Single bunch stability to monopole excitation

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(Received 18 January 1999; published 14 April 1999)

We study single bunch stability with respect to monopole longitudinal oscillations in electron storage rings. Our analysis is different from the standard approach based on the linearized Vlasov equation. Rather, we reduce the full nonlinear Fokker-Planck equation to a Schrödinger-like equation which is subsequently analyzed by perturbation theory. We show that the Haissinski solution [Nuovo Cimento Soc. Ital. Fis. **18B**, 72 (1973)] may become unstable with respect to monopole oscillations and derive a stability criterion in terms of the ring impedance. We then discuss this criterion and apply it to a broadband resonator impedance model. [S1098-4402(99)00034-8]

PACS numbers: 29.20.Dh, 29.27.–a

I. INTRODUCTION

Single bunch longitudinal instability is one of the factors limiting the performance of electron storage rings. Theoretical analysis of this instability is usually based on the Fokker-Planck equation for the particle distribution function. This equation includes the effects of both dynamic (Hamiltonian) and stochastic forces. The Hamiltonian part describes the synchrotron motion while radiation terms account for the (much slower) effects of the synchrotron radiation and define the rms beam size at low intensity. The steady state solution of the Fokker-Planck equation was first obtained in 1973 by Haissinski [1] and since then it was confirmed in numerous experiments held below the instability threshold. Unfortunately, apart from a few limiting cases, finding other possible solutions of the Fokker-Planck equation that could account for the instability turned out to be quite difficult. This is why much of the analysis to explain the instability was done utilizing the linearized Vlasov equation technique, where the Fokker-Planck equation was linearized with respect to the Haissinski solution. In this approach the Haissinski solution is also used to introduce the action-angle variables that make the Haissinski Hamiltonian independent of angle, which results in great simplification of further analysis.

The linearized Vlasov equation technique naturally leads to the concept of azimuthal phase space modes that are basically the components of the perturbation to the Haissinski solution with certain azimuthal symmetry. The first three of such modes are sketched in Fig. 1. Note that in this figure and throughout the rest of the paper we assume the simplest phase space topology, where action-angle variables can be defined uniformly across the whole plane. In other words, we neglect the possibility of several potential well minima.

As seen from Fig. 1, the monopole mode is quite special because, in contrast to other modes, its physical space projection does not change significantly on the time scale of a synchrotron period. This argues that radiation rather than Hamiltonian forces define the dynamics of this mode. Also, by definition of action-angle variables, the unperturbed Haissinski solution has monopole azimuthal structure. These two features of the monopole mode explain why it is omitted from the standard linearized Vlasov analysis. Indeed, in that approach the sole effect of radiation terms in the Fokker-Planck equation is that they define the Haissinski solution which subsequently cancels them out, so that only Hamiltonian terms remain in the linearized Vlasov equation. The possibility that a perturbation is monopole, but with radial structure different from the Haissinski solution, is neglected.

In this paper we are exploring the possibility that an instability can be associated with the monopole mode. Rather than extending the linearized Vlasov technique we find it more convenient to transform the Fokker-Planck equation to a Schrödinger-like equation and then analyze the latter using the Haissinski solution as a basis. The advantages of this approach are that it is tractable and it conveniently allows us to use some well-known facts about Schrödinger equation solutions.

The only essential approximation that we make in this paper is that we assume that the monopole mode can be considered separately from other azimuthal modes. This is by no means general. On the contrary, it is known that, for example, some collective instabilities result from azimuthal mode coupling; hence, concentrating on one mode in that case would be inappropriate. However, at lower intensity, when incoherent frequency shifts are small compared to the synchrotron frequency, the cross talk between different azimuthal modes is negligible. Whether a monopole mode or any other single azimuthal mode can become unstable at this low intensity is, in our opinion, a quantitative question that depends on the exact measure of the storage ring impedance. In fact, there are computer simulations for model impedances [2] showing that radial modes that belong to one azimuthal mode become unstable before any significant azimuthal mode coupling occurs. Of course, if a monopole mode does become unstable by itself, its independence from the other modes applies only to the initial stage of instability.

