Estimation of emittance growth with simple orbit correction in long linacs

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Estimating emittance growth is important for evaluation of performance of linear accelerators, especially where stable and low emittance beam is required, such as linear colliders. Usually, estimation of emittance growth is performed using tracking simulations, including the Monte Carlo method, which tend to take a long time. We have developed a much faster and simpler method of quantitative estimation. Formulas of orbit and emittance due to random misalignment without corrections were reported in our previous paper [K. Kubo,Phys. Rev. ST Accel. Beams **14**, 014401 (2011)]. Here, formulas of emittance growth after a simple orbit correction are reported. This method is valid for very long linacs with many components, where statistical treatment is justified. It is shown that the results from the method agree well with tracking simulations for the International Linear Collider main linac.

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I. INTRODUCTION

In a linear accelerator with misalignment of its components, certain orbit corrections need to be applied, where the beam is steered based on measured beam positions along the beam line. Usually, estimation of emittance growth after such corrections needs tracking simulations, including the Monte Carlo method, which tend to take a long time. We have developed a much faster and simpler method of quantitative estimation. For deriving the formulas given here, we extended the method for the formulas for orbit change and emittance growth without corrections, which were reported before [1].

This method is valid for very long linacs with many components, where statistical treatment of errors is justified. It is shown that the results from the method agree well with tracking simulations for the International Linear Collider (ILC) main linac.

We assume a linear accelerator consists of quadrupole magnets, steering magnets, beam position monitors (BPMs), and accelerating cavities. One quadrupole magnet is assumed to have a steering magnet and a BPM attached.

In the following sections, we derive formulas of emittance growth due to dispersive effect, after a simple orbit correction (one to one steering), with transverse offset error of BPMs. Then, results are compared with tracking simulations for the ILC main linac.

We do not consider effects of a wakefield in this report. We also do not consider transverse kicks by accelerating cavities, which can be caused by tilting misalignment of cavities and geometrical asymmetries of couplers attached to cavities. Steering magnets and beam offset at quadrupole magnets are the only considered sources of transverse kicks.

For simplicity, the linac is assumed to be straight, though it is following the curvature of the earth in actual ILC design. Also for simplicity, we only consider one transverse direction, denoting y. It is straightforward to include the other direction.

Some past studies on analytic and semianalytic estimations of orbit change and emittance growth are introduced in our previous paper [1]. Reference [2] extensively studied analytic estimation of emittance growth in high-energy linear accelerators. Though it gave some formulas, its assumptions are not valid in the case of the ILC main linac. Our method and formulas for emittance growth are new, which can be applied to the ILC main linac.

In this paper, only a simple orbit correction (one to one steering) is assumed, though more complicated beam based orbit corrections will be applied in actual accelerators, such as dispersion-free steering (DFS) [3,4], ballistic steering [5], and so on. Extending our formulas for these corrections will need further studies. However, an idea of extension of our method assuming DFS is briefly discussed later.

II. EMITTANCE GROWTH BY DISPERSIVE EFFECT

A. Emittance with dispersion

First, we derive an expression for emittance growth due to dispersion, similar as in our past paper [1].

A dispersive effect is estimated from orbit difference between particles with different energies. Let us assume that position and angle deviation depend on energy deviation are as follows:

$$\delta y = \sum_n \eta_n (\delta E/E)^n, \qquad \delta y' = \sum_n \eta'_n (\delta E/E)^n, \quad (1)$$

where $\delta E/E$ is relative energy deviation of a particle, η_n and η'_n are the nth order dispersion and angle dispersion, respectively.

Square of emittance with the deviations is expressed as

$$\begin{aligned}\epsilon^2 &= \overline{(y-\bar{y})^2} \times \overline{(y'-\bar{y}')^2} - [\overline{(y-\bar{y})(y'-\bar{y}')}]^2 \\ &= \overline{(y_0+\delta y-\bar{y}_0-\overline{\delta y})^2} \times \overline{(y'_0+\delta y'-\bar{y}'_0-\overline{\delta y'})^2} \\ &\quad - [\overline{(y_0+\delta y-\bar{y}_0-\overline{\delta y})(y'_0+\delta y'-\bar{y}'_0-\overline{\delta y'})}]^2,\end{aligned} \tag{2}$$

where y_0 and y'_0 are position and angle without the deviations due to the dispersion, and the overlines denote average over all particles in the beam.

After some manipulations, we will have

$$\epsilon^2 = \epsilon_0^2 + (\Delta\epsilon^2)_1 + (\Delta\epsilon^2)_2, \tag{3}$$

where

$$(\Delta\epsilon^2)_1 = \epsilon_0 \sum_{m,n=1}^{\infty} \mathcal{H}_{mn}[\overline{(\delta E/E)^{m+n}} - \overline{(\delta E/E)^m}\,\overline{(\delta E/E)^n}] \tag{4}$$

and

$$\begin{aligned}(\Delta\epsilon^2)_2 &= \frac{1}{2}\sum_{k,l,m,n=1}^{\infty}(\eta_k\eta'_l-\eta_l\eta'_k)(\eta_m\eta'_n-\eta_n\eta'_m) \\ &\quad \times [\overline{(\delta E/E)^{k+m}} - \overline{(\delta E/E)^k}\,\overline{(\delta E/E)^m}] \\ &\quad \times [\overline{(\delta E/E)^{l+n}} - \overline{(\delta E/E)^l}\,\overline{(\delta E/E)^n}],\end{aligned} \tag{5}$$

where ϵ_0 is the emittance without dispersion, and we defined

$$\mathcal{H}_{nm} \equiv \gamma_y\eta_m\eta_n + \alpha_y(\eta_m\eta'_n+\eta_m\eta'_n) + \beta_y\eta'_m\eta'_n, \tag{6}$$

where α_y, β_y, and γ_y are the Twiss parameters, and we used

$$\overline{(y_0-\bar{y}_0)^2} = \epsilon_0\beta_y \tag{7}$$

$$\overline{(y_0-\bar{y}_0)(y'_0-\bar{y}'_0)} = -\epsilon_0\alpha_y \tag{8}$$

$$\overline{(y'_0-\bar{y}'_0)^2} = \epsilon_0\gamma_y. \tag{9}$$

For evaluating emittance growth in the following sections, we calculate $\mathcal{H}_{mn}$ and $(\eta_k\eta'_l-\eta_l\eta'_k)(\eta_m\eta'_n-\eta_n\eta'_m)$ at the end of the linac.

In the following evaluations, we assume the energy deviation has the normal distribution and for a positive integer N,

$$\begin{aligned}\overline{(\delta E/E)^{2N}} &= \frac{(2N-1)!}{2^{N-1}(N-1)!}(\sigma_E/E)^{2N} \\ \overline{(\delta E/E)^{2N-1}} &= 0,\end{aligned} \tag{10}$$

where σ_E is the energy spread. With this assumption, only terms of $m+n=$ even contribute $(\Delta\epsilon^2)_1$ and only terms of $k+l+m+n=$ even contribute $(\Delta\epsilon^2)_2$.

