Response matrices in strongly coupled storage rings with a radio-frequency system constraining the revolution time

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We calculate the orbit response coefficients for an arbitrarily coupled storage ring subject to maintaining the orbit length constant due to the presence of a radio-frequency system.

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I. INTRODUCTION

The orbit correction system in a storage ring, where the information from beam position monitors is used to calculate steering magnet excitations in order to place the beam in the center of the beam pipe, is a central pillar of accelerator operation. The central quantity in the corrections system is the response matrix C^{ij} that relates the beam position on a beam position monitor (BPM) labeled i to the excitation of a steering magnet, labeled j. A further use of the response matrix is the determination of the beam optics by carefully analyzing the deviation of measured from computer-calculated response matrix elements [1,2]. The response matrix can be calculated from the knowledge of the magnet configuration and the corresponding transfer matrices in the following way

$$C^{ij} = R^{ij}(1 - R^{jj})^{-1} \tag{1}$$

where R^{ij} is the 4×4 transfer matrix from the steering magnet to the BPM and R^{jj} is the transfer matrix for one turn starting at the location of the steering magnet. This method works for arbitrarily coupled rings, but ignores that the length of the closed orbit changes when exciting a steering magnet at a location with nonzero dispersion.

In the presence a rf system, however, the revolution time is forced to stay constant in order to maintain synchronization with the rf system. The beam automatically adjusts to this by slightly varying the beam energy and thus the closed orbit obtains a contribution proportional to the dispersion in the ring. A careful analysis [2–4] shows that for a ring without transverse betatron coupling the response coefficients C^{ij}_{12} are given by

$$C^{ij}_{12} = [R^{ij}(1 - R^{jj})^{-1}]_{12} - \frac{D^i D^j}{\eta C} \tag{2}$$

where the transfer matrices R are the 2×2 horizontal transfer matrices corresponding to those referred to above. The extra term with the horizontal dispersion D^j at the steering magnet and D^i the BPM takes into account the change of energy and assures that the orbit length stays constant. C is the circumference of the ring and η the phase slip factor. This expression is only valid for rings without betatron coupling and in the remainder of this report we generalize it to arbitrarily coupled accelerators.

II. COUPLED RESPONSE COEFFICIENTS

We start by considering the 6×6 transfer matrix R^{jj} that starts at the location of the steering magnet and do not take synchrotron radiation into account. The requirement that the variation of the revolution time does not change when applying a perturbation $|v\rangle = (v_x, v_{x'}, v_y, v_{y'})^t$ imposes the following constraint on the closed-orbit vector $|x\rangle = (x, x', y, y')^t$, arrival time τ and energy δ

$$\begin{pmatrix} x \\ x' \\ y \\ y' \\ \tau \\ \delta \end{pmatrix} = \begin{pmatrix} A & B & C \\ D & E & F \\ G & H & I \end{pmatrix} \begin{pmatrix} x \\ x' \\ y \\ y' \\ \tau \\ \delta \end{pmatrix} + \begin{pmatrix} v_x \\ v_{x'} \\ v_y \\ v_{y'} \\ 0 \\ 0 \end{pmatrix} \tag{3}$$

where we omit the superscript jj to simplify the notation and choose to express the transfer matrix in terms of 2×2 sub-matrices $A, B, \ldots, I$. Note that the superscript t denotes transpose of a vector or matrix. The top left 4×4 section with A, B, D, E contains the 4×4 coupled one-turn transfer matrix $\hat{R}$ starting at the steering magnet. The requirement to leave the revolution time unchanged when changing the steering magnet by making for example $v_{x'}$ or $v_{y'}$ nonzero is given by forcing the fifth and sixth component of the vector on the left-hand side to be equal. Equation (3) can be solved in a straightforward way by calculating $(1 - R^{jj})^{-1}$, as is done in [5], and probably a number of other codes, but in order to investigate it

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further and seek a generalization of Eq. (2) for coupled rings, we write it in the following way by borrowing from quantum mechanics for the notation

$$|x\rangle = \hat{R}|x\rangle + \begin{pmatrix} C \\ F \end{pmatrix}\begin{pmatrix} \tau \\ \delta \end{pmatrix} + |v\rangle$$
$$\begin{pmatrix} \tau \\ \delta \end{pmatrix} = (G,H)|x\rangle + I\begin{pmatrix} \tau \\ \delta \end{pmatrix}. \tag{4}$$

The first equation corresponds to the first four rows of the previous equation and the second equation to the two bottom rows. A consequence of the requirement to maintain the revolution time is that the arrival time τ and the energy δ have to adjust according to

$$(1-I)\begin{pmatrix} \tau \\ \delta \end{pmatrix} = (G,H)|x\rangle. \tag{5}$$

Solving this equation for the τ and δ we find in the case that the matrix on the left-hand side is invertible

$$\begin{pmatrix} \tau \\ \delta \end{pmatrix} = (1-I)^{-1}(G,H)|x\rangle \tag{6}$$

and

$$\begin{pmatrix} \tau \\ \delta \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ -1/R_{56} & 0 \end{pmatrix}(G,H)|x\rangle \tag{7}$$

in case the matrix $(1-I)$ is not invertible. In the following we will use the first expression from Eq. (6) with the tacit assumption to use the matrix from Eq. (7) instead of $(1-I)^{-1}$ in case the matrix is not invertible.