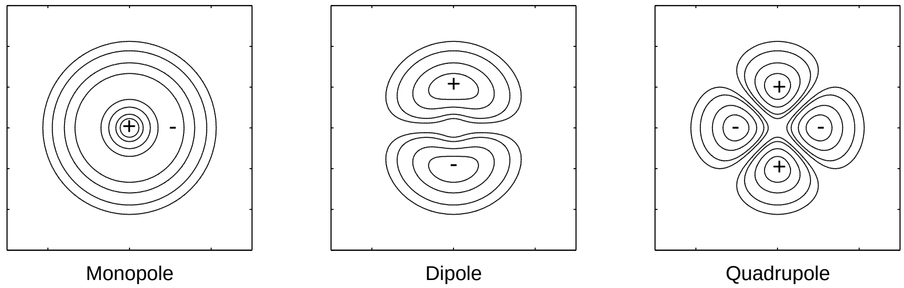

FIG. 1. Example contour plots of the lowest three azimuthal modes.

The nonlinear stage may be rather complex and it falls beyond the scope of this paper.

Another argument to justify an independent consideration of the monopole mode refers to separate time scales in this problem. Specifically, if we assume that the monopole mode instability develops slowly compared to the filamentation time, then the whole beam effectively maintains its initial monopole structure. Quantitatively, this amounts to a requirement that the incoherent frequency spread characteristic of the Haissinski equilibrium be much higher than the radiation damping rate. Such a condition is not unusual for many electron storage rings.

II. NOTATION AND BASIC EQUATIONS

For a relativistic bunch, $\gamma = E/mc^2 \gg 1$, longitudinal dynamics is conveniently described in dimensionless variables

$$x = z/\sigma_0, \qquad p = -\delta/\delta_0, \qquad \tilde{\tau} = \omega_{s0} t\,, \tag{1}$$

where z is the position of a particle with respect to the bunch centroid ($z > 0$ in the head of a bunch), δ is the relative energy spread $\Delta E/E$, and ω_{s0} is the synchrotron frequency. The subscript "0" refers to zero-current equilibrium quantities, related by $\omega_{s0}\sigma_0/c = |\alpha|\delta_0$, where α is the momentum compaction. The Fokker-Planck equation for the distribution function $\rho(x, p, \tilde{\tau})$ in these variables can be written (e.g., [3]) as

$$\frac{\partial \rho}{\partial \tilde{\tau}} + \{H, \rho\}_{p,x} = \frac{\gamma_d}{\omega_{s0}} \frac{\partial}{\partial p}\left(\frac{\partial \rho}{\partial p} + p\rho\right), \tag{2}$$

where $\{\cdots\}$ denotes the Poisson brackets, $H(x, p, \tilde{\tau})$ is the self-consistent Hamiltonian

$$H(x, p, \tilde{\tau}) \equiv \frac{p^2}{2} + \frac{x^2}{2} + \Lambda \int dx'\, dp'\, \rho(x', p', \tilde{\tau}) S(x' - x)\,, \tag{3}$$

and ρ is normalized as $\int dp\, dx\, \rho(x, p, \tilde{\tau}) = 1$. We have neglected the nonlinearities of the rf potential well and defined the parameter Λ as

$$\Lambda \equiv \frac{N r_0}{C \gamma \alpha \delta_0^2}\,, \tag{4}$$

where N is the number of particles in a bunch, r_0 is the classical electron radius, and C is the ring circumference. We have also defined a dimensionless function

$$S(x) \equiv \sigma_0 \int_0^x dx'\, W(\sigma_0 x') \tag{5}$$

in terms of the wakefield $W(z)$ for two particles separated by distance z. Causality was assumed in the form $W(z) = 0$ for $z < 0$.

The Fokker-Planck equation (2) has a steady state Haissinski solution [1]

$$\rho_H(x, p) = Z_H e^{-H_H(x,p)}, \tag{6}$$

where

$$H_H(x, p) = \frac{p^2}{2} + \frac{x^2}{2} + \Lambda \int dx'\, dp'\, \rho_H(x', p') S(x' - x)\,, \tag{7}$$

and Z_H is a normalizing factor. Explicit forms of ρ_H and H_H can be obtained numerically.

Canonical transformation from x, p to action-angle variables J, ϕ can be defined to make the Hamiltonian H_H phase independent, $H_H(x, p) \rightarrow H_H(J)$.

