suffice to describe the entire bunch train region $0 < T < \tau_b$. We instead consider a disturbance with slowly varying envelope of the form [2]

$$b(Z,T) = b_0 \exp[g_0(Z,T)] \exp(i\omega_\beta Z - i\omega_i T)\,, \tag{5}$$

and similarly for $c(Z,T)$. Assuming that $g_0(Z,T)$ is nearly constant over the betatron wavelength and the ion oscillation period, Eq. (1) becomes

$$\begin{aligned} (2i\omega_\beta \partial_z g_0 + KT)b &= KTc\,, \\ (-2i\omega_i \partial_T g_0)c &= \omega_i^2 b\,. \end{aligned} \tag{6}$$

Eliminating $c(Z,T)$ yields

$$\left(\partial_z g_0 + \frac{KT}{2i\omega_\beta} \right) \partial_T g_0 = \frac{KT\omega_i}{4\omega_\beta}\,. \tag{7}$$

Provided that $|\partial_Z g_0| \gg KT/2\omega_\beta$, Eq. (7) reduces to

$$(\partial_Z g_0)\partial_T g_0 = \frac{KT\omega_i}{4\omega_\beta}\,. \tag{8}$$

We look for a solution of the form $g_0(Z,T) = \alpha(Z)\beta(T) + \text{const}$. Substituting into Eq. (8) yields

$$\alpha(Z)\partial_Z \alpha(Z) = \frac{\omega_i KT}{4\omega_\beta \beta(T)\partial_T \beta(T)} = k\,, \tag{9}$$

where k is also a constant. Equation (9) can be solved for $k = \frac{1}{2}$ to yield $\alpha(Z) = \pm Z^{1/2}$ and $\beta(T) = \pm (K\omega_i/2\omega_\beta)^{1/2} T$, so that we have solutions with $g_0(0,0) = 0$ given by

$$g_0(Z,T) = \pm \sqrt{\frac{K\omega_i}{2\omega_\beta}}\, Z^{1/2} T\,. \tag{10}$$

The positive solution, previously obtained in Ref. [2], describes growth. Equation (10) is valid for $|\partial_Z g_0| \gg KT/2\omega_\beta$; i.e.,

$$Z \ll \frac{\omega_i \omega_\beta}{2K}\,. \tag{11}$$

Equation (11) may also be written as

$$4\left[\frac{\omega_e^2(T)}{2\omega_\beta} \right] Z \ll \omega_i T\,, \tag{11a}$$

suggesting that the range of validity of Eq. (10) is limited by BNS damping, i.e., the dephasing between the ion and electron oscillations when the betatron frequency shift $\omega_e^2(T)/2\omega_\beta$ increases toward the tail of the electron beam.

Our derivation was also based on the assumption that g_0 is nearly constant over a betatron wavelength and ion oscillation period, i.e., $|\partial_Z g_0| \ll \omega_\beta$ and $|\partial_T g_0| \ll \omega_i$. This further limits the validity of Eq. (10) to

$$\left(\frac{KT}{4\omega_\beta^2} \right)^2 Z_0 \ll Z \ll \frac{Z_0}{4}\,, \tag{12}$$

where

$$Z_0 \equiv \frac{2\omega_i \omega_\beta}{K}\,. \tag{13}$$

At the lower limit of this validity range, $\partial_Z g_0 \sim \omega_\beta$, indicating a growth rate on the order of the betatron frequency. Because of the resonant interaction with a single ion bounce frequency, a betatron damping rate on the order of the betatron frequency would be needed to prevent growth of the disturbance throughout the above range in Z.

For $|\partial_Z g_0| \ll KT/2\omega_\beta$, Eq. (7) becomes

$$\left(\frac{KT}{2i\omega_\beta} \right) \partial_T g_0 = \frac{KT\omega_i}{4\omega_\beta}\,, \tag{14}$$

with solution $g_0(Z,T) = (i\omega_i T/2) + f(Z)$, which does not obey the assumption that $g_0(Z,T)$ is nearly constant over the time scale of ion oscillations. Thus, we have no solution for large values of Z.