III. ONE TO ONE STEERING WITH BPM OFFSET

In this section, we consider a simple orbit correction, so called "one to one steering." We assume every quadrupole magnet in a linac has a steering magnet and a BPM within a very short distance. (Let us call a set of a quadrupole magnet, a steering magnet and a BPM "magnet unit".) In the one to one steering, the beam is steered using steering magnets to make the reading of every BPM zero. Since length of a magnet unit is assumed to be very small, kick by a steering magnet and transverse offset of a quadrupole magnet have an equivalent effect, and misalignment of quadrupole magnets has no important effect. Here, we estimate emittance growth due to random and independent offset error of BPM. The beam orbit follows the offset error of each BPM.

Any effects of accelerating cavities to transverse beam motion (such as tilting of cavity and wakefield) are ignored here.

Let a_i be offset error of the ith BPM with respect to the designed straight beam line. Kick angle of the beam center (particles with design energy), due to this offset error, at the ith magnet unit after one to one steering will be

$$\theta_i = \frac{a_{i+1}-2a_i+a_{i-1}}{L_{qq}}, \tag{11}$$

where L_{qq} is the distance between magnet units, which is assumed to be constant along the linac.

The deviation of the kick angle due to relative energy deviation $\delta E/E$ at the ith magnet unit is

$$\begin{aligned}\delta\theta_i &= \theta_i\left(\frac{1}{1+(\delta E/E)_i}-1\right) \\ &\approx \theta_i[-(\delta E/E)_i + (\delta E/E)_i^2/2 + \cdots].\end{aligned} \tag{12}$$

IV. DERIVATION OF FORMULAS

A. Expressions of dispersion

Position and angle deviations at the end of the linac up to the first order of the energy deviation are

$$\delta y_{f1} = \eta_1(\delta E/E)_f \approx \sum_i R_{12}(i\to f)\delta\theta_i, \tag{13}$$

$$\delta y'_{f1} = \eta'_1(\delta E/E)_f \approx \sum_i R_{22}(i\to f)\delta\theta_i, \tag{14}$$

then

$$\eta_1 = \sum_i(-\theta_i)R_{12}(i\to f)\frac{E_f}{E_i}, \tag{15}$$

$$\eta_1' = \sum_i (-\theta_i) R_{22}(i \to f) \frac{E_f}{E_i}, \tag{16}$$

where $R_{mn}(i \to f)$ is the m-n element of the transfer matrix from the ith magnet unit to the end of the linac, E_i is the designed beam energy at the ith magnet unit, and E_f is the final beam energy. δE for each particle is assumed to be constant along the linac.

The transfer matrix can be expressed as

$$R_{12}(i \to f) = \sqrt{\frac{E_i}{E_f}} \sqrt{\beta_i \beta_f} \sin\phi_{i,f} \tag{17}$$

and

$$R_{22}(i \to f) = \sqrt{\frac{E_i}{E_f}} \sqrt{\frac{\beta_i}{\beta_f}} (\cos\phi_{i,f} - \alpha_f \sin\phi_{i,f}), \tag{18}$$

where $\phi_{i,f}$ is the betatron phase advance from the ith magnet unit to the linac end.

We can take the second order term in Eq. (12) and have the second order dispersion and angle dispersion from this term as

$$\eta_{2a} = \sum_i \theta_i R_{12}(i \to f) \frac{E_f^2}{2E_i^2}, \tag{19}$$

$$\eta_{2a}' = \sum_i \theta_i R_{22}(i \to f) \frac{E_f^2}{2E_i^2}. \tag{20}$$

For the second order dispersion, we need to include another effect as follows. If the first order dispersion at the i_2th quadrupole magnet is η_{1,i_2}, its position dependent field induces second order dispersion. The position deviation at the magnet is $\eta_{1,i_2}(\delta E/E)_{i_2}$ and its quadrupole field kicks the particle as

$$\begin{aligned} &k_{i_2} \eta_{1,i_2} (\delta E/E)_{i_2} \frac{1}{1 + (\delta E/E)_{i_2}} \\ &\quad \approx k_{i_2} \eta_{1,i_2} (\delta E/E)_{i_2} - k_{i_2} \eta_{1,i_2} (\delta E/E)_{i_2}^2. \end{aligned} \tag{21}$$

The first term is already included, for the first order dispersions downstream, in Eqs. (15) and (16). The second term is the deviation of the kick angle at the magnet proportional to $(\delta E)^2$, which contributes second order dispersions downstream.

η_{1,i_2} is obtained by replacing f in Eq. (15) by i_2, and we will have the second order dispersion and angle dispersion at the linac end from this effect as

$$\eta_{2b} = \sum_{i_1,i_2} k_{i_2} (-\theta_{i_1}) R_{12}(i_1 \to i_2) R_{12}(i_2 \to f) \frac{E_f^2}{E_{i_1} E_{i_2}}, \tag{22}$$

$$\eta_{2b}' = \sum_{i_1,i_2} k_{i_2} (-\theta_{i_1}) R_{12}(i_1 \to i_2) R_{22}(i_2 \to f) \frac{E_f^2}{E_{i_1} E_{i_2}}, \tag{23}$$

where the summation is over magnet units with the constraint $i_1 < i_2$, or the i_2th magnet unit is downstream of the i_1th magnet unit.

Comparing Eq. (22) and (23) with Eqs. (19) and (20), η_{2b} and η_{2b}' have additional summations over the magnet units. Since we assume a long linac with a large number of magnets, η_{2b} and η_{2b}' are expected to be much larger than η_{2a} and η_{2a}', respectively, and we ignore η_{2a} and η_{2a}' in the following evaluations. Further calculations show that this approximation is good if the energy gain between two magnet units is much smaller than the initial beam energy, which is true in the case of the ILC main linac.

For higher order dispersions, we consider that $(n-1)$th order dispersion at a quadrupole magnet induces nth order dispersion downstream. Then, we have expressions for nth order dispersion and angle dispersion as

$$\begin{aligned} \eta_n \approx &\sum_{i_1,\ldots,i_n} (-\theta_{i_1}) \prod_{p=2}^{n} [k_{i_p} R_{12}(i_{p-1} \to i_p)] R_{12}(i_n \to f) \\ &\times \frac{E_f^n}{\prod_{p=1}^{n} E_{i_p}}, \end{aligned} \tag{24}$$

$$\begin{aligned} \eta_n' \approx &\sum_{i_1,\ldots,i_n} (-\theta_{i_1}) \prod_{p=2}^{n} [k_{i_p} R_{22}(i_{p-1} \to i_p)] R_{12}(i_n \to f) \\ &\times \frac{E_f^n}{\prod_{p=1}^{n} E_{i_p}}, \end{aligned} \tag{25}$$

where the summation is over magnet units with the constraint $i_1 < i_2 < \cdots < i_n$.