Haissinski particle density ρ_H in these variables depends only on J, and arbitrary distribution function $\rho(J, \phi, \tilde{\tau}) = \rho_H(J) + \delta\rho(J, \phi, \tilde{\tau})$ can be expanded in azimuthal harmonics

$$\rho(J, \phi, \tilde{\tau}) = \rho_H(J) + \sum_{m=-\infty}^{\infty} \delta\rho_m(J, \tilde{\tau}) e^{im\phi}. \tag{8}$$

Similarly,

$$\begin{aligned} H(J, \phi, \tilde{\tau}) &= H_H(x, p) \\ &\quad + \Lambda \int dx'\, dp'\, \delta\rho(x', p', \tilde{\tau}) S(x' - x) \\ &= H_H(J) + \sum_{k=-\infty}^{\infty} \delta H_k(J, \tilde{\tau}) e^{ik\phi}, \end{aligned} \tag{9}$$

where

$$\delta H_k(J, \tilde{\tau}) \equiv \Lambda \sum_m \int dJ'\, S_{k,m}(J, J') \delta\rho_k(J', \tilde{\tau})\,, \tag{10}$$

and

$$S_{k,m}(J,J') \equiv \frac{1}{2\pi}\int d\phi\, d\phi'\, e^{im\phi'-ik\phi} \times S(x(J',\phi') - x(J,\phi)) \,. \qquad (11)$$

In the J, ϕ variables the form of the left-hand side of the Fokker-Planck equation is unchanged, appearing as in Eq. (2). The right-hand side can be obtained using the invariance of the Poisson brackets [4]. Namely, for any $F \equiv F(x,p)$,

$$\frac{\partial F(x,p)}{\partial p} = \{x,F\}_{x,p} = \{x,F\}_{\phi,J} \equiv \frac{\partial}{\partial J}\left(\frac{\partial x}{\partial \phi}F\right) - \frac{\partial}{\partial \phi}\left(\frac{\partial x}{\partial J}F\right). \qquad (12)$$

Hence, for the zeroth Fourier harmonic $\rho_0(J,\tilde{\tau}) \equiv \langle \rho(J,\phi,\tilde{\tau})\rangle$, where $\langle\cdots\rangle$ defines phase averaging, we have

$$\frac{\partial \rho_0}{\partial \tilde{\tau}} + \{H,\rho_0\}_{J,\phi} = \frac{\partial}{\partial J}\left(\frac{\partial x}{\partial \phi}\tilde{F}\right), \qquad (13)$$

where $\tilde{F} \equiv \partial\rho/\partial p + p\rho$. On the other hand, for any Hamiltonian $H = p^2/2 + U(x,\tilde{\tau})$, the canonical momentum can be found as $p = \{x,H\}_{\phi,J}$. Therefore, if we neglect nonzero azimuthal modes by assuming $H = H(J,\tilde{\tau})$, $\rho = \rho_0(J,\tilde{\tau})$, we get

$$\frac{\partial \rho}{\partial \tau} = \frac{\partial}{\partial J}\left\{\left\langle\left(\frac{\partial x}{\partial \phi}\right)^2\right\rangle\left[\frac{\partial \rho}{\partial J} + \omega(J,\tau)\rho\right]\right\}, \qquad (14)$$

where

$$\omega(J,\tau) \equiv \frac{\partial H}{\partial J} = \omega_H(J) + \Lambda\int dJ'\,\delta\rho_0(J',\tau)\,\frac{\partial S_{0,0}}{\partial J} \,. \qquad (15)$$

As discussed earlier, the monopole mode should not change much on the time scale of a synchrotron period. This is why time was renormalized above to the radiation damping constant

$$\tau \equiv \gamma_d t = (\gamma_d/\omega_{s0})\tilde{\tau} \,. \qquad (16)$$

For the x derivative in Eq. (14), we can write

$$\left\langle\left(\frac{\partial x}{\partial \phi}\right)^2\right\rangle = \oint\left(\frac{\dot{x}}{\omega(J,\tau)}\right)^2 d\tau \simeq \frac{1}{\omega(J,\tau)}\oint p\,dx \equiv \frac{J}{\omega(J,\tau)} \simeq \frac{J}{\omega_H(J)}\,, \qquad (17)$$

where integration is performed over one synchrotron period and the last equality assumes small deviation from the Haissinski solution. Finally, introducing the diffusion coefficient as

$$D(J) \equiv \frac{J}{\omega_H(J)}\,, \qquad (18)$$

we can rewrite Eq. (14) as

$$\frac{\partial \rho}{\partial \tau} = \frac{\partial}{\partial J}\left\{D(J)\left[\frac{\partial \rho}{\partial J} + \omega(J,\tau)\rho\right]\right\}. \qquad (19)$$

Note that in our derivation we allow arbitrary time dependence of $H(J,\tau)$.