A more complete result may be obtained if Eq. (5) is replaced with the ansatz

$$b(Z,T) = b_0 \exp[g_0(Z,T)] \times \exp\left[i\left(\omega_\beta + \frac{\omega_e^2(T)}{2\omega_\beta} \right) Z - i\omega_i T \right]. \tag{15}$$

For $g_0(Z,T) = \text{const}$, the betatron motion described by Eq. (15) includes the incoherent betatron frequency shift

$\omega_e^2(T)/2\omega_\beta$ that results from stationary ions. The oscillation frequency in the laboratory depends upon Z, and is given by

$$\tilde{\omega}(Z) \equiv \omega_i - \frac{KZ}{2\omega_\beta}, \tag{16}$$

which equals the ion bounce frequency for $Z = 0$, and equals 0 for $Z = Z_0$. Provided that $g_0(Z,T)$ is nearly constant over a betatron wavelength and ion oscillation period, Eq. (1) becomes

$$\begin{aligned}(2i\omega_\beta\partial_Z g_0)b &= KTc\,,\\ [-2i\tilde{\omega}(Z)\partial_T g_0 - \tilde{\omega}^2(Z) + \omega_i^2]c &= \omega_i^2 b\,.\end{aligned} \tag{17}$$

Eliminating $c(Z,T)$ yields

$$(2i\omega_\beta\partial_Z g_0)[-2i\tilde{\omega}(Z)\partial_T g_0 - \tilde{\omega}^2(Z) + \omega_i^2] = KT\omega_i^2\,. \tag{18}$$

When $|\partial_T g_0| \gg |[\omega_i^2 - \tilde{\omega}^2(Z)]/[2\tilde{\omega}(Z)]|$ and $\tilde{\omega}(Z) \approx \omega_i$, Eq. (18) reduces to Eq. (8), so that $g_0(Z,T)$ is again given by

$$g_0(Z,T) = \pm\sqrt{\frac{K\omega_i}{2\omega_\beta}}\,Z^{1/2}T\,, \tag{19}$$

valid in this case for

$$\left(\frac{KT}{4\omega_\beta^2}\right)^2 Z_0 \ll Z \ll Z_0 \equiv \frac{2\omega_i\omega_\beta}{K}\,. \tag{20}$$

For large values of Z where $|\partial_T g_0| \ll |[\omega_i^2 - \tilde{\omega}^2(Z)]/2\tilde{\omega}(Z)|$, Eq. (18) yields

$$\partial_Z g_0 = \frac{2i\omega_i^2\omega_\beta T}{KZ^2}\,, \tag{21}$$

with solution

$$g_0(Z,T) = \frac{-2i\omega_i^2\omega_\beta T}{KZ} + f(T)\,, \tag{22}$$

valid for slowly varying $f(T)$ and

$$Z \gg 2Z_0 = \frac{4\omega_i\omega_\beta}{K}\,. \tag{23}$$

For large Z, $|\exp[g_0(Z,T)]|$ is unchanged with increasing Z, indicating that the disturbance has saturated. For $Z = Z_0 = 2\omega_i\omega_\beta/K$, a rough estimate of $g_0(Z,T)$ may be obtained from Eq. (19), which gives $g_0(Z,T) = \omega_i T$.

Thus, a disturbance in which ions initially oscillate at the ion bounce frequency grows with increasing propagation time Z until Z reaches $Z_0 = 2\omega_i\omega_\beta/K$, at which point the disturbance has been amplified by a factor $\sim\exp(\omega_i T)$. For much larger Z, the disturbance has saturated from BNS damping and an approximately constant level of disturbance is maintained.

The saturation of the fast beam-ion instability from BNS damping when $Z \approx 2\omega_i\omega_\beta/K$, with a total amplification of the disturbance by the factor $\sim\exp(\omega_i T)$, is confirmed by several examples of growing disturbances that have been previously calculated (see Figs. 3, 4, 6, and 7 of Ref. [8] and Ref. [9]). For large oscillation amplitudes comparable to the beam height, nonlinear saturation may also limit the instability growth [3,4,6,9].

