Accelerator-feasible N-body nonlinear integrable system

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Nonlinear N-body integrable Hamiltonian systems, where N is an arbitrary number, have attracted the attention of mathematical physicists for the last several decades, following the discovery of some number of these systems. This paper presents a new integrable system, which can be realized in facilities such as particle accelerators. This feature makes it more attractive than many of the previous such systems with singular or unphysical forces.

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I. INTRODUCTION

Integrable systems with many or infinite number of degrees of freedom attract attention due to their unique properties. For example, the Korteweg–de Vries equation is equivalent to a Hamiltonian system with infinite degrees of freedom with special waves, called solitons, particlelike scattering of them, and the possibility to solve the correspondent nonlinear equations analytically [1]. A few N-particle systems were found for special interaction forces between particles (see, e.g., [2]). The practical realizations and applications of such systems are questionable because of very special features of such forces. Here we describe a new N-particle integrable system and show the practical possibilities of how to realize such a system in a particle accelerator. This system has N integrals of motion, which are independent and their Poisson brackets vanish.

II. INTEGRABLE MANY-BODY SYSTEM

Let us consider the following 1D mapping for the ith particle, among N identical particles. The proposed mapping consists of two stages: a purely linear transformation and a particle interaction stage,

$$\begin{pmatrix} x_i \\ p_i \end{pmatrix}_n = \begin{pmatrix} 1 & 0 \\ K_n & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} x_i \\ p_i \end{pmatrix}_{n-1}. \tag{1}$$

One can rewrite it as

$$\begin{aligned} x_{i,n} &= p_{i,n-1}, \\ p_{i,n} &= -x_{i,n-1} + K_n x_{i,n}, \end{aligned} \tag{2}$$

where $x_{i,n}$ and $p_{i,n}$ are the coordinate and the momentum, respectively, of particle $i = [1\ldots N]$ at the nth mapping step. The K_n is the particle interaction term as follows:

$$K_n = \frac{b}{a\sigma_n^2 + 1}, \tag{3}$$

where $\sigma_n^2 = \frac{1}{N}\sum_i (x_{i,n})^2$ is the mean squared position of all particles and a and b are arbitrary parameters. This system can be viewed as a sequence of thin lenses, acting identically on each particle, separated by a linear map. However, the focal length of each lens depends on the sum of all coordinates squared; therefore, the system motion depends on the particle's initial conditions, and it is indeed a nonlinear system. For $N = 1$ (a single particle), the mapping (2) becomes the well-known nonlinear single-particle 1D integrable McMillan mapping [3] with $\sigma_n^2 = x_n^2$. This mapping has the following integral of motion (a conserved quantity):

$$I = a x_n^2 p_n^2 - b x_n p_n + x_n^2 + p_n^2. \tag{4}$$

It was shown in Ref. [4] that an $N = 2$ mapping is also integrable; i.e., it has two independent commuting (having vanishing Poisson brackets) integrals of motion. One is the so-called angular momentum $M = x_{1,n}p_{2,n} - x_{2,n}p_{1,n}$, and the second one is similar to (4):

$$\begin{aligned} I = {} & \frac{a}{4}(x_{1,n}p_{1,n} + x_{2,n}p_{2,n})^2 - \frac{b}{2}(x_{1,n}p_{1,n} + x_{2,n}p_{2,n}) \\ & + \frac{1}{2}(x_{1,n}^2 + x_{2,n}^2 + p_{1,n}^2 + p_{2,n}^2). \end{aligned} \tag{5}$$

We have found that the mapping (1), (2) is integrable for any N; i.e., it has N independent commuting integrals of motion. We first notice that all moments of the following type (for $i, j = [1...N]$ and $i \neq j$):

$$M_{i,j} = x_{i,n}p_{j,n} - x_{j,n}p_{i,n} \tag{6}$$

are conserved. This gives $\frac{N(N-1)}{2}$ integrals of motion. However, only some of them have vanishing Poisson brackets, while for integrability one needs N independent commuting integrals of motion (see, e.g., [5] for Poisson brackets definitions and conditions for integrability). Below, we will prove that the mapping has N independent commuting integrals of motion. The first $N-1$ commuting integrals of motion can be constructed by using moments (6):

$$I_j = \sum_{i=1}^{j} (M_{i,j+1})^2, \tag{7}$$

where $j = [1, 2 \ldots N-1]$. The proof can be established by induction. Assume $k-1$ invariants commute, and then take the kth integral. It consists of a sum of all terms with the new $(k+1)$th variable, which is absent from all previous integrals. We omit the index n for brevity and take any term from previous integrals, $(x_i p_j - p_i x_j)^2$, and the two terms from the last (kth) integral $(x_i p_{k+1} - p_i x_{k+1})^2 + (x_j p_{k+1} - p_j x_{k+1})^2$ with $i, j \le k$. Now we take the Poisson brackets of these terms and check piecewise that they are equal to zero:

$$\begin{aligned} &-4(x_i p_j - p_i x_j)(x_i p_{k+1} - p_i x_{k+1}) \\ &\quad + 4(x_i p_j - p_i x_j)(x_i p_{k+1} - p_i x_{k+1}) -, \\ &4(x_i p_j - p_i x_j)(x_j p_{k+1} - p_j x_{k+1}) \\ &\quad + 4(x_i p_j - p_i x_j)(x_j p_{k+1} - p_j x_{k+1}) = 0. \end{aligned} \tag{8}$$