III. TRANSFORMATION TO A SCHRÖDINGER-LIKE EQUATION

The Fokker-Planck equation (19) has a standard one-dimensional form that permits transformation to a Schrödinger-like equation [5]. Let us introduce a new independent variable

$$y \equiv y(J) = \int_0^J dJ'/\sqrt{D(J')} \qquad (20)$$

and two functions

$$f(y,\tau) \equiv \frac{1}{\sqrt{D(J(y))}}\, e^{\Phi(y,\tau)/2}\rho(J(y),\tau)\,, \qquad (21)$$

$$\Phi(y,\tau) \equiv H(J(y),\tau) - (1/2)\ln D(J(y))\,, \qquad (22)$$

where $J(y)$ on the right-hand side of Eqs. (21) and (22) is given implicitly by Eq. (20). Now the Fokker-Planck equation (19) takes the form

$$\frac{\partial f}{\partial \tau} = \frac{\partial^2 f}{\partial y^2} - U_S(y,\tau)f + \frac{1}{2}\dot{\Phi}(y,\tau)f\,, \qquad (23)$$

where

$$U_S(y,\tau) \equiv [\Phi'(y,\tau)/2]^2 - \Phi''(y,\tau)/2\,, \qquad (24)$$

and the dot and the prime denote partial derivatives with respect to τ and y, respectively. This equation is strongly nonlinear since, according to Eq. (15), Φ is also related to f by

$$\Phi(y,\tau) = \Lambda\int dy'\,S_{0,0}(J,J') \times [e^{-\Phi(y',\tau)/2}f(y',\tau) - e^{-\Phi_H(y')/2}f_H(y')]\,, \qquad (25)$$

where

$$f_H(y) \equiv Z_H e^{-\Phi_H(y)/2},$$

$$\Phi_H(y) \equiv H_H(J(y)) + \frac{1}{2}\ln\left(\frac{\omega_H(J(y))}{J(y)}\right), \qquad (26)$$

$$\int dy e^{-\Phi_H(y)/2}f_H(y) = 1\,.$$

Note that $f_H(y)$ is the steady state solution of Eq. (23) and it corresponds to the Haissinski solution, as can be checked by direct substitution. However, it is not obvious how to proceed to other, time-dependent solutions of Eq. (23). On the other hand, if the $\dot{\Phi}$ term is neglected, the analysis of Eq. (23) can be made by analogy with more familiar quantum mechanical problems. Indeed, without that term, Eq. (23) can be thought of as a Schrödinger equation for a particle in the potential well $U_S(y,\tau)$.[1] Whether the $\dot{\Phi}$ term is negligible for a general case remains unclear. However, since this term is zero for the Haissinski solution, one can safely neglect it for solutions that are close to f_H. This includes, for example, the important case of the early time behavior of a system initialized with the Haissinski distribution at $\tau = 0$. In the next two sections we will follow this approach. Namely, we will first analyze Eq. (23) with the last term neglected, and then account for it by perturbation theory.

IV. SCHRÖDINGER EQUATION ANALYSIS

After neglecting the $\dot{\Phi}$ term, Eq. (23) reads

$$\frac{\partial f}{\partial \tau} = \frac{\partial^2 f}{\partial y^2} - U(y,\tau) f\,. \tag{27}$$

First, we solve a linear problem for which $\omega(J) = \omega^0$ is a constant. In this case, $y = 2\sqrt{\omega^0 J}$ and the Schrödinger potential is simply

$$U_S^0(y) = \frac{y^2}{16} - \frac{1}{2} - \frac{1}{4y^2}\,, \tag{28}$$

which makes Eq. (27) a solvable eigenvalue problem. It is easy to check that the solution is

$$f^0(y,\tau) = \sum_{m=0}^{\infty} \psi_m^0(y) e^{-\lambda_m^0 \tau}, \tag{29}$$

where

$$\lambda_m^0 = m, \qquad m = 0,1,2,\ldots, \tag{30}$$

$$\psi_m^0(y) = (y/2)^{1/2} e^{-y^2/8} L_m(y^2/4)\,, \tag{31}$$

and L_m denotes the Laguerre polynomial of order m. As expected, the linear problem does not have any unstable solutions. Any initial distribution exponentially approaches the Haissinski solution $\psi_0^0(y)$ on the time scale defined by radiation damping.