III. A SPREAD IN ION BOUNCE FREQUENCIES

Let us now consider the case where there is a spread in ion bounce frequencies, for an ion density that grows linearly with the passage of the bunch train. In a spread-frequency model [10,16], ions with the bounce frequency $\omega_{ij} > 0$ may be described by an ion channel with vertical position $c_j(Z,T)$. Letting $c(Z,T)$ denote the average vertical postion of all ions, we have

$$\begin{aligned}\left[\frac{\partial^2}{\partial Z^2} + \omega_\beta^2 + \omega_e^2(T)\right]b(Z,T) &= \omega_e^2(T)c(Z,T)\,,\\ \left[\frac{\partial^2}{\partial T^2} + \omega_{ij}^2\right]c_j(Z,T) &= \omega_{ij}^2 b(Z,T)\,.\end{aligned} \tag{24}$$

As in Eq. (1), we have neglected the ion damping term $(1/T)\partial_T c_j(Z,T)$ in the left-hand side of the lower equation in Eq. (24), which results from an ion density that increases linearly in T, described by Eq. (4). We consider a disturbance of the form

$$b(Z,T) = b_0\exp[g(Z,T)] \times \exp\left[i\left(\omega_\beta + \frac{\omega_e^2(T)}{2\omega_\beta}\right)Z - i\omega_i T\right], \tag{25}$$

where ω_i is a typical ion bounce frequency. We obtain a result similar to Eq. (17)

$$\begin{aligned}(2i\omega_\beta\partial_Z g)b &= KTc\,,\\ [-2i\tilde{\omega}(Z)\partial_T g - \tilde{\omega}^2(Z) + \omega_{ij}^2]c_j &= \omega_{ij}^2 b\,.\end{aligned} \tag{26}$$

in which $\tilde{\omega}(Z)$ is again given by Eq. (16).

For a distribution over positive ion bounce angular frequencies $f_i(\eta)$, normalized so that its integral is 1, the second line of Eq. (26) becomes

$$c = b\int_0^\infty \frac{\eta^2 f_i(\eta)\,d\eta}{-2i\tilde{\omega}(Z)\partial_T g - \tilde{\omega}^2(Z) + \eta^2}\,, \tag{27}$$

so that eliminating $c(Z,T)$ from Eq. (26) yields

$$\partial_Z g = \frac{KT}{2i\omega_\beta}\int_0^\infty \frac{\eta^2 f_i(\eta)\,d\eta}{-2i\tilde{\omega}(Z)\partial_T g - \tilde{\omega}^2(Z) + \eta^2}\,. \tag{28}$$

For a sufficiently broad distribution of ion bounce frequencies, we consider the weak growth limit $[\mathrm{Re}(\partial_T g) \to 0]$ given by the Plemelj formula [17] for $2\,\mathrm{sgn}[\tilde{\omega}(Z)]\,\mathrm{Im}(\partial_T g) \leq |\tilde{\omega}(Z)|$,

$$\partial_Z g = \frac{KT}{2i\omega_\beta}\left[\mathrm{P}\int_0^\infty d\eta\,\frac{f_i(\eta)\eta^2}{\eta^2 - \hat{\omega}^2(Z,T)} + i\,\frac{\pi}{2}\,\mathrm{sgn}[\tilde{\omega}(Z)\,\mathrm{Re}(\partial_T g)]\,|\hat{\omega}(Z,T)|\,f_i(|\hat{\omega}(Z,T)|)\right], \tag{29}$$

(where P denotes principal value), where

$$\hat{\omega}(Z,T) \equiv [\tilde{\omega}^2(Z) - 2\tilde{\omega}(Z)\,\mathrm{Im}\partial_T g(Z,T)]^{1/2}. \tag{30}$$

When $2|\mathrm{Im}(\partial_T g)| \ll |\tilde{\omega}(Z)|$, we have $\hat{\omega}(Z,T) \approx |\tilde{\omega}(Z)|$, and Eq. (29) yields

$$\partial_Z g = \frac{KT}{2i\omega_\beta}\left[\mathrm{P}\int_0^\infty d\eta\,\frac{f_i(\eta)\eta^2}{\eta^2 - \tilde{\omega}^2(Z)} + i\,\frac{\pi}{2}\,\mathrm{sgn}[\mathrm{Re}(\partial_T g)]\tilde{\omega}(Z)f_i(|\tilde{\omega}(Z)|)\right]. \tag{31}$$

Equation (31) also approximates Eq. (28) when $\tilde{\omega}(Z) \approx 0$, i.e., for $Z \approx Z_0$.