Using expressions for the transfer matrix like Eqs. (17) and (18),

$$\begin{aligned} \eta_n \approx &\sum_{i_1,\ldots,i_n} (-\theta_{i_1}) \frac{\sqrt{\beta_{i_1}}}{\sqrt{E_{i_1}}} \prod_{p=2}^{n} \left(\frac{k_{i_p} \beta_{i_p} \sin\phi_{i_{p-1},i_p}}{E_{i_p}} \right) \\ &\times E_f^{n-(1/2)} \sqrt{\beta_f} \sin\phi_{i_n,f}, \end{aligned} \tag{26}$$

$$\begin{aligned} \eta_n' \approx &\sum_{i_1,\ldots,i_n} (-\theta_{i_1}) \frac{\sqrt{\beta_{i_1}}}{\sqrt{E_{i_1}}} \prod_{p=2}^{n} \left(\frac{k_{i_p} \beta_{i_p} \sin\phi_{i_{p-1},i_p}}{E_{i_p}} \right) E_f^{n-(1/2)} \\ &\times \frac{1}{\sqrt{\beta_f}} (\cos\phi_{i_n,f} - \alpha \sin\phi_{i_n,f}), \end{aligned} \tag{27}$$

where $\phi_{i,j}$ denotes the betatron phase advance from the ith magnet unit to the jth unit.

From these, we have

$$\begin{aligned} \mathcal{H}_{nm} \approx &\sum_{i_1,\ldots,i_m} \sum_{j_1,\ldots,j_n} (\theta_{i_1} \theta_{j_1}) \frac{\sqrt{\beta_{i_1} \beta_{j_1}}}{\sqrt{E_{i_1} E_{j_1}}} \\ &\times \prod_{p=2}^{m} \left(\frac{k_{i_p} \beta_{i_p} \sin\phi_{i_{p-1},i_p}}{E_{i_p}} \right) \prod_{q=2}^{n} \left(\frac{k_{j_q} \beta_{j_q} \sin\phi_{j_{q-1},j_q}}{E_{j_q}} \right) \\ &\times \cos\phi_{i_m,j_n} E_f^{m+n-1}, \end{aligned} \tag{28}$$

and

$$\eta_m\eta_n' - \eta_m'\eta_n \approx \sum_{i_1,\ldots,i_m}\sum_{j_1,\ldots,j_n}(\theta_{i_1}\theta_{j_1})\frac{\sqrt{\beta_{i_1}\beta_{j_1}}}{\sqrt{E_{i_1}E_{j_1}}} \times \prod_{p=2}^{m}\left(\frac{k_{i_p}\beta_{i_p}\sin\phi_{i_{p-1},i_p}}{E_{i_p}}\right) \times \prod_{q=2}^{n}\left(\frac{k_{j_q}\beta_{j_q}\sin\phi_{j_{q-1},j_q}}{E_{j_q}}\right) \times \sin\phi_{i_m,j_n}E_f^{m+n-1}. \tag{29}$$

B. Expected emittance growth

We will evaluate the expected value (average of many linacs) of $\mathcal{H}_{mn}$, assuming the offset of error of each BPM is independent. We use

$$\langle a_i a_j\rangle = \delta_{ij}a^2, \tag{30}$$

where $\langle\,\rangle$ denote average over many sets of BPM offset errors and a the root mean square (rms) of the error.

Then, from Eqs. (11) and (28), expected $\mathcal{H}_{mn}$ due to BPM offset error can be expressed as

$$\langle\mathcal{H}_{mn}\rangle \approx \frac{a^2}{L_{qq}^2}\sum_{i_1,\ldots,i_m}\sum_{j_1,\ldots,j_n}\Delta_{i_1,j_1}\frac{\sqrt{\beta_{i_1}\beta_{j_1}}}{\sqrt{E_{i_1}E_{j_1}}} \times \prod_{p=2}^{m}\left(\frac{k_{i_p}\beta_{i_p}\sin\phi_{i_{p-1},i_p}}{E_{i_p}}\right) \times \prod_{q=2}^{n}\left(\frac{k_{j_q}\beta_{j_q}\sin\phi_{j_{q-1},j_q}}{E_{j_q}}\right)\cos\phi_{i_m,j_n}E_f^{m+n-1}. \tag{31}$$

Here we defined

$$\Delta_{i,j} \equiv \delta_{i-2,j} - 4\delta_{i-1,j} + 6\delta_{i,j} - 4\delta_{i+1,j} + \delta_{i+2,j}, \tag{32}$$

where $\delta_{i,j}$ is the Kronecker delta.

Also, from Eqs. (11) and (29), we have

$$\langle(\eta_k\eta_l' - \eta_l\eta_k')(\eta_m\eta_n' - \eta_n\eta_m')\rangle \approx \frac{a^4}{L_{qq}^4}\sum_{g_1,\ldots,g_k}\sum_{h_1,\ldots,h_l}\sum_{i_1,\ldots,i_m}\sum_{j_1,\ldots,j_n}(\Delta_{g_1,h_1}\Delta_{i_1,j_1} + \Delta_{g_1,i_1}\Delta_{h_1,j_1} + \Delta_{g_1,j_1}\Delta_{h_1,i_1})\frac{\sqrt{\beta_{g_1}\beta_{h_1}\beta_{i_1}\beta_{j_1}}}{\sqrt{E_{g_1}E_{h_1}E_{i_1}E_{j_1}}} \times \prod_{p=2}^{k}\left(\frac{k_{g_p}\beta_{g_p}\sin\phi_{g_{p-1},g_p}}{E_{g_p}}\right)\prod_{q=2}^{l}\left(\frac{k_{h_q}\beta_{h_q}\sin\phi_{h_{q-1},h_q}}{E_{h_q}}\right)\prod_{r=2}^{m}\left(\frac{k_{i_r}\beta_{i_r}\sin\phi_{i_{r-1},i_r}}{E_{i_r}}\right) \times \prod_{s=2}^{n}\left(\frac{k_{j_s}\beta_{j_s}\sin\phi_{j_{s-1},j_s}}{E_{j_s}}\right)\sin\phi_{g_k,h_l}\sin\phi_{i_m,j_n}E_f^{k+l+m+n-2}. \tag{33}$$

Here, we used an approximation,

$$\langle a_{g_1}a_{h_1}a_{i_1}a_{j_1}\rangle \approx \langle a_{g_1}a_{h_1}\rangle\langle a_{i_1}a_{j_1}\rangle + \langle a_{g_1}a_{i_1}\rangle\langle a_{h_1}a_{j_1}\rangle + \langle a_{g_1}a_{j_1}\rangle\langle a_{h_1}a_{i_1}\rangle. \tag{34}$$

Using Eqs. (3), (10), (31), and (33), we can calculate emittance growth, up to desired order of energy spread.

C. Analytic formula

Though the expressions in the previous subsection are suitable for numerical calculation using a computer, we can a derive simpler formula using some further approximations as follows.

Assumptions and approximations used in this article are listed in Table I.