For the general case, $\omega(J) \neq$ const, asymptotic behavior of the solution of Eq. (23) is described by the solutions to the linear problem Eqs. (28) and (31). Namely, $f(y)$ scales as $\sqrt{y}$ at small y and goes to 0 as $P(y)e^{-y^2/8}$ at large y, where $P(y)$ is a polynomial. Similarly, $U(y)$ is quadratic at large y and has a $-1/4y^2$ singularity as y approaches 0.

Suppose that a bunch is described by the Haissinski distribution at $\tau = 0$. By the foregoing arguments, the behavior of this system for small τ can be obtained from Eq. (27). In this case,

$$f(y,\tau) = \sum_m \psi_m(y) e^{-\lambda_m \tau}, \tag{32}$$

where $\psi_m(y)$ are the eigenfunctions of the equation

$$\frac{\partial^2 \psi_m}{\partial y^2} - U_S(y,\tau)\psi_m = -\lambda_m \psi_m\,. \tag{33}$$

Therefore, stability of the initial state depends on whether Eq. (33) has at least one negative eigenvalue $\lambda_m < 0$. Let us look at the possibility that such a negative eigenvalue exists.

First, the eigenfunctions with the asymptotic behavior described above are orthogonal and they can be normalized by

$$\int \psi_n(y)\psi_m(y)\,dy = \delta_{n,m}\,. \tag{34}$$

Hence, an eigenvalue of Eq. (3) is given by

$$\lambda_n = -\int dy\, \psi_n \left[\frac{\partial^2}{\partial y^2} - U_S(y,\tau)\right]\psi_n\,. \tag{35}$$

In spite of a singularity of $U_S(y,\tau)$ at $y = 0$, all the eigenvalues of Eq. (33) are bounded from below. Indeed, at small distances where $\psi_n \propto \sqrt{y}$, the second derivative in Eq. (35) gives a $1/4y^2$ term which cancels a similar term in $U_S(y)$. In fact, it can be shown that all the eigenvalues are higher than the average of the "effective potential" $V_S(y) \equiv U_S(y) + 1/4y^2$.

As an example, we consider a broadband resonator impedance model with shunt impedance R_s, resonance frequency ω_R, and quality factor Q. The function $S(z)$ in this case is (e.g., [6])

$$S(x) = \frac{I}{\Lambda Q \zeta} \sin(\sigma \zeta x) e^{-\sigma x/2Q}, \tag{36}$$

where we defined

$$I \equiv 4\pi \Lambda R_s / Z_0\,, \tag{37}$$

$$\sigma \equiv \omega_R \sigma_0 / c\,, \tag{38}$$

$$\zeta \equiv \sqrt{1 - 1/(2Q)^2}\,, \tag{39}$$

and Z_0 is the impedance of free space. The frequency $\omega_H(J)$ and the effective potential $V_S(y)$ for the parameters $Q = 1$ and $\sigma = 3$ are shown in Fig. 2 for two values of intensity $I = 1$ and $I = -1$. Negative intensity corresponds to negative momentum compaction, and the importance of this case will be discussed later. For these parameters, the effective potential indeed has a minimum where $V_S(y) < 0$. At first sight, we could expect a mode trapped near the bottom of the potential well with the eigenvalue negative at large current. This, however, is not true since $\psi_0 = f_H$ is the solution of Eq. (33) with $\lambda_0 = 0$. Because $f_H(y)$ does not have zeros, this solution has the lowest eigenvalue and the rest of λ_m is positive.

[1]Of course, the similarity is quite formal. The problem is purely classical, so no $\hbar$ appears anywhere. Also, there is no imaginary constant in front of the time derivative.