For $Z \approx 0$, $\tilde{\omega}(Z) \approx \omega_i$, so that

$$\partial_Z g = \frac{KT}{2i\omega_\beta}\left[\mathrm{P}\int_0^\infty d\eta\,\frac{f_i(\eta)\eta^2}{\eta^2 - \omega_i^2} + i\,\frac{\pi}{2}\,\mathrm{sgn}[\mathrm{Re}(\partial_T g)]\omega_i f_i(\omega_i)\right], \tag{32}$$

which integrates to give solutions with $g(0,0) = 0$,

$$g(Z,T) = \frac{KTZ}{2i\omega_\beta}\left[\mathrm{P}\int_0^\infty d\eta\,\frac{f_i(\eta)\eta^2}{\eta^2 - \omega_i^2} \pm i\,\frac{\pi}{2}\,\omega_i f_i(\omega_i)\right]. \tag{33}$$

For a normal distribution of ion bounce frequencies with mean ω_i and standard deviation σ_ω, $(\pi/2)\omega_i f_i(\omega_i) = (\pi/8)^{1/2}\omega_i/\sigma_\omega$. In this case, the assumptions $2|\mathrm{Im}(\partial_T g)| \ll |\tilde{\omega}(Z)|$ and $f_i(\tilde{\omega}(Z)) \approx f_i(\omega_i)$ are obeyed for $Z \ll (\sigma_\omega/\omega_i)Z_0$. Thus, we have solutions for $Z \ll (\sigma_\omega/\omega_i)Z_0$, one of which displays exponential growth in both Z and T. The instability growth is given by

$$\exp[\mathrm{Re}(g(Z,T))] = \exp\!\left(\frac{KT}{2\omega_\beta}\sqrt{\frac{\pi}{8}}\,\frac{\omega_i}{\sigma_\omega}\,Z\right) = \exp\!\left(\frac{\omega_e^2(T)}{2\omega_\beta}\sqrt{\frac{\pi}{8}}\,\frac{\omega_i}{\sigma_\omega}\,Z\right), \tag{34}$$

where $\omega_e^2(T)/2\omega_\beta$ is the incoherent betatron frequency shift resulting from the ions. For a "broad" distribution of ion bounce frequencies with standard deviation comparable to the mean, $(\pi/8)^{1/2}\omega_i/\sigma_\omega$ is approximately equal to 1, and the growth rate in Z approximately equals the incoherent betatron frequency shift due to ions. A growth rate comparable to the incoherent betatron frequency shift has been previously estimated for a large spread in ion bounce frequencies [18].

Because we neglected the ion damping term $(1/T)\partial_T c_j(Z,T)$ in the lower equation in Eq. (24), which describes an instantaneous ion damping rate $\sim 1/T$, the unstable solution of Eqs. (32)–(34) is valid only where the growth rate in T exceeds $\sim 1/T$; i.e., $(KZ/2\omega_\beta)[(\pi/8)^{1/2}\omega_i/\sigma_\omega] \gg 1/T$. This criterion may be written as $Z \gg Z_0/[(\omega_i T)(\pi/8)^{1/2}\omega_i/\sigma_\omega]$; combined with the requirement $Z \ll (\sigma_\omega/\omega_i)Z_0$, we have the region of validity for exponential growth in Z described by Eqs. (32)–(34),

$$\frac{1}{\omega_i T}\,\frac{\sigma_\omega}{\omega_i}\,Z_0 \ll Z \ll \frac{\sigma_\omega}{\omega_i}\,Z_0\,. \tag{35}$$

For $Z \approx (\sigma_\omega/\omega_i)Z_0$, Eq. (34) gives an approximate value for the amplification of the disturbance during the exponential growth regime of $\exp[\mathrm{Re}(g(Z,T))] \sim \exp(\omega_i T)$. The approximate amplification is independent of the ion frequency spread σ_ω.

For $Z \approx 2Z_0$, $\tilde{\omega}(Z) \approx -\omega_i$. For an unstable disturbance that has grown in both Z and T for $Z < Z_0$, $\mathrm{Re}(\partial_T g) > 0$. Equation (31) gives, for $Z \approx 2Z_0$,

$$\mathrm{Re}(\partial_Z g) = -\frac{KT}{2\omega_\beta}\left[\frac{\pi}{2}\,\omega_i f_i(\omega_i)\right] < 0\,, \tag{36}$$

indicating that the magnitude of the disturbance decreases with increasing Z for $Z \approx 2Z_0$. This consequence of BNS damping when $Z \sim 2Z_0$ has been previously calculated for the case of a single ion bounce frequency (see Figs. 4, 6, and 7 of Ref. [8] and Ref. [9]).