The expression for τ and δ we now insert into the first of Eq. (4) and find

$$|x\rangle = \hat{R}|x\rangle + \begin{pmatrix} C \\ F \end{pmatrix}(1-I)^{-1}(G,H)|x\rangle + |v\rangle$$
$$= \left[\hat{R} + \begin{pmatrix} C \\ F \end{pmatrix}(1-I)^{-1}(G,H)\right]|x\rangle + |v\rangle \tag{8}$$

which, after solving for the closed orbit vector $|x\rangle$, becomes

$$|x\rangle = \left[1 - \hat{R} - \begin{pmatrix} C \\ F \end{pmatrix}(1-I)^{-1}(G,H)\right]^{-1}|v\rangle = \tilde{C}^{jj}|v\rangle \tag{9}$$

where the previous equation defines the response matrix of the closed orbit $\tilde{C}^{jj}$ at the location of the steering magnet j due to the perturbation $|v\rangle$ of the magnet. The response at the BPM i is trivially computed by left multiplying $\tilde{C}^{jj}$ with the 4×4 transfer matrix $\hat{R}_{ij}$ from the steering magnet to the BPM. Finally we find that the general response matrix, containing the constraint of maintaining the revolution time fixed is given by $\tilde{C}^{ij} = \hat{R}^{ij}\tilde{C}^{jj}$.

III. DISPERSION-LIKE QUANTITIES

The expression for the response matrix in Eq. (9) is suitable for numerical calculations of the response matrices, but gives little insight into the physics. We therefore use it as the starting point toward a generalized version of Eq. (2) and rewrite Eq. (9) in the following form

$$\left[1 - (1-\hat{R})^{-1}\begin{pmatrix} C \\ F \end{pmatrix}(1-I)^{-1}(G,H)\right]|x\rangle = (1-\hat{R})^{-1}|v\rangle \tag{10}$$

and introduce the dispersion-like quantities $\tilde{C}$ and $\tilde{F}$, which are defined by

$$\begin{pmatrix} \tilde{C} \\ \tilde{F} \end{pmatrix} = (1-\hat{R})^{-1}\begin{pmatrix} C \\ F \end{pmatrix}. \tag{11}$$

The rational is, of course, that the second column of the 4×2 matrix C and F contains the matrix elements $R_{16}, R_{26}, R_{36}, R_{46}$ and left-multiplying with the inverse of $(1-R)$ results in the column vector containing the coupled dispersions D_x, D'_x, D_y, D'_y which populate the second column of $\tilde{C}$ and $\tilde{F}$. These definitions result in

$$\left[1 - \begin{pmatrix} \tilde{C} \\ \tilde{F} \end{pmatrix}(1-I)^{-1}(G,H)\right]|x\rangle = (1-\hat{R})^{-1}|v\rangle. \tag{12}$$

The matrix in the square brackets has the form of the sum of unity plus an outer product which permits us to write the inverse as

$$\left[1 - \begin{pmatrix} \tilde{C} \\ \tilde{F} \end{pmatrix}(1-I)^{-1}(G,H)\right]^{-1}$$
$$= \left[1 + \begin{pmatrix} \tilde{C} \\ \tilde{F} \end{pmatrix}Q(1-I)^{-1}(G,H)\right] \tag{13}$$

where the 2×2 matrix Q is given by

$$Q = [1 - (1-I)^{-1}(G\tilde{C} + H\tilde{F})]^{-1}. \tag{14}$$

Details about the algebra involved as well as the numerical validity are deferred to the Supplemental material [6].

Left-multiplying Eq. (12) with the inverse of the matrix in the square brackets and after some reordering we arrive at

$$|x\rangle = \left[(1-\hat{R})^{-1} + \begin{pmatrix} \tilde{C} \\ \tilde{F} \end{pmatrix}Q(1-I)^{-1}(G,H)(1-\hat{R})^{-1}\right]|v\rangle. \tag{15}$$

Finally, we need to express the matrices G and H in terms of the dispersion-like quantities $\tilde{C}$ and $\tilde{F}$. It is easy to show that the symplecticity of the transfer matrix implies conditions among the sub-matrices $A, B, \ldots, I$. In particular, the matrices G and H are given in terms of the matrices in the two top rows by

$$G = \frac{1}{\det I}[ISC^tSA + ISF^tSD]$$
$$H = \frac{1}{\det I}[ISC^tSB + ISF^tSE] \tag{16}$$

where S is the 2×2 symplectic matrix

$$S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \tag{17}$$

and we also introduce S_4, the 4×4 symplectic matrix containing two matrices S on the diagonal and zeros elsewhere. The two equations for G and H can be written more compactly as

$$(G, H) = \frac{IS}{\det I}(C^t, F^t)S_4\hat{R} = \frac{IS}{\det I}(\tilde{C}^t, \tilde{F}^t)(1 - \hat{R}^t)S_4\hat{R} \tag{18}$$