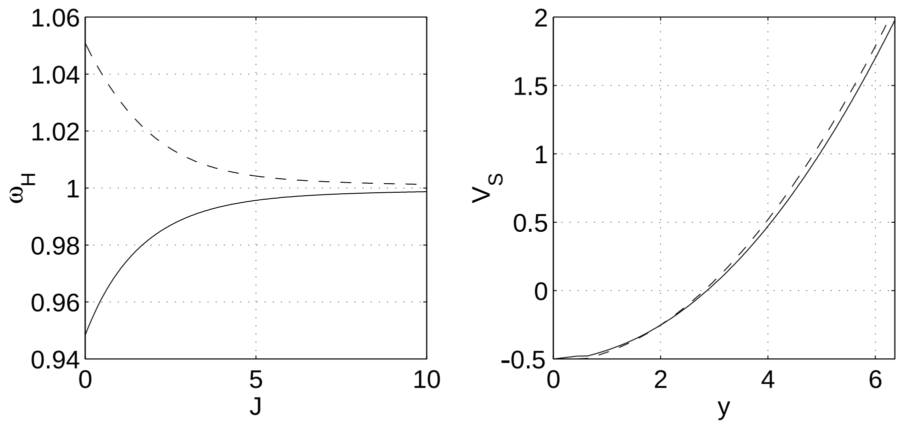

FIG. 2. The incoherent frequency $\omega_H(J)$ and the corresponding effective potential $V_S(y)$ (right) shown for broadband $Q = 1$ resonator impedance with $\sigma = 3$ and two values of intensity $I = 1$ and $I = -1$ (dashed line).

Therefore, in this approximation, the Haissinski solution is stable. How much this conclusion depends on the assumption that the $\dot{\Phi}$ term in Eq. (23) is negligible can be analyzed by perturbation technique.

V. PERTURBATION THEORY

Equation (23), and the condition of self-consistency [Eq. (25)], is a strongly nonlinear system of equations. We want to analyze it with perturbation theory, assuming small deviation from the Haissinski solution. Let us introduce perturbations as

$$\begin{aligned} v(y,\tau) &\equiv H(J(y),\tau) - H_H(J(y)), \\ \psi(y,\tau) &\equiv f(y,\tau) - f_H(y,\tau). \end{aligned} \tag{40}$$

This gives $\Phi = \Phi_H(y) + v(y,\tau)$, and

$$v(y,\tau) = \Lambda \int dy'\, S_{0,0}(J(y),J(y'))e^{-\Phi_H(y')/2} \times [\psi(y',\tau) - (1/2)v(y',\tau)f_H(y')]. \tag{41}$$

The perturbation $\psi(y,\tau)$ is normalized by the condition

$$\int dy\, \psi(y,\tau)e^{-\Phi_H(y)/2} = \frac{Z_H}{2}\int dy\, v(y,\tau)e^{-\Phi_H(y)}, \tag{42}$$

and satisfies the equation

$$\frac{\partial\psi}{\partial\tau} = \frac{\partial^2\psi}{\partial y^2} - U_H(y)\psi + \frac{1}{2}f_H[\dot{v} + v'' - \Phi_H' v']. \tag{43}$$

Let us expand ψ and v in series over orthogonal eigenfunctions ψ_n,

$$\begin{aligned} \psi(y,\tau) &= \sum_{m=0}^{\infty} C_n(\tau)\psi_n, \\ v(y,\tau) &= \frac{2}{Z_H}\sum_{m=0}^{\infty} D_n(\tau)e^{\Phi_H(y)/2}\psi_n. \end{aligned} \tag{44}$$

We assume that eigenfunctions ψ_n satisfy the equation

$$\psi_n''(y) - U_H(y)\psi_n(y) = -\lambda_n\psi_n(y). \tag{45}$$

The above expansion includes the eigenfunction $\psi_0(y) = \sqrt{Z_H}\, e^{-\Phi_H(y)/2}$, with $\lambda_0 = 0$. Note that, by the same argument as above, the remaining eigenvalues are positive.

The linearized Fokker-Planck equation (43) and the condition of self-consistency (41) lead to the following system:

$$\dot{C}_n = \dot{D}_n - \lambda_n(C_n + D_n), \tag{46}$$

$$D_n = \Lambda\sum_k \kappa_{n,k}(C_k - D_k), \tag{47}$$

where

$$\kappa_{n,k} \equiv \int dy\, \psi_n(y)e^{-\Phi_H(y)/2} \times \int dy'\, \psi_k(y')e^{-\Phi_H(y')/2}S_{0,0}(J(y),J(y')). \tag{48}$$

Because of orthogonality, the normalization condition (42) gives just

$$C_0(\tau) = D_0(\tau). \tag{49}$$

Since $\lambda_0 = 0$, this is automatically satisfied by Eq. (46) with the initial condition $C_0(0) = D_0(0)$.