For $Z \gg 2Z_0$, $\tilde{\omega}(Z) \approx -KZ/2\omega_\beta$ obeys $|\tilde{\omega}(Z)| \gg \omega_i$. When Z is sufficiently large that there are no ions with bounce frequency $|\tilde{\omega}(Z)|$ so that $f_i(|\tilde{\omega}(Z)|) = 0$, Eq. (31) gives

$$\partial_Z g \approx \frac{KT}{2i\omega_\beta}\left[\frac{\omega_i^2}{-(\frac{KZ}{2\omega_\beta})^2}\right], \tag{37}$$

with solution

$$g(Z,T) \approx \frac{-2i\omega_i^2\omega_\beta T}{KZ} + f(T)\,, \tag{38}$$

in which $f(T)$ is a slowly varying function of T. The assumed inequality $2|\mathrm{Im}(\partial_T g)| \ll |\tilde{\omega}(Z)|$ is obeyed, so we have a valid solution for $Z \gg 2Z_0$. This solution, whose magnitude remains constant with increasing Z, is the same as in the case without a spread in ion bounce frequencies, obtained in Eq. (22).

For a broad spread in ion frequencies with standard deviation comparable to the mean, the instability grows exponentially in Z for $Z \ll Z_0$, and the growth rate in Z at time T behind the head of the bunch train is approximately $\omega_e^2(T)/2\omega_\beta$, the ion-induced incoherent betatron frequency shift. In this case, instability growth may be prevented for all of the bunch train ($0 < T < \tau_b$) when the betatron damping rate exceeds the incoherent betatron frequency shift induced by ions at the tail of the bunch train.

As was the case with no spread in ion bounce frequencies, a disturbance in which ions initially oscillate at a typical ion bounce frequency ω_i grows with increasing

Z until Z reaches $Z_0 = 2\omega_i\omega_\beta/K$, at which point the disturbance has been amplified by a factor $\sim\exp(\omega_i T)$. For $Z \sim 2Z_0$, the disturbance decreases with increasing Z because of BNS damping. For larger Z, the disturbance has saturated and an approximately constant level of disturbance is maintained. For large oscillation amplitudes comparable to the beam height, nonlinear saturation may also limit the instability growth [3,4,6,9].

When the instability "rise time" Z_{rise} is defined as the propagation time required for an initial disturbance to grow by a factor of e at the tail of the bunch train, Eq. (10) yields $Z_{\mathrm{rise}} = 2\omega_\beta/(K\omega_i\tau_b^2)$ for a single ion bounce frequency [2], valid for $\omega_i\tau_b \gg 1$. For a sufficiently large spread in ion bounce frequencies so that the weak growth assumption of Eq. (29) applies (e.g., $\sigma_\omega/\omega_i \sim 1$), Eq. (34) yields $Z_{\mathrm{rise}} = (2\omega_\beta/K\tau_b)[(8/\pi)^{1/2}\sigma_\omega/\omega_i]$. The rise time is increased by a factor of $(8/\pi)^{1/2}\sigma_\omega\tau_b$ by the spread in ion bounce frequencies.

For a broad distribution of ion bounce frequencies with standard deviation comparable to the mean, the rise time is increased by a factor of $\omega_i\tau_b \gg 1$. Accordingly, the rate of betatron damping (or feedback) required to prevent instability is greatly reduced by the ion frequency spread.

IV. SUMMARY

The fast beam-ion instability has been considered for a single ion bounce frequency and for a spread in ion bounce frequencies. In both cases, a disturbance in which ions initially oscillate at a typical ion bounce frequency ω_i grows with increasing propagation time Z until Z reaches $\sim Z_0 \equiv 2\omega_i\omega_\beta/K$, at which point the disturbance has been amplified by a factor $\sim\exp(\omega_i T)$. For $Z \sim 2Z_0$, BNS damping may cause the magnitude of the disturbance to decrease with increasing Z. For larger Z, the disturbance has saturated and an approximately constant level of disturbance is maintained.

For a broad distribution of ion bounce frequencies whose standard deviation is comparable to the mean, the instability grows exponentially in propagation time Z for $Z \ll Z_0$; therefore, it grows exponentially with the propagation distance $z = \nu Z$ of the bunch train. The growth rate in Z at time T behind the head of the bunch train is approximately $\omega_e^2(T)/(2\omega_\beta)$, the ion-induced incoherent betatron frequency shift. Instability growth may be prevented for all of the bunch train ($0 < T < \tau_b$) when the rate of betatron damping (or feedback) exceeds the incoherent betatron frequency shift induced by ions at the tail of the bunch train.

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