It is easy to show that all other terms from this last integral commute with the term $(x_i p_j - p_i x_j)^2$, since they do not contain any variables in common, thus proving that integrals (7) commute. The independence of integrals (7) follows from the fact that each sum in (7) always includes one new coordinate compared to the previous integrals and cannot be expressed via them in any way and vice versa. Notice that the last integral I_{N-1} can be rewritten in a familiar way:

$$I_{N-1} = \sum_{i=1}^{N} x_i^2 \sum_{i=1}^{N} p_i^2 - \left(\sum_{i=1}^{N} x_i p_i\right)^2, \tag{9}$$

which is actually the so-called emittance $\varepsilon = \frac{\sqrt{I_{N-1}}}{N}$.

We will now show that there exists an additional integral of the type (5) for any N. We prove the results of McMillan first on our way to the final expression. Let us take a quadratic form I, symmetric in x and p:

$$I = a x^2 p^2 - bxp + x^2 + p^2, \tag{10}$$

with a and b from (1). After the linear transformation in Eq. (1), but before the nonlinear kick, it will transform into

$$I = a x^2 p^2 + bxp + x^2 + p^2. \tag{11}$$

Then, we apply a nonlinear kick (2), which changes the sign of the momentum in the expression (10), if the invariant is written in the new variables. In Ref. [6], this kick is called the sign reversal function for obvious algebraic reasons (if $I = ap^2 + bp + c$, $p = \bar{p} - b/a$, then $I = a\bar{p}^2 - b\bar{p} + c$). The expression (11) in the new variables becomes

$$I = ax^2 p^2 - bxp + x^2 + p^2. \tag{12}$$

One can see that (10) and (12) coincide—this is the proof for the McMillan invariant (4) with $N = 1$. We will now take an invariant

$$I = \frac{a}{N^2}\left(\sum_i x_{i,n} p_{i,n}\right)^2 - \frac{b}{N}\sum_i x_{i,n} p_{i,n} + \frac{1}{N}\sum_i x_{i,n}^2 + \frac{1}{N}\sum_i p_{i,n}^2. \tag{13}$$

It is symmetric with respect to the substitution $x_{i,n} \leftrightarrow p_{i,n}$. One can note that after the nonlinear kick, all momenta p_i receive an increment equal to $\frac{bx_i}{a\sum_i x_i^2/N+1}$.

Take the formula (13) and add the previous expression to the corresponding momenta. After omitting the index n and the terms with the sum of the coordinates squared (since they do not change in a thin lens) for brevity, it yields

$$\begin{aligned} I &= \frac{a}{N}\left(\sum_i x_i p_i + \frac{b\sum_i x_i^2}{\frac{a}{N}\sum_i x_i^2 + 1}\right)^2 - b\sum_i x_i p_i - \sum_i \frac{b^2 x_i^2}{\frac{a}{N}\sum_i x_i^2 + 1} + \sum_i\left(p_i + \frac{bx_i}{\frac{a}{N}\sum_i x_i^2 + 1}\right)^2 \\ &= \frac{a}{N}\left(\sum_i x_i p_i\right)^2 + \frac{a}{N}\sum_i \frac{2bp_i x_i \sum_i x_i^2}{\frac{a}{N}\sum_i x_i^2 + 1} + \frac{a}{N}\left[\frac{b\sum_i x_i^2}{\frac{a}{N}\sum_i x_i^2 + 1}\right]^2 - b\sum_i x_i p_i - \frac{b^2 \sum_i x_i^2}{\frac{a}{N}\sum_i x_i^2 + 1} \\ &\quad + \sum_i (p_i)^2 + \frac{b^2 \sum_i x_i^2}{(\frac{a}{N}\sum_i x_i^2 + 1)^2} + \sum_i \frac{2bp_i x_i}{\frac{a}{N}\sum_i x_i^2 + 1} \\ &= \frac{a}{N}\left(\sum_i x_i p_i\right)^2 + b\sum_i x_i p_i + \sum_i (p_i)^2. \end{aligned} \tag{14}$$

One can see that this expression (14) is the same as (13) (without omitted insignificant terms), while only the momenta sign is opposite. The same conclusion as in the 1D case follows: the expression (13) is an invariant of the map (1), because it is symmetric in coordinates and momenta, the linear part of the map changes the sign of momenta, and the kick restores it; thus, the expression restores its form after the transformation (1). This integral (13) is independent of integrals (7), because they are comprised only of cross products of momenta and coordinates, but the integral (13) has contributions from the radial momentum as well (see, for example, the first two terms). It commutes with all integrals (7), because it consists of a function of "radial variables" (the first two terms) and two terms symmetric with respect to any rotation in any direction and, as such, always commutes with any angular momentum. This concludes our proof that the mapping is integrable.

With the exception of a matched distribution (described below), which results in a perfectly linear motion with a constant K_n, this mapping is nonlinear; i.e., particle trajectories depend on the initial amplitudes of all particles through a nonlinear interaction term, Eq. (3). Figure 1 shows an example of phase-space trajectories for four particles ($N = 4$).

It is now easy to understand how to extend this system to a higher-dimension space. A system of N particles in the M-dimensional space is obtained from (1) by replacing x_i by a vector with the dimension M. The kick to each momentum will be proportional to its conjugate coordinate, and the sum of coordinates squared in (1) has to be replaced by the sum of particle coordinates squared over all N particles. It is easy to see that the new system corresponds to the one-dimensional one with $N \times M$ particles, and therefore all approaches work in the same manner, and it is exactly integrable in the same sense as the one-dimensional system with N particles.

III. ENVELOPE EQUATIONS

Let us now describe the evolution of the N-particle envelope σ_n, defined as the rms position of all particles. Following Ref. [7], we first note that the interaction term (3) acts as an identical lens (that depends on the N-particle envelope) on all particles and, therefore, results in a piecewise linear mapping (which depends on the particle positions) for each step. We have already demonstrated above (9) that the so-called emittance

$$\varepsilon = \sqrt{\langle x_n^2\rangle\langle p_n^2\rangle - \langle x_n p_n\rangle^2} \tag{15}$$

is a conserved quantity ($\langle\ldots\rangle$ indicates averaging over N particles). This also means that one can write the mapping for the envelope much like for the radial position and the radial momentum of a single particle in Ref. [4]. First, we will define the envelope momentumlike variable

$$\eta_n = \frac{1}{\sigma_n N}\sum_i x_{i,n}p_{i,n}. \tag{16}$$

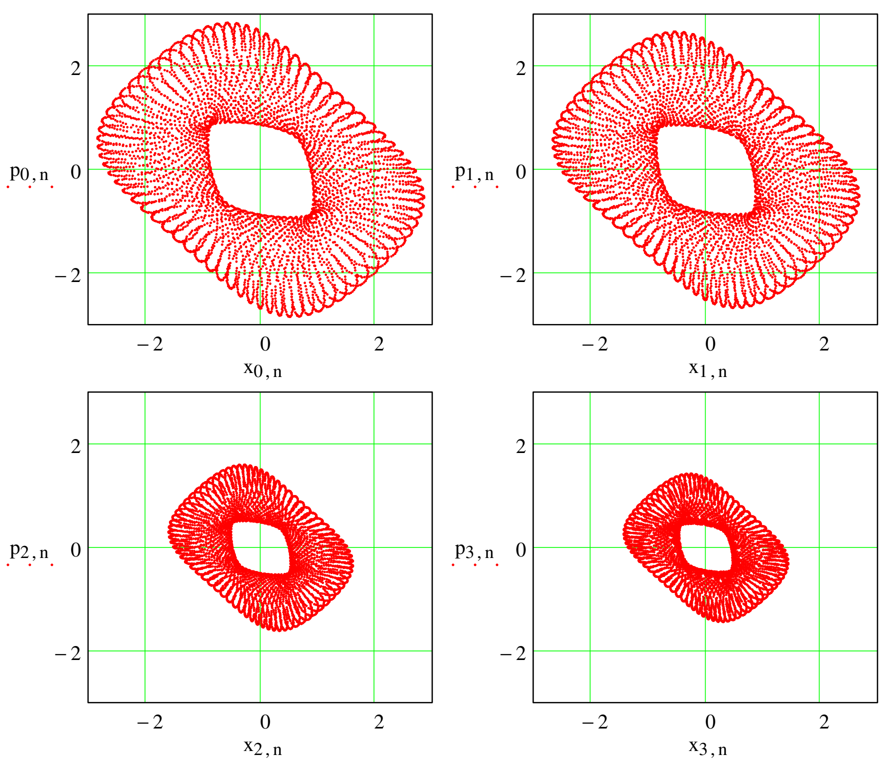

FIG. 1. Phase-space trajectories of four particles with random initial conditions.

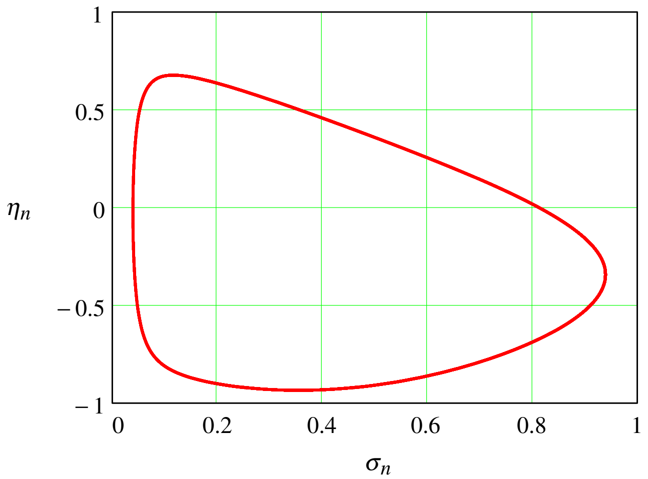

FIG. 2. An example of the envelope mapping for four particles.

Now, the mapping for the envelope can be written as

$$\sigma_n = \sqrt{\eta_{n-1}^2 + \frac{\varepsilon^2}{\sigma_{n-1}^2}},$$
$$\eta_n = -\frac{\sigma_{n-1}\eta_{n-1}}{\sigma_n} + K_n\sigma_n. \qquad (17)$$

This is also an integrable mapping. The integral (similar to one in Ref. [4] that uses slightly different variables) can be written as

$$I = (a\sigma_n^2 + 1)\eta_n^2 - b\sigma_n\eta_n + \sigma_n^2 + \frac{\varepsilon^2}{\sigma_n^2}, \qquad (18)$$

which is actually the same integral as (13), after we express the emittance via (15). Our interpretation of this result is that the envelope mapping couples the emittance and the additional integral of motion (18), thus making the mapping integrable. It is interesting to point out that while the preservation of emittance in a linear channel appears to be trivial, the existence of an additional integral of motion is far from trivial. Contrary to the rms (or envelope) equations for the Kapchinsky-Vladimirsky space charge distribution, which are chaotic in general even for the azimuthally symmetric beams [8], the equations (17) have only regular analytic solutions. Figure 2 shows an example of the envelope mapping for four particles with some random initial distribution.

IV. MATCHED DISTRIBUTION

In this section, we will consider a matched distribution, i.e., the distribution in which the rms position σ and the envelope momentum η remain constant. From Eq. (17), we have

$$\sigma^2 = \eta^2 + \frac{\varepsilon^2}{\sigma^2}, \qquad \eta = \frac{1}{2}\frac{b\sigma}{a\sigma^2 + 1}. \qquad (19)$$

It is obvious that solutions to Eqs. (19) can be obtained for any emittance value, although some may restrict the allowed a and b values.

In the limit of the large number of particles ($N \gg 1$), we would obtain the Vlasov equation as in a common plasma approach. For the smooth Vlasov equation, there exists a distribution which maps onto itself by mapping (2). We would call such a distribution "matched" because the interaction term $K_n = K$ remains constant, and, thus, the mapping becomes perfectly linear. Such a linear mapping preserves the so-called Courant-Snyder invariant [9]

$$J = x^2 - Kxp + p^2. \qquad (20)$$

Thus, it is obvious that any matched particle distribution function $f(J)$ would remain unchanged under the mapping (2), provided the rms position $\sigma = \sqrt{\int x^2 f(J)dxdp}$ satisfies the following equation:

$$K = \frac{b}{a\sigma^2 + 1}. \qquad (21)$$

V. PRACTICAL REALIZATION

Finally, let us describe how to realize such a mapping in a particle accelerator. The beam dynamics, described by (2), is fairly straightforward to realize in a circular accelerator. One needs to measure the beam rms position σ, which is a variable in Eqs. (2), and apply a quadrupole kick with the coefficient of $\frac{b}{a\sigma^2+1}$ to the circulating beam [the second term in the second line of the map (2)], where σ of the beam has to be taken at the quadrupole position. It can be achieved, for example, by placing the profile measurement device and the quadrupole such that the betatron phase advance between them is an integer of π. The quadrupole kicks themselves have to be separated by $\pi/2$ betatron phase advance [it corresponds to the linear part of (2)]. This is, in a sense, a standard feedback, but the beam parameter of interest is its rms position, not the displacement.

VI. SUMMARY

An example of a rare N-body integrable nonlinear system has been found and described. Its unique properties allow one to create such a system experimentally in accelerators, opening a new venue for various applications to charged particle beams.

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