Looking for exponentially varying solutions, $C_n \equiv a_n e^{\mu\tau}, D_n \equiv b_n e^{\mu\tau}$, we transform the system of Eqs. (46) and (47) to the matrix equation

$$b_n = -2\Lambda\sum_k \kappa_{n,k}\frac{b_k\lambda_k}{\mu + \lambda_k}. \tag{50}$$

The solution of this equation is given by the roots of the determinant for the matrix

$$M \equiv \delta_{n,k} + 2\Lambda\frac{\kappa_{n,k}\lambda_k}{\mu + \lambda_k}. \tag{51}$$

Positive roots $\mu > 0$ would mean instability to monopole excitation of a bunch.

Since the matrix M is infinite, it is unclear how to find its determinant in the general case. However, it is easy to see that off diagonal terms of $\kappa_{n,k}$ are small, while the diagonal terms quickly converge to zero. This is why we expect that a good approximation for the roots μ can be found by truncating the matrix M. If we truncate matrix M to the lowest nontrivial rank

$$M^{(2)} = \begin{pmatrix} 1 & 2\Lambda\kappa_{1,1}\frac{\lambda_1}{\mu+\lambda_1} \\ 0 & 1 + 2\Lambda\kappa_{1,1}\frac{\lambda_1}{\mu+\lambda_1} \end{pmatrix}, \tag{52}$$

then a zero determinant occurs for

$$\mu = -\lambda_1(1 + 2\Lambda\kappa_{1,1}). \tag{53}$$

Because $\lambda_1 > 0$, this root is positive when

$$2\Lambda\kappa_{1,1} < -1, \tag{54}$$

and this may be viewed as the criterion for the onset of the monopole instability.

It is easy to see, however, that usually $\kappa_{1,1} > 0$. Indeed, let us introduce the impedance $Z(\nu)$ so that

$$W(z) = \int \frac{d\nu}{2\pi} Z(\nu) e^{-i\nu z/c}. \tag{55}$$

Then

$$\kappa_{n,k} = i\,\frac{2\sigma_0}{Z_0}\int \frac{d\nu}{2\pi}\,\frac{Z(\nu)}{\nu}\,F_n(\nu)F_k^*(\nu), \tag{56}$$

where

$$F_n(\nu) \equiv \int dJ \left(\frac{\omega_H(J)}{J}\right)^{1/4} e^{-H_H(J)/2}\psi_n(y(J)) \times \int d\phi\, e^{i\nu\sigma_0 x(J,\phi)/c}. \tag{57}$$

Now $\kappa_{1,1}$ depends on $|F_1(\nu)|^2$ which is a positive and even function of ν. Hence, $\kappa_{1,1}$ is given by the odd part of the impedance, $\mathrm{Im}\,Z(\nu)$, which is negative for inductive impedance and positive for capacitive impedance. As a result, for the most common case of inductive impedance and positive momentum compaction, $\Lambda\kappa_{1,1} > 0$ and the Haissinski solution is stable.

However, the situation is not that simple for negative momentum compaction or in the case of capacitive impedance. Each of these conditions has been proposed by various authors to get shorter bunches and also as a remedy against longitudinal instabilities. For illustration, we continue our example of the broadband resonator model for $Q = 1$. Using Eqs. (56) and (57), we numerically compute the quantity $2\Lambda\kappa_{1,1}$ as a function of σ at intensity $I = -1$. The result, together with the threshold value given by Eq. (54), is plotted in Fig. 3. It shows that a bunch is monopole unstable at this intensity provided its zero current length exceeds about one-twelfth of the resonator wavelength. Note that, according to Fig. 2, this intensity is not high at all since, for example, for $\sigma = 3$, it leads only to about a 5% increase in the incoherent frequency spread.