where, in the second of the equations, we use Eq. (11) to express C and F in terms of the dispersion-like quantities $\tilde{C}$ and $\tilde{F}$. Inserting this expression in Eq. (15) and some algebra, we arrive at

$$|x\rangle = \left[(1 - \hat{R})^{-1} - \begin{pmatrix} \tilde{C} \\ \tilde{F} \end{pmatrix} Q(1 - I)^{-1}\frac{IS}{\det I}(\tilde{C}^t, \tilde{F}^t)S_4\right]|v\rangle$$
$$= C^{jj}|v\rangle \tag{19}$$

which expresses the response coefficients in terms of dispersion-like quantities $\tilde{C}$ and $\tilde{F}$.

IV. PHYSICAL INTERPRETATION

To make the connection to the conventionally used expression Eq. (2) we assume that the matrix I is only given in terms of the R_{56} and that the first four rows of the fifth column of the transfer matrix contain zeros. This implies that we have

$$\begin{pmatrix} \tilde{C} \\ \tilde{F} \end{pmatrix} = (1 - \hat{R})^{-1}\begin{pmatrix} C \\ F \end{pmatrix}$$
$$= (1 - \hat{R})^{-1}\begin{pmatrix} 0 & R_{16} \\ 0 & R_{26} \\ 0 & R_{36} \\ 0 & R_{46} \end{pmatrix}$$
$$= \begin{pmatrix} 0 & D_x \\ 0 & D'_x \\ 0 & D_y \\ 0 & D'_y \end{pmatrix} \tag{20}$$

which follows from the requirement that the dispersion $|D\rangle = (D_x, D'_x, D_y, D'_y)^t$ is the periodic solution of the equation

$$|D\rangle = \hat{R}|D\rangle + \begin{pmatrix} R_{16} \\ R_{26} \\ R_{36} \\ R_{46} \end{pmatrix}. \tag{21}$$

The matrix Q that appears in Eq. (19) and is given by Eq. (14) evaluates to

$$Q = \left[1 - \begin{pmatrix} 0 & 0 \\ -1/R_{56} & 0 \end{pmatrix}\left\{\begin{pmatrix} R_{51} & R_{52} \\ R_{61} & R_{62} \end{pmatrix}\begin{pmatrix} 0 & D_x \\ 0 & D'_x \end{pmatrix} + \begin{pmatrix} R_{53} & R_{54} \\ R_{63} & R_{64} \end{pmatrix}\begin{pmatrix} 0 & D_y \\ 0 & D'_y \end{pmatrix}\right\}\right]^{-1}$$
$$= \begin{pmatrix} 1 & 0 \\ 0 & R_{56}/(R_{56} + R_{51}D_x + R_{52}D'_x + R_{53}D_y + R_{54}D'_y) \end{pmatrix} \tag{22}$$

where we expressed $(1 - I)^{-1}$ by the form from Eq. (7) following the discussion from there. Inserting Q in Eq. (19) and after some straightforward algebra we find the expression for the response matrix in terms of the dispersion as

$$\tilde{C}^{jj} = (1 - \hat{R})^{-1} - \frac{1}{\eta C}\begin{pmatrix} D_x \\ D'_x \\ D_y \\ D'_y \end{pmatrix}\begin{pmatrix} -D'_x, & D_x, & -D'_y, & D_y \end{pmatrix} \tag{23}$$

where the two vectors with the dispersions need to be evaluated as an outer matrix product. Here we also introduce the circumference C and use the generalization of the phase slip factor η

$$-\eta C = R_{56} + R_{51}D_x + R_{52}D'_x + R_{53}D_y + R_{54}D'_y \quad (24)$$

which agrees with the expression used in MADX [7]. Equation (23) is the generalization of Eq. (2) for a fully coupled ring. Note that the dispersions appearing in Eqs. (23) and (24) are the coupled dispersions following from Eq. (21). The response coefficient to a different location can be calculated by left-multiplying $\tilde{C}^{jj}$ by $\hat{R}^{ij}$, the 4×4 transfer matrix from the location of the steering magnet, labeled j to the other, labeled i. The column vector with the dispersion at location of the steerer is thus propagated to the dispersion at the location i.

V. CONCLUSIONS

We calculate the orbit response coefficients for arbitrarily coupled storage rings in case the revolution time is constrained by a radio-frequency system and exciting a steering magnet causes a variation of the orbit length. The computations only involve manipulations of 6×6 transfer matrices that are usually available from codes such as MADX [7]. This makes the method easy to implement numerically. Expressing the additional term in terms of the coupled dispersion we arrived in Eq. (23) at a generalized version of the well-known result from Eq. (2).

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[6] See Supplemental Material at http://link.aps.org/supplemental/10.1103/PhysRevSTAB.18.054001 for the longer calculations leading to Eq. 13, 16, and 19 as well as code to show that the different response matrices yield equal numerical results.

[7] The MAD-X Home Page at http://cern.ch/madx.