The assumption “energy deviation, δE, is constant for each particle” is essential for our formula. It means energy gain of all particles are the same and the energy spread is dominantly determined as the initial condition. We will discuss possible modification of this assumption later.

We also assume the linac is with a uniform FODO lattice (iteration of focusing quadrupole magnet—drift space—defocusing quadrupole magnet—drift space), where beta function at every other magnet unit is β_F (at focusing quadrupole magnet) and at the others β_D (at defocusing quadrupole magnet), and betatron phase advance between magnet units is constant ($=\phi_{qq}$). We write the strength of quadrupole magnets as $k_F(<0)$ (focusing quadrupole magnet) and $k_D(>0)$ (defocusing quadrupole magnet).

Then, taking the average over many magnet units, we can make replacements,

$$\begin{aligned}\beta_i, \sqrt{\beta_i\beta_{i+2}} &\to (\beta_F + \beta_D)/2(\equiv\bar{\beta}),\\ \sqrt{\beta_i\beta_{i+1}} &\to \sqrt{\beta_F\beta_D}(\equiv\tilde{\beta}),\\ k_i\beta_i &\to (k_F\beta_F + k_D\beta_D)/2(\equiv\overline{k\beta}).\end{aligned} \tag{35}$$

TABLE I. List of assumptions and approximations for analytic formula.

Energy deviation, δE, is constant for each particle
Same mean square of offset error for all BPM, following a normal distribution
Uniform FODO lattice
Large number of components, for taking averages
Same energy gain in all cavities
Initial beam energy (E_0) $\gg$ energy gain between magnets (E_{qq})

We also assume that the energy gain between two magnet units is small compared with the beam energy and take approximation $E_{i\pm 2,i\pm 1} \approx E_i$ for any i. The summations can be replaced by integrations as

$$\sum_{i_1,\ldots,i_n} \to \frac{1}{E_{qq}^n} \int_{E_0}^{E_f} dE_{i_1} \int_{E_{i_1}}^{E_f} dE_{i_2} \cdots \int_{E_{i_{n-1}}}^{E_f} dE_{i_n}, \tag{36}$$

where E_{qq} is the energy gain between magnet units and E_0 the initial beam energy.

Then, the energy part in Eq. (31) becomes

$$\sum_{i_1,\ldots,i_m} \sum_{j_1,\ldots,j_n} \frac{\Delta_{i_1,j_1}}{\sqrt{E_{i_1}E_{j_1}}} \prod_{p=2}^{m}\left(\frac{1}{E_{i_p}}\right) \prod_{q=2}^{n}\left(\frac{1}{E_{j_q}}\right) E_f^{m+n-1} \to \frac{E_f^{m+n-1}}{E_{qq}^{m+n-1}} \int_{E_0}^{E_f} \frac{dE_{i_1}}{E_{i_1}} \int_{E_{i_1}}^{E_f} \frac{dE_{1_2}}{E_{1_2}} \cdots \int_{E_{i_{m-1}}}^{E_f} \frac{dE_{i_m}}{E_{i_m}} \int_{E_{i_1}}^{E_f} \frac{dE_{j_2}}{E_{j_2}} \cdots \int_{E_{j_{n-1}}}^{E_f} \frac{dE_{j_n}}{E_{j_n}} \Delta_{i_1,j_1}$$
$$= \frac{\Delta_{i_1,j_1}}{(m-1)!(n-1)!(m+n+1)} \left(\frac{E_f \log\frac{E_f}{E_0}}{E_{qq}}\right)^{m+n-1}. \tag{37}$$

Similarly, for the energy part in Eq. (33), we have

$$\sum_{g_1,\ldots,g_k} \sum_{h_1,\ldots,h_l} \sum_{i_1,\ldots,i_m} \sum_{j_1,\ldots,j_n} \frac{\Delta_{g_1,h_1}\Delta_{i_1,j_1}}{\sqrt{E_{g_1}E_{h_1}E_{i_1}E_{j_1}}} \prod_{p=2}^{k}\left(\frac{1}{E_{g_p}}\right) \prod_{q=2}^{l}\left(\frac{1}{E_{h_q}}\right) \prod_{r=2}^{m}\left(\frac{1}{E_{i_r}}\right) \prod_{s=2}^{n}\left(\frac{1}{E_{j_s}}\right) E_f^{k+l+m+n-2}$$
$$\to \frac{\Delta_{g_1,h_1}\Delta_{i_1,j_1}}{(k-1)!(l-1)!(m-1)!(n-1)!(k+l+1)(m+n+1)} \left(\frac{E_f \log\frac{E_f}{E_0}}{E_{qq}}\right)^{k+l+m+n-2}. \tag{38}$$

For the part of trigonometric functions, considering summations over many different phases, we ignore summation of oscillating terms as

$$\sum_j \sin\phi_{i,j}\sin\phi_{j,k} = \sum_j \sin(\psi_j - \psi_i)\sin(\psi_k - \psi_j) = \sum_j \tfrac{1}{2}[-\cos(\psi_k - \psi_i) + \cos(\psi_k + \psi_i - 2\psi_j)] \to -\tfrac{1}{2}\cos\phi_{i,k}\sum_j \tag{39}$$

and

$$\sum_j \sin\phi_{i,j}\cos\phi_{j,k} = \sum_j \sin(\psi_j - \psi_i)\cos(\psi_k - \psi_j) \quad = \sum_j \tfrac{1}{2}[\sin(\psi_k - \psi_i) - \sin(\psi_k + \psi_i - 2\psi_j)] \to \tfrac{1}{2}\sin\phi_{i,k}\sum_j. \tag{40}$$

Here, ψ_j denotes the betatron phase at the jth magnet unit.

Applying the above replacements subsequently to the part of trigonometric functions in Eq. (31),

$$\sum_{i_1,\ldots,i_m} \sum_{j_1,\ldots,j_n} \prod_{p=2}^{m}(\sin\phi_{i_{p-1},i_p}) \prod_{q=2}^{n}(\sin\phi_{j_{q-1},j_q})\cos\phi_{i_m,j_n} \to -(-1)^{\frac{n+m}{2}}\left(\frac{1}{2}\right)^{m+n-2}\cos\phi_{i_1,j_1} \sum_{i_1,\ldots,i_m}\sum_{j_1,\ldots,j_n} \quad (\text{if } n+m = \text{even}). \tag{41}$$

Note that $\mathcal{H}_{mn}$ with odd $m+n$ does not contribute to the emittance growth, from Eqs. (3) and (10).

Similarly, for the part of trigonometric functions in Eq. (33) we use

$$\sum_{g_1,\ldots,g_k} \sum_{h_1,\ldots,h_l} \prod_{p=2}^{k}(\sin\phi_{g_{p-1},g_p}) \prod_{q=2}^{l}(\sin\phi_{h_{q-1},h_q})\sin\phi_{g_k,h_l} \to \begin{cases} -(-1)^{\frac{k+l}{2}}\left(\frac{1}{2}\right)^{k+l-2}\sin\phi_{g_1,h_1}\sum_{g_1,\cdots,g_k}\sum_{h_1,\cdots,h_l} & [(k,l) = (\text{odd},\text{odd})] \\ -(-1)^{\frac{k+l-1}{2}}\left(\frac{1}{2}\right)^{k+l-2}\cos\phi_{g_1,h_1}\sum_{g_1,\cdots,g_k}\sum_{h_1,\cdots,h_l} & [(k,l) = (\text{odd},\text{even})] \\ (-1)^{\frac{k+l-1}{2}}\left(\frac{1}{2}\right)^{k+l-2}\cos\phi_{g_1,h_1}\sum_{g_1,\cdots,g_k}\sum_{h_1,\cdots,h_l} & [(k,l) = (\text{even},\text{odd})] \\ (-1)^{\frac{k+l}{2}}\left(\frac{1}{2}\right)^{k+l-2}\sin\phi_{g_1,h_1}\sum_{g_1,\cdots,g_k}\sum_{h_1,\cdots,h_l} & [(k,l) = (\text{even},\text{even})]. \end{cases} \tag{42}$$

Now we can derive an expression for the average of $\mathcal{H}_{mn}$ as

$$\langle \mathcal{H}_{mn} \rangle \approx 4\frac{a^2}{L_{qq}^2} F(m,n) \left(\frac{E_f \log\frac{E_f}{E_0}}{E_{qq}} \right)^{m+n-1} (\overline{k\beta})^{m+n-2} [\bar{\beta}(1+\cos^2\phi_{qq}) - 2\tilde{\beta}\cos\phi_{qq}] \quad (n+m = \text{even}), \tag{43}$$

where we defined

$$F(m,n) \equiv \left(\frac{1}{2}\right)^{m+n-2} \frac{-(-1)^{\frac{m+n}{2}}}{(m+n-1)(m-1)!(n-1)!}, \tag{44}$$

and used

$$\sum_{i_1,j_1} \Delta_{i_1,j_1} \sqrt{\beta_{i_1}\beta_{j_1}} \cos\phi_{i_1,j_1} \to 4[\bar{\beta}(1+\cos^2\phi_{qq}) - 2\tilde{\beta}\cos\phi_{qq}] \sum_{i_1,j_1}. \tag{45}$$

From a little more manipulations, we also have

$$\langle (\eta_k\eta_l' - \eta_l\eta_k')(\eta_m\eta_n' - \eta_n\eta_m') \rangle \approx 0 \quad [\text{if } (k+l, m+n) = (\text{even}, \text{even})] \tag{46}$$

and

$$\begin{aligned} \langle (\eta_k\eta_l' - \eta_l\eta_k')(\eta_m\eta_n' - \eta_n\eta_m') \rangle \approx{}& (-1)^{k+n} 8\frac{a^4}{L_{qq}^4} \left(\frac{E_f \log\frac{E_f}{E_0}}{E_{qq}} \right)^{k+l+m+n-2} (\overline{k\beta})^{k+l+m+n-4} \{2F(k,l)F(m,n) \\ &+ F(k,n)F(m,l) + F(k,m)F(l,n)\} [\bar{\beta}(1+\cos^2\phi_{qq}) - 2\tilde{\beta}\cos\phi_{qq}]^2 \\ &[\text{if } (k+l, m+n) = (\text{odd}, \text{odd})], \end{aligned} \tag{47}$$

where we used Eq. (45) and

$$\Delta_{g_1,i_1}\Delta_{h_1,j_1}\sqrt{\beta_{g_1}\beta_{h_1}\beta_{i_1}\beta_{j_1}} \sin\phi_{g_1,h_1} \sin\phi_{i_1,j_1} = -\Delta_{g_1,j_1}\Delta_{h_1,i_1}\sqrt{\beta_{g_1}\beta_{h_1}\beta_{i_1}\beta_{j_1}} \sin\phi_{g_1,h_1} \sin\phi_{i_1,j_1} \tag{48}$$

and

$$\begin{aligned} \sum_{g_1,h_1,i_1,j_1} \Delta_{g_1,i_1}\Delta_{h_1,j_1}\sqrt{\beta_{g_1}\beta_{h_1}\beta_{i_1}\beta_{j_1}} \cos\phi_{g_1,h_1} \cos\phi_{i_1,j_1} &= \sum_{g_1,h_1,i_1,j_1} \Delta_{g_1,j_1}\Delta_{h_1,i_1}\sqrt{\beta_{g_1}\beta_{h_1}\beta_{i_1}\beta_{j_1}} \cos\phi_{g_1,h_1} \cos\phi_{i_1,j_1} \\ &\to 8[\bar{\beta}(1+\cos^2\phi_{qq}) - 2\tilde{\beta}\cos\phi_{qq}]^2. \end{aligned} \tag{49}$$

Note that only terms of $k+l+m+n = \text{even}$ contribute the emittance growth, from Eqs. (3) and (10).

Now, we have an analytical expression of average of emittance growth, from Eq. (3), substituting Eqs. (10), (43), and (47) for $\overline{(\delta E/E)^N}$, $\mathcal{H}_{mn}$, and $(\eta_k\eta_l' - \eta_l\eta_k') \times (\eta_m\eta_n' - \eta_n\eta_m')$, respectively.

D. Effect of second order dispersion to orbit correction

So far, we used Eq. (11) as the kick angle at each magnet unit, assuming the orbit of the on-momentum (designed energy) particle is steered as going through centers of all BPMs. However, orbit correction is based on the position of center of mass of the beam, which will be distorted by even order dispersions as

$$\bar{y} = \bar{y}_0 + \sum_N \eta_{2N} \overline{(\delta E/E)^{2N}}. \tag{50}$$

From here, we consider the lowest order of this effect ($N = 1$).

Additional kick angle due to second order dispersion at the ith magnet unit is, similar to Eq. (11), expressed as

$$\Delta\theta_i = \frac{-\Delta y_{i+1} + 2\Delta y_i - \Delta y_{i-1}}{L_{qq}}, \tag{51}$$

where deviation of the center of mass of the beam at i is

$$\Delta y_i \approx \eta_{2,i}(\sigma_E/E_i)^2. \tag{52}$$

Similarly to Eqs. (15) and (16), this additional kick induces additional dispersion and angle dispersion at the end of the linac as

$$\Delta\eta_1 = \sum_i (-\Delta\theta_i) R_{12}(i \to f) \frac{E_f}{E_i}, \tag{53}$$

$$\Delta\eta_1' = \sum_i (-\Delta\theta_i) R_{22}(i \to f) \frac{E_f}{E_i}. \tag{54}$$

And, from Eq. (22) (replacing f by i_3, $i_3 \pm 1$),

$$\Delta\eta_1 = \sum_{i_1,i_2,i_3} \frac{1}{L_{qq}}(-\theta_{i_1})R_{12}(i_1 \to i_2)\frac{\sigma_E^2 E_f}{E_{i_1}E_{i_2}E_{i_3}} k_{i_2}[-R_{12}(i_2 \to i_3+1) + 2R_{12}(i_2 \to i_3) - R_{12}(i_2 \to i_3-1)]R_{12}(i_3 \to f), \tag{55}$$

$$\Delta\eta_1' = \sum_{i_1,i_2,i_3} \frac{1}{L_{qq}}(-\theta_{i_1})R_{12}(i_1 \to i_2)\frac{\sigma_E^2 E_f}{E_{i_1}E_{i_2}E_{i_3}} k_{i_2}[-R_{12}(i_2 \to i_3+1) + 2R_{12}(i_2 \to i_3) - R_{12}(i_2 \to i_3-1)]R_{22}(i_3 \to f). \tag{56}$$

For the lowest order of energy spread to the emittance growth, we modify $\mathcal{H}_{11}$ as

$$\begin{aligned}\mathcal{H}_{11} + \Delta\mathcal{H}_{11} &\equiv \gamma_y(\eta_1+\Delta\eta_1)^2 + 2\alpha_y(\eta_1+\Delta\eta_1)(\eta_1'+\Delta\eta_1') + \beta_y(\eta_1'+\Delta\eta_1')^2 \\ &\approx \mathcal{H}_{11} + 2[\gamma_y\eta_1\Delta\eta_1 + \alpha_y(\eta_1\Delta\eta_1' + \eta_1'\Delta\eta_1) + \beta_y\eta_1'\Delta\eta_1'].\end{aligned} \tag{57}$$

Equation (3), which is expressing emittance growth, should have an additional term as

$$(\Delta\epsilon^2)_3 = \epsilon_0\Delta\mathcal{H}_{11}(\sigma_E/E_f)^2. \tag{58}$$

From the above expressions, the average of $\Delta\mathcal{H}_{11}$ can be evaluated as

$$\begin{aligned}\langle\Delta\mathcal{H}_{11}\rangle &\approx 2\sum_{i_1,i_2,i_3,j}\frac{\langle\theta_{i_1}\theta_j\rangle}{L_{qq}}\frac{\sqrt{\beta_{i_1}\beta_j}}{E_{i_1}E_j}\frac{k_{i_2}\beta_{i_2}}{E_{i_2}E_{i_3}}\sigma_E^2E_f\sin\phi_{i_1,i_2}\sin\phi_{i_2,i_3}\Big(-\sqrt{\beta_{i_3}\beta_{i_3-1}}\cos\phi_{j,i_3-1} + 2\beta_{i_3}\cos\phi_{j,i_3} \\ &\quad - \sqrt{\beta_{i_3}\beta_{i_3+1}}\cos\phi_{j,i_3+1}\Big) \\ &= 2\sum_{i_1,i_2,i_3,j}\frac{a^2}{L_{qq}^3}\Delta_{i_1,j}\frac{\sqrt{\beta_{i_1}\beta_j}}{E_{i_1}E_j}\frac{k_{i_2}\beta_{i_2}}{E_{i_2}E_{i_3}}\sigma_E^2E_f\sin\phi_{i_1,i_2}\sin\phi_{i_2,i_3}\Big(-\sqrt{\beta_{i_3}\beta_{i_3-1}}\cos\phi_{j,i_3-1} + 2\beta_{i_3}\cos\phi_{j,i_3} \\ &\quad - \sqrt{\beta_{i_3}\beta_{i_3+1}}\cos\phi_{j,i_3+1}\Big).\end{aligned} \tag{59}$$

Using further approximations and some manipulations, we have an expression for the expected $\Delta\mathcal{H}_{11}$ as

$$\langle\Delta\mathcal{H}_{11}\rangle \approx -\frac{8}{3}\frac{a^2}{L_{qq}^3}\frac{\sigma_E^2}{E_f^2}\left(\frac{\log\frac{E_f}{E_0}}{E_{qq}}\right)^3\overline{k\beta}(\bar\beta - \tilde\beta\cos\phi_{qq})[\bar\beta(1+\cos^2\phi_{qq}) - 2\tilde\beta\cos\phi_{qq}]. \tag{60}$$

E. Summary of the formula

Let us write down the obtained formula, which is from Eqs. (3), (10), (43), (47), and (60). Defining

$$\begin{aligned}G(m,n) &\equiv \frac{(m+n-1)!}{2^{(m+n)/2-1}[(m+n)/2-1]!} - \frac{(m-1)!}{2^{m/2-1}(m/2-1)!}\frac{(n-1)!}{2^{n/2-1}(n/2-1)!} \quad [(m,n) = (\text{even},\text{even})] \\ &\equiv \frac{(m+n-1)!}{2^{(m+n)/2-1}[(m+n)/2-1]!} \quad [(m,n) = (\text{odd},\text{odd})] \\ &\equiv 0 \quad (m+n = \text{odd}),\end{aligned} \tag{61}$$

the formula is

$$\langle\epsilon^2\rangle - \epsilon_0^2 \approx (\Delta\epsilon^2)_1 + (\Delta\epsilon^2)_2 + (\Delta\epsilon^2)_3, \tag{62}$$

$$(\Delta\epsilon^2)_1 = \epsilon_0\sum_{m,n(m+n=\text{even})}4\frac{a^2}{L_{qq}^2}F(m,n)G(m,n)\left(\sigma_E\frac{\log\frac{E_f}{E_0}}{E_{qq}}\right)^{m+n-1}\frac{\sigma_E}{E_f}(\overline{k\beta})^{m+n-2}[\bar\beta(1+\cos^2\phi_{qq}) - 2\tilde\beta\cos\phi_{qq}], \tag{63}$$

$$(\Delta\epsilon^2)_2 = 4\sum_{k,l,(k+l=\mathrm{odd})}\sum_{m,n,(m+n=\mathrm{odd})}(-1)^{k+n}\frac{a^4}{L_{qq}^4}G(k,m)G(l,n)\left(\frac{\sigma_E\log\frac{E_f}{E_0}}{E_{qq}}\right)^{k+l+m+n-2}\frac{\sigma_E^2}{E_f^2}(\overline{k\beta})^{k+l+m+n-4}\{2F(k,l)F(m,n) + F(k,n)F(m,l) + F(k,m)F(l,n)\}[\bar{\beta}(1+\cos^2\phi_{qq}) - 2\tilde{\beta}\cos\phi_{qq}]^2, \tag{64}$$

$$(\Delta\epsilon^2)_3 = -\epsilon_0\frac{8}{3}\frac{a^2}{L_{qq}^3}\frac{\sigma_E}{E_f}\left(\sigma_E\frac{\log\frac{E_f}{E_0}}{E_{qq}}\right)^3\overline{k\beta}(\bar{\beta} - \tilde{\beta}\cos\phi_{qq})[\bar{\beta}(1+\cos^2\phi_{qq}) - 2\tilde{\beta}\cos\phi_{qq}]. \tag{65}$$

The function F is defined as Eq. (44).

$(\Delta\epsilon^2)_1$ and $(\Delta\epsilon^2)_2$ are summations of series proportional to $[\sigma_E\log(E_f/E_0)/E_{qq}]^{2N}$ $(N = 1, 2, \ldots)$. Therefore, in cases $\sigma_E\log(E_f/E_0)/E_{qq} \ll 1$, taking some lower terms is a good approximation. For the ILC main linac this factor is about 0.23.

F. Possibility of modifications

For the formulas, we made some assumptions as in Table I. Let us consider if we can extend the formula to more general cases.

One assumption we used for the analytic formulas is a simple FODO lattice. It will not be difficult applying our method for other lattice designs. For example, in the case β_F and β_D are not constant but simple functions of beam energy, we will obtain formulas with modified energy integrations. For more complicated lattice designs, instead of using analytic formulas, we can use Eqs. (31), (33), and (59), for numerical calculations. Such calculations do not need Monte Carlo simulations and will be much faster than tracking simulations.

The essential approximation for our formula is "energy deviation, δE, is constant for each particle." It means energy gain of all particles are the same and the energy spread is dominantly determined as the initial condition. This is a good approximation for the ILC main linac using superconducting accelerating cavities, with weak longitudinal wakefield and relatively low rf frequency, which make longitudinal position dependent energy gain very small. However, this assumption may not be used for most accelerators using normal conducting cavities, because of strong wakefield and high rf frequencies (significant slope of rf voltage in a beam bunch).

We can consider cases in which the opposite extreme assumption is valid, where the initial energy spread can be ignored and the energy spread is dominantly induced in the acceleration. The deviation of the energy gain of a particle (from longitudinal wakefield and rf voltage slope) will depend only on the relative longitudinal position in a beam bunch, and we may assume δE of a particle at any location is proportional to the design energy at the location. Modification of our formulas with such an assumption is straightforward, by replacing δE at location i by $(\delta E/E)E_i$, assuming $(\delta E/E)$ is constant for each particle. We will have different dependences on the final beam energy.

The evaluation will be more complicated if the initial energy spread and the induced energy spread are comparable. In such cases, it will be difficult to derive simple analytic formulas. However, we will still have formulas for numerical calculations by replacing δE at location i by $\delta E_{\mathrm{init}} + (\delta E/E)_{\mathrm{ind}}E_i$, where δE_{init} (initial energy deviation) and $(\delta E/E)_{\mathrm{ind}}$ (induced relative energy deviation) are constant for each particle.

G. Possibility of extension for DFS

In this paper, we assumed only a simple orbit correction method (one to one steering). Extending our formulas for more complicated corrections will need further studies. However, there is an idea of extension of our method for dispersion-free steering (DFS) as follows.

Assuming DFS, the first order dispersion will be measured at every BPM location. Therefore, the first order dispersion will be corrected with accuracy determined by the error of dispersion measurement. Residual first order dispersion at a quadrupole magnet will induce the second order dispersion downstream (just like orbit offset at a quadrupole magnet induces the first order dispersion). Then, the second and higher order dispersion at the end of a linac assuming DFS will be expressed as a function of the errors of dispersion measurement. This is just like the formula that we obtained, which is expressing the first and higher order dispersion assuming one to one steering as a function of the BPM offset errors. In such a way, a formula of emittance growth including the higher order dispersion will be derived.

V. COMPARISON WITH TRACKING SIMULATIONS

For checking validity of the derived formulas, tracking simulations are performed for the ILC main linac, setting similar conditions to Ref. [1]. The simulation code SLEPT [6] was used. Relevant parameters of the ILC main linac are listed in Table II [7]. For simplicity, we simulated a straight linac, while the actual ILC main linac is designed to be curved following the earth's curvature.

Though the code SLEPT could include various sources of emittance dilution, the following effects were turned off in the tracking simulations for comparing with the formulas; transverse wakefield, tilting misalignment of cavities, and transverse kicks in accelerating cavities due to geometrical

TABLE II. Relevant parameters of the ILC main linac.

Initial beam energy	E_0	15 GeV
Final beam energy	E_f	250 GeV
Initial energy spread	σ_E	0.16 GeV
Initial normalized emittance	$(\gamma\epsilon_y)$	2×10^{-8} m
β at focusing magnet	β_F	$\sim$140 m
β at defocusing magnet	β_D	$\sim$40 m
Strength of focusing magnet	k_F	-0.0286 m^{-1}
Strength of defocusing magnet	k_D	0.0320 m^{-1}
Acceleration per cavity	V_c	$\sim$32 MV
Energy gain between quad magnets	E_{qq}	$eV_c \times 26$

asymmetries of couplers. On the other hand, effects of bunch length, longitudinal position dependent energy differences which are induced during acceleration, were included in the tracking simulations, while our formulas ignore the effects. Also, it is obvious that the tracking simulation included many orders of dispersion (the highest order is the same as the number of quadrupole magnets in the linac).

For taking average, the simulation for each condition was performed with 1000 different sets of random numbers (random seeds).

Root mean square (rms) of BPM offset error was set as 50 μm as nominal error.

First, we compared contributions of several lowest orders of σ_E^2 to $(\Delta\epsilon^2)_1$ and $(\Delta\epsilon^2)_2$. Figure 1 shows contributions of $(\Delta\epsilon^2)_1$ and $(\Delta\epsilon^2)_2$ of each order of σ_E^2, to the $\langle(\gamma\epsilon)^2\rangle$ (square of normalized emittance) growth. It shows that it will be good enough to take three lowest orders for each of $(\Delta\epsilon^2)_1$ and $(\Delta\epsilon^2)_2$. The figure also shows the contribution of $(\Delta\epsilon^2)_3$ (we only evaluated the lowest order), which cannot be ignored. In the following calculations, we take up to σ_E^6 order for $(\Delta\epsilon^2)_1$ and up to σ_E^{10} order for $(\Delta\epsilon^2)_2$.

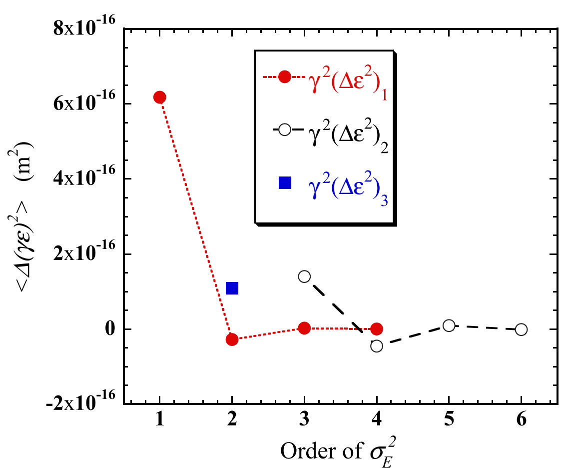

FIG. 1. Contribution of each order of σ_E^2 of $(\Delta\epsilon^2)_1$, $(\Delta\epsilon^2)_2$, and $(\Delta\epsilon^2)_3$ to expected growth of square of normalized emittance $[\langle(\gamma\epsilon)^2\rangle]$, with BPM offset error 50 μm (rms) after one to one correction in the ILC main linac.

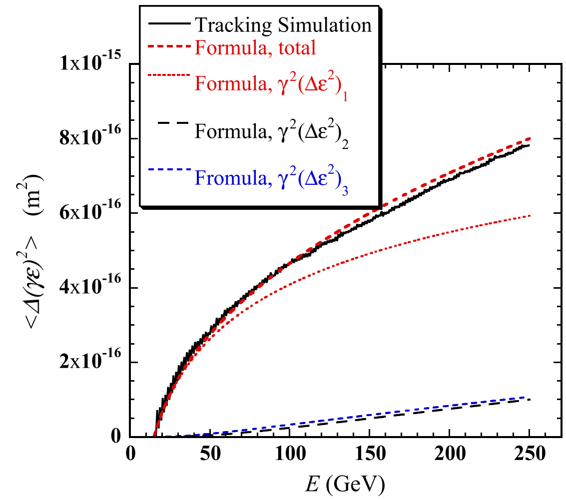

FIG. 2. Average growth of square of normalized emittance $[\langle(\gamma\epsilon)^2\rangle]$ with BPM offset error 50 μm (rms) after one to one correction, as a function of beam energy along the ILC main linac. Results of tracking simulation and the analytic formula. Contributions of $(\Delta\epsilon^2)_1$, $(\Delta\epsilon^2)_2$, and $(\Delta\epsilon^2)_3$ are also shown.

In Fig. 2, we compare results of tracking simulations and the formula for square of emittance growth along the linac, starting from the beam energy 15 GeV. The contribution of each of $(\Delta\epsilon^2)_1$, $(\Delta\epsilon^2)_2$, and $(\Delta\epsilon^2)_3$ is also shown.

Figure 3 shows the square of emittance growth as a function of initial beam energy. For higher initial beam energy, the downstream part of the linac is used, keeping the energy gain per length constant. The final beam energy is 250 GeV.

We also performed tracking simulations with different initial energy spread and rms of BPM misalignment. Figure 4 shows the average square of emittance growth at

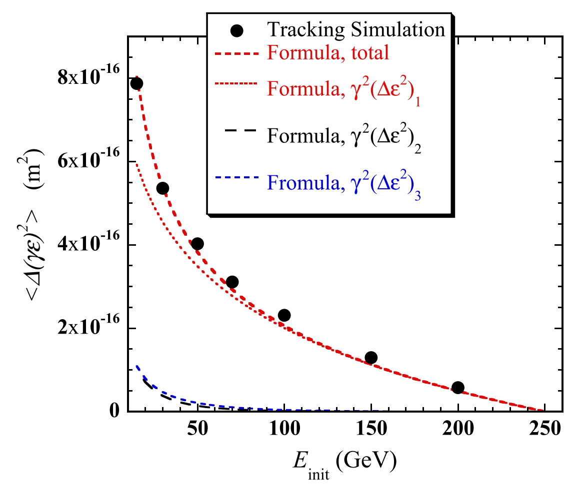

FIG. 3. Average growth of square of emittance $(\langle(\gamma\epsilon)^2\rangle)$ at the linac end, with BPM offset error 50 μm (rms) after one to one correction, as a function of the initial beam energy for the downstream part of the ILC main linac. Results of tracking simulation (circle), and the analytic formula. Contributions of $(\Delta\epsilon^2)_1$, $(\Delta\epsilon^2)_2$, and $(\Delta\epsilon^2)_3$ are also shown.

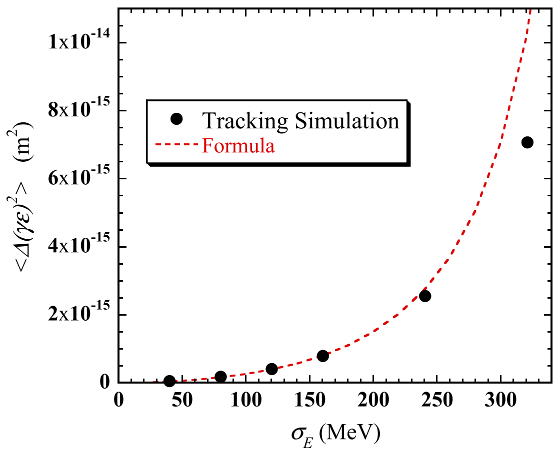

FIG. 4. Average growth of square of emittance $[\langle(\gamma\epsilon)^2\rangle]$ at the linac end, with BPM offset error 50 μm (rms) after one to one correction, as a function of the initial beam energy spread. Results of tracking simulation (circle), and the analytic formula.

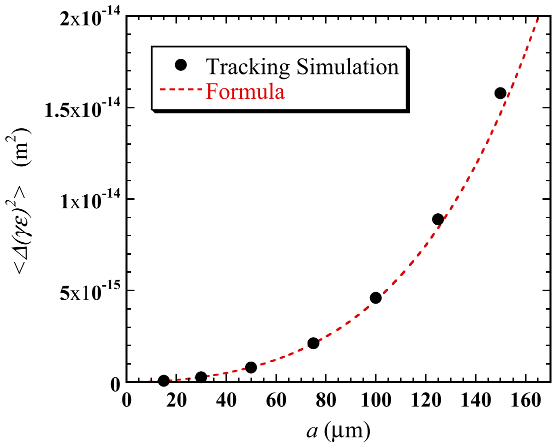

FIG. 5. Average growth of square of emittance $[\langle(\gamma\epsilon)^2\rangle]$ at the linac end after one to one correction as a function of BPM offset error (rms). Results of tracking simulation (circle), and the analytic formula.

the linac end as a function of the initial energy spread, comparing with the result of the formulas. Figure 5 shows the average square of emittance growth at the linac end as a function of the rms of BPM misalignment, comparing with result of the formulas.

The only significant discrepancy between the tracking simulations and the formulas can be seen for the highest energy spread point of Fig. 4. This discrepancy is understandable because, for this point, the quantity $\sigma_E \log(E_f/E_0)/E_{qq} \approx 1$, which no longer satisfies the condition that it be small compared to 1.

Except for this, all of the results confirmed good agreement between the analytic formulas and the tracking simulations.

VI. SUMMARY AND DISCUSSIONS

Formulas of emittance growth after a simple orbit correction (one to one steering) with random offset of beam position monitors were derived. They were from extensions of the formulas of orbit distortion and emittance growth due to random misalignment of quadrupole magnets and random tilting of accelerating cavities without corrections, which had been reported in our past paper [1]. The results were compared with tracking simulations, showing good agreement.

In the formulas, we expand expected growth of emittance square by σ_E^{2N} ($N = 1, 2, 3, \ldots$). Taking lower orders will be a good approximation in cases $[\sigma_E \log(E_f/E_0)/E_{qq}]^2 \ll 1$.

For our formulas, we made some assumptions shown in Table I. The possibility of modifications of the assumptions was discussed.

The possibility of extension of our method for a more complicated correction (DFS) was also discussed.

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