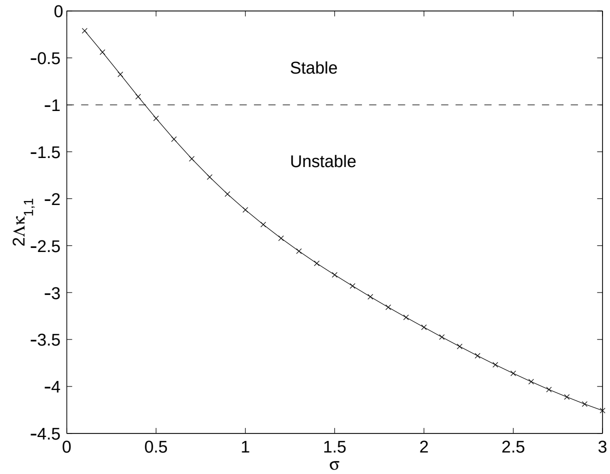

FIG. 3. Illustration for the monopole instability criterion [Eq. (54)] for broadband $(Q = 1)$ resonator impedance for negative momentum compaction and intensity $I = -1$. Numerical calculation was done using Eqs. (56) and (57), where $\omega(J)$ was assumed constant and ψ_1 was taken from the solution to the linear problem (31).

VI. DISCUSSION

We have investigated single bunch stability with respect to longitudinal monopole oscillations. These oscillations may become unstable as a result of an imbalance between radiation excitation and damping. Since this phenomenon falls beyond the coverage of the linearized Vlasov approach, we chose to employ a different technique that has not been used for instability analysis. This technique involves the transformation of the phase-averaged Fokker-Planck equation to a Schrödinger equation with an additional term arising from the self-consistent potential. This Schrödinger equation is analyzed similar to quantum mechanics and the effects of the additional term are found by perturbation analysis.

Utilizing this technique we have obtained a simple criterion, Eq. (54), for the onset of monopole instability. According to this criterion, monopole mode instability does not appear in the most common case of storage ring operation with positive momentum compaction when the impedance is largely inductive. However, for the case of negative momentum compaction, $\alpha < 0$, as we have illustrated in Fig. 3, bunches may become monopole unstable at modest intensity. We expect a similar behavior for the somewhat rare case of predominantly capacitive impedance and $\alpha > 0$.

As discussed in the Introduction, the essential assumption we make in our analysis is that the monopole mode can be considered separately from the rest of the azimuthal modes. This assumes the absence of azimuthal mode coupling which implies some limitations on the growth rate of the instability. It also assumes that the other azimuthal modes are stable by themselves. It is interesting that, since the monopole instability criterion Eq. (54) effectively includes only the imaginary part of impedance, the second assumption is rather relaxed. Indeed, as follows from the linearized Vlasov analysis, the azimuthal modes, other than monopole, become unstable due to the asymmetry in the Haissinski potential that comes from the real part of impedance. (We omit a somewhat exotic possibility of multiple minima in the Haissinski potential.) Therefore, monopole mode instability can exist when the remaining azimuthal modes are stable.

It is conceivable that the monopole instability could be one of the factors that prevents high current operation of storage rings with negative momentum compaction. Many attempts of such operation have been tried to shorten a bunch and to avoid various instabilities (e.g., [7,8]), often the so-called microwave instability (e.g., [6]). Unfortunately, since usually only the static bunch shape and/or the energy spread measurements are reported, it is hard to infer what particular instability was the limitation. However, in some cases, it appears that there is something other than the microwave instability, because the threshold increase predicted for this instability (e.g., [9]) is not observed. It would be nice to find a concrete evidence of monopole instability in either future experiments or in the log books from past experiments. Such evidence might include, for example, growth of the longitudinal beam size, in the absence of synchrotron sidebands to the revolution harmonics of a beam position monitor signal.

Finally, we hope that the technique described in this paper can be applied to other problems in accelerator physics that lead to the one-dimensional Fokker-Planck equation. This includes, for example, microwave instability, beam-beam interaction in collider rings, and even halo formation in rings and linacs. It would be especially interesting if, for any of these problems, along with a steady state solution with $\lambda = 0$, there exists an exponentially growing solution of the Schrödinger equation that has a negative eigenvalue $\lambda < 0$. This could qualitatively explain the relaxation oscillation behavior seen in many numerical and real life experiments.

ACKNOWLEDGMENTS

We thank Alex Chao and Bob Siemann for insightful discussions and support of this study, and also Albert Hoffman for pointing out the relevance of the negative momentum compaction case to this problem. Many thanks are also due to Dave Whittum for reading this manuscript and providing valuable comments. This work was supported by the Department of Energy Contract No. DE-AC03-76SF00